LECTURE NOTES ON INTERMEDIATE FLUID MECHANICS

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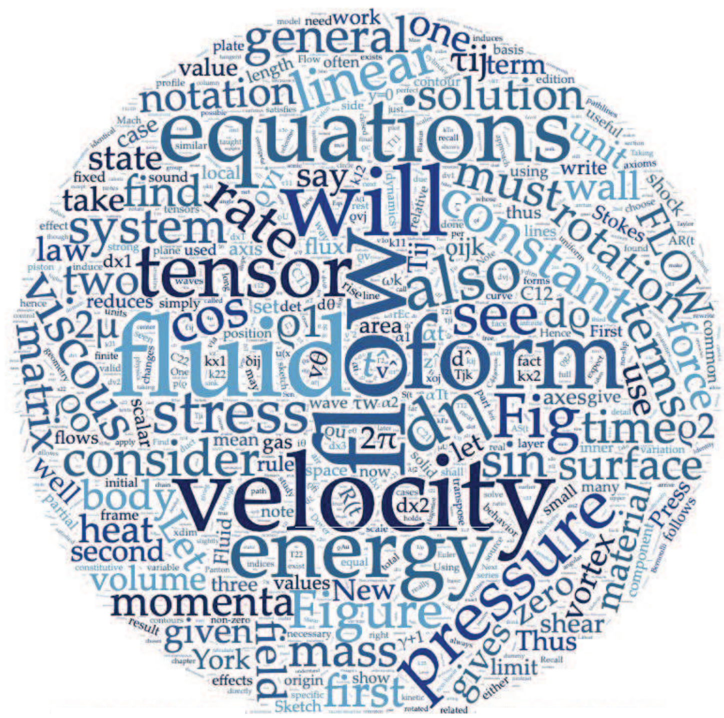

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Preface

These are lecture notes for AME 60635, Intermediate Fluid Mechanics, taught in the Department of Aerospace and Mechanical Engineering of the University of Notre Dame. Most students are beginning graduate students and advanced engineering undergraduates. Typically they have completed one undergraduate fluids course as well as courses in linear algebra and differential equations. The course provides a survey of continuum fluid mechanics. Part I gives an extensive development of the compressible Navier-Stokes equations. Part II focuses on their solution in various limits: vorticity dynamics, compressible flow, potential flow, and viscous laminar flow. The emphasis is on fluid physics and the mathematics necessary to efficiently describe the physics. The notes make no attempt to address three important topics: 1) discrete computational models of the continuum physics, 2) turbulent fluid mechanics, or 3) molecular dynamics. The notes do provide the foundation for later courses that address computational fluid dynamics (CFD) as well as turbulence; courses that address the molecular nature of fluids are enlightened by understanding of the continuum limit.

While there is rigor in the development, it is not absolute. The student should call on other sources for a full description. Much of the development and notation follows Panton (2013), who gives a clear presentation. Other material is drawn from a variety of sources. A detailed bibliography is provided. The notes, along with information on the course, can be found at `https://www3.nd.edu/~powers/ame.60635`. At this stage, anyone is free to duplicate the notes.

The notes have been transposed from written notes I composed in developing this course in 1992 and a related course in viscous fluid flow in 1991. Many enhancements have been made, and thanks go to many students and faculty who have pointed out errors. It is likely that there are more waiting to be discovered; I would be happy to hear from you regarding these or suggestions for improvement.

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Part I

Governing equations: development

Chapter 1

Philosophy of rational continuum mechanics

see Truesdell, Chapter 1,
see Panton, Chapter 1,
see Paolucci, Chapter 1,
see Eringen, Introduction.

We will study in these notes the mechanics of a *fluid*, defined as a material that continuously deforms under the influence of an applied shear stress. Such a definition allows both liquids and gases to be considered fluids. We seek to present an approach to *fluid mechanics* founded on the general principles of *rational continuum mechanics*. These general principles apply to all continuous materials: solids, liquids, and gases. The first four chapters will be quite general and may be applied to all continuous materials. The remaining chapters are specific to fluids.

There are many paths to understanding fluid mechanics, and good arguments can be made for each. A typical first undergraduate class will combine a mix of basic equations, coupled with strong physical motivations, and allows the student to develop a knowledge that is of great practical value, often driven strongly by intuition. Such an approach works well within the confines of the intuition we develop in everyday life. It often fails when the engineer moves into unfamiliar territory. For example, lack of fundamental understanding of high Mach number flows led to many aircraft and rocket failures in the 1950's. In such cases, a return to the formalism of a careful theory, one that clearly exposes the strengths and weaknesses of all assumptions, is invaluable in both understanding the true fluid physics, and applying that knowledge to engineering design.

Probably the most formal of approaches is that of the school of thought advocated most clearly by Truesdell,[1] who forcefully advocated for rational continuum mechanics. Truesdell developed a broadly based theory that encompassed all materials that could be regarded

[1]Clifford Ambrose Truesdell, III, 1919-2000, American continuum mechanician and natural philosopher. Taught at Indiana and Johns Hopkins Universities.

as continua, including solids, liquids, and gases, in the limit when averaging volumes were sufficiently large so that the micro- and nanoscopic structure of these materials was unimportant. For fluids (both liquid and gas), such length scales are often on the order of microns, while for solids, it may be somewhat smaller, depending on the type of molecular structure. The difficulty of the Truesdellian approach is that it is burdened with a difficult notation and tends to become embroiled in proofs and philosophy, that while ultimately useful, can preclude learning basic fluid mechanics in the time scale of the human lifetime.

In this course, we will attempt to steer between the fallible pragmatism of undergraduate fluid mechanics and the harsh formalism of the Truesdellian school. The material will pay some due homage to rational continuum mechanics and will be geared towards a basic understanding of fluid behavior. We shall first spend some time carefully developing the governing equations for a compressible viscous fluid. We shall then study representative solutions of these equations in a wide variety of physically motivated limits in order to understand how the basic conservation principles of mass, linear momenta,[2] angular momenta, and energy, coupled with constitutive relations, influence the behavior of fluids.

1.1 Mechanics

Mechanics is the broad superset of the topic matter of this course. Mechanics is the science that seeks an explanation for the motion of bodies based upon models grounded in well defined axioms. Axioms, as in geometry, are statements that cannot be proved; they are useful insofar as they give rise to results that are consistent with our empirical observations. A hallmark of science has been the struggle to identify the smallest set of axioms that are sufficient to describe our universe. When we find an axiom to be inconsistent with observation, it must be modified or eliminated. A familiar example of this is the Michelson-[3] Morley[4] experiment, that motivated Einstein[5] to modify the Newtonian[6] axioms of conservation of mass and energy into a conservation of mass-energy.

In Truesdell's exposition on mechanics, he suggests the following hierarchy:

- *bodies* exist,

[2]Throughout these notes, we use the less common plural "momenta" as a reminder that in our three-dimensional world, there are three scalar components of the singular "momentum."

[3]Albert Abraham Michelson, 1852-1931, Prussian born American physicist, graduate of the U.S. Naval Academy and faculty member at Case School of Applied Science, Clark University, and University of Chicago.

[4]Edward Williams Morley, 1838-1923, New Jersey-born American physical chemist, graduate of Williams College, professor of chemistry at Western Reserve College.

[5]Albert Einstein, 1879-1955, German physicist who developed the theory of relativity and made fundamental contributions to quantum mechanics and Brownian motion in fluid mechanics; spent later life in the United States.

[6]Sir Isaac Newton, 1642-1727, English physicist and mathematician and chief figure of the scientific revolution of the seventeenth and eighteenth centuries. Developed calculus, theories of gravitation and motion of bodies, and optics. Educated at Cambridge University and holder of the Lucasian chair at Cambridge. In civil service as Warden of the Mint, he became the terror of counterfeiters, sending many to the gallows.

- bodies are assigned to *place*,
- *geometry* is the theory of place,
- change of place in *time* is the *motion* of the body,
- a description of the motion of a body is *kinematics*,
- motion is the consequence of *forces*,
- study of forces on a body is *dynamics*.

There are many subsets of mechanics, e.g. statistical mechanics, quantum mechanics, continuum mechanics, fluid mechanics, or solid mechanics. Auto mechanics, while a legitimate topic for study, does not generally fall into the class of mechanics we consider here, though the intersection of the two sets is not the empty set.

1.2 Continuum mechanics

Early mechanicians, such as Newton, dealt primarily with point masses and finite collections of particles. In one sense this is because such systems are the easiest to study, and it makes more sense to grasp the simple before the complex. External motivation was also present in the 18th century, that had a martial need to understand the motion of cannonballs and a theological need to understand the motion of planets. The discipline that considers systems of this type is often referred to as classical mechanics. Mathematically, such systems are generally characterized by a finite number of ordinary differential equations, and the properties of each particle (e.g. position, velocity) are taken to be functions of time only.

Continuum mechanics, generally attributed to Euler,[7] considers instead an infinite number of particles. In continuum mechanics every physical property (e.g. velocity, density, pressure) is taken to be a function of both time and space. There is an infinitesimal property variation from point to point in space. While variations are generally continuous, finite numbers of surfaces of discontinuous property variation are allowed. This models, for example, the contact between one continuous body and another. Point discontinuities are not allowed, however. Finite valued material properties are required. Mathematically, such systems are characterized by a finite number of partial differential equations in which the properties of the continuum material are functions of both space and time. It is possible to show that a partial differential equation can be thought of as an infinite number of ordinary differential equations, so this is consistent with our model of a continuum as an infinite number of particles.

[7]Leonhard Euler, 1707-1783, Swiss-born mathematician and physicist who served in the court of Catherine I of Russia in St. Petersburg, regarded by many as one of the greatest mechanicians.

1.3 Rational continuum mechanics

The modifier "rational" was first applied by Truesdell to continuum mechanics to distinguish the formal approach advocated by his school, from less formal, though mainly not irrational, approaches to continuum mechanics. Rational continuum mechanics is developed with tools similar to those that Euclid[8] used for his geometry: formal definitions, axioms, and theorems, all accompanied by careful language and proofs. This course will generally follow the less formal, albeit still rigorous, approach of Panton (2013), including the adoption of much of Panton's notation.

1.4 Notions from Newtonian continuum mechanics

The following are useful notions from Newtonian continuum mechanics. Here we use Newtonian to distinguish our mechanics from Einsteinian or relativistic mechanics. Newton himself did not study continuum mechanics; however, notions from his studies of the mechanics of discrete sets of point masses extend to the mechanics of continua.

- *Space* is three-dimensional and independent of time.
- An *inertial frame* is a reference frame in which the laws of physics are invariant; further, a body in an inertial frame with zero net force acting upon it does not accelerate.
- A *Galilean*[9] *transformation* specifies how to transform from one inertial frame to another inertial frame moving at constant velocity relative to the original frame. If a second inertial frame has constant velocity $\mathbf{v}_o = u_o\mathbf{i} + v_o\mathbf{j} + w_o\mathbf{k}$ relative to the original inertial frame, the Galilean transformation $(x, y, z, t) \to (x', y', z', t')$ is as follows

$$x' = x - u_o t, \tag{1.1}$$
$$y' = y - v_o t, \tag{1.2}$$
$$z' = z - w_o t, \tag{1.3}$$
$$t' = t. \tag{1.4}$$

  This must be accompanied with a transformation of the velocities

$$u' = u - u_o, \tag{1.5}$$
$$v' = v - v_o, \tag{1.6}$$
$$w' = w - w_o. \tag{1.7}$$

[8]Euclid, Greek geometer of profound influence who taught in Alexandria, Egypt, during the reign of Ptolemy I Soter, who ruled 323-283 BC.

[9]Galileo Galilei, 1564-1642, Pisa-born Italian astronomer, physicist, and developer of experimental methods, first employed a pendulum to keep time, builder and user of telescopes used to validate the Copernican view of the universe, developer of the principle of inertia and relative motion.

- *Control volumes* are useful; we will study three varieties:
  - Fixed: constant in space,
  - Material: no flux of mass through boundaries, can deform,
  - Arbitrary: can move, can deform, can have different fluid contained within.
- *Control surfaces* enclose control volumes; they have the same three varieties:
  - Fixed,
  - Material,
  - Arbitrary.
- *Density* is a material property, not used in classical mechanics, that only considers point masses. We can define density ρ as

$$\rho = \lim_{V \to 0} \frac{\sum_{i=1}^{N} m_i}{V}. \tag{1.8}$$

Here V is the volume of the space considered, N is the number of particles contained within the volume, and m_i is the mass of the ith particle. We can define a length scale L associated with the volume V to be $L = V^{1/3}$. In commonly encountered physical scenarios, we expect the density to vary with distance on a macroscale, approach a limiting value at the microscale, and become ill-defined below a cutoff scale below which molecular effects are important. That is to say, when V becomes too small, such that only a few molecules are contained within it, we expect wild oscillations in ρ.

We will in fact assume that matter can be modeled as a *continuum*: the limit in which discrete changes from molecule to molecule can be ignored and distances and times over which we are concerned are much larger than those of the molecular scale. This will enable the use of calculus in our continuum thermodynamics.

Continuum mechanics will treat macroscopic effects only and ignore individual molecular effects. For example molecules bouncing off a wall exchange momentum with the wall and induce pressure. We could use Newtonian mechanics for each particle collision to calculate the net force on the wall. Instead our approach amounts to considering the *average* over space and time of the net effect of millions of collisions on a wall.

The continuum theory can break down in important applications where the length and time scales are of comparable magnitude to molecular time scales. Important applications where the continuum assumption breaks down include

  - rarefied gas dynamics of the outer atmosphere (relevant for low orbit space vehicles), and
  - nano-scale heat transfer (relevant in cooling of computer chips).

To get some idea of the scales involved, we note that for air at atmospheric pressure and temperature that the time and distance between molecular collisions provides the limits of the continuum. Under these conditions, we observe for air that

- length > 0.1 μm, and
- time > 0.1 ns,

will be sufficient to admit the continuum assumption. For denser gases, these cutoff scales are smaller. For lighter gases, these cutoff scales are larger. A sketch of a possible density variation in a gas near atmospheric pressure is given in Fig. 1.1.

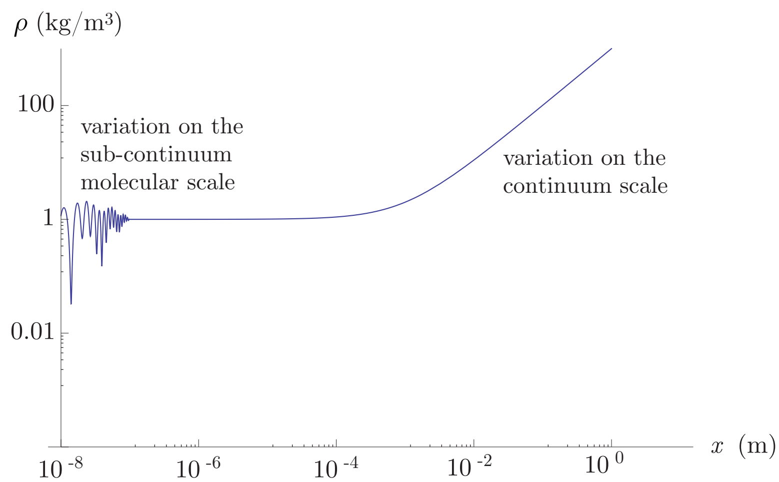

Figure 1.1: Sketch of possible density variation of a gas near atmospheric pressure.

Details of collision theory can be found in advanced texts such as that of Vincenti and Kruger (1965), pp. 12-26. They show for air that the mean free path λ is well modeled by the equation:

$$\lambda = \frac{\mathcal{M}}{\sqrt{2}\pi \mathcal{N} \rho d^2}. \tag{1.9}$$

Here $\mathcal{M}$ is the molecular mass, $\mathcal{N}$ is Avogadro's number, and d is the molecular diameter.

Example 1.1

Find the variation of mean free path with density for air.

We turn to Vincenti and Kruger (1965) for numerical parameter values, that are seen to be $\mathcal{M} = 28.9$ kg/kmole, $\mathcal{N} = 6.02252 \times 10^{23}$ molecule/mole, $d = 3.7 \times 10^{-10}$ m. Thus,

$$\lambda = \frac{\left(28.9 \ \frac{\text{kg}}{\text{kmole}}\right)\left(1 \ \frac{\text{kmole}}{1000 \ \text{mole}}\right)}{\sqrt{2}\pi \left(6.02252 \times 10^{23} \ \frac{\text{molecule}}{\text{mole}}\right) \rho \left(3.7 \times 10^{-10} \ \text{m}\right)^2}, \tag{1.10}$$

$$= \frac{7.8895 \times 10^{-8} \ \frac{\text{kg}}{\text{molecule m}^2}}{\rho}. \tag{1.11}$$

The unit "molecule" is not really a dimension, but really is literally a "unit," that may well be thought of as dimensionless. Thus, we can safely say

$$\lambda = \frac{7.8895 \times 10^{-8} \ \frac{\text{kg}}{\text{m}^2}}{\rho}. \tag{1.12}$$

A plot of the variation of mean free path λ as a function of ρ is given in Fig. 1.2. Vincenti

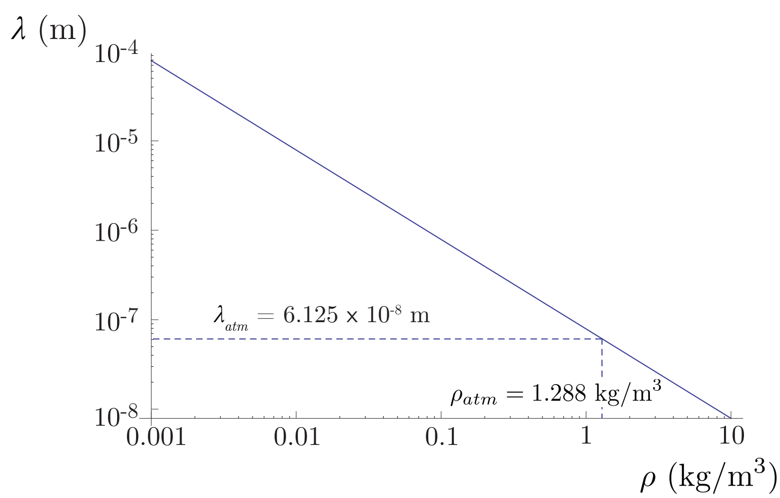

Figure 1.2: Mean free path length, λ, as a function of density, ρ, for air.

and Kruger (1965) go on to consider an atmosphere with density of $\rho = 1.288$ kg/m^3. For this density

$$\lambda = \frac{7.8895 \times 10^{-8} \ \frac{\text{kg}}{\text{m}^2}}{1.288 \ \frac{\text{kg}}{\text{m}^3}}, \tag{1.13}$$

$$= 6.125 \times 10^{-8} \ \text{m}, \tag{1.14}$$

$$= 6.125 \times 10^{-2} \mu\text{m}. \tag{1.15}$$

Vincenti and Kruger (1965) also show the mean molecular speed under these conditions is roughly $c = 500$ m/s, so the mean time between collisions, τ, is

$$\tau \sim \frac{\lambda}{c} = \frac{6.125 \times 10^{-8} \ \text{m}}{500 \ \frac{\text{m}}{\text{s}}} = 1.225 \times 10^{-10} \ \text{s}. \tag{1.16}$$

Density is an example of a *scalar* property. We shall have more to say later about scalars. For now we say that a scalar property associates a single number with each point in time and space. We can think of this by writing the usual notation $\rho(x, y, z, t)$, that indicates ρ has functional variation with position and time.

- Other properties are not scalar, but are *vector* properties. For example the velocity vector

$$\mathbf{v}(x, y, z, t) = u(x, y, z, t)\mathbf{i} + v(x, y, z, t)\mathbf{j} + w(x, y, z, t)\mathbf{k}, \tag{1.17}$$

associates three scalars u, v, w with each point in space and time. We will see that a vector can be characterized as a scalar associated with a particular direction in space. Here we use a boldfaced notation for a vector. This is known as Gibbs[10] notation. We will soon study an alternate notation, developed by Einstein, and known as Cartesian[11] index notation.

- Other properties are not scalar or vector, but are what is know as tensorial. The relevant properties are called *tensors*. The best known example is the stress tensor, whose physics and mathematics will be fully described in Ch. 4.2.2. One can think of a tensor as a quantity that associates a vector with a plane inclined at a selected angle passing through a given point in space. An example is the viscous stress tensor $\boldsymbol{\tau}$, that is best expressed as a three by three matrix with nine components:

$$\boldsymbol{\tau}(x, y, z, t) = \begin{pmatrix} \tau_{xx}(x, y, z, t) & \tau_{xy}(x, y, z, t) & \tau_{xz}(x, y, z, t) \\ \tau_{yx}(x, y, z, t) & \tau_{yy}(x, y, z, t) & \tau_{yz}(x, y, z, t) \\ \tau_{zx}(x, y, z, t) & \tau_{zy}(x, y, z, t) & \tau_{zz}(x, y, z, t) \end{pmatrix}. \tag{1.18}$$

[10]Josiah Willard Gibbs, 1839-1903, American physicist and chemist with a lifelong association with Yale University who made fundamental contributions to vector analysis, statistical mechanics, thermodynamics, and chemistry. Studied in Europe in the 1860s. Probably one of the few great American scientists of the nineteenth century.

[11]René Descartes, 1596-1650, French mathematician and philosopher of great influence. A great doubter of existence who nevertheless concluded, “I think, therefore I am.” Developed analytic geometry.

Chapter 2

Some necessary mathematics

see Panton, Chapter 3,
see Aris, Chapters 1-3, 7, Appendices A and B,
see Hughes and Gaylord, Appendix,
see Segel, Chapters 1 and 2,
see Yih, Appendix 2,
see Paolucci, Chapter 2,
see Powers and Sen, Chapters 1 and 2.

Here we outline some fundamental mathematical principles that are necessary to understand continuum mechanics as it will be presented here.

2.1 Vectors and Cartesian tensors

2.1.1 Gibbs and Cartesian index notation

Gibbs notation for vectors and tensors is the most familiar from undergraduate courses. It typically uses boldface, arrows, underscores, or overbars to denote a vector or a tensor. Unfortunately, it also hides some of the structures that are actually present in the equations. Einstein realized this in developing the theory of general relativity and developed a useful alternate, index notation. In these notes we will focus on what is known as Cartesian index notation, that is restricted to Cartesian coordinate systems. Einstein also developed a more general index system for non-Cartesian systems. We will briefly touch on this in our summaries of our equations later in this chapter but refer the reader to books such as that of Aris (1962) for a full exposition. While it can seem difficult at the outset, in the end many agree that the use of index notation actually simplifies many common notions in fluid mechanics. Moreover, its use in the archival literature is widespread, so to be conversant in fluid mechanics, one must know index notation. Table 2.1 summarizes the correspondences between Gibbs, Cartesian index, and matrix notation. Here we adopt a convention for the Gibbs notation, that we will find at times conflicts with other conventions, in which italics

Quantity	Common Parlance	Gibbs	Cartesian Index	Matrix
zeroth order tensor	scalar	a	a	$\left(a \right)$
first order tensor	vector	$\mathbf{a}$	a_i	$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$
second order tensor	tensor	A	a_{ij}	$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$
third order tensor	tensor	$\overline{\mathsf{A}}$	a_{ijk}	-
fourth order tensor	tensor	$\overline{\overline{\mathsf{A}}}$	a_{ijkl}	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	-

Table 2.1: Scalar, vector, and tensor notation conventions.

font (a) indicates a scalar, bold font ($\mathbf{a}$) indicates a vector, upper case sans serif (A) indicates a second order tensor,[1] over-lined upper case sans serif ($\overline{\mathsf{A}}$) indicates a third order tensor, double over-lined upper case sans serif ($\overline{\overline{\mathsf{A}}}$) indicates a fourth order tensor. In Cartesian index notation, their is no need to use anything except italics, as all terms are thought of as scalar components of a more expansive structure, with the structure indicated by the presence of subscripts.

The essence of the Cartesian index notation is as follows. We can represent a three-dimensional vector $\mathbf{a}$ as a linear combination of scalars and orthonormal basis vectors:

$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}. \tag{2.1}$$

We choose now to associate the subscript 1 with the x direction, the subscript 2 with the y direction, and the subscript 3 with the z direction. Further, we replace the orthonormal basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. Then the vector $\mathbf{a}$ is represented by

$$\mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 = \sum_{i=1}^{3} a_i \mathbf{e}_i = a_i \mathbf{e}_i = a_i = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}. \tag{2.2}$$

Following Einstein, we have adopted the convention that a summation is understood to exist when two indices, known as dummy indices, are repeated, and have further left the explicit representation of basis vectors out of our final version of the notation. We have also included

[1]Following longstanding fluid mechanics tradition, we will break this convention for the viscous stress tensor, $\boldsymbol{\tau}$.

a representation of **a** as a 3×1 column vector. We adopt the standard that all vectors can be thought of as column vectors. Often in matrix operations, we will need row vectors. They will be formed by taking the transpose, indicated by a superscript T, of a column vector. In the interest of clarity, full consistency with notions from matrix algebra, as well as transparent translation to the conventions of necessarily meticulous (as well as popular) software tools such as `MATLAB`, we will scrupulously use the transpose notation. This comes at the expense of a more cluttered set of equations at times. We also note that most authors do not *explicitly* use the transpose notation, but its use is implicit.

2.1.2 Rotation of axes

The Cartesian index notation is developed to be valid under transformations from one Cartesian coordinate system to another Cartesian coordinate system. It is not applicable to either general orthogonal systems (such as cylindrical or spherical) or non-orthogonal systems. It is straightforward, but tedious, to develop a more general system to handle generalized coordinate transformations, and Einstein did just that as well. For our purposes however, the simpler Cartesian index notation will suffice.

We will consider a coordinate transformation that is a simple rotation of axes. This transformation preserves all angles; hence, right angles in the original Cartesian system will be right angles in the rotated, but still Cartesian system. It also preserves lengths of geometric features, with no stretching. We will require, ultimately, that whatever theory we develop must generate results in which physically relevant quantities such as temperature, pressure, density, and velocity magnitude, are independent of the particular set of coordinates with which we choose to describe the system. To motivate this, let us consider a two-dimensional rotation from an unprimed system to a primed system. So, we seek a transformation that maps $(x_1, x_2)^T \to (x_1', x_2')^T$. We will rotate the unprimed system counterclockwise through an angle α to achieve the primed system.[2] The rotation is sketched in Fig. 2.1. It is easy to show that the angle $\beta = \pi/2 - \alpha$. Here a point P is identified by a particular set of coordinates (x_1^*, x_2^*). One of the keys to all of continuum mechanics is realizing that while the location (or velocity, or stress, ...) of P may be represented differently in various coordinate systems, ultimately it must represent the same physical reality. Straightforward geometry shows the following relation between the primed and unprimed coordinate systems for x_1'

$$x_1^{*'} = x_1^* \cos\alpha + x_2^* \cos\beta. \tag{2.3}$$

More generally, we can say for an arbitrary point that

$$x_1' = x_1 \cos\alpha + x_2 \cos\beta. \tag{2.4}$$

[2]This is an example of a so-called *alias transformation*. In such a transformation, the coordinate axes transform, but the underlying object remains unchanged. So a vector may be considered to be invariant, but its representation in different coordinate systems may be different. Alias transformations are most common in continuum mechanics. In contrast, an *alibi transformation* is one in which the coordinate axes remain fixed, but the object transforms. This mode of thought is most common in fields such as robotics. In short, alias rotates the axes, but not the body; alibi rotates the body, but not the axes.

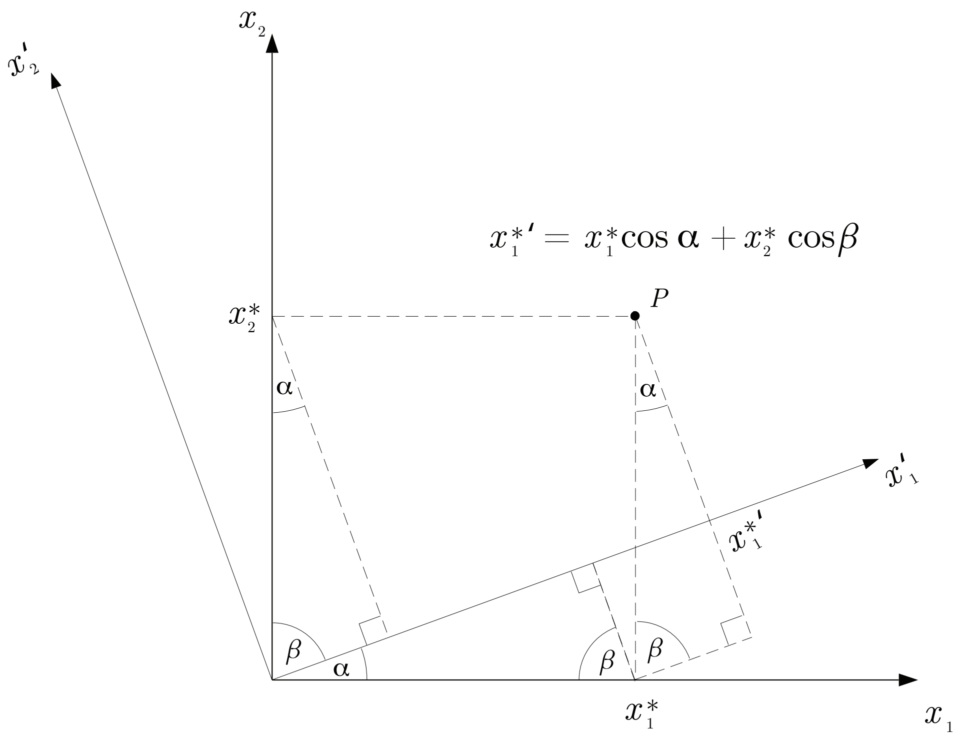

Figure 2.1: Sketch of coordinate transformation that is a rotation of axes.

We adopt the following notation

- (x_1, x'_1) denotes the angle between the x_1 and x'_1 axes,
- (x_2, x'_2) denotes the angle between the x_2 and x'_2 axes,
- (x_3, x'_3) denotes the angle between the x_3 and x'_3 axes,
- (x_1, x'_2) denotes the angle between the x_1 and x'_2 axes,
- $\vdots$

Thus, in two-dimensions, we have

$$x'_1 = x_1 \cos(x_1, x'_1) + x_2 \cos(x_2, x'_1). \tag{2.5}$$

In three dimensions, this extends to

$$x'_1 = x_1 \cos(x_1, x'_1) + x_2 \cos(x_2, x'_1) + x_3 \cos(x_3, x'_1). \tag{2.6}$$

Extending this analysis to calculate x'_2 and x'_3 gives

$$x'_2 = x_1 \cos(x_1, x'_2) + x_2 \cos(x_2, x'_2) + x_3 \cos(x_3, x'_2), \tag{2.7}$$
$$x'_3 = x_1 \cos(x_1, x'_3) + x_2 \cos(x_2, x'_3) + x_3 \cos(x_3, x'_3). \tag{2.8}$$

These can be written in matrix form as

$$(x_1' \quad x_2' \quad x_3') = (x_1 \quad x_2 \quad x_3) \begin{pmatrix} \cos(x_1, x_1') & \cos(x_1, x_2') & \cos(x_1, x_3') \\ \cos(x_2, x_1') & \cos(x_2, x_2') & \cos(x_2, x_3') \\ \cos(x_3, x_1') & \cos(x_3, x_2') & \cos(x_3, x_3') \end{pmatrix}. \tag{2.9}$$

If we use the shorthand notation, for example, that $\ell_{11} = \cos(x_1, x_1')$, $\ell_{12} = \cos(x_1, x_2')$, etc., we have

$$\underbrace{(x_1' \quad x_2' \quad x_3')}_{\mathbf{x}'^T} = \underbrace{(x_1 \quad x_2 \quad x_3)}_{\mathbf{x}^T} \underbrace{\begin{pmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{pmatrix}}_{\mathsf{Q}}. \tag{2.10}$$

In Gibbs notation, defining the matrix of ℓ's to be Q[3], and recalling that all vectors are taken to be column vectors, we can alternatively say[4]

$$\mathbf{x}'^T = \mathbf{x}^T \cdot \mathsf{Q}. \tag{2.11}$$

Taking the transpose of both sides and recalling the useful identities that $(\mathsf{A} \cdot \mathbf{b})^T = \mathbf{b}^T \cdot \mathsf{A}^T$ and $(\mathsf{A}^T)^T = \mathsf{A}$, we can also say

$$\mathbf{x}' = \mathsf{Q}^T \cdot \mathbf{x}. \tag{2.12}$$

We call $\mathsf{Q} = \ell_{ij}$ the matrix of direction cosines and $\mathsf{Q}^T = \ell_{ji}$ the rotation matrix. It can be shown that coordinate systems that satisfy the right hand rule require further that

$$\det \mathsf{Q} = 1. \tag{2.13}$$

Matrices Q that have $|\det \mathsf{Q}| = 1$ are associated with volume-preserving transformations. Matrices Q that have $\det \mathsf{Q} > 0$, are orientation-preserving transformations. Matrices Q that have $\det \mathsf{Q} = 1$ are thus volume- and orientation-preserving, and can be thought of a rotations. A matrix that had determinant -1 would be volume-preserving but not orientation-preserving. It could be considered as a reflection. A matrix Q composed of orthonormal column vectors, with $|\det \mathsf{Q}| = 1$ (thus either rotation or reflection matrices) is commonly known as *orthogonal*, though perhaps "orthonormal" would have been a more descriptive

[3]Panton (2013) has a different notation for the direction cosines ℓ_{ij} and employs Q for a different purpose; our usage is probably more common in the broader literature.

[4]The more commonly used alternate convention of not explicitly using the transpose notation for vectors would instead have our $\mathbf{x}'^T = \mathbf{x}^T \cdot \mathsf{Q}$ written as $\mathbf{x}' = \mathbf{x} \cdot \mathsf{Q}$. In fact, our use of the transpose notation is strictly viable only for Cartesian coordinate systems, while many will allow Gibbs notation to represent vectors in non-Cartesian coordinates, for which the transpose operation is ill-suited. However, realizing that these notes will primarily focus on Cartesian systems, and that such operations relying on the transpose are useful notions from linear algebra, it will be employed in an overly liberal fashion in these notes. The alternate convention still typically applies, where necessary, the transpose notation for tensors, so it would also hold that $\mathbf{x}' = \mathsf{Q}^T \cdot \mathbf{x}$.

nomenclature. Another way to think of the matrix of direction cosines $\ell_{ij} = \mathsf{Q}$ is as a matrix of orthonormal basis vectors in its columns:

$$\ell_{ij} = \mathsf{Q} = \begin{pmatrix} \vdots & \vdots & \vdots \\ \mathbf{n}^{(1)} & \mathbf{n}^{(2)} & \mathbf{n}^{(3)} \\ \vdots & \vdots & \vdots \end{pmatrix}. \tag{2.14}$$

In a result that is both remarkable and important, it can be shown that *the transpose of an orthogonal matrix is its inverse*:

$$\mathsf{Q}^T = \mathsf{Q}^{-1}. \tag{2.15}$$

Thus, we have

$$\mathsf{Q} \cdot \mathsf{Q}^T = \mathsf{Q}^T \cdot \mathsf{Q} = \mathsf{I}. \tag{2.16}$$

The equation $\mathbf{x}'^T = \mathbf{x}^T \cdot \mathsf{Q}$ is really a set of three linear equations. For instance, the first is

$$x_1' = x_1\ell_{11} + x_2\ell_{21} + x_3\ell_{31}. \tag{2.17}$$

More generally, we could say that

$$x_j' = x_1\ell_{1j} + x_2\ell_{2j} + x_3\ell_{3j}. \tag{2.18}$$

Here j is a so-called "free index," that for three-dimensional space takes on values $j = 1, 2, 3$. Some rules of thumb for free indices are

- A free index can appear only once in each additive term.
- One free index (e.g. k) may replace another (e.g. j) as long as it is replaced in each additive term.

We can simplify Eq. (2.18) further by writing

$$x_j' = \sum_{i=1}^{3} x_i \ell_{ij}. \tag{2.19}$$

This is commonly written in the following form:

$$x_j' = x_i \ell_{ij}. \tag{2.20}$$

We again note that it is to be understood that whenever an index is repeated, as has the index i here, that a summation from $i = 1$ to $i = 3$ is to be performed and that i is the "dummy index." Some rules of thumb for dummy indices are

- dummy indices can appear *only twice* in a given additive term,
- a pair of dummy indices, say i, i, can be exchanged for another, say j, j, in a given additive term with no need to change dummy indices in other additive terms.

We define the Kronecker[5] delta, δ_{ij} as

$$\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \tag{2.21}$$

This is effectively the identity matrix I:

$$\delta_{ij} = \mathsf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.22}$$

Direct substitution proves that what is effectively the law of cosines can be written as

$$\ell_{ij}\ell_{kj} = \delta_{ik}. \tag{2.23}$$

This is also equivalent to Eq. (2.16), $\mathsf{Q} \cdot \mathsf{Q}^T = \mathsf{I}$.

Example 2.1

Show for the two-dimensional system described in Fig. 2.1 that $\ell_{ij}\ell_{kj} = \delta_{ik}$ holds.

Expanding for the two-dimensional system, we get

$$\ell_{i1}\ell_{k1} + \ell_{i2}\ell_{k2} = \delta_{ik}. \tag{2.24}$$

First, take $i = 1, k = 1$. We get then

$$\begin{aligned} \ell_{11}\ell_{11} + \ell_{12}\ell_{12} = \delta_{11} &= 1, & (2.25) \\ \cos\alpha\cos\alpha + \cos(\alpha + \pi/2)\cos(\alpha + \pi/2) &= 1, & (2.26) \\ \cos\alpha\cos\alpha + (-\sin(\alpha))(-\sin(\alpha)) &= 1, & (2.27) \\ \cos^2\alpha + \sin^2\alpha &= 1. & (2.28) \end{aligned}$$

This is obviously true. Next, take $i = 1, k = 2$. We get then

$$\begin{aligned} \ell_{11}\ell_{21} + \ell_{12}\ell_{22} = \delta_{12} &= 0, & (2.29) \\ \cos\alpha\cos(\pi/2 - \alpha) + \cos(\alpha + \pi/2)\cos(\alpha) &= 0, & (2.30) \\ \cos\alpha\sin\alpha - \sin\alpha\cos\alpha &= 0. & (2.31) \end{aligned}$$

This is obviously true. Next, take $i = 2, k = 1$. We get then

$$\begin{aligned} \ell_{21}\ell_{11} + \ell_{22}\ell_{12} = \delta_{21} &= 0, & (2.32) \\ \cos(\pi/2 - \alpha)\cos\alpha + \cos\alpha\cos(\pi/2 + \alpha) &= 0, & (2.33) \\ \sin\alpha\cos\alpha + \cos\alpha(-\sin\alpha) &= 0. & (2.34) \end{aligned}$$

This is obviously true. Next, take $i = 2, k = 2$. We get then

$$\begin{aligned} \ell_{21}\ell_{21} + \ell_{22}\ell_{22} = \delta_{22} &= 1, & (2.35) \\ \cos(\pi/2 - \alpha)\cos(\pi/2 - \alpha) + \cos\alpha\cos\alpha &= 1, & (2.36) \\ \sin\alpha\sin\alpha + \cos\alpha\cos\alpha &= 1. & (2.37) \end{aligned}$$

Again, this is obviously true.

[5]Leopold Kronecker, 1823-1891, German mathematician, critic of set theory, who stated "God made the integers; all else is the work of man."

Using this, we can easily find the inverse transformation back to the unprimed coordinates via the following operations:

$$\ell_{kj}x'_j = \ell_{kj}x_i\ell_{ij}, \tag{2.38}$$
$$= \ell_{ij}\ell_{kj}x_i, \tag{2.39}$$
$$= \delta_{ik}x_i, \tag{2.40}$$
$$\ell_{kj}x'_j = x_k, \tag{2.41}$$
$$\ell_{ij}x'_j = x_i, \tag{2.42}$$
$$x_i = \ell_{ij}x'_j. \tag{2.43}$$

The Kronecker delta is also known as the substitution tensor as it has the property that application of it to a vector simply substitutes one index for another:

$$x_k = \delta_{ki}x_i. \tag{2.44}$$

For students familiar with linear algebra, it is easy to show that the matrix of direction cosines, ℓ_{ij}, is a rotation matrix. Each of its columns is a vector that is orthogonal to the other column vectors. Additionally, each column vector is itself normal. Such a matrix has a Euclidean norm of unity, and three eigenvalues that have magnitude of unity. Its determinant is $+1$, that renders it a rotation; in contrast a reflection matrix would have determinant of -1. Operation of a rotation matrix on a vector rotates it, but does not stretch it.

2.1.3 Vectors

Three scalar quantities v_i where $i = 1, 2, 3$ are scalar components of a *vector* if they transform according to the following rule

$$v'_j = v_i\ell_{ij}, \tag{2.45}$$

under a rotation of axes characterized by direction cosines ℓ_{ij}. In Gibbs notation, we would say

$$\mathbf{v}'^T = \mathbf{v}^T \cdot \mathsf{Q}, \tag{2.46}$$

or alternatively

$$\mathbf{v}' = \mathsf{Q}^T \cdot \mathbf{v}. \tag{2.47}$$

We can also say that a vector associates a scalar with a chosen direction in space by an expression that is linear in the direction cosines of the chosen direction.

Example 2.2

Consider the set of scalars that describe the velocity in a two-dimensional Cartesian system:

$$v_i = \begin{pmatrix} v_x \\ v_y \end{pmatrix}, \tag{2.48}$$

where we return to the typical x, y coordinate system. Determine if v_i is a vector.

In a rotated coordinate system, using the same notation of Fig. 2.1, we find that

$$v'_x = v_x \cos\alpha + v_y \cos(\pi/2 - \alpha) = v_x \cos\alpha + v_y \sin\alpha, \quad (2.49)$$
$$v'_y = v_x \cos(\pi/2 + \alpha) + v_y \cos\alpha = -v_x \sin\alpha + v_y \cos\alpha. \quad (2.50)$$

This is linear in the direction cosines, and satisfies the definition for a vector.

Example 2.3

Do two arbitrary scalars, say the quotient of pressure and density and the product of specific heat and temperature, $(p/\rho, c_v T)^T$, form a vector?

If this quantity is a vector, then we can say

$$v_i = \begin{pmatrix} p/\rho \\ c_v T \end{pmatrix}. \quad (2.51)$$

This pair of numbers has an obvious physical meaning in our unrotated coordinate system. If the system were a calorically perfect ideal gas (CPIG), the first component would represent the difference between the enthalpy and the internal energy, and the second component would represent the internal energy. And if we rotate through an angle α, we arrive at a transformed quantity of

$$v'_1 = \frac{p}{\rho}\cos\alpha + c_v T \cos(\pi/2 - \alpha), \quad (2.52)$$
$$v'_2 = \frac{p}{\rho}\cos(\pi/2 + \alpha) + c_v T \cos(\alpha). \quad (2.53)$$

This quantity does not have any known physical significance, and so it seems that these quantities do not form a vector.

We have the following vector algebra

- Addition
  - $w_i = u_i + v_i$ (Cartesian index notation)
  - $\mathbf{w} = \mathbf{u} + \mathbf{v}$ (Gibbs notation)
- Dot product (inner product)
  - $u_i v_i = b$ (Cartesian index notation)
  - $\mathbf{u}^T \cdot \mathbf{v} = b$ (Gibbs notation)
  - both notations require $u_1 v_1 + u_2 v_2 + u_3 v_3 = b$.

While u_i and v_i have scalar components that change under a rotation of axes, their inner product (or dot product) is a true scalar and is invariant under a rotation of axes.

Example 2.4

Demonstrate invariance of the dot product $\mathbf{u}^T \cdot \mathbf{v} = b$ by subjecting vectors $\mathbf{u}$ and $\mathbf{v}$ to a rotation.

Under rotation, our vectors transform as $\mathbf{u}' = \mathsf{Q}^T \cdot \mathbf{u}$, $\mathbf{v}' = \mathsf{Q}^T \cdot \mathbf{v}$. Thus $\mathsf{Q} \cdot \mathbf{u}' = \mathsf{Q} \cdot \mathsf{Q}^T \cdot \mathbf{u} = \mathbf{u}$. and $\mathsf{Q} \cdot \mathbf{v}' = \mathsf{Q} \cdot \mathsf{Q}^T \cdot \mathbf{v} = \mathbf{v}$. Then consider the dot product

$$\mathbf{u}^T \cdot \mathbf{v} = b, \tag{2.54}$$

$$(\mathsf{Q} \cdot \mathbf{u}')^T \cdot (\mathsf{Q} \cdot \mathbf{v}') = b, \tag{2.55}$$

$$\mathbf{u}'^T \cdot \underbrace{\mathsf{Q}^T \cdot \mathsf{Q}}_{=\mathsf{I}} \cdot \mathbf{v}' = b, \tag{2.56}$$

$$\mathbf{u}'^T \cdot \mathsf{I} \cdot \mathbf{v}' = b, \tag{2.57}$$

$$\mathbf{u}'^T \cdot \mathbf{v}' = b. \tag{2.58}$$

The inner product is invariant under rotation.

Here we have in the Gibbs notation explicitly noted that the transpose is part of the inner product. Most authors in fact assume the inner product of two vectors implies the transpose and do not write it explicitly, writing the inner product simply as $\mathbf{u} \cdot \mathbf{v} \equiv \mathbf{u}^T \cdot \mathbf{v}$.

2.1.4 Tensors

2.1.4.1 Definition

A second order tensor, or a rank two tensor, is nine scalar components that under a rotation of axes transformation according to the following rule:

$$T'_{ij} = \ell_{ki}\ell_{lj}T_{kl}. \tag{2.59}$$

We could also write this in an expanded form as

$$T'_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} \ell_{ki}\ell_{lj}T_{kl} = \sum_{k=1}^{3}\sum_{l=1}^{3} \ell_{ik}^T T_{kl}\ell_{lj}. \tag{2.60}$$

In these expressions, i and j are both free indices; while k and l are dummy indices. The notation ℓ_{ik}^T is unusual and rarely used. It does allow us to see the correspondence to GIbbs notation. The Gibbs notation for this transformation is easily shown to be

$$\mathsf{T}' = \mathsf{Q}^T \cdot \mathsf{T} \cdot \mathsf{Q}. \tag{2.61}$$

Analogously to our conclusion for a vector, we say that a tensor associates a vector with each direction in space by an expression that is linear in the direction cosines of the chosen direction. For a given tensor T_{ij}, the first subscript is associated with the face of a unit cube (hence the mnemonic device, *first-face*); the second subscript is associated with the vector components for the vector on that face.

Tensors can also be expressed as matrices. All rank two tensors are two-dimensional matrices, but not all matrices are rank two tensors, as they do not necessarily satisfy the transformation rules. We can say

$$T_{ij} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}. \tag{2.62}$$

The first row vector, $(T_{11} \quad T_{12} \quad T_{13})$, is the vector associated with the 1 face. The second row vector, $(T_{21} \quad T_{22} \quad T_{23})$, is the vector associated with the 2 face. The third row vector, $(T_{31} \quad T_{32} \quad T_{33})$, is the vector associated with the 3 face.

Example 2.5

Consider how the equation $\mathsf{A}\cdot\mathbf{x} = \mathbf{b}$ transforms under rotation.

Using

$$\mathsf{A}' = \mathsf{Q}^T\cdot\mathsf{A}\cdot\mathsf{Q}, \tag{2.63}$$
$$\mathbf{x}' = \mathsf{Q}^T\cdot\mathbf{x}, \tag{2.64}$$
$$\mathbf{b}' = \mathsf{Q}^T\cdot\mathbf{b}, \tag{2.65}$$

we see that by pre-multiplying all equations by Q, and post-multiplying the tensor equation by Q^T that

$$\mathsf{A} = \mathsf{Q}\cdot\mathsf{A}'\cdot\mathsf{Q}^T, \tag{2.66}$$
$$\mathbf{x} = \mathsf{Q}\cdot\mathbf{x}', \tag{2.67}$$
$$\mathbf{b} = \mathsf{Q}\cdot\mathbf{b}', \tag{2.68}$$

giving us

$$\underbrace{\mathsf{Q}\cdot\mathsf{A}'\cdot\mathsf{Q}^T}_{\mathsf{A}}\cdot\underbrace{\mathsf{Q}\cdot\mathbf{x}'}_{\mathbf{x}} = \underbrace{\mathsf{Q}\cdot\mathbf{b}'}_{\mathbf{b}}, \tag{2.69}$$
$$\mathsf{Q}\cdot\mathsf{A}'\cdot\mathbf{x}' = \mathsf{Q}\cdot\mathbf{b}', \tag{2.70}$$
$$\mathsf{Q}^T\cdot\mathsf{Q}\cdot\mathsf{A}'\cdot\mathbf{x}' = \mathsf{Q}^T\cdot\mathsf{Q}\cdot\mathbf{b}', \tag{2.71}$$
$$\mathsf{A}'\cdot\mathbf{x}' = \mathbf{b}'. \tag{2.72}$$

Obviously, the form is invariant under rotation.

We also have the following items associated with tensors.

2.1.4.2 Alternating unit tensor

The alternating unit tensor, a tensor of rank 3, ϵ_{ijk} will soon be seen to be useful, especially when we introduce the vector cross product. It is defined as follows

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if} \quad ijk = 123, 231, \quad \text{or} \quad 312, \\ 0 & \text{if any two indices identical,} \\ -1 & \text{if} \quad ijk = 321, 213, \quad \text{or} \quad 132. \end{cases} \tag{2.73}$$

Another way to remember this is to start with the sequence 123, that is positive. A sequential permutation, say from 123 to 231, retains the positive nature. A trade, say from 123 to 213, gives a negative value.

An identity that will be used extensively

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}, \tag{2.74}$$

can be proved a number of ways, including tedious direct substitution for all values of i, j, k, l, m.

2.1.4.3 Some secondary definitions

2.1.4.3.1 Transpose The transpose of a second rank tensor, denoted by a superscript T, is found by exchanging elements about the diagonal. In shorthand index notation, this is simply

$$(T_{ij})^T = T_{ji}. \tag{2.75}$$

Written out in full, if

$$T_{ij} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}, \tag{2.76}$$

then

$$T_{ij}^T = T_{ji} = \begin{pmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{pmatrix}. \tag{2.77}$$

2.1.4.3.2 Symmetric A tensor D_{ij} is symmetric iff

$$D_{ij} = D_{ji}, \tag{2.78}$$

$$\mathsf{D} = \mathsf{D}^T. \tag{2.79}$$

A symmetric tensor has only six independent scalars. We will reserve D for tensors that are symmetric. We will see that D is associated with the deformation of a fluid element.

2.1.4.3.3 Anti-symmetric A tensor R_{ij} is anti-symmetric iff

$$R_{ij} = -R_{ji}, \tag{2.80}$$
$$\mathsf{R} = -\mathsf{R}^T. \tag{2.81}$$

An anti-symmetric tensor must have zeroes on its diagonal and only three independent scalars on off-diagonal elements. We will reserve R for tensors that are anti-symmetric. We will see that R is associated with the rotation of a fluid element. But R is *not* a rotation matrix.

2.1.4.3.4 Decomposition An arbitrary tensor T_{ij} can be separated into a symmetric and anti-symmetric pair of tensors:

$$T_{ij} = \frac{1}{2}T_{ij} + \frac{1}{2}T_{ij} + \frac{1}{2}T_{ji} - \frac{1}{2}T_{ji}. \tag{2.82}$$

Rearranging, we get

$$T_{ij} = \underbrace{\frac{1}{2}\left(T_{ij} + T_{ji}\right)}_{\text{symmetric}} + \underbrace{\frac{1}{2}\left(T_{ij} - T_{ji}\right)}_{\text{anti-symmetric}}. \tag{2.83}$$

The first term must be symmetric, and the second term must be anti-symmetric. This is easily seen by considering applying this to any matrix of actual numbers. If we define the symmetric part of the matrix T_{ij} by the following notation

$$T_{(ij)} = \frac{1}{2}\left(T_{ij} + T_{ji}\right), \tag{2.84}$$

and the anti-symmetric part of the same matrix by the following notation

$$T_{[ij]} = \frac{1}{2}\left(T_{ij} - T_{ji}\right), \tag{2.85}$$

we then have

$$T_{ij} = T_{(ij)} + T_{[ij]}. \tag{2.86}$$

2.1.4.4 Tensor inner product

The tensor inner product of two tensors T_{ij} and S_{ji} is defined as follows

$$T_{ij}S_{ji} = a, \tag{2.87}$$

where a is a scalar. In Gibbs notation, we would say

$$\mathsf{T} : \mathsf{S} = a. \tag{2.88}$$

It is easily shown, and will be important in upcoming derivations, that the tensor inner product of any symmetric tensor D with any anti-symmetric tensor R is the scalar *zero*:

$$D_{ij}R_{ji} = 0, \tag{2.89}$$
$$\mathsf{D}:\mathsf{R} = 0. \tag{2.90}$$

Example 2.6

For all 2×2 matrices, prove the tensor inner product of general symmetric and anti-symmetric tensors is zero.

Take

$$\mathsf{D} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad \mathsf{R} = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}. \tag{2.91}$$

By definition then

$$\mathsf{D}:\mathsf{R} = D_{ij}R_{ji} = D_{11}R_{11} + D_{12}R_{21} + D_{21}R_{12} + D_{22}R_{22}, \tag{2.92}$$
$$= a(0) + b(-d) + bd + c(0), \tag{2.93}$$
$$= 0. \quad \text{QED.} \tag{2.94}$$

The theorem is proved.[6] The proof can be extended to arbitrary square matrices.

Further, if we decompose a tensor into its symmetric and anti-symmetric parts, $T_{ij} = T_{(ij)} + T_{[ij]}$ and take $T_{(ij)} = D_{ij} = \mathsf{D}$ and $T_{[ij]} = R_{ij} = \mathsf{R}$, so that $\mathsf{T} = \mathsf{D} + \mathsf{R}$, we note the following common term can be expressed as a tensor inner product with a dyadic product:

$$x_iT_{ij}x_j = \mathbf{x}^T \cdot \mathsf{T} \cdot \mathbf{x}, \tag{2.95}$$
$$x_i(T_{(ij)} + T_{[ij]})x_j = \mathbf{x}^T \cdot (\mathsf{D} + \mathsf{R}) \cdot \mathbf{x}, \tag{2.96}$$
$$x_iT_{(ij)}x_j = \mathbf{x}^T \cdot \mathsf{D} \cdot \mathbf{x}, \tag{2.97}$$
$$T_{(ij)}x_ix_j = \mathsf{D} : \mathbf{x}\mathbf{x}^T. \tag{2.98}$$

2.1.4.5 Dual vector of a tensor

We define the dual vector, d_i, of a tensor T_{jk} as follows[7]

$$d_i = \frac{1}{2}\epsilon_{ijk}T_{jk} = \frac{1}{2}\underbrace{\epsilon_{ijk}T_{(jk)}}_{=0} + \frac{1}{2}\epsilon_{ijk}T_{[jk]}. \tag{2.99}$$

[6] The common abbreviation QED at the end of the proof stands for the Latin *quod erat demonstrandum*, "that which was to be demonstrated."

[7] There is a lack of uniformity in the literature in this area. First, note this definition differs from that given by Panton (2013) by a factor of 1/2. It is closer, but not identical, to the approach found in Aris (1962), p. 25.

The term ϵ_{ijk} is anti-symmetric for any fixed i; for example for $i = 1$, we have

$$\epsilon_{1jk} = \begin{pmatrix} \epsilon_{111} & \epsilon_{112} & \epsilon_{113} \\ \epsilon_{121} & \epsilon_{122} & \epsilon_{123} \\ \epsilon_{131} & \epsilon_{132} & \epsilon_{133} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{2.100}$$

Thus, when its tensor inner product is taken with the symmetric $T_{(jk)}$, the result must be the scalar zero. Hence, we also have

$$d_i = \frac{1}{2}\epsilon_{ijk}T_{[jk]}. \tag{2.101}$$

Let us find the inverse relation for d_i, Starting with Eq. (2.99), we take the inner product of d_i with ϵ_{ilm} to get

$$\epsilon_{ilm}d_i = \frac{1}{2}\epsilon_{ilm}\epsilon_{ijk}T_{jk}. \tag{2.102}$$

Employing Eq. (2.74) to eliminate the ϵ's in favor of δ's, we get

$$\begin{aligned} \epsilon_{ilm}d_i &= \frac{1}{2}\left(\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}\right)T_{jk}, && (2.103) \\ &= \frac{1}{2}(T_{lm} - T_{ml}), && (2.104) \\ &= T_{[lm]}. && (2.105) \end{aligned}$$

Hence,

$$T_{[lm]} = \epsilon_{ilm}d_i. \tag{2.106}$$

Note that

$$T_{[lm]} = \epsilon_{1lm}d_1 + \epsilon_{2lm}d_2 + \epsilon_{3lm}d_3 = \begin{pmatrix} 0 & d_3 & -d_2 \\ -d_3 & 0 & d_1 \\ d_2 & -d_1 & 0 \end{pmatrix}. \tag{2.107}$$

And we can write the decomposition of an arbitrary tensor as the sum of its symmetric part and a factor related to the dual vector associated with its anti-symmetric part:

$$\underbrace{T_{ij}}_{\text{arbitrary tensor}} = \underbrace{T_{(ij)}}_{\text{symmetric part}} + \underbrace{\epsilon_{kij}d_k}_{\text{anti-symmetric part}}. \tag{2.108}$$

2.1.4.6 Tensor product: two tensors

The tensor product between two arbitrary tensors yields a third tensor. For second order tensors, we have the tensor product in Cartesian index notation as

$$S_{ij}T_{jk} = P_{ik}. \tag{2.109}$$

Note that j is a dummy index, i and k are free indices, and that the free indices in each additive term are the same. In that sense they behave somewhat as dimensional units, that must be the same for each term. In Gibbs notation, the equivalent tensor product is written as

$$\mathsf{S}\cdot\mathsf{T}=\mathsf{P}. \tag{2.110}$$

In contrast to the tensor inner product, that has two pairs of dummy indices and two dots, the tensor product has one pair of dummy indices and one dot. The tensor product is equivalent to matrix multiplication in matrix algebra.

An important property of tensors is that, in general, the tensor product *does not commute*, $\mathsf{S}\cdot\mathsf{T}\neq\mathsf{T}\cdot\mathsf{S}$. In the most formal manifestation of Cartesian index notation, one should also not commute the elements, and the dummy indices should appear next to another in adjacent terms as shown. However, it is of no great consequence to change the order of terms so that we can write $S_{ij}T_{jk}=T_{jk}S_{ij}$. That is in Cartesian index notation, elements do commute. *But*, in Cartesian index notation, the order of the indices is extremely important, and it is this order that does not commute: $S_{ij}T_{jk}\neq S_{ji}T_{jk}$ in general. The version presented for $S_{ij}T_{jk}$ in Eq. (2.109), in which the dummy index j is juxtaposed between each term, is slightly preferable as it maintains the order we find in the Gibbs notation.

Example 2.7

For two general 2×2 tensors, S and T, find the tensor inner product.

The tensor inner product is

$$\mathsf{S}\cdot\mathsf{T}=\begin{pmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{pmatrix}\begin{pmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{pmatrix}=\begin{pmatrix}S_{11}T_{11}+S_{12}T_{21} & S_{11}T_{12}+S_{12}T_{22}\\ S_{21}T_{11}+S_{22}T_{21} & S_{21}T_{12}+S_{22}T_{22}\end{pmatrix}. \tag{2.111}$$

Compare with the commutation:

$$\mathsf{T}\cdot\mathsf{S}=\begin{pmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{pmatrix}\begin{pmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{pmatrix}=\begin{pmatrix}S_{11}T_{11}+S_{21}T_{12} & S_{12}T_{11}+S_{22}T_{12}\\ S_{11}T_{21}+S_{21}T_{22} & S_{12}T_{21}+S_{22}T_{22}\end{pmatrix}. \tag{2.112}$$

Clearly $\mathsf{S}\cdot\mathsf{T}\neq\mathsf{T}\cdot\mathsf{S}$. It can be shown that if both S and T are symmetric, that $\mathsf{S}\cdot\mathsf{T}=(\mathsf{T}\cdot\mathsf{S})^T$.

2.1.4.7 Vector product: vector and tensor

The product of a vector and tensor, again that does not in general commute, comes in two flavors, pre-multiplication and post-multiplication, both important, and given in Cartesian index and Gibbs notation next:

2.1.4.7.1 Pre-multiplication

$$u_j = v_iT_{ij}=T_{ij}v_i, \tag{2.113}$$

$$\mathbf{u}^T = \mathbf{v}^T\cdot\mathsf{T}\neq\mathsf{T}\cdot\mathbf{v}. \tag{2.114}$$

In the Cartesian index notation here, the first form is preferred as it has a correspondence with the Gibbs notation, but both are correct representations given our summation convention.

2.1.4.7.2 Post-multiplication

$$w_i = T_{ij} v_j = v_j T_{ij}, \tag{2.115}$$
$$\mathbf{w} = \mathsf{T} \cdot \mathbf{v} \neq \mathbf{v}^T \cdot \mathsf{T}. \tag{2.116}$$

2.1.4.8 Dyadic product: two vectors

As opposed to the inner product between two vectors, that yields a scalar, we also have the dyadic product, that yields a tensor. In Cartesian index and Gibbs notation, we have

$$T_{ij} = u_i v_j = v_j u_i, \tag{2.117}$$
$$\mathsf{T} = \mathbf{u}\mathbf{v}^T \neq \mathbf{v}\mathbf{u}^T. \tag{2.118}$$

Notice there is no dot in the dyadic product; the dot is reserved for the inner product.

Example 2.8

Find the dyadic product between two general two-dimensional vectors. Show the dyadic product does not commute in general, and find the condition under which it does commute.

Take

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{2.119}$$

Then

$$\mathbf{u}\mathbf{v}^T = u_i v_j = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{pmatrix}. \tag{2.120}$$

Compare this to the commuted operation, $\mathbf{v}\mathbf{u}^T$:

$$\mathbf{v}\mathbf{u}^T = v_i u_j = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \begin{pmatrix} u_1 & u_2 \end{pmatrix} = \begin{pmatrix} v_1 u_1 & v_1 u_2 \\ v_2 u_1 & v_2 u_2 \end{pmatrix}. \tag{2.121}$$

By inspection, we see the operations in general do not commute. They do commute if $v_2/v_1 = u_2/u_1$. So in order for the dyadic product to commute, $\mathbf{u}$ and $\mathbf{v}$ must be parallel.

It is easily seen that the dyadic product $\mathbf{v}\mathbf{v}^T$ is a symmetric tensor. For the two-dimensional system, we would have

$$\mathbf{v}\mathbf{v}^T = v_i v_j = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} v_1 v_1 & v_1 v_2 \\ v_2 v_1 & v_2 v_2 \end{pmatrix}. \tag{2.122}$$

2.1.4.9 Contraction

We contract a general tensor, that has all of its subscripts different, by setting one subscript to be the same as the other. A single contraction will reduce the order of a tensor by two. For example the contraction of the second order tensor T_{ij} is T_{ii}, that indicates a sum is to be performed:

$$T_{ii} = T_{11} + T_{22} + T_{33}, \tag{2.123}$$

$$\text{tr}\ \mathsf{T} = T_{11} + T_{22} + T_{33}. \tag{2.124}$$

So, in this case the contraction yields a scalar. In matrix algebra, this particular contraction is the trace of the matrix.

2.1.4.10 Vector cross product

The vector cross product is defined in Cartesian index and Gibbs notation as

$$w_i = \epsilon_{ijk} u_j v_k, \tag{2.125}$$

$$\mathbf{w} = \mathbf{u} \times \mathbf{v}. \tag{2.126}$$

Expanding for $i = 1, 2, 3$ gives

$$w_1 = \epsilon_{123} u_2 v_3 + \epsilon_{132} u_3 v_2 = u_2 v_3 - u_3 v_2, \tag{2.127}$$

$$w_2 = \epsilon_{231} u_3 v_1 + \epsilon_{213} u_1 v_3 = u_3 v_1 - u_1 v_3, \tag{2.128}$$

$$w_3 = \epsilon_{312} u_1 v_2 + \epsilon_{321} u_2 v_1 = u_1 v_2 - u_2 v_1. \tag{2.129}$$

2.1.4.11 Vector associated with a plane

We often have to select a vector that is associated with a particular direction. Now for any direction we choose, there exists an associated unit vector and normal plane. Recall that our notation has been defined so that the first index is associated with a face or direction, and the second index corresponds to the components of the vector associated with that face. If we take n_i to be a unit normal vector associated with a given direction and normal plane, and we have been given a tensor T_{ij}, the vector t_j associated with that plane is given in Cartesian index and Gibbs notation by

$$t_j = n_i T_{ij}, \tag{2.130}$$

$$\mathbf{t}^T = \mathbf{n}^T \cdot \mathsf{T}, \tag{2.131}$$

$$\mathbf{t} = \mathsf{T}^T \cdot \mathbf{n}. \tag{2.132}$$

A sketch of a Cartesian element with the tensor components sketched on the proper face is shown in 2.2.

Example 2.9

Find the vector associated with the 1 face, $\mathbf{t}^{(1)}$, as shown in Fig. 2.2,

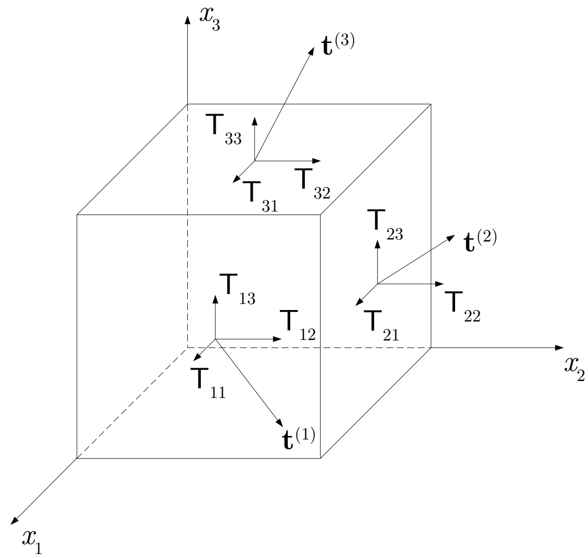

Figure 2.2: Sample Cartesian element that is aligned with coordinate axes, along with tensor components and vectors associated with each face.

We first choose the unit normal associated with the x_1 face, that is the vector $n_i = (1, 0, 0)^T$. The associated vector is found by doing the actual summation

$$t_j = n_i T_{ij} = n_1 T_{1j} + n_2 T_{2j} + n_3 T_{3j}. \tag{2.133}$$

Now $n_1 = 1$, $n_2 = 0$, and $n_3 = 0$, so for this problem, we have

$$t_j^{(1)} = T_{1j}. \tag{2.134}$$

2.2 Solution of linear algebra equations

We briefly discuss the solution of linear algebra equations of the form

$$A_{ij} x_j = b_i, \tag{2.135}$$

$$\mathsf{A} \cdot \mathbf{x} = \mathbf{b}. \tag{2.136}$$

Full details can be found in any text addressing linear algebra, e.g. Powers and Sen (2015). Let us presume that A is a known square matrix of dimension $N \times N$, $\mathbf{x}$ is an unknown column vector of length N, and $\mathbf{b}$ is a known column vector of length N. The following can be proved:

- A unique solution for $\mathbf{x}$ exists iff $\det \mathsf{A} \neq 0$.
- If $\det \mathsf{A} = 0$, solutions for $\mathbf{x}$ may or may not exist; if they exist, they are not unique.
- *Cramer's*[8] *rule*, a method involving the ratio of determinants discussed in linear algebra texts, can be used to find $\mathbf{x}$; other methods exist, such as Gaussian elimination.

Let us consider a few examples for $N = 2$.

Example 2.10

Use Cramer's rule to solve a general linear algebra problem with $N = 2$.

Consider then

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{2.137}$$

The solution from Cramer's rule involves the ratio of determinants. We get

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}. \tag{2.138}$$

If $b_1, b_2 \neq 0$ and $\det \mathsf{A} = a_{11}a_{22} - a_{12}a_{21} \neq 0$, there is a unique nontrivial solution for $\mathbf{x}$. If $b_1 = b_2 = 0$ and $\det \mathsf{A} = a_{11}a_{22} - a_{12}a_{21} \neq 0$, we must have $x_1 = x_2 = 0$. Obviously, if $\det \mathsf{A} = a_{11}a_{22} - a_{12}a_{21} = 0$, we cannot use Cramer's rule to compute as it involves division by zero. But we can salvage a non-unique solution if we also have $b_1 = b_2 = 0$, as we shall see.

Example 2.11

Find any and all solutions for

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.139}$$

Certainly $(x_1, x_2)^T = (0, 0)^T$ is a solution. But maybe there are more. Cramer's rule gives

$$x_1 = \frac{\begin{vmatrix} 0 & 2 \\ 0 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}} = \frac{0}{0}, \qquad x_2 = \frac{\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}} = \frac{0}{0}. \tag{2.140}$$

This is indeterminate! But the more robust Gaussian elimination process allows us to use row operations (multiply the top row by -2 and add to the bottom row) to rewrite the original equation as

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.141}$$

[8] Gabriel Cramer, 1704-1752, Swiss mathematician at University of Geneva.

By inspection, we get an infinite number of solutions, given by the one-parameter family of equations

$$x_1 = -2s, \qquad x_2 = s, \qquad s \in \mathbb{R}^1. \tag{2.142}$$

We could also eliminate s and say that $x_1 = -2x_2$. The solutions are linearly dependent. In terms of the language of vectors, we find the solution to be a vector of fixed direction, with arbitrary magnitude. In terms of a unit normal vector, we could write the solution as

$$\mathbf{x} = s \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}, \qquad s \in \mathbb{R}^1. \tag{2.143}$$

Example 2.12

Find any and all solutions for

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.144}$$

Cramer's rule gives

$$x_1 = \frac{\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}} = \frac{4}{0}, \qquad x_2 = \frac{\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}} = \frac{-2}{0}. \tag{2.145}$$

There is no value of $\mathbf{x}$ that satisfies $\mathsf{A} \cdot \mathbf{x} = \mathbf{b}$![9]

2.3 Eigenvalues, eigenvectors, and tensor invariants

For a given tensor T_{ij}, it is possible to select a plane for which the vector from T_{ij} associated with that plane points in the same direction as the normal associated with the chosen plane. In fact for a three-dimensional element, it is possible to choose three planes for which the vector associated with the given planes is aligned with the unit normal associated with those planes. We can think of this as finding a rotation as sketched in Fig. 2.3.

[9] There is however, in some sense a *best* solution, that is, an $\mathbf{x}$ of minimum length that also minimizes $||\mathsf{A} \cdot \mathbf{x} - \mathbf{b}||$. Using the pseudoinverse procedure described in Powers and Sen (2015), Ch. 7, we find there exists a non-unique set of $\mathbf{x} = (1/25 - 2s, 2/25 + s)^T,\ s \in \mathbb{R}^1$, for which for all values of s, the so-called error norm takes on the same minimum value, $e = ||\mathsf{A} \cdot \mathbf{x} - \mathbf{b}|| = 2/\sqrt{5}$. For $s = 0$, we then get the "best" $\mathbf{x} = (1/25, 2/25)^T$ in that this $\mathbf{x}$ minimizes the error and is itself of minimum length.

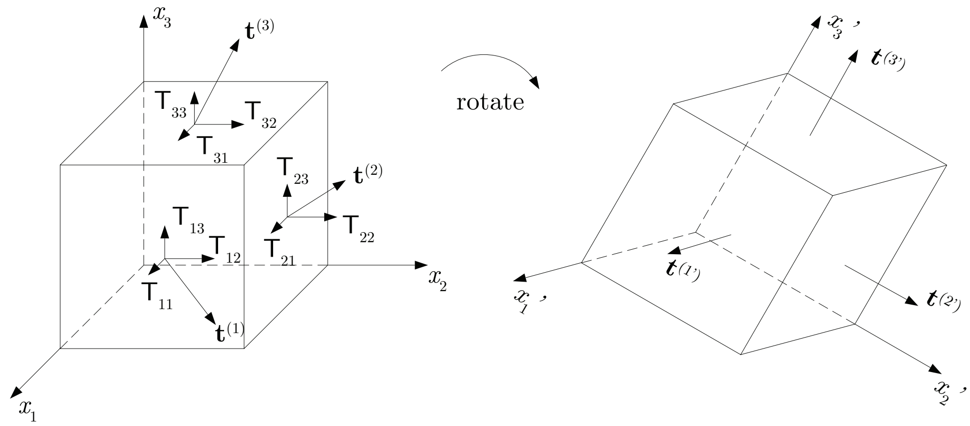

Figure 2.3: Sample Cartesian element that is rotated so that its faces have vectors that are aligned with the unit normals associated with the faces of the element.

Mathematically, we can enforce this condition by requiring that

$$\underbrace{n_i T_{ij}}_{\text{vector associated with chosen direction}} = \underbrace{\lambda n_j}_{\text{scalar multiple of chosen direction}} . \tag{2.146}$$

Here λ is an as of yet unknown scalar. The vector n_i could be a unit vector, but does not have to be. We can rewrite this as

$$n_i T_{ij} = \lambda n_i \delta_{ij}. \tag{2.147}$$

In Gibbs notation, this becomes $\mathbf{n}^T \cdot \mathsf{T} = \lambda \mathbf{n}^T \cdot \mathsf{I}$. In mathematics, this is known as a left eigenvalue problem. Solutions n_i that are non-trivial are known as left eigenvectors. We can also formulate this as a right eigenvalue problem by taking the transpose of both sides to obtain $\mathsf{T}^T \cdot \mathbf{n} = \lambda \mathsf{I} \cdot \mathbf{n}$. Here we have used the fact that $\mathsf{I}^T = \mathsf{I}$. We note that the left eigenvectors of T are the right eigenvectors of T^T. Eigenvalue problems are quite general and arise whenever an operator operates on a vector to generate a vector that leaves the original unchanged except in magnitude.

We can rearrange to form

$$n_i \left(T_{ij} - \lambda \delta_{ij}\right) = 0. \tag{2.148}$$

In matrix notation, this can be written as

$$\left(n_1 \quad n_2 \quad n_3 \right) \begin{pmatrix} T_{11} - \lambda & T_{12} & T_{13} \\ T_{21} & T_{22} - \lambda & T_{23} \\ T_{31} & T_{32} & T_{33} - \lambda \end{pmatrix} = \left(0 \quad 0 \quad 0 \right). \tag{2.149}$$

A trivial solution to this equation is $(n_1, n_2, n_3) = (0, 0, 0)$. But this is not interesting. As suggested by our understanding of Cramer's rule, we can get a non-unique, non-trivial solution if we enforce the condition that the determinant of the coefficient matrix be zero.

As we have an unknown parameter λ, we have sufficient degrees of freedom to accomplish this. So, we require

$$\begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{vmatrix} = 0. \tag{2.150}$$

We know from linear algebra that such an equation for a third order matrix gives rise to a characteristic polynomial for λ of the form[10]

$$\lambda^3 - I_T^{(1)}\lambda^2 + I_T^{(2)}\lambda - I_T^{(3)} = 0, \tag{2.151}$$

where $I_T^{(1)}, I_T^{(2)}, I_T^{(3)}$ are scalars that are functions of all the scalars T_{ij}. The I_T's are known as the *invariants* of the tensor T_{ij}. They can be shown, following a detailed analysis, to be given by

$$I_T^{(1)} = T_{ii} = \text{tr } \mathsf{T}, \tag{2.152}$$

$$I_T^{(2)} = \frac{1}{2}(T_{ii}T_{jj} - T_{ij}T_{ji}) = \frac{1}{2}\left((\text{tr } \mathsf{T})^2 - \text{tr } (\mathsf{T}\cdot\mathsf{T})\right) = (\det \mathsf{T})\left(\text{tr } \mathsf{T}^{-1}\right), \tag{2.153}$$

$$= \frac{1}{2}\left(T_{(ii)}T_{(jj)} + T_{[ij]}T_{[ij]} - T_{(ij)}T_{(ij)}\right), \tag{2.154}$$

$$I_T^{(3)} = \epsilon_{ijk}T_{1i}T_{2j}T_{3k} = \det \mathsf{T}. \tag{2.155}$$

Here "det" denotes the determinant. It can also be shown that if $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}$ are the three eigenvalues, then the invariants can also be expressed as

$$I_T^{(1)} = \lambda^{(1)} + \lambda^{(2)} + \lambda^{(3)}, \tag{2.156}$$

$$I_T^{(2)} = \lambda^{(1)}\lambda^{(2)} + \lambda^{(2)}\lambda^{(3)} + \lambda^{(3)}\lambda^{(1)}, \tag{2.157}$$

$$I_T^{(3)} = \lambda^{(1)}\lambda^{(2)}\lambda^{(3)}. \tag{2.158}$$

In general these eigenvalues, and consequently, the eigenvectors are complex. Additionally, in general the eigenvectors are non-orthogonal. If, however, the matrix we are considering is symmetric, that is often the case in fluid mechanics, it can be formally proven that all the eigenvalues are real and all the eigenvectors are real and orthogonal. If for instance, our tensor is the stress tensor, we will show that it is symmetric in the absence of external couples. The eigenvectors of the stress tensor can form the basis for an intrinsic coordinate system that has its axes aligned with the principal stress on a fluid element. The eigenvalues themselves give the value of the principal stress. This is actually a generalization of the familiar Mohr's[11] circle from solid mechanics.

Example 2.13

Find the principal axes and principal values of stress if the stress tensor is

$$T_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}. \tag{2.159}$$

[10]We employ a slightly more common form here than the similar Eq. (3.10.4) of Panton (2013).

[11]Christian Otto Mohr, 1835-1918, Holstein-born German civil engineer, railroad and bridge designer.

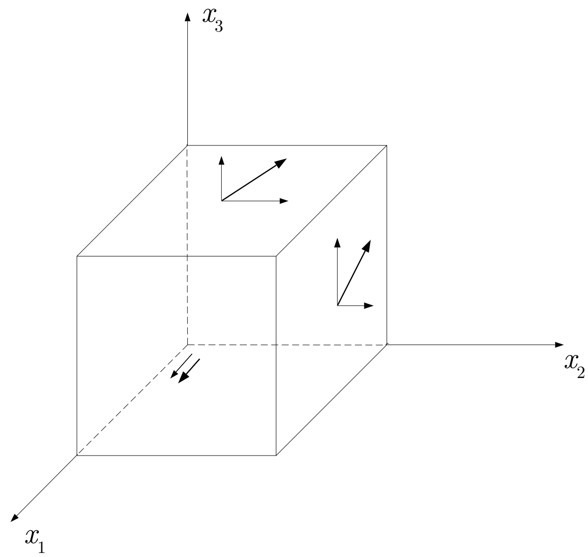

Figure 2.4: Sketch of stresses being applied to a cubical fluid element. The thinner lines with arrows are the components of the stress tensor; the thicker lines on each face represent the vector associated with the particular face.

A sketch of these stresses is shown on the fluid element in Fig. 2.4. We take the eigenvalue problem

$$n_i T_{ij} = \lambda n_j, \tag{2.160}$$

$$= \lambda n_i \delta_{ij}, \tag{2.161}$$

$$n_i \left(T_{ij} - \lambda \delta_{ij}\right) = 0. \tag{2.162}$$

This becomes for our problem

$$\begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}. \tag{2.163}$$

For a non-trivial solution for n_i, we must have

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{vmatrix} = 0. \tag{2.164}$$

This gives rise to the polynomial equation

$$(1-\lambda)\left((1-\lambda)(1-\lambda) - 4\right) = 0. \tag{2.165}$$

This has three solutions

$$\lambda = 1, \qquad \lambda = -1, \qquad \lambda = 3. \tag{2.166}$$

Notice all eigenvalues are real, that we expect because the tensor is symmetric.

Now let us find the eigenvectors (aligned with the principal axes of stress) for this problem First, it can easily be shown that when the vector product of a vector with a tensor commutes when the tensor is symmetric. Although this is not a crucial step, we will use it to write the eigenvalue problem in a slightly more familiar notation:

$$n_i\,(T_{ij} - \lambda\delta_{ij}) = 0 \Longrightarrow (T_{ij} - \lambda\delta_{ij})\,n_i = 0, \qquad \text{because scalar components commute.} \tag{2.167}$$

Because of symmetry, we can now commute the indices to get

$$(T_{ji} - \lambda\delta_{ji})\,n_i = 0, \qquad \text{because indices commute if symmetric.} \tag{2.168}$$

Expanding into matrix notation, we get

$$\begin{pmatrix} T_{11}-\lambda & T_{21} & T_{31} \\ T_{12} & T_{22}-\lambda & T_{32} \\ T_{13} & T_{23} & T_{33}-\lambda \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.169}$$

We have taken the transpose of T. Substituting for T_{ji} and considering the eigenvalue $\lambda = 1$, we get

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.170}$$

We get two equations $2n_2 = 0$, and $2n_3 = 0$; thus, $n_2 = n_3 = 0$. We can satisfy all equations with an arbitrary value of n_1. It is always the case that an eigenvector will have an arbitrary magnitude and a well-defined direction. Here we will choose to normalize our eigenvector and take $n_1 = 1$, so that the eigenvector is

$$n_j = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{for} \qquad \lambda = 1. \tag{2.171}$$

Geometrically, this means that the original 1 face already has an associated vector that is aligned with its normal vector.

Now consider the eigenvector associated with the eigenvalue $\lambda = -1$. Again substituting into the original equation, we get

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.172}$$

This is simply the system of equations

$$2n_1 = 0, \tag{2.173}$$
$$2n_2 + 2n_3 = 0, \tag{2.174}$$
$$2n_2 + 2n_3 = 0. \tag{2.175}$$

Clearly $n_1 = 0$. We could take $n_2 = 1$ and $n_3 = -1$ for a non-trivial solution. Alternatively, let's normalize and take

$$n_j = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}. \tag{2.176}$$

Finally consider the eigenvector associated with the eigenvalue $\lambda = 3$. Again substituting into the original equation, we get

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.177}$$

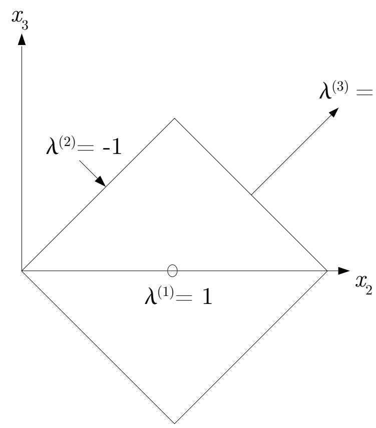

Figure 2.5: Sketch of fluid element rotated to be aligned with axes of principal stress, along with magnitude of principal stress. The 1 face projects out of the page.

This is the system of equations

$$-2n_1 = 0, \quad (2.178)$$
$$-2n_2 + 2n_3 = 0, \quad (2.179)$$
$$2n_2 - 2n_3 = 0. \quad (2.180)$$

Clearly again $n_1 = 0$. We could take $n_2 = 1$ and $n_3 = 1$ for a non-trivial solution. Once again, let us normalize and take

$$n_j = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}. \quad (2.181)$$

In summary, the three eigenvectors and associated eigenvalues are

$$n_j^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{for} \quad \lambda^{(1)} = 1, \quad (2.182)$$
$$n_j^{(2)} = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} \end{pmatrix} \quad \text{for} \quad \lambda^{(2)} = -1, \quad (2.183)$$
$$n_j^{(3)} = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \quad \text{for} \quad \lambda^{(3)} = 3. \quad (2.184)$$

The eigenvectors are mutually orthogonal, as well as normal. We say they form an orthonormal set of vectors. Their orthogonality, as well as the fact that all the eigenvalues are real can be shown to be a direct consequence of the symmetry of the original tensor. A sketch of the principal stresses on the element rotated so that it is aligned with the principal axes of stress is shown on the fluid element in Fig. 2.5. The three orthonormal eigenvectors when cast into a matrix, form an orthogonal matrix Q, and calculation reveals that $\det \mathsf{Q} = 1$, so that it is a rotation matrix.

$$\mathsf{Q} = \begin{pmatrix} \vdots & \vdots & \vdots \\ \mathbf{n}^{(1)} & \mathbf{n}^{(2)} & \mathbf{n}^{(3)} \\ \vdots & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}. \quad (2.185)$$

Example 2.14

For a given stress tensor, that we will take to be symmetric though the theory applies to non-symmetric tensors as well,

$$T_{ij} = \mathsf{T} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 1 \end{pmatrix}, \tag{2.186}$$

find the three basic tensor invariants of stress $I_T^{(1)}$, $I_T^{(2)}$, and $I_T^{(3)}$, and show they are truly invariant when the tensor is subjected to a rotation with direction cosine matrix of

$$\ell_{ij} = \mathsf{Q} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{2.187}$$

Calculation reveals that

$$\det \mathsf{Q} = 1, \tag{2.188}$$

and that $\mathsf{Q} \cdot \mathsf{Q}^T = \mathsf{I}$, so that Q^T is a rotation matrix. The eigenvalues of T, that are the principal values of stress, are easily calculated to be

$$\lambda^{(1)} = 5.28675, \qquad \lambda^{(2)} = -3.67956, \qquad \lambda^{(3)} = 3.39281. \tag{2.189}$$

The three invariants of T_{ij} are

$$I_T^{(1)} = \text{tr } \mathsf{T} = \text{tr} \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 1 \end{pmatrix} = 1 + 3 + 1 = 5, \tag{2.190}$$

$$I_T^{(2)} = \frac{1}{2}\left((\text{tr } \mathsf{T})^2 - \text{tr } (\mathsf{T} \cdot \mathsf{T})\right), \tag{2.191}$$

$$= \frac{1}{2}\left(\left(\text{tr} \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 1 \end{pmatrix}\right)^2 - \text{tr} \left(\begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 1 \end{pmatrix}\right)\right), \tag{2.192}$$

$$= \frac{1}{2}\left(5^2 - \text{tr} \begin{pmatrix} 21 & 4 & 6 \\ 4 & 14 & 4 \\ 6 & 4 & 18 \end{pmatrix}\right), \tag{2.193}$$

$$= \frac{1}{2}(25 - 21 - 14 - 18), \tag{2.194}$$

$$= -14, \tag{2.195}$$

$$I_T^{(3)} = \det \mathsf{T} = \det \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 1 \end{pmatrix} = -66. \tag{2.196}$$

Now when we rotate the tensor T, we get a transformed tensor given by

$$\mathsf{T}' = \mathsf{Q}^T \cdot \mathsf{T} \cdot \mathsf{Q} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad (2.197)$$

$$= \begin{pmatrix} 4.10238 & 2.52239 & 1.60948 \\ 2.52239 & -0.218951 & -2.91291 \\ 1.60948 & -2.91291 & 1.11657 \end{pmatrix}. \quad (2.198)$$

We then seek the tensor invariants of T'. Leaving out some of the details, that are the same as those for calculating the invariants of the T, we find the invariants indeed are invariant:

$$I_T^{(1)} = 4.10238 - 0.218951 + 1.11657 = 5, \quad (2.199)$$

$$I_T^{(2)} = \frac{1}{2}(5^2 - 53) = -14, \quad (2.200)$$

$$I_T^{(3)} = -66. \quad (2.201)$$

Finally, we verify that the stress invariants are indeed related to the principal values (the eigenvalues of the stress tensor) as follows

$$I_T^{(1)} = \lambda^{(1)} + \lambda^{(2)} + \lambda^{(3)} = 5.28675 - 3.67956 + 3.39281 = 5, \quad (2.202)$$

$$I_T^{(2)} = \lambda^{(1)}\lambda^{(2)} + \lambda^{(2)}\lambda^{(3)} + \lambda^{(3)}\lambda^{(1)}, \quad (2.203)$$

$$= (5.28675)(-3.67956) + (-3.67956)(3.39281) + (3.39281)(5.28675) = -14, \quad (2.204)$$

$$I_T^{(3)} = \lambda^{(1)}\lambda^{(2)}\lambda^{(3)} = (5.28675)(-3.67956)(3.39281) = -66. \quad (2.205)$$

Example 2.15

For a given two-dimensional stress tensor, which here we will take to be asymmetric,

$$T_{ij} = \mathsf{T} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}, \quad (2.206)$$

find the two basic tensor invariants of stress $I_T^{(1)}$ and $I_T^{(2)}$ and show they are truly invariant when the tensor is subjected to a rotation with direction cosine matrix of

$$\ell_{ij} = \mathsf{Q} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (2.207)$$

Calculation reveals that $\det \mathsf{Q} = 1$ and that $\mathsf{Q} \cdot \mathsf{Q}^T = \mathsf{I}$, so that Q^T is a rotation matrix. The eigenvalue problem induces the condition

$$\begin{vmatrix} T_{11} - \lambda & T_{12} \\ T_{21} & T_{22} - \lambda \end{vmatrix} = 0. \quad (2.208)$$

This give the characteristic polynomial

$$(T_{11}-\lambda)(T_{22}-\lambda)-T_{12}T_{21} = 0, \tag{2.209}$$
$$\lambda^2-(T_{11}+T_{22})\lambda+(T_{11}T_{22}-T_{12}T_{21}) = 0, \tag{2.210}$$
$$\lambda^2-I_T^{(1)}\lambda+I_T^{(2)} = 0. \tag{2.211}$$

Here we have for the two-dimensional system, the two invariants

$$I_T^{(1)} = T_{11}+T_{22}=\lambda^{(1)}+\lambda^{(2)}=\text{tr }\mathsf{T}, \tag{2.212}$$
$$I_T^{(2)} = T_{11}T_{22}-T_{12}T_{21}=\lambda^{(1)}\lambda^{(2)}=\det\mathsf{T}. \tag{2.213}$$

For this system, the eigenvalues of T are easily calculated to be

$$\lambda^{(1)}=2+\sqrt{2}, \qquad \lambda^{(2)}=2-\sqrt{2}. \tag{2.214}$$

The two invariants of T are

$$I_T^{(1)} = \text{tr }\mathsf{T}=\text{tr}\begin{pmatrix}2&1\\2&2\end{pmatrix}=2+2=4, \tag{2.215}$$
$$I_T^{(2)} = \det\mathsf{T}=\det\begin{pmatrix}2&1\\2&2\end{pmatrix}=2(2)-1(2)=2. \tag{2.216}$$

Now when we rotate the tensor T, we get a transformed tensor given by

$$\mathsf{T}'=\mathsf{Q}^T\cdot\mathsf{T}\cdot\mathsf{Q}=\begin{pmatrix}\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}2&1\\2&2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\\frac{1}{2}&\frac{7}{2}\end{pmatrix}. \tag{2.217}$$

By inspection, we see the invariants of T' are indeed the same as those of T:

$$I_T^{(1)} = \frac{1}{2}+\frac{7}{2}=4, \tag{2.218}$$
$$I_T^{(2)} = \left(\frac{1}{2}\right)\left(\frac{7}{2}\right)-\left(\frac{-1}{2}\right)\left(\frac{1}{2}\right)=2. \tag{2.219}$$

Finally, we also see

$$I_T^{(1)} = \lambda^{(1)}+\lambda^{(2)}=(2+\sqrt{2})+(2-\sqrt{2})=4, \tag{2.220}$$
$$I_T^{(2)} = \lambda^{(1)}\lambda^{(2)}=\left(2+\sqrt{2}\right)\left(2-\sqrt{2}\right)=2. \tag{2.221}$$

2.4 Grad, div, curl, etc.

Thus far, we have mainly dealt with the algebra of vectors and tensors. Now let us consider the calculus. For now, let us consider variables that are a function of the spatial vector x_i. We shall soon allow variation with time t also. We will typically encounter quantities such as

- $\phi(x_i)\rightarrow$ a scalar function of the position vector,
- $v_j(x_i)\rightarrow$ a vector function of the position vector, or
- $T_{jk}(x_i)\rightarrow$ a tensor function of the position vector.

2.4.1 Gradient operator

The gradient operator, sometimes denoted by "grad," is motivated as follows. Consider $\phi(x_i)$, that when written in full is

$$\phi(x_i) = \phi(x_1, x_2, x_3). \tag{2.222}$$

Taking a derivative using the chain rule gives

$$d\phi = \frac{\partial \phi}{\partial x_1}\, dx_1 + \frac{\partial \phi}{\partial x_2}\, dx_2 + \frac{\partial \phi}{\partial x_3}\, dx_3. \tag{2.223}$$

Following Panton (2013), we define a non-traditional, but useful further notation ∂_i for the partial derivative

$$\partial_i \equiv \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_1}\mathbf{e}_1 + \frac{\partial}{\partial x_2}\mathbf{e}_2 + \frac{\partial}{\partial x_3}\mathbf{e}_3 = \nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}. \tag{2.224}$$

So the chain rule is actually

$$d\phi = \partial_1 \phi\ dx_1 + \partial_2 \phi\ dx_2 + \partial_3 \phi\ dx_3, \tag{2.225}$$

that is written using our summation convention as

$$d\phi = \partial_i \phi\ dx_i. \tag{2.226}$$

After commuting so as to juxtapose the i subscript, we have

$$d\phi = dx_i\ \partial_i \phi. \tag{2.227}$$

In Gibbs notation, we say

$$d\phi = d\mathbf{x}^T \cdot \nabla \phi = d\mathbf{x}^T \cdot \text{grad } \phi. \tag{2.228}$$

We can also take the transpose of both sides, recalling that the transpose of a scalar is the scalar itself, to obtain

$$(d\phi)^T = \left(d\mathbf{x}^T \cdot \nabla \phi\right)^T, \tag{2.229}$$

$$d\phi = (\nabla \phi)^T \cdot d\mathbf{x}, \tag{2.230}$$

$$d\phi = \nabla^T \phi \cdot d\mathbf{x}. \tag{2.231}$$

Here we expand ∇^T as $\nabla^T = (\partial_1, \partial_2, \partial_3)$. When ∂_i or ∇ operates on a scalar, it is known as the gradient operator. The gradient operator operating on a scalar function gives rise to a vector function.

We next describe the gradient operator operating on a vector. For vectors in Cartesian index and Gibbs notation, we have, following a similar analysis[12]

$$dv_i = dx_j \partial_j v_i = \partial_j v_i \; dx_j, \tag{2.232}$$

$$d\mathbf{v}^T = d\mathbf{x}^T \cdot \nabla \mathbf{v}^T, \tag{2.233}$$

$$d\mathbf{v} = (\nabla \mathbf{v}^T)^T \cdot d\mathbf{x}, \tag{2.234}$$

$$d\mathbf{v} = (\text{grad } \mathbf{v})^T \cdot d\mathbf{x}. \tag{2.235}$$

Here the quantity $\partial_j v_i$ is the gradient of a vector, that is a tensor. So the gradient operator operating on a vector raises its order by one. The Gibbs notation with transposes suggests properly that the gradient of a vector can be expanded as

$$\nabla \mathbf{v}^T = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} \partial_1 v_1 & \partial_1 v_2 & \partial_1 v_3 \\ \partial_2 v_1 & \partial_2 v_2 & \partial_2 v_3 \\ \partial_3 v_1 & \partial_3 v_2 & \partial_3 v_3 \end{pmatrix}. \tag{2.236}$$

Lastly we consider the gradient operator operating on a tensor. For tensors in Cartesian index notation, we have, following a similar analysis

$$dT_{ij} = dx_k \; \partial_k T_{ij} = \partial_k T_{ij} \; dx_k. \tag{2.237}$$

Here the quantity $\partial_k T_{ij}$ is a third order tensor. So the gradient operator operating on a tensor raises its order by one as well. The Gibbs notation is not straightforward as it can involve something akin to the transpose of a three-dimensional matrix.

2.4.2 Divergence operator

The contraction of the gradient operator on either a vector or a tensor is known as the divergence, sometimes denoted by "div." For the divergence of a vector, we have

$$\partial_i v_i = \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3 = \nabla^T \cdot \mathbf{v} = \text{div } \mathbf{v}. \tag{2.238}$$

The divergence of a vector is a scalar. A vector field that is divergence-free, $\nabla^T \cdot \mathbf{v} = 0$, is defined as *solenoidal*.

For the divergence of a second order tensor, we have

$$\partial_i T_{ij} = \partial_1 T_{1j} + \partial_2 T_{2j} + \partial_3 T_{3j} = \nabla^T \cdot \mathsf{T} = \text{div } \mathsf{T}. \tag{2.239}$$

The divergence operator operating on a tensor gives rise to a row vector. We will sometimes have to transpose this row vector in order to arrive at a column vector, e.g. we will have need for the column vector $\left(\nabla^T \cdot \mathsf{T}\right)^T$. We note that, as with the vector inner product, most texts assume the transpose operation is understood and write the divergence of a vector or tensor simply as $\nabla \cdot \mathbf{v}$ or $\nabla \cdot \mathsf{T}$.

[12] A more common approach, not using the transpose notation, would be to say here for the Gibbs notation that $d\mathbf{v} = d\mathbf{x} \cdot \nabla \mathbf{v}$. However, this is only works if we consider $d\mathbf{v}$ to be a row vector, as $d\mathbf{x} \cdot \nabla \mathbf{v}$ is a row vector. All in all, while at times clumsy, the transpose notation allows a for great deal of clarity and consistency with matrix algebra.

2.4.3 Curl operator

The curl operator is the derivative analog to the cross product. We write it in the following three ways:

$$\omega_i = \epsilon_{ijk}\partial_j v_k, \quad (2.240)$$
$$\boldsymbol{\omega} = \nabla \times \mathbf{v}, \quad (2.241)$$
$$\boldsymbol{\omega} = \text{curl } \mathbf{v}. \quad (2.242)$$

Expanding for $i = 1, 2, 3$ gives

$$\omega_1 = \epsilon_{123}\partial_2 v_3 + \epsilon_{132}\partial_3 v_2 = \partial_2 v_3 - \partial_3 v_2, \quad (2.243)$$
$$\omega_2 = \epsilon_{231}\partial_3 v_1 + \epsilon_{213}\partial_1 v_3 = \partial_3 v_1 - \partial_1 v_3, \quad (2.244)$$
$$\omega_3 = \epsilon_{312}\partial_1 v_2 + \epsilon_{321}\partial_2 v_1 = \partial_1 v_2 - \partial_2 v_1. \quad (2.245)$$

2.4.4 Laplacian operator

The Laplacian[13] operator can operate on a scalar, vector, or tensor function. It is a simple combination of first the gradient followed by the divergence. It yields a function of the same order as that which it operates on. For its most common operation on a scalar, it is denoted as follows:

$$\partial_i\partial_i\phi = \nabla^T \cdot \nabla\ \phi = \nabla^2\phi = \text{div grad } \phi. \quad (2.246)$$

In viscous fluid flow, we will have occasion to have the Laplacian operate on vector:

$$\partial_i\partial_i v_j = \left(\nabla^T \cdot \nabla\ \mathbf{v}^T\right)^T = \left(\nabla^2\mathbf{v}^T\right)^T = \nabla^2\mathbf{v} = \text{div grad } \mathbf{v}. \quad (2.247)$$

2.4.5 Time derivative

Following Panton (2013), we will employ a useful but unusual notation for the partial derivative with respect to time:

$$\partial_o \equiv \frac{\partial}{\partial t}, \quad (2.248)$$

that will be used extensively later.

2.4.6 Relevant theorems

We will use several theorems that are developed in vector calculus. Here we give the simplest of motivations, and simply present them. The reader should consult a standard mathematics text for detailed derivations.

[13]Pierre-Simon Laplace, 1749-1827, Normandy-born French astronomer of humble origin. Educated at Caen, taught in Paris at École Militaire.

2.4.6.1 Fundamental theorem of calculus

The fundamental theorem of calculus is as follows

$$\int_{x=a}^{x=b} f(x)\ dx = \int_{x=a}^{x=b} \left(\frac{d\phi}{dx}\right)\ dx = \phi(b) - \phi(a). \tag{2.249}$$

It effectively says that to find the integral of a function $f(x)$, that is the area under the curve, it suffices to find a function ϕ, whose derivative is f, and evaluate ϕ at each endpoint, and take the difference to find the area under the curve.

2.4.6.2 Gauss's theorem

Gauss's[14] theorem is the analog of the fundamental theorem of calculus extended to volume integrals. It applies to tensor functions of arbitrary order and is as follows:

$$\int_R \partial_i \left(T_{ijk\dots}\right)\ dV = \int_S n_i T_{ijk\dots}\ dS. \tag{2.250}$$

Here R is an arbitrary volume, dV is the element of volume, S is the surface that bounds V, n_i is the outward unit normal to S, and $T_{jk..}$ is an arbitrary tensor function. The surface integral is analogous to evaluating the function at the end points in the fundamental theorem of calculus. In Gibbs notation, we have

$$\int_R \nabla^T \cdot \mathsf{T}\ dV = \int_S \mathbf{n}^T \cdot \mathsf{T}\ dS. \tag{2.251}$$

If we take $T_{jk\dots}$ to be the scalar of unity (whose derivative must be zero), Gauss's theorem reduces to

$$\int_S n_i\ dS = 0. \tag{2.252}$$

That is the unit normal to the surface integrated over the surface, cancels to zero when the entire surface is included. We will use Gauss's theorem extensively. It allows us to convert sometimes difficult volume integrals into easier interpreted surface integrals. It is often useful to use this theorem as a means of toggling back and forth from one form to another.

Example 2.16

Demonstrate the validity of Gauss's theorem for the tensor field

$$\mathsf{T} = \begin{pmatrix} x_1 & x_2 & x_1 \\ x_2 & x_2 & x_3 \\ x_3 & x_3 & x_2 \end{pmatrix}, \tag{2.253}$$

[14] Carl Friedrich Gauss, 1777-1855, Brunswick-born German mathematician, considered the founder of modern mathematics. Worked in astronomy, physics, crystallography, optics, biostatistics, and mechanics. Studied and taught at Göttingen.

where the volume under consideration is the unit cube defined on the domain $x_1 \in [0,1]$, $x_2 \in [0,1]$, $x_3 \in [0,1]$.

We first note that T is asymmetric. We easily see that

$$\nabla^T \cdot \mathsf{T} = \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_1 \\ x_2 & x_2 & x_3 \\ x_3 & x_3 & x_2 \end{pmatrix}, \quad (2.254)$$

$$= \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}. \quad (2.255)$$

Integrating the constant row vector over the unit cube, we find

$$\int_V \nabla^T \cdot \mathsf{T} \, dV = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}. \quad (2.256)$$

Then, we can evaluate the surface integral on each of the six faces and perform a set of six surface integrals. Leaving out the details, we do so, and find

$$\int_S \mathbf{n}^T \cdot \mathsf{T} \, dS = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}. \quad (2.257)$$

This verifies Gauss's theorem for this case. For asymmetric tensors such as our T, we need to be careful about commuting operators. For example, for this problem

$$\int_S \mathsf{T} \cdot \mathbf{n} \, dS = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}. \quad (2.258)$$

2.4.6.3 Stokes' theorem

Stokes'[15] theorem is as follows.

$$\int_S n_i \epsilon_{ijk} \partial_j v_k \, dS = \oint_C v_i \alpha_i \, ds. \quad (2.259)$$

Once again S is a bounding surface and n_i is its outward unit normal. The integral with the circle through it denotes a closed contour integral with respect to arc length s, and α_i is the unit tangent vector to the bounding curve C.

In Gibbs notation, it is written as

$$\int_S \mathbf{n}^T \cdot \nabla \times \mathbf{v} \, dS = \oint_C \mathbf{v}^T \cdot \boldsymbol{\alpha} \, ds. \quad (2.260)$$

[15]Sir George Gabriel Stokes, 1819-1903, Irish-born British physicist and mathematician, holder of the Lucasian chair of Mathematics at Cambridge University, developed, simultaneously with Navier, the governing equations of fluid motion, in a form that was more robust than that of Navier.

2.4.6.4 A useful identity

It is easy to show that a useful identity involving ∇, $\mathbf{v}$ and $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is as follows:

$$v_j \partial_j v_i = \partial_i \left(\frac{1}{2} v_j v_j \right) - \epsilon_{ijk} v_j \omega_k, \tag{2.261}$$

$$\left(\mathbf{v}^T \cdot \nabla \right) \mathbf{v} = \nabla \left(\frac{1}{2} \mathbf{v}^T \cdot \mathbf{v} \right) - \mathbf{v} \times \boldsymbol{\omega}. \tag{2.262}$$

This is easily proved by considering the right hand side of Eq. (2.261), expanding, and using Eqs. (2.241) and then (2.74):

$$\partial_i \left(\frac{1}{2} v_j v_j \right) - \epsilon_{ijk} v_j \omega_k = v_j \partial_i v_j - \epsilon_{ijk} v_j \underbrace{\epsilon_{klm} \partial_l v_m}_{=\omega_k}, \tag{2.263}$$

$$= v_j \partial_i v_j - \epsilon_{kij} \epsilon_{klm} v_j \partial_l v_m, \tag{2.264}$$

$$= v_j \partial_i v_j - \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right) v_j \partial_l v_m, \tag{2.265}$$

$$= \underbrace{v_j \partial_i v_j - v_j \partial_i v_j}_{=0} + v_j \partial_j v_i, \tag{2.266}$$

$$= v_j \partial_j v_i, \qquad \text{QED.} \tag{2.267}$$

2.4.6.5 Leibniz's rule: general transport theorem for arbitrary regions

Leibniz's[16] rule relates time derivatives of integral quantities to a form that distinguishes changes that are happening within the boundaries to changes due to fluxes through boundaries. This is the foundation of the so-called *control volume* approach. Using the nomenclature of Whitaker (1968), p. 92, we also call Leibniz's rule the *general transport theorem*. Leibniz's rule applied to an arbitrary tensorial function is as follows:

$$\frac{d}{dt} \int_{AR(t)} T_{jk...}(x_i, t) \, dV = \int_{AR(t)} \partial_o T_{jk...} \, dV + \int_{AS(t)} n_l w_l T_{jk...} \, dS. \tag{2.268}$$

- $AR(t) \rightarrow$ arbitrary region that is a time-dependent volume,
- $AS(t) \rightarrow$ bounding surface of the arbitrary moving volume,
- $w_l \rightarrow$ velocity vector of points on the moving surface,
- $n_l \rightarrow$ unit normal to moving surface.

[16] Gottfried Wilhelm von Leibniz, 1646-1716, Leipzig-born German philosopher and mathematician. Invented calculus independent of Newton and employed a superior notation to that of Newton.

Say we have the special case in which $T_{jk...} = 1$; then Leibniz's rule reduces to

$$\frac{d}{dt}\int_{AR(t)} dV = \int_{AR(t)} \partial_o(1)\ dV + \int_{AS(t)} n_k w_k(1)\ dS, \quad (2.269)$$

$$\frac{dV_{AR}}{dt} = \int_{AS(t)} n_k w_k\ dS. \quad (2.270)$$

This simply says the total volume of the region, that we call V_{AR}, changes in response to net motion of the bounding surface.

2.4.6.5.1 Material region: Reynolds transport theorem In the special case where the volume contains the same fluid particles, the velocity of the boundary is the fluid particle velocity, $w_l = v_l$, and our general transport theorem becomes, again using the nomenclature of Whitaker (1968), p. 92, the *Reynolds*[17] *transport theorem*:

$$\frac{d}{dt}\int_{MR(t)} T_{jk...}(x_i,t)\ dV = \int_{MR(t)} \partial_o T_{jk...}\ dV + \int_{MS(t)} n_l v_l T_{jk...}\ dS. \quad (2.271)$$

The term $MR(t)$ and $MS(t)$ denote the time-dependent *material region* and *material surface* to denote that the geometry in question always contains the same material particles.

2.4.6.5.2 Fixed region In the special case where the volume is fixed in time, the velocity of the boundary is zero, $w_l = 0$, and our general transport theorem becomes

$$\frac{d}{dt}\int_{FR} T_{jk...}(x_i,t)\ dV = \int_{FR} \partial_o T_{jk...}\ dV. \quad (2.272)$$

In this case there is no time-dependency of the fixed region FR.

2.4.6.5.3 Scalar function In the special case where $T_{jk...}$ is a scalar function f, Leibniz's rule reduces to

$$\frac{d}{dt}\int_{AR(t)} f(x_i,t)\ dV = \int_{AR(t)} \partial_o f(x_i,t)\ dV + \int_{AS(t)} n_l w_l f(x_i,t)\ dS. \quad (2.273)$$

Further, considering one-dimensional cases only, we can then say

$$\frac{d}{dt}\int_{x=a(t)}^{x=b(t)} f(x,t)\ dx = \int_{x=a(t)}^{x=b(t)} \partial_o f\ dx + \frac{db}{dt} f(b(t),t) - \frac{da}{dt} f(a(t),t). \quad (2.274)$$

As in the fundamental theorem of calculus, for the one-dimensional case, we do not have to evaluate a surface integral; instead, we simply must consider the function at its endpoints. Here db/dt and da/dt are the velocities of the bounding surface and analogous to w_k. The terms $f(b(t),t)$ and $f(a(t),t)$ are equivalent to evaluating $T_{jk..}$ on $AS(t)$.

[17]Osborne Reynolds, 1842-1912, Belfast-born British engineer and physicist, educated in mathematics at Cambridge, first professor of engineering at Owens College, Manchester, did fundamental experimental work in fluid mechanics and heat transfer.

2.5 General coordinate transformations

Here we introduce, following Aris (1962), Powers and Sen (2015), Ch. 1.6, and many others, some standard notation from tensor analysis for general coordinate transformations. This extends our analysis of Sec. 2.1.2, that was confined to simple rotation transformations. In this notation, both sub- and superscripts are needed to distinguish between what are known as *covariant* and *contravariant* vectors, that are really different mathematical representations of the same quantity, just cast onto different basis vectors. The basis vectors may or may not be orthonormal. They may not even be orthogonal. All they must be is linearly independent. In brief, we start with a general transformation from the non-Cartesian coordinate, defined here as x^i, to the Cartesian coordinate, defined here as ξ^k:

$$\xi^1 = \xi^1(x^1, x^2, x^3), \tag{2.275}$$
$$\xi^2 = \xi^2(x^1, x^2, x^3), \tag{2.276}$$
$$\xi^3 = \xi^3(x^1, x^2, x^3). \tag{2.277}$$

As an example, this form includes the transformation from a non-Cartesian cylindrical coordinate system to a Cartesian system; this will be taken up in detail in Ch. 7.1. We could also say

$$\xi^k = \xi^k(x^i), \qquad \boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x}). \tag{2.278}$$

Local linearization of the transformation gives

$$\begin{pmatrix} d\xi^1 \\ d\xi^2 \\ d\xi^3 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial \xi^1}{\partial x^1} & \frac{\partial \xi^1}{\partial x^2} & \frac{\partial \xi^1}{\partial x^3} \\ \frac{\partial \xi^2}{\partial x^1} & \frac{\partial \xi^2}{\partial x^2} & \frac{\partial \xi^2}{\partial x^3} \\ \frac{\partial \xi^3}{\partial x^1} & \frac{\partial \xi^3}{\partial x^2} & \frac{\partial \xi^3}{\partial x^3} \end{pmatrix}}_{\mathsf{J}} \begin{pmatrix} dx^1 \\ dx^2 \\ dx^3 \end{pmatrix}, \tag{2.279}$$

$$d\boldsymbol{\xi} = \mathsf{J} \cdot d\mathbf{x}. \tag{2.280}$$

Here the local *Jacobian matrix* J is defined as

$$\mathsf{J} = \begin{pmatrix} \frac{\partial \xi^1}{\partial x^1} & \frac{\partial \xi^1}{\partial x^2} & \frac{\partial \xi^1}{\partial x^3} \\ \frac{\partial \xi^2}{\partial x^1} & \frac{\partial \xi^2}{\partial x^2} & \frac{\partial \xi^2}{\partial x^3} \\ \frac{\partial \xi^3}{\partial x^1} & \frac{\partial \xi^3}{\partial x^2} & \frac{\partial \xi^3}{\partial x^3} \end{pmatrix} = \frac{\partial \xi^k}{\partial x^i}. \tag{2.281}$$

The chain rule for partial differentiation can be used to represent the gradient as

$$\underbrace{\begin{pmatrix} \frac{\partial}{\partial x^1} \\ \frac{\partial}{\partial x^2} \\ \frac{\partial}{\partial x^3} \end{pmatrix}}_{\nabla_{\mathbf{x}}} = \underbrace{\begin{pmatrix} \frac{\partial \xi^1}{\partial x^1} & \frac{\partial \xi^2}{\partial x^1} & \frac{\partial \xi^3}{\partial x^1} \\ \frac{\partial \xi^1}{\partial x^2} & \frac{\partial \xi^2}{\partial x^2} & \frac{\partial \xi^3}{\partial x^2} \\ \frac{\partial \xi^1}{\partial x^3} & \frac{\partial \xi^2}{\partial x^3} & \frac{\partial \xi^3}{\partial x^3} \end{pmatrix}}_{\mathsf{J}^T} \underbrace{\begin{pmatrix} \frac{\partial}{\partial \xi^1} \\ \frac{\partial}{\partial \xi^2} \\ \frac{\partial}{\partial \xi^3} \end{pmatrix}}_{\nabla_{\boldsymbol{\xi}}}, \tag{2.282}$$

$$\nabla_{\mathbf{x}} = \mathsf{J}^T \cdot \nabla_{\boldsymbol{\xi}}. \tag{2.283}$$

Inverting, we find

$$\nabla_{\boldsymbol{\xi}} = \left(\mathsf{J}^T\right)^{-1} \cdot \nabla_{\mathbf{x}}. \tag{2.284}$$

This can be directly compared with Eq. (2.280). In the special case for which the transformation is a rotation, we have $\mathsf{J} = \mathsf{Q}$ and thus $\left(\mathsf{J}^T\right)^{-1} = \left(\mathsf{Q}^T\right)^{-1} = \mathsf{Q}$. In this case, we recover the simpler $d\boldsymbol{\xi} = \mathsf{Q} \cdot d\mathbf{x}$ and $\nabla_{\boldsymbol{\xi}} = \mathsf{Q} \cdot \nabla_{\mathbf{x}}$.

For Cartesian systems, we must have the classical formula for differential distance ds:

$$ds^2 = \left(d\xi^1\right)^2 + \left(d\xi^2\right)^2 + \left(d\xi^3\right)^2 = d\boldsymbol{\xi}^T \cdot d\boldsymbol{\xi}. \tag{2.285}$$

The differential distance must be an invariant in either coordinate representation, so we must have

$$ds^2 = d\boldsymbol{\xi}^T \cdot d\boldsymbol{\xi} = (\mathsf{J} \cdot d\mathbf{x})^T \cdot (\mathsf{J} \cdot d\mathbf{x}) = d\mathbf{x}^T \cdot \underbrace{\mathsf{J}^T \cdot \mathsf{J}}_{\mathsf{G}} \cdot \mathbf{x} = d\mathbf{x}^T \cdot \mathsf{G} \cdot d\mathbf{x}. \tag{2.286}$$

Here we have defined the *metric tensor*

$$\mathsf{G} = \mathsf{J}^T \cdot \mathsf{J}, \tag{2.287}$$

$$g_{ij} = \frac{\partial \xi^k}{\partial x^i} \frac{\partial \xi^k}{\partial x^j}. \tag{2.288}$$

One can also show that

$$g^{ij} = \frac{1}{2} \epsilon^{imn} \epsilon^{jpq} g_{mp} g_{nq}, \tag{2.289}$$

$$g_{ik} g^{kj} = \delta_i^j. \tag{2.290}$$

We also have the *Jacobian determinant* J defined as

$$J = \sqrt{g} = \det \frac{\partial \xi^k}{\partial x^i} = \det \mathsf{J}. \tag{2.291}$$

A vector's contravariant representation is given by v^i. Its covariant representation is given by v_i. The metric tensor links one representation to the other via

$$v_j = v^i g_{ij}, \qquad v^i = g^{ij} v_j. \tag{2.292}$$

In the remaining paragraphs of this chapter, we present some slightly modified text first presented by Powers and Sen (2015) in their Ch. 1.6.5 to better understand the nature of vectors in terms of linear combinations of covariant and contravariant basis vectors. The only requirement we place on the basis vectors is linear independence: they must point in different directions. They need not be unit vectors, and their lengths may differ from one another. Consider the non-orthogonal basis vectors $\mathbf{e}_1, \mathbf{e}_2$, aligned with the x^1 and x^2 directions shown in Fig. 2.6a. The vector $\mathbf{v}$ can then be written as

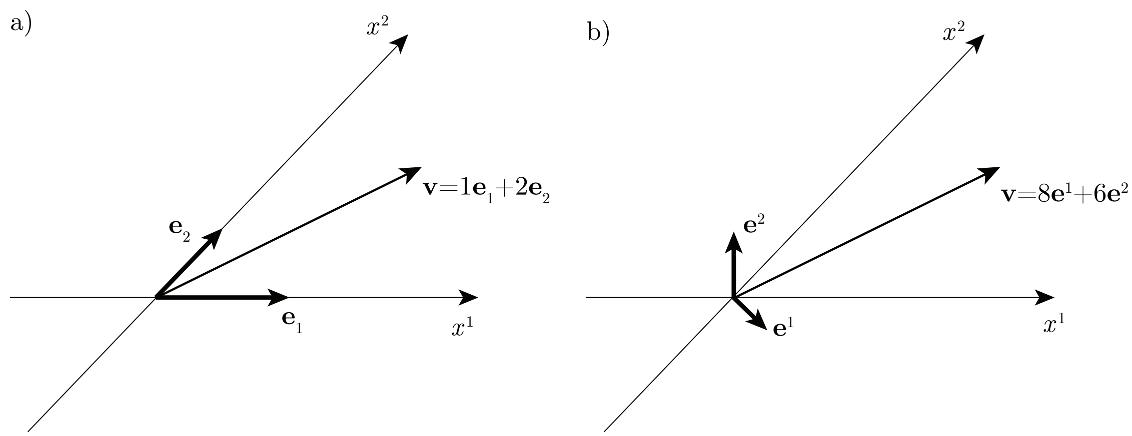

Figure 2.6: Example of a vector $\mathbf{v}$ represented in a non-orthogonal coordinate system along with its a) basis vectors, and b) dual basis vectors. (Figure adapted from Fig. 1.16 of Powers and Sen (2015)).

$$\mathbf{v} = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 \tag{2.293}$$

Here v^1 and v^2 are the contravariant components of $\mathbf{v}$. And the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are the contravariant basis vectors, even though they are subscripted. The entity $\mathbf{v}$ is best thought of as either an entity unto itself or perhaps as a column vector whose components are Cartesian. In matrix form, we can think of $\mathbf{v}$ as

$$\mathbf{v} = v^1\begin{pmatrix}\vdots\\ \mathbf{e}_1\\ \vdots\end{pmatrix} + v^2\begin{pmatrix}\vdots\\ \mathbf{e}_2\\ \vdots\end{pmatrix} = \underbrace{\begin{pmatrix}\vdots & \vdots\\ \mathbf{e}_1 & \mathbf{e}_2\\ \vdots & \vdots\end{pmatrix}}_{\mathsf{J}}\begin{pmatrix}v^1\\ v^2\end{pmatrix}. \tag{2.294}$$

The matrix of basis vectors really acts as a local Jacobian matrix, J, which relates the Cartesian and non-orthogonal representations of $\mathbf{v}$.

Vectors which compose a *dual* or *reciprocal* basis have two characteristics: they are orthogonal to the all the original basis vectors with different indices, and the dot product of each dual vector with respect to the original vector of the same index must be unity. The covariant basis vectors $\mathbf{e}^1, \mathbf{e}^2$ are dual to $\mathbf{e}_1, \mathbf{e}_2$, as shown in Fig. 2.6b. Specifically, we have $\mathbf{e}^{1^T} \cdot \mathbf{e}_2 = 0$, $\mathbf{e}^{2^T} \cdot \mathbf{e}_1 = 0$, $\mathbf{e}^{1^T} \cdot \mathbf{e}_1 = 1$, and $\mathbf{e}^{2^T} \cdot \mathbf{e}_2 = 1$. In matrix form, this is

$$\underbrace{\begin{pmatrix}\cdots & \mathbf{e}^{1T} & \cdots\\ \cdots & \mathbf{e}^{2T} & \cdots\end{pmatrix}}_{\mathsf{J}^{-1}}\underbrace{\begin{pmatrix}\vdots & \vdots\\ \mathbf{e}_1 & \mathbf{e}_2\\ \vdots & \vdots\end{pmatrix}}_{\mathsf{J}} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mathsf{I}. \tag{2.295}$$

Obviously, the matrix of dual vectors can be formed by inverting the matrix of the original basis vectors. We can also represent $\mathbf{v}$ as

$$\mathbf{v} = v_1 \mathbf{e}^1 + v_2 \mathbf{e}^2. \tag{2.296}$$

In matrix form, we can think of $\mathbf{v}$ as

$$\mathbf{v} = v_1 \begin{pmatrix} \vdots \\ \mathbf{e}^1 \\ \vdots \end{pmatrix} + v_2 \begin{pmatrix} \vdots \\ \mathbf{e}^2 \\ \vdots \end{pmatrix} = \underbrace{\begin{pmatrix} \vdots & \vdots \\ \mathbf{e}^1 & \mathbf{e}^2 \\ \vdots & \vdots \end{pmatrix}}_{(\mathsf{J}^{-1})^T} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{2.297}$$

We might also say

$$\mathbf{v}^T = (\, v_1 \quad v_2 \,) \underbrace{\begin{pmatrix} \cdots & \mathbf{e}^{1T} & \cdots \\ \cdots & \mathbf{e}^{2T} & \cdots \end{pmatrix}}_{\mathsf{J}^{-1}}. \tag{2.298}$$

Because the magnitude of $\mathbf{v}$ is independent of its coordinate system, we can say

$$\mathbf{v}^T \cdot \mathbf{v} = (\, v_1 \quad v_2 \,) \underbrace{\underbrace{\begin{pmatrix} \cdots & \mathbf{e}^{1T} & \cdots \\ \cdots & \mathbf{e}^{2T} & \cdots \end{pmatrix}}_{\mathsf{J}^{-1}} \underbrace{\begin{pmatrix} \vdots & \vdots \\ \mathbf{e}_1 & \mathbf{e}_2 \\ \vdots & \vdots \end{pmatrix}}_{\mathsf{J}}}_{\mathsf{I}} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \tag{2.299}$$

and thus

$$\mathbf{v}^T \cdot \mathbf{v} = (\, v_1 \quad v_2 \,) \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = v_i v^i. \tag{2.300}$$

Now we can also transpose Eq. (2.294) to obtain

$$\mathbf{v}^T = (\, v^1 \quad v^2 \,) \underbrace{\begin{pmatrix} \cdots & \mathbf{e}_1^T & \cdots \\ \cdots & \mathbf{e}_2^T & \cdots \end{pmatrix}}_{\mathsf{J}^T}. \tag{2.301}$$

Now combining this with Eq. (2.294) to form $\mathbf{v}^T \cdot \mathbf{v}$, we also see

$$\mathbf{v}^T \cdot \mathbf{v} = (\, v^1 \quad v^2 \,) \underbrace{\begin{pmatrix} \cdots & \mathbf{e}_1^T & \cdots \\ \cdots & \mathbf{e}_2^T & \cdots \end{pmatrix}}_{\mathsf{J}^T} \underbrace{\begin{pmatrix} \vdots & \vdots \\ \mathbf{e}_1 & \mathbf{e}_2 \\ \vdots & \vdots \end{pmatrix}}_{\mathsf{J}} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}. \tag{2.302}$$

Now with the metric tensor $g_{ij} = \mathsf{G} = \mathsf{J}^T \cdot \mathsf{J}$, we have

$$\mathbf{v}^T \cdot \mathbf{v} = \begin{pmatrix} v^1 & v^2 \end{pmatrix} \cdot \mathsf{G} \cdot \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}. \tag{2.303}$$

One can now compare Eq. (2.303) with Eq. (2.300) to infer the covariant components v_1 and v_2. For the same vector $\mathbf{v}$, the covariant components are different than the contravariant components. Thus, for example,

$$v_i = \begin{pmatrix} v^1 & v^2 \end{pmatrix} \cdot \mathsf{G} = v^j g_{ij} = g_{ij} v^j. \tag{2.304}$$

Deducing from Eq. (2.290) that $g^{ij} = \mathsf{G}^{-1}$, we also see

$$g^{ij} v_i = v^j. \tag{2.305}$$

In Cartesian coordinates, a basis and its dual are the same, and so also are the contravariant and covariant components of a vector. For this reason Cartesian vectors and tensors are usually written with only subscripts.

Example 2.17

Consider the vector $\mathbf{v}$ whose Cartesian representation is

$$\mathbf{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}. \tag{2.306}$$

Consider also the set of non-orthogonal basis vectors

$$\mathbf{e}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{2.307}$$

Find the contravariant and covariant components, v^i and v_i, of $\mathbf{v}$.

Here the Jacobian matrix is

$$\mathsf{J} = \begin{pmatrix} \vdots & \vdots \\ \mathbf{e}_1 & \mathbf{e}_2 \\ \vdots & \vdots \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.308}$$

The contravariant components v^i are given by solving

$$\mathbf{v} = \mathsf{J} \cdot \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \tag{2.309}$$

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}. \tag{2.310}$$

Inverting, we find

$$\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{2.311}$$

Thus, we have

$$\mathbf{v} = 1\mathbf{e}_1 + 2\mathbf{e}_2, \tag{2.312}$$

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 1\begin{pmatrix} 2 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{2.313}$$

The covariant basis vectors are given by

$$\mathsf{J}^{-1} = \begin{pmatrix} \cdots & \mathbf{e}^{1T} & \cdots \\ \cdots & \mathbf{e}^{2T} & \cdots \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}. \tag{2.314}$$

Thus, we have

$$\mathbf{e}^1 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \qquad \mathbf{e}^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.315}$$

The metric tensor is given by

$$g_{ij} = \mathsf{G} = \mathsf{J}^T \cdot \mathsf{J} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}. \tag{2.316}$$

We can get the covariant components in many ways. Let us choose

$$v_i = g_{ij}v^j = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}. \tag{2.317}$$

Thus, we have the covariant representation of $\mathbf{v}$ as

$$\mathbf{v} = 8\mathbf{e}^1 + 6\mathbf{e}^2, \tag{2.318}$$

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 8\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} + 6\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.319}$$

In Cartesian coordinates we have

$$\mathbf{v}^T \cdot \mathbf{v} = (\,4 \quad 2\,)\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 20. \tag{2.320}$$

This is invariant under coordinate transformation as in our non-orthogonal coordinate system, we have

$$v_i v^i = (\,8 \quad 6\,)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 20. \tag{2.321}$$

The vectors represented in Fig. 2.6 are proportional to those of this problem.

The *Christoffel*[18] *symbols* are given by

$$\Gamma^m_{ij} = \frac{1}{2}g^{mk}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right). \tag{2.322}$$

[18]Elwin Bruno Christoffel, 1829-1900, German mathematician and physicist.

As discussed in detail by Powers and Sen (2015), such notions as covariant derivatives of contravariant vectors can be defined and can be shown to take the form

$$\nabla_j v^i = \frac{\partial v^i}{\partial x^j} + \Gamma^i_{jl} v^l. \quad (2.323)$$

Here the notation ∇_j is that for a generalized covariant derivative. It can be shown that rotational transformations from Cartesian systems have zero Christoffel symbols. However, more general transformations, such as Cartesian to cylindrical coordinates, have non-trivial Christoffel symbols. Their physical manifestation are terms such as centripetal and Coriolis[19] accelerations, as will be demonstrated in detail in Ch. 7.1.1. These terms are not based on the derivative of the vector, but are related to the vector itself.

[19]Gaspard Gustave de Coriolis, 1792-1843, Paris-born mathematician, taught with Navier, introduced the terms "work" and "kinetic energy" with modern scientific meaning, wrote on the mathematical theory of billiards.

Chapter 3

Kinematics

see Panton, Chapter 4,
see Whitaker, Chapter 3,
see Aris, Chapter 4.

The previous chapter was in many ways a discussion of geometry or place. Here we will consider kinematics, the study of motion in space. Here we will pay no regard to the forces that cause the motion. If we knew the position of every fluid particle as a function of time, then we could in principle also describe the velocity and acceleration of each particle. We could also make statements about how groups of particles translate, rotate, and deform. This is the essence of kinematics. Fluid motion is generally a highly nonlinear phenomenon. In this chapter, we will develop tools, using a local linear analysis, to break down the most complex fluid flows to a summation of fundamental motions.

3.1 Lagrangian description

A Lagrangian[1] description is similar to a classical description of motion in that each fluid particle is effectively labeled and tracked in terms of its initial position x_j^o and time $\hat{t}$. We take the position vector of a particle r_i to be

$$r_i = \tilde{r}_i(x_j^o, \hat{t}). \tag{3.1}$$

The velocity v_i of a particular particle is the time derivative of its position, holding x_j^o fixed:

$$v_i = \left.\frac{\partial \tilde{r}_i}{\partial \hat{t}}\right|_{x_j^o}. \tag{3.2}$$

[1]Joseph-Louis Lagrange (originally Giuseppe Luigi Lagrangia), 1736-1813, Italian born, Italian-French mathematician. Worked on celestial mechanics and the three body problem. Worked in Berlin and Paris. Part of the committee that formulated the metric system.

The acceleration a_i of a particular particle is the second time derivative of its position, holding x_j^o fixed:

$$a_i = \left.\frac{\partial^2 \tilde{r}_i}{\partial \hat{t}^2}\right|_{x_j^o}. \tag{3.3}$$

We can also write other variables as functions of time and initial position, for example, we could have for pressure $p(x_j^o, \hat{t})$.

The Lagrangian description has important pedagogical value, but is only occasionally used in practice, except maybe where it can be useful to illustrate a particular point. In solid mechanics, it is often critically important to know the location of each solid element, and it is the method of choice.

3.2 Eulerian description

It is more common in fluid mechanics to use the Eulerian description of fluid motion. In this description, all variables are taken to be functions of time and *local* position, rather than initial position. Here, we will take the local position to be given by the position vector $x_i = r_i$. A general transformation from one coordinate system (x_i, t) to another $(x_i^o, \hat{t})$ can take the general form

$$t = t(\hat{t}, x_j^o), \tag{3.4}$$
$$x_i = x_i(\hat{t}, x_j^o). \tag{3.5}$$

At this point, we can extend and adapt the analysis introduced in Ch. 2.5. While that discussion was focused on spatial coordinate transformations, there is no reason it cannot be extended to so-called space-time systems such as we have here. The chain rule tells us

$$\begin{pmatrix} dt \\ dx_i \end{pmatrix} = \underbrace{\begin{pmatrix} \left.\frac{\partial t}{\partial \hat{t}}\right|_{x_j^o} & \left.\frac{\partial t}{\partial x_j^o}\right|_{\hat{t}} \\ \left.\frac{\partial x_i}{\partial \hat{t}}\right|_{x_j^o} & \left.\frac{\partial x_i}{\partial x_j^o}\right|_{\hat{t}} \end{pmatrix}}_{\mathsf{J}} \begin{pmatrix} d\hat{t} \\ dx_j^o \end{pmatrix}. \tag{3.6}$$

This has the Jacobian matrix J of

$$\mathsf{J} = \begin{pmatrix} \left.\frac{\partial t}{\partial \hat{t}}\right|_{x_j^o} & \left.\frac{\partial t}{\partial x_j^o}\right|_{\hat{t}} \\ \left.\frac{\partial x_i}{\partial \hat{t}}\right|_{x_j^o} & \left.\frac{\partial x_i}{\partial x_j^o}\right|_{\hat{t}} \end{pmatrix}. \tag{3.7}$$

We will consider the transformation from Lagrangian coordinates to Eulerian coordinates is given by the more specific form

$$t = \hat{t}, \tag{3.8}$$
$$x_i = \tilde{r}_i(\hat{t}, x_j^o). \tag{3.9}$$

This has the more specific Jacobian matrix

$$\mathsf{J} = \begin{pmatrix} 1 & 0 \\ \left.\frac{\partial \tilde{r}_i}{\partial \hat{t}}\right|_{x_j^o} & \left.\frac{\partial \tilde{r}_i}{\partial x_j^o}\right|_{\hat{t}} \end{pmatrix}. \tag{3.10}$$

We have

$$J = \det \mathsf{J} = \det \left.\frac{\partial \tilde{r}_i}{\partial x_j^o}\right|_{\hat{t}}. \tag{3.11}$$

As long as $J > 0$, the transformation is non-singular and orientation-preserving. At $\hat{t} = 0$, we have $x_i = \tilde{r}_i = x_i^o$, Thus at $\hat{t} = 0$, we get $J = \det \delta_{ij} = 1$. So at the initial state, the transformation is volume- and orientation-preserving.

3.3 Material derivatives

The material derivative is the derivative following a fluid particle. It is also known as the substantial derivative or the total derivative. It is trivial in Lagrangian coordinates, because by definition, a Lagrangian description tracks a fluid particle. It is not as straightforward in the Eulerian viewpoint.

Consider a scaler fluid property such as temperature T, that is a function of position and time. We can characterize the position and time in either an Eulerian or Lagrangian fashion. Let the Lagrangian representation be $T = T_L(x_i^o, \hat{t})$ and the Eulerian representation be $T = T_E(x_i, t)$. Now both formulations must give the same result at the same time and position; applying our transformation between the two systems thus yields

$$T = T_L(x_i^o, \hat{t}) = T_E(x_i = \tilde{r}_i(x_j^o, \hat{t}), t = \hat{t}). \tag{3.12}$$

Now from basic calculus we have

$$dx_i = \left.\frac{\partial \tilde{r}_i}{\partial \hat{t}}\right|_{x_j^o} d\hat{t} + \left.\frac{\partial \tilde{r}_i}{\partial x_j^o}\right|_{\hat{t}} dx_j^o. \tag{3.13}$$

From basic calculus, we also have

$$dT_L = \left.\frac{\partial T_L}{\partial \hat{t}}\right|_{x_j^o} d\hat{t} + \left.\frac{\partial T_L}{\partial x_j^o}\right|_{\hat{t}} dx_j^o, \tag{3.14}$$

$$dT_E = \left.\frac{\partial T_E}{\partial t}\right|_{x_i} dt + \left.\frac{\partial T_E}{\partial x_i}\right|_{t} dx_i. \tag{3.15}$$

Now, we must have $dT = dT_L = dT_E$ for the same fluid particle, so making appropriate substitutions, we get

$$\left.\frac{\partial T_L}{\partial \hat{t}}\right|_{x_j^o} d\hat{t} + \left.\frac{\partial T_L}{\partial x_j^o}\right|_{\hat{t}} dx_j^o = \left.\frac{\partial T_E}{\partial t}\right|_{x_i} dt + \left.\frac{\partial T_E}{\partial x_i}\right|_{t} \underbrace{\left(\left.\frac{\partial \tilde{r}_i}{\partial \hat{t}}\right|_{x_j^o} d\hat{t} + \left.\frac{\partial \tilde{r}_i}{\partial x_j^o}\right|_{\hat{t}} dx_j^o \right)}_{dx_i}. \tag{3.16}$$

For the variation of T of a particular particle, we hold x_j^o fixed, so that $dx_j^o = 0$. Using also the fact that $\hat{t} = t$, so $d\hat{t} = dt$, and dividing by $d\hat{t}$, we get

$$\left.\frac{\partial T_L}{\partial \hat{t}}\right|_{x_j^o} = \left.\frac{\partial T_E}{\partial t}\right|_{x_i} + \left.\frac{\partial T_E}{\partial x_i}\right|_t \left.\frac{\partial \tilde{r}_i}{\partial \hat{t}}\right|_{x_j^o}, \tag{3.17}$$

and using the definition of fluid particle velocity, Eq. (3.2), we get

$$\left.\frac{\partial T_L}{\partial \hat{t}}\right|_{x_j^o} = \left.\frac{\partial T_E}{\partial t}\right|_{x_i} + v_i \left.\frac{\partial T_E}{\partial x_i}\right|_t. \tag{3.18}$$

Removing the operands T, T_L, and T_E, and recognizing that holding x_j^o fixed is the same as holding x_i^o fixed, we can write the derivative following a particle in the following manner as an operator

$$\left.\frac{\partial}{\partial \hat{t}}\right|_{x_i^o} = \left.\frac{\partial}{\partial t}\right|_{x_i} + v_i \left.\frac{\partial}{\partial x_i}\right|_t = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} + \mathbf{v}^T \cdot \nabla = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} + \mathbf{v}^T \cdot \text{grad} \equiv \frac{D}{Dt} \equiv \frac{d}{dt}. \tag{3.19}$$

We will generally use the following shorthand for d/dt, the derivative following a particle:

$$\frac{d}{dt} = \partial_o + v_i \partial_i. \tag{3.20}$$

Here we have invoked our shorthand for the spatial gradient operator, ∂_i, Eq. (2.224, and for the partial derivative with respect to time, ∂_o, Eq. (2.248).

We can achieve the same result in a different fashion involving the chain rule. The chain rule gives

$$\begin{pmatrix} \left.\frac{\partial T}{\partial \hat{t}}\right|_{x_j^o} \\ \left.\frac{\partial T}{\partial x_j^o}\right|_{\hat{t}} \end{pmatrix} = \underbrace{\begin{pmatrix} \left.\frac{\partial t}{\partial \hat{t}}\right|_{x_j^o} & \left.\frac{\partial x_i}{\partial \hat{t}}\right|_{x_j^o} \\ \left.\frac{\partial t}{\partial x_j^o}\right|_{\hat{t}} & \left.\frac{\partial x_i}{\partial x_j^o}\right|_{\hat{t}} \end{pmatrix}}_{=\mathsf{J}^T} \begin{pmatrix} \left.\frac{\partial T}{\partial t}\right|_{x_i} \\ \left.\frac{\partial T}{\partial x_i}\right|_t \end{pmatrix}. \tag{3.21}$$

We recognize J^T as the transpose of the Jacobian matrix of the transformation. Invoking our transformation, Eqs. (3.9,3.8), we get

$$\begin{pmatrix} \left.\frac{\partial T}{\partial \hat{t}}\right|_{x_j^o} \\ \left.\frac{\partial T}{\partial x_j^o}\right|_{\hat{t}} \end{pmatrix} = \begin{pmatrix} 1 & \left.\frac{\partial \tilde{r}_i}{\partial \hat{t}}\right|_{x_j^o} \\ 0 & \left.\frac{\partial \tilde{r}_i}{\partial x_j^o}\right|_{\hat{t}} \end{pmatrix} \begin{pmatrix} \left.\frac{\partial T}{\partial t}\right|_{x_i} \\ \left.\frac{\partial T}{\partial x_i}\right|_t \end{pmatrix} = \begin{pmatrix} 1 & v_i \\ 0 & \left.\frac{\partial \tilde{r}_i}{\partial x_j^o}\right|_{\hat{t}} \end{pmatrix} \begin{pmatrix} \left.\frac{\partial T}{\partial t}\right|_{x_i} \\ \left.\frac{\partial T}{\partial x_i}\right|_t \end{pmatrix}. \tag{3.22}$$

Thus, we get the equivalent, after again recognizing that holding x_j^o fixed is the same as holding x_i^o fixed:

$$\left.\frac{\partial T}{\partial \hat{t}}\right|_{x_i^o} = \left.\frac{\partial T}{\partial t}\right|_{x_i} + v_i \left.\frac{\partial T}{\partial x_i}\right|_t. \tag{3.23}$$

We also find from inverting Eq. (3.21), that one gets the relationship

$$\begin{pmatrix} \left.\frac{\partial}{\partial t}\right|_{x_i} \\ \left.\frac{\partial}{\partial x_i}\right|_{t} \end{pmatrix} = \left(\mathsf{J}^T\right)^{-1} \begin{pmatrix} \left.\frac{\partial}{\partial \hat{t}}\right|_{x_j^o} \\ \left.\frac{\partial}{\partial x_j^o}\right|_{\hat{t}} \end{pmatrix}. \tag{3.24}$$

This extends the earlier-developed Eq. (2.284).

Example 3.1

The one-dimensional unsteady motion of a set of fluid particles is given by

$$\tilde{r}(x^o, \hat{t}) = x^o \left(1 + \left(\frac{\hat{t}}{\tau}\right)^2\right), \tag{3.25}$$

where τ is a constant. Heat is transferred to the particles in such a way that its temperature evolution is governed by the equation

$$\left.\frac{\partial T}{\partial \hat{t}}\right|_{x^o} = \frac{T_o x^o}{L\tau}, \qquad T(0, x^o) = T_o \left(\frac{x^o}{L}\right)\left(1 - \frac{x^o}{L}\right). \tag{3.26}$$

Here T_o and L are constant reference temperature and length, respectively. Analyze the fluid motion and temperature evolution in an Eulerian frame.

First note that when $\hat{t} = 0$ that $\tilde{r} = x^o$, as required. The transformation to Eulerian coordinates is given by

$$x = x^o \left(1 + \left(\frac{\hat{t}}{\tau}\right)^2\right), \tag{3.27}$$

$$t = \hat{t}. \tag{3.28}$$

The velocity and acceleration of a fluid particle are

$$v = \left.\frac{\partial \tilde{r}}{\partial \hat{t}}\right|_{x^o} = 2x^o \frac{\hat{t}}{\tau^2}, \tag{3.29}$$

$$a = \left.\frac{\partial^2 \tilde{r}}{\partial \hat{t}^2}\right|_{x^o} = \frac{2x^o}{\tau^2}. \tag{3.30}$$

We can integrate the temperature evolution equation to get

$$T(\hat{t}, x^o) = \frac{T_o x^o \hat{t}}{L\tau} + f(x^o). \tag{3.31}$$

Here $f(x^o)$ is an arbitrary function of x^o, that can be evaluated with the initial condition so that

$$T(\hat{t}, x^o) = T_o \frac{x^o}{L}\frac{\hat{t}}{\tau} + T_o \left(\frac{x^o}{L}\right)\left(1 - \frac{x^o}{L}\right), \tag{3.32}$$

$$= T_o \frac{x^o}{L}\left(\frac{\hat{t}}{\tau} + 1 - \frac{x^o}{L}\right). \tag{3.33}$$

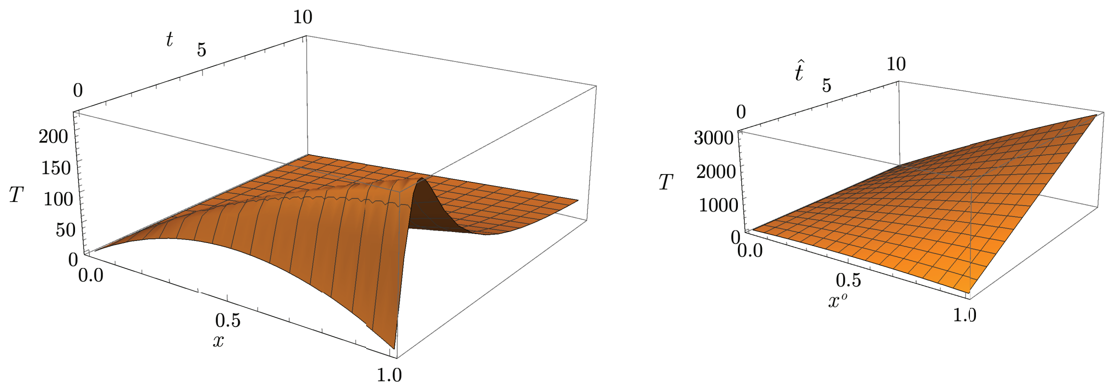

Figure 3.1: Eulerian plot $T(x,t)$ and Lagrangian plot of $T(x^o,\hat{t})$ for example problem originally described in Lagrangian coordinates, with $\tau = 1$, $L = 1$, $T_o = 300$.

Because of the simple nature of the flow field, the transformation to Eulerian coordinates is easy and seen by inspection to give

$$T(t,x) = T_o \frac{x/L}{1+\left(\frac{t}{\tau}\right)^2}\left(\frac{t}{\tau} + 1 - \frac{x/L}{1+\left(\frac{t}{\tau}\right)^2}\right). \tag{3.34}$$

For $\tau = 1$, $L = 1$, $T_o = 300$, plots of $T(x,t)$ and $T(x^o,\hat{t})$ are shown in Fig. 3.1.

Example 3.2

Find the material derivative of $T(t,x_1,x_2,x_3)$ using a simplistic alternative approach based on the definition of the derivative.

If $T = T(t,x_1,x_2,x_3)$, then by definition of the total derivative, we have

$$dT = \frac{\partial T}{\partial t}\,dt + \frac{\partial T}{\partial x_1}\,dx_1 + \frac{\partial T}{\partial x_2}\,dx_2 + \frac{\partial T}{\partial x_3}\,dx_3. \tag{3.35}$$

Let us scale by dt and then say

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial T}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial T}{\partial x_3}\frac{dx_3}{dt}. \tag{3.36}$$

On fluid particle path, we have $dx_1/dt = v_1$, $dx_2/dt = v_2$, $dx_3/dt = v_3$, so we get

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x_1}v_1 + \frac{\partial T}{\partial x_2}v_2 + \frac{\partial T}{\partial x_3}v_3, \tag{3.37}$$

$$= \frac{\partial T}{\partial t} + v_1\frac{\partial T}{\partial x_1} + v_2\frac{\partial T}{\partial x_2} + v_3\frac{\partial T}{\partial x_3}, \tag{3.38}$$

$$= \partial_o T + v_i \partial_i T, \tag{3.39}$$

$$= \frac{\partial T}{\partial t} + \mathbf{v}^T \cdot \nabla T, \tag{3.40}$$

$$= \left(\frac{\partial}{\partial t} + \mathbf{v}^T \cdot \nabla\right) T. \tag{3.41}$$

3.4 Streamlines

Streamlines are lines that are everywhere instantaneously parallel to velocity vectors. If a differential vector dx_k is parallel to a velocity vector v_j, then the cross product of the two vectors must be zero; hence, for a streamline we must have

$$\epsilon_{ijk} v_j \; dx_k = 0. \tag{3.42}$$

In Gibbs notation, we would say

$$\mathbf{v} \times d\mathbf{x} = \mathbf{0}. \tag{3.43}$$

Recalling that the cross product can be interpreted as a determinant, we get this condition to reduce to

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ dx_1 & dx_2 & dx_3 \end{vmatrix} = \mathbf{0}. \tag{3.44}$$

Expanding the determinant gives

$$\mathbf{e}_1(v_2 \; dx_3 - v_3 \; dx_2) + \mathbf{e}_2(v_3 \; dx_1 - v_1 \; dx_3) + \mathbf{e}_3(v_1 \; dx_2 - v_2 \; dx_1) = \mathbf{0}. \tag{3.45}$$

Because the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are linearly independent, the coefficient on each must be zero, giving rise to

$$v_2 \; dx_3 = v_3 \; dx_2, \qquad \Rightarrow \qquad \frac{dx_3}{v_3} = \frac{dx_2}{v_2}, \tag{3.46}$$

$$v_3 \; dx_1 = v_1 \; dx_3, \qquad \Rightarrow \qquad \frac{dx_1}{v_1} = \frac{dx_3}{v_3}, \tag{3.47}$$

$$v_1 \; dx_2 = v_2 \; dx_1, \qquad \Rightarrow \qquad \frac{dx_2}{v_2} = \frac{dx_1}{v_1}. \tag{3.48}$$

Combining, we get

$$\frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3}. \tag{3.49}$$

At a fixed instant in time, $t = t_o$, we set the terms in Eq. (3.49) all equal to an arbitrary differential parameter $d\tau$ to obtain

$$\frac{dx_1}{v_1(x_1, x_2, x_3; t = t_o)} = \frac{dx_2}{v_2(x_1, x_2, x_3; t = t_o)} = \frac{dx_3}{v_3(x_1, x_2, x_3; t = t_o)} = d\tau. \tag{3.50}$$

Here τ should not be thought of as time, but just as a dummy parameter. Streamlines are only defined at a fixed time. While they will generally look different at different times, in the process of actually integrating to obtain them, time does not enter into the calculation. We then divide each equation by $d\tau$ and find they are equivalent to a system of differential equations of the autonomous form

$$\frac{dx_1}{d\tau} = v_1(x_1, x_2, x_3; t = t_o), \qquad x_1(\tau = 0) = x_{1o}, \tag{3.51}$$

$$\frac{dx_2}{d\tau} = v_2(x_1, x_2, x_3; t = t_o), \qquad x_2(\tau = 0) = x_{2o}, \tag{3.52}$$

$$\frac{dx_3}{d\tau} = v_3(x_1, x_2, x_3; t = t_o), \qquad x_3(\tau = 0) = x_{3o}. \tag{3.53}$$

After integration, that in general must be done numerically, we find

$$x_1(\tau; t_o, x_{1o}), \tag{3.54}$$

$$x_2(\tau; t_o, x_{2o}), \tag{3.55}$$

$$x_3(\tau; t_o, x_{3o}), \tag{3.56}$$

where we let the parameter τ vary over whatever domain we choose.

3.5 Pathlines

The pathlines are the locus of points traversed by a particular fluid particle. For an Eulerian description of motion where the velocity field is known as a function of space and time $v_j(x_i, t)$, we can get the pathlines by integrating the following set of three non-autonomous ordinary differential equations, with the associated initial conditions:

$$\frac{dx_1}{dt} = v_1(x_1, x_2, x_3, t), \qquad x_1(t = t_o) = x_{1o}, \tag{3.57}$$

$$\frac{dx_2}{dt} = v_2(x_1, x_2, x_3, t), \qquad x_2(t = t_o) = x_{2o}, \tag{3.58}$$

$$\frac{dx_3}{dt} = v_3(x_1, x_2, x_3, t), \qquad x_3(t = t_o) = x_{3o}. \tag{3.59}$$

In general these are non-linear equations, and often require full numerical solution, that gives us

$$x_1(t; x_{1o}), \tag{3.60}$$

$$x_2(t; x_{2o}), \tag{3.61}$$

$$x_3(t; x_{3o}). \tag{3.62}$$

3.6 Streaklines

A streakline is the locus of points that have passed through a particular point at some past time $t = \hat{t}$. Streaklines can be found by integrating a similar set of equations to those for pathlines.

$$\frac{dx_1}{dt} = v_1(x_1, x_2, x_3, t), \qquad x_1(t = \hat{t}) = x_{1o}, \tag{3.63}$$

$$\frac{dx_2}{dt} = v_2(x_1, x_2, x_3, t), \qquad x_2(t = \hat{t}) = x_{2o}, \tag{3.64}$$

$$\frac{dx_3}{dt} = v_3(x_1, x_2, x_3, t), \qquad x_3(t = \hat{t}) = x_{3o}. \tag{3.65}$$

After integration, that is generally done numerically, we get

$$x_1(t; x_{1o}, \hat{t}), \tag{3.66}$$

$$x_2(t; x_{2o}, \hat{t}), \tag{3.67}$$

$$x_3(t; x_{3o}, \hat{t}). \tag{3.68}$$

Then, if we fix time t and the particular point in which we are interested $(x_{1o}, x_{2o}, x_{3o})^T$, we get a parametric representation of a streakline

$$x_1(\hat{t}), \tag{3.69}$$

$$x_2(\hat{t}), \tag{3.70}$$

$$x_3(\hat{t}). \tag{3.71}$$

Example 3.3

If $v_1 = 2x_1 + t$, $v_2 = x_2 - 2t$, find a) the streamline through the point $(1, 1)^T$ at $t = 1$, b) the pathline for the fluid particle that is at the point $(1, 1)^T$ at $t = 1$, and c) the streakline through the point $(1, 1)^T$ at $t = 1$.

a) streamline

For the streamline, we have the following set of differential equations,

$$\frac{dx_1}{d\tau} = 2x_1 + t|_{t=1}, \qquad x_1(\tau = 0) = 1, \tag{3.72}$$

$$\frac{dx_2}{d\tau} = x_2 - 2t|_{t=1}, \qquad x_2(\tau = 0) = 1. \tag{3.73}$$

Here it is inconsequential where the parameter τ has its origin, as long as *some* value of τ corresponds to a streamline through $(1, 1)^T$, so we have taken the origin for $\tau = 0$ to be the point $(1, 1)^T$. These equations at $t = 1$ are

$$\frac{dx_1}{d\tau} = 2x_1 + 1, \qquad x_1(\tau = 0) = 1, \tag{3.74}$$

$$\frac{dx_2}{d\tau} = x_2 - 2, \qquad x_2(\tau = 0) = 1. \tag{3.75}$$

Solving, we get

$$x_1 = \frac{3}{2}e^{2\tau} - \frac{1}{2}, \tag{3.76}$$

$$x_2 = -e^{\tau} + 2. \tag{3.77}$$

Solving for τ, we find

$$\tau = \frac{1}{2} \ln \left(\frac{2}{3} \left(x_1 + \frac{1}{2} \right) \right). \tag{3.78}$$

So, eliminating τ and writing $x_2(x_1)$, we get the streamline to be

$$x_2 = 2 - \sqrt{\frac{2}{3} \left(x_1 + \frac{1}{2} \right)}. \tag{3.79}$$

b) pathline

For the pathline we have the following equations

$$\frac{dx_1}{dt} = 2x_1 + t, \qquad x_1(t=1) = 1, \tag{3.80}$$

$$\frac{dx_2}{dt} = x_2 - 2t, \qquad x_2(t=1) = 1. \tag{3.81}$$

These have solution

$$x_1 = \frac{7}{4} e^{2(t-1)} - \frac{t}{2} - \frac{1}{4}, \tag{3.82}$$

$$x_2 = -3e^{t-1} + 2t + 2. \tag{3.83}$$

It is algebraically difficult to eliminate t so as to write $x_2(x_1)$ explicitly. However, the analysis certainly gives a parametric representation of the pathline, that can be plotted in x_1, x_2 space.

c) streakline

For the streakline we have the following equations

$$\frac{dx_1}{dt} = 2x_1 + t, \qquad x_1(t=\hat{t}) = 1, \tag{3.84}$$

$$\frac{dx_2}{dt} = x_2 - 2t, \qquad x_2(t=\hat{t}) = 1. \tag{3.85}$$

These have solution

$$x_1 = \frac{5+2\hat{t}}{4} e^{2(t-\hat{t})} - \frac{t}{2} - \frac{1}{4}, \tag{3.86}$$

$$x_2 = -(1+2\hat{t})e^{t-\hat{t}} + 2t + 2. \tag{3.87}$$

We evaluate the streakline at $t = 1$ and get

$$x_1 = \frac{5+2\hat{t}}{4} e^{2(1-\hat{t})} - \frac{3}{4}, \tag{3.88}$$

$$x_2 = -(1+2\hat{t})e^{1-\hat{t}} + 4. \tag{3.89}$$

Once again, it is algebraically difficult to eliminate $\hat{t}$ so as to write $x_2(x_1)$ explicitly. However, the analysis gives a parametric representation of the streakline, that can be plotted in x_1, x_2 space.

A plot of the streamline, pathline, and streakline for this problem is shown in Fig. 3.2. At the point $(1,1)^T$, all three intersect with the same slope. This can also be deduced from the equations governing streamlines, pathlines, and streaklines.

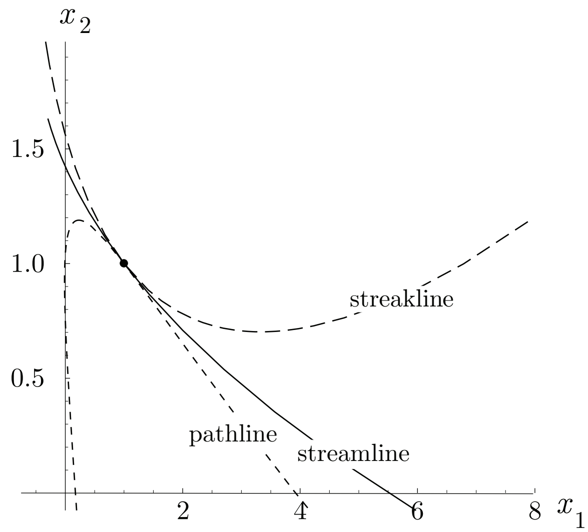

Figure 3.2: Streamline, pathlines, and streaklines for unsteady flow of example problem.

3.7 Kinematic decomposition of motion

In general the motion of a fluid is non-linear in nearly all respects. Certainly, it is common for particle pathlines to have a non-linear path; however, this is not actually a hallmark of non-linearity in that linear theories of fluid motion routinely predict pathlines with finite curvature. More to the point, we cannot in general use the method of superposition to add one flow to another to generate a third. One fundamental source of non-linearity is the non-linear operator $v_i \partial_i$, that we will see appears in most of our governing equations.

However, the local behavior of fluids is nearly always dominated by linear effects. By analyzing only the linear effects induced by small changes in velocity, that we will associate with the velocity gradient, we will learn about the richness of fluid motion. In the linear analysis, we will see that a fluid particle's motion can be described as a summation of

- linear translation,
- rotation as a solid body, and
- straining:
  - extensional, and
  - shear.

Both types of straining can be thought of as deformation rates. We use the word "straining" in contrast to "strain" to distinguish fluid and flexible solid behavior. Generally it is the rate of change of strain (that is the "straining") that has most relevance for a fluid, while it is the actual strain that has the most relevance for a flexible solid. This is because the stress in a flexible solid responds to strain, while the stress in a fluid responds to a strain rate.

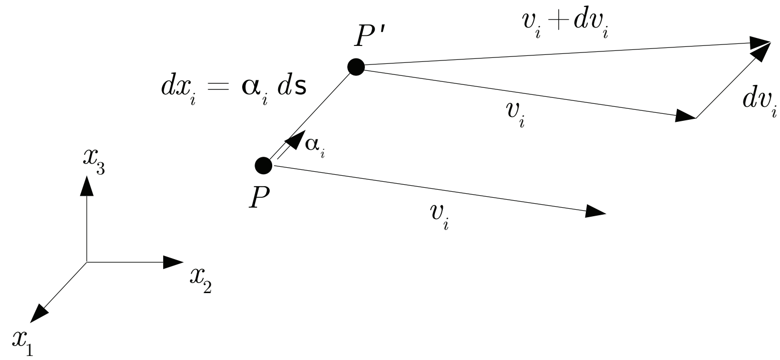

Figure 3.3: Sketch of fluid particle P in motion with velocity v_i and nearby neighbor particle P' with velocity $v_i + dv_i$.

Nevertheless, while strain itself is associated with equilibrium configurations of a flexible solid, when its motion is decomposed, strain rate is relevant. In contrast, a rigid solid can be described by only a sum of linear translation and rotation. A point mass only translates; it cannot rotate or strain.

$$\begin{aligned}
\text{fluid motion} &= \text{translation} + \text{rotation} + \underbrace{\text{extensional straining} + \text{shear straining}}_{\text{straining}}, \\
\text{flexible solid motion} &= \text{translation} + \text{rotation} + \underbrace{\text{extensional straining} + \text{shear straining}}_{\text{straining}}, \\
\text{rigid solid motion} &= \text{translation} + \text{rotation}, \\
\text{point mass motion} &= \text{translation}.
\end{aligned}$$

Let us consider in detail the configuration shown in Fig. 3.3. Here we have a fluid particle at point P with coordinates x_i and velocity v_i. A small distance $dr_i = dx_i$ away is the fluid particle at point P', with coordinates $x_i + dx_i$. This particle moves with velocity $v_i + dv_i$. We can describe the difference in location by the product of a unit tangent vector α_i and a scalar differential distance magnitude $d\mathsf{s}$:

$$dr_i = dx_i = \alpha_i \; d\mathsf{s}. \tag{3.90}$$

Note that α_i is in general not aligned with the velocity vector, and the differential distance $d\mathsf{s}$ is not associated with the arc length along a particle path. Later in Sec. 3.13, we will select an alignment with the particle path, and thus choose $\alpha_i = \alpha_{ti}$ and $d\mathsf{s} = ds$, where α_{ti} is the unit tangent to the particle path and ds is the arc length.

3.7.1 Translation

We have the motion at P' to be $v_i + dv_i$. Obviously, the first term v_i represents translation.

3.7.2 Solid body rotation and straining

What remains is dv_i, and we shall see that it is appropriate to characterize this term as a solid body rotation combined with straining. We have from the definition of a total derivative that

$$\begin{aligned} dv_j &= dx_i\ \partial_i v_j, && (3.91)\\ d\mathbf{v}^T &= d\mathbf{x}^T \cdot \nabla \mathbf{v}^T, && (3.92)\\ d\mathbf{v} &= \left(\nabla \mathbf{v}^T\right)^T \cdot d\mathbf{x}, && (3.93)\\ d\mathbf{v} &= \mathsf{L}^T \cdot d\mathbf{x}. && (3.94) \end{aligned}$$

Here

$$\partial_i v_j = \nabla \mathbf{v}^T \equiv \mathsf{L}, \tag{3.95}$$

is the *velocity gradient tensor*. We can break $\partial_i v_j = \mathsf{L}$ into a symmetric and anti-symmetric part and say then

$$dv_j = \underbrace{dx_i\ \partial_{(i} v_{j)}}_{\text{shear and extensional straining}} + \underbrace{dx_i\ \partial_{[i} v_{j]}}_{\text{rotation}} \tag{3.96}$$

We also will find it useful to decompose the velocity gradient tensor L into a deformation tensor, D:

$$\mathsf{D} = D_{ij} \equiv \partial_{(i} v_{j)}, \tag{3.97}$$

a rotation tensor R:

$$\mathsf{R} = R_{ij} \equiv \partial_{[i} v_{j]}. \tag{3.98}$$

This yields

$$\mathsf{L} = \mathsf{D} + \mathsf{R}. \tag{3.99}$$

Thus,

$$\begin{aligned} dv_j &= dx_i\ D_{ij} + dx_i\ R_{ij} = (\alpha_i D_{ij} + \alpha_i R_{ij})\ d\mathsf{s}, && (3.100)\\ d\mathbf{v}^T &= d\mathbf{x}^T \cdot \mathsf{D} + d\mathbf{x}^T \cdot \mathsf{R} = \left(\boldsymbol{\alpha}^T \cdot \mathsf{D} + \boldsymbol{\alpha}^T \cdot \mathsf{R}\right) d\mathsf{s}, && (3.101)\\ d\mathbf{v} &= \mathsf{D} \cdot d\mathbf{x} + \mathsf{R}^T \cdot d\mathbf{x} = \left(\mathsf{D} \cdot \boldsymbol{\alpha} + \mathsf{R}^T \cdot \boldsymbol{\alpha}\right) d\mathsf{s}. && (3.102) \end{aligned}$$

Let

$$\begin{aligned} dv_j^{(s)} &= dx_i\ \partial_{(i} v_{j)} = \alpha_i \partial_{(i} v_{j)}\ d\mathsf{s}, && (3.103)\\ d\mathbf{v}^{(s)T} &= d\mathbf{x}^T \cdot \mathsf{D} = \boldsymbol{\alpha}^T \cdot \mathsf{D}\ d\mathsf{s}, && (3.104)\\ d\mathbf{v}^{(s)} &= \mathsf{D} \cdot d\mathbf{x} = \mathsf{D} \cdot \boldsymbol{\alpha}\ d\mathsf{s}. && (3.105) \end{aligned}$$

We will see this is associated with straining, both by shear and extension. We will call the symmetric tensor $\partial_{(i}v_{j)} = \mathsf{D}$ the strain rate or deformation tensor.

Further, let

$$dv_j^{(r)} = dx_i \ \partial_{[i}v_{j]} = \alpha_i \partial_{[i}v_{j]} \ d\mathsf{s}, \tag{3.106}$$

$$d\mathbf{v}^{(r)T} = d\mathbf{x}^T \cdot \mathsf{R} = \boldsymbol{\alpha}^T \cdot \mathsf{R} \ d\mathsf{s}, \tag{3.107}$$

$$d\mathbf{v}^{(r)} = \mathsf{R}^T \cdot d\mathbf{x} = \mathsf{R}^T \cdot \boldsymbol{\alpha} \ d\mathsf{s}. \tag{3.108}$$

We will see this is associated with rotation as a solid body, with $\partial_{[i}v_{j]} = \mathsf{R}$ as the rotation tensor.

3.7.2.1 Solid body rotation

Let us examine $dv_j^{(r)}$. First, we define the *vorticity vector* ω_k as the curl of the velocity field

$$\omega_k = \epsilon_{kij}\partial_i v_j, \tag{3.109}$$

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}. \tag{3.110}$$

Let us now split the velocity gradient $\partial_i v_j$ into its symmetric and anti-symmetric parts and recast the vorticity vector as

$$\omega_k = \underbrace{\epsilon_{kij}\partial_{(i}v_{j)}}_{=0} + \epsilon_{kij}\partial_{[i}v_{j]}. \tag{3.111}$$

The first term on the right side is zero because it is the tensor inner product of an anti-symmetric and symmetric tensor. In what remains, we see that half of the vorticity ω_k is actually the dual vector, Ω_k, associated with the anti-symmetric $\partial_{[i}v_{j]}$. See Ch. 2.1.4.5.

$$\omega_k = \epsilon_{kij}\partial_{[i}v_{j]} = \nabla \times \mathbf{v}, \tag{3.112}$$

$$\Omega_k = \frac{1}{2}\omega_k = \frac{1}{2}\epsilon_{kij}\partial_{[i}v_{j]} = \frac{1}{2}\nabla \times \mathbf{v}. \tag{3.113}$$

Using Eq. (2.106) to invert Eq. (3.113), we find

$$\partial_{[i}v_{j]} = \epsilon_{kij}\Omega_k = \frac{1}{2}\epsilon_{kij}\omega_k. \tag{3.114}$$

Thus, we have

$$dv_j^{(r)} = dx_i \ \frac{1}{2}\epsilon_{kij}\omega_k, \tag{3.115}$$

$$= \epsilon_{kij}\left(\frac{\omega_k}{2}\right) dx_i, \tag{3.116}$$

$$= \epsilon_{jki}\left(\frac{\omega_k}{2}\right) dx_i, \tag{3.117}$$

$$= \frac{1}{2}\boldsymbol{\omega} \times d\mathbf{r} \qquad \text{and if} \qquad \boldsymbol{\Omega} = \frac{\boldsymbol{\omega}}{2}, \tag{3.118}$$

$$= \underbrace{\boldsymbol{\Omega} \times d\mathbf{r}}_{\text{Solid body rotation of one point about another}}. \tag{3.119}$$

By introducing this definition for $\mathbf{\Omega}$, we see this term takes on the exact form for the differential velocity due to solid body rotation of P' about P from classical rigid body kinematics. Hence, we give it the same interpretation.

3.7.2.2 Straining

Next we consider the remaining term, that we will associate with straining. First, let us further decompose this into what will be seen to be an extensional *(es)* straining and a shear straining *(ss)*:

$$dv_k^{(s)} = \underbrace{dv_k^{(es)}}_{\text{extension}} + \underbrace{dv_k^{(ss)}}_{\text{shear}}, \tag{3.120}$$

$$d\mathbf{v}^{(s)} = d\mathbf{v}^{(es)} + d\mathbf{v}^{(ss)}. \tag{3.121}$$

3.7.2.2.1 Extensional straining Let us define the extensional straining to be the component of straining in the direction of dx_j. To do this, we need to project $dv_j^{(s)}$ onto the unit vector α_j, then point the result in the direction of that same unit vector;

$$dv_k^{(es)} = \underbrace{\left(\alpha_j dv_j^{(s)}\right)}_{\text{projection of straining}} \alpha_k. \tag{3.122}$$

Now using the definition of $dv_j^{(s)}$, Eq. (3.103), we get

$$dv_k^{(es)} = \left(\alpha_j \underbrace{\left(\alpha_i \partial_{(i} v_{j)} \ d\mathsf{s}\right)}_{=dv_j^{(s)}}\right) \alpha_k, \tag{3.123}$$

$$= \left(\alpha_i \partial_{(i} v_{j)} \alpha_j\right) \alpha_k \ d\mathsf{s}, \tag{3.124}$$

$$d\mathbf{v}^{(es)} = \left(\boldsymbol{\alpha}^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}\right) \boldsymbol{\alpha} \ d\mathsf{s}. \tag{3.125}$$

Now, because $\alpha_i \alpha_j$ is symmetric, we can be led to a useful result. Consider the series of operations involving the velocity gradient, in general asymmetric, and a scalar quantity, $\mathcal{D}$:

$$\mathcal{D} = \boldsymbol{\alpha}^T \cdot \mathsf{L} \cdot \boldsymbol{\alpha}, \tag{3.126}$$

$$= \boldsymbol{\alpha}^T \cdot (\mathsf{D} + \mathsf{R}) \cdot \boldsymbol{\alpha}, \tag{3.127}$$

$$= \boldsymbol{\alpha}^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha} + \underbrace{\boldsymbol{\alpha}^T \cdot \mathsf{R} \cdot \boldsymbol{\alpha}}_{=0}, \tag{3.128}$$

$$= \boldsymbol{\alpha}^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}. \tag{3.129}$$

Thus, we can recast Eq. (3.125) as

$$d\mathbf{v}^{(es)} = \left(\boldsymbol{\alpha}^T \cdot \mathsf{L} \cdot \boldsymbol{\alpha}\right) \boldsymbol{\alpha} \ d\mathsf{s}. \tag{3.130}$$

3.7.2.2.2 Shear straining What straining that is not aligned with the axis connecting P and P' must then be normal to that axis, and is easily visualized to represent a shearing between the two points. Hence the shear straining is

$$\begin{aligned} dv_j^{(ss)} &= dv_j^{(s)} - dv_j^{(es)}, &(3.131)\\ &= \left(\partial_{(j}v_{i)}\alpha_i - \alpha_i\partial_{(i}v_{k)}\alpha_k\alpha_j\right)\ d\mathsf{s}, &(3.132)\\ &= \left(\partial_{(j}v_{i)}\alpha_i - \alpha_p\partial_{(p}v_{k)}\alpha_k\underbrace{\delta_{ji}\alpha_i}_{\alpha_j}\right)\ d\mathsf{s}, &(3.133)\\ &= \left(\partial_{(j}v_{i)} - \left(\alpha_p\partial_{(p}v_{k)}\alpha_k\right)\delta_{ji}\right)\alpha_i\ d\mathsf{s}, &(3.134)\\ d\mathbf{v}^{(ss)} &= \left(\mathsf{D} - \left(\boldsymbol{\alpha}^T\cdot\mathsf{D}\cdot\boldsymbol{\alpha}\right)\mathsf{I}\right)\cdot\boldsymbol{\alpha}\ d\mathsf{s}. &(3.135)\end{aligned}$$

3.7.2.2.3 Principal axes of strain rate We recall from our earlier discussion of Ch. 2.3 that the principal axes of stress are those axes for which the force associated with a given axis points in the same direction as that axis. We can extend this idea to straining, but develop it in a slightly different, but ultimately equivalent fashion based on notions from linear algebra. We first recall that most[2] arbitrary asymmetric square matrices L can be decomposed into a diagonal form as follows:

$$\mathsf{L} = \mathsf{P}\cdot\Lambda\cdot\mathsf{P}^{-1}. \tag{3.136}$$

Here P is a matrix of the same dimension as L that has in its columns the right eigenvectors of L. When L is symmetric, it can be shown that its eigenvalues are guaranteed to be real, and its eigenvectors are guaranteed to be orthogonal. Further, because the eigenvectors can always be scaled by a constant and remain eigenvectors, we can choose to scale them in such a way that they are all normalized. In such a case in which the matrix P has orthonormal columns, the matrix is orthogonal, and we call it Q, as discussed on p. 27. So, when L is symmetric, such as when $\mathsf{L} = \mathsf{D}$, the symmetric part of the velocity gradient, we also have the following decomposition

$$\mathsf{D} = \mathsf{Q}\cdot\Lambda\cdot\mathsf{Q}^{-1}. \tag{3.137}$$

Orthogonal matrices have the property that their transpose is equal to their inverse, Eq. (2.15), and so we also have the even more useful

$$\mathsf{D} = \mathsf{Q}\cdot\Lambda\cdot\mathsf{Q}^{T}. \tag{3.138}$$

[2]Some matrices, that often do not have enough linearly independent eigenvectors, cannot be diagonalized; however, the argument can be extended through use of the singular value decomposition. The singular value decomposition can also be used to effectively diagonalize asymmetric matrices; however, in that case it can be shown there is no equivalent interpretation of the principal axes. Consequently, we will quickly focus the discussion on symmetric matrices.

Geometrically Q can be constructed to be equivalent to a matrix of direction cosines; as we have seen before when so done, its transpose Q^T is a rotation matrix that rotates but does not stretch a vector when it operates on the vector.

Now let us consider the straining component of the velocity difference; taking the symmetric $\partial_{(i}v_{j)} = \mathsf{D}$, that we further assume to be a constant for this analysis, we rewrite Eq. (3.105) using Gibbs notation as

$$\begin{aligned} \left(d\mathbf{v}^{(s)}\right)^T &= d\mathbf{x}^T \cdot \mathsf{D}, &(3.139)\\ d\mathbf{v}^{(s)} &= \mathsf{D}^T \cdot d\mathbf{x}, &(3.140)\\ d\mathbf{v}^{(s)} &= \mathsf{D} \cdot d\mathbf{x}, \qquad \text{because } \mathsf{D} \text{ is symmetric}, &(3.141)\\ d\mathbf{v}^{(s)} &= \mathsf{Q} \cdot \Lambda \cdot \mathsf{Q}^T \cdot d\mathbf{x}. &(3.142)\end{aligned}$$

Now let us select what amounts to a special axes rotation via matrix multiplication by the orthogonal matrix Q^T:

$$\begin{aligned} \mathsf{Q}^T \cdot d\mathbf{v}^{(s)} &= \mathsf{Q}^T \cdot \mathsf{Q} \cdot \Lambda \cdot \mathsf{Q}^T \cdot d\mathbf{x}, &(3.143)\\ &= \mathsf{Q}^{-1} \cdot \mathsf{Q} \cdot \Lambda \cdot \mathsf{Q}^T \cdot d\mathbf{x}, &(3.144)\\ &= \Lambda \cdot \mathsf{Q}^T \cdot d\mathbf{x}, &(3.145)\\ d\underbrace{\left(\mathsf{Q}^T \cdot \mathbf{v}^{(s)}\right)}_{=\mathbf{v}'^{(s)}} &= \Lambda \cdot d\underbrace{\left(\mathsf{Q}^T \cdot \mathbf{x}\right)}_{=\mathbf{x}'}, \text{ because } \mathsf{D} \text{ and thus } \mathsf{Q}^T \text{ are assumed constant}. &(3.146)\end{aligned}$$

Recall from the definition of vectors, Eq. (2.45), that $\mathsf{Q}^T \cdot \mathbf{v}^{(s)} = \mathbf{v}'^{(s)}$ and $\mathsf{Q}^T \cdot \mathbf{x} = \mathbf{x}'$. That is, these are the representations of the vectors in a specially rotated coordinate system, so we have

$$d\mathbf{v}'^{(s)} = \Lambda \cdot d\mathbf{x}'. \tag{3.147}$$

Now because Λ is diagonal, we see that a perturbation in $\mathbf{x}'$ confined to any one of the rotated coordinate axes induces a change in velocity that lies in the same direction as that coordinate axis. For instance on the $1'$ axis, we have $dv'^{(s)}_1 = \Lambda_{11} dx'_1$. That is to say that *in this specially rotated frame, all straining is extensional; there is no shear straining.*

3.7.2.2.4 Extensional strain rate quadric Let us study in some more detail the scalar that is the magnitude of the velocity difference due to extensional strain rate, given by Eq. (3.129):

$$\mathcal{D} = \boldsymbol{\alpha}^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}. \tag{3.148}$$

For a given D, this is a quadratic equation for the components of $\boldsymbol{\alpha}$. However, it is subject to the constraint

$$\boldsymbol{\alpha}^T \cdot \boldsymbol{\alpha} = 1. \tag{3.149}$$

For a two-dimensional system, for example, this becomes

$$\mathcal{D} = D_{11}\alpha_1^2 + 2D_{12}\alpha_1\alpha_2 + D_{22}\alpha_2^2. \tag{3.150}$$

In (α_1, α_2) space and fixed $\mathcal{D}$, this may form an ellipse, hyperbola, or circle, depending on numerical values of D_{ij}. However, we have the constraint

$$\alpha_1^2 + \alpha_2^2 = 1. \tag{3.151}$$

One may imagine that as α_1 is varied for a given D that α_2 will also vary, as will $\mathcal{D}$, and that there could be extreme values of $\mathcal{D}$, depending on α_1.

Because D is symmetric, we can decompose it into a diagonal form and then say

$$\begin{aligned} \mathcal{D} &= \boldsymbol{\alpha}^T \cdot \underbrace{\mathsf{Q} \cdot \Lambda \cdot \mathsf{Q}^T}_{\mathsf{D}} \cdot \boldsymbol{\alpha}, && (3.152) \\ &= \boldsymbol{\alpha}^T \cdot \mathsf{Q} \cdot \Lambda \cdot \mathsf{Q}^T \cdot \boldsymbol{\alpha}, && (3.153) \\ &= (\mathsf{Q}^T \cdot \boldsymbol{\alpha})^T \cdot \Lambda \cdot \mathsf{Q}^T \cdot \boldsymbol{\alpha}. && (3.154) \end{aligned}$$

Then, defining a rotated coordinate system by $\boldsymbol{\alpha}' = \mathsf{Q}^T \cdot \boldsymbol{\alpha}$, we see

$$\mathcal{D} = \boldsymbol{\alpha}'^T \cdot \Lambda \cdot \boldsymbol{\alpha}'. \tag{3.155}$$

Because of the diagonal form of Λ, this is easily written in full as

$$\mathcal{D} = \lambda^{(1)}\alpha_1'^2 + \lambda^{(2)}\alpha_2'^2 + \lambda^{(3)}\alpha_3'^2. \tag{3.156}$$

These are subject to the constraint that $\boldsymbol{\alpha}'$ is a unit vector; thus,

$$1 = \alpha_1'^2 + \alpha_2'^2 + \alpha_3'^2. \tag{3.157}$$

One can formally show through techniques of calculus of variations that $\mathcal{D}$ has a maximum given by the maximum eigenvalue and a minimum given by the minimum eigenvalue.

Example 3.4

Analyze $\mathcal{D}$, the magnitude of the velocity difference attributable to extensional strain in a selected direction $\boldsymbol{\alpha}$ for the deformation tensor

$$\mathsf{D} = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}. \tag{3.158}$$

We first note the eigenvalues of D are given by the roots of the characteristic polynomial

$$(1-\lambda)(5-\lambda) - 1 = 0. \tag{3.159}$$

This yields

$$\lambda^{(1)} = 3 + \sqrt{5} = 5.23607, \qquad \lambda^{(2)} = 3 - \sqrt{5} = 0.76932. \tag{3.160}$$

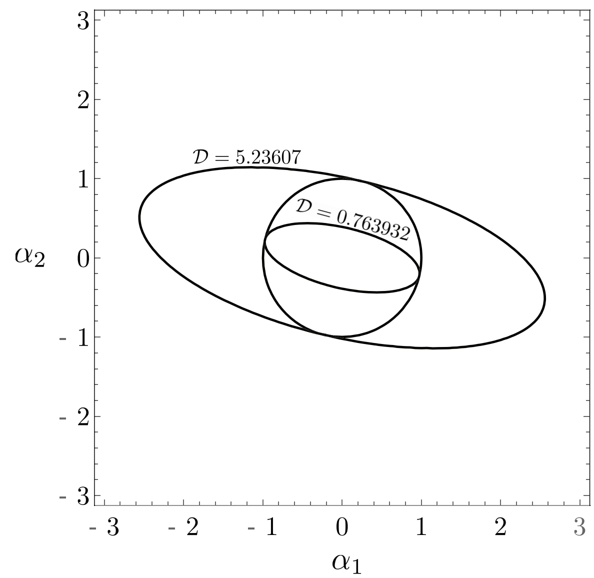

Figure 3.4: Two special contours of $\mathcal{D}$ along with the unit circle illustrating the extreme values of $\mathcal{D}$ as well as the orientation of the axes along with the extreme values of extensional strain are realized.

We have

$$\mathcal{D} = D_{11}\alpha_1^2 + 2D_{12}\alpha_1\alpha_2 + D_{22}\alpha_2^2, \quad (3.161)$$
$$= \alpha_1^2 + 2\alpha_1\alpha_2 + 5\alpha_2^2. \quad (3.162)$$

We can plot contours for which $\mathcal{D}$ is constant in the (α_1, α_2) plane and get an infinite family of curves. However, we also have a constraint, namely $\alpha_1^2 + \alpha_2^2 = 1$.

Our Eq. (3.156) suggests that the eigenvalues may well be special values of the contours of $\mathcal{D}$, and so we examine those two contours:

$$5.23607 = \alpha_1^2 + 2\alpha_1\alpha_2 + 5\alpha_2^2, \quad (3.163)$$
$$0.76932 = \alpha_1^2 + 2\alpha_1\alpha_2 + 5\alpha_2^2. \quad (3.164)$$

These two curves, along with the unit circle $\alpha_1^2 + \alpha_2^2 = 1$ are plotted in Fig. 3.4. An infinite family contours of $\mathcal{D}$ exist. Many of them will also intersect the unit circle, and so are candidate solutions. However the special contours we selected are extreme values. For intersection with the unit circle, we require

$$\mathcal{D} \in [\lambda_{min}, \lambda_{max}], \quad (3.165)$$
$$\in [0.76392, 5.23607]. \quad (3.166)$$

It is easily shown by computing the eigenvectors of D and casting their normalized values into the columns of the orthogonal matrix Q that the diagonal decomposition of D is

$$\mathsf{D} = \mathsf{Q} \cdot \Lambda \cdot \mathsf{Q}^T, \quad (3.167)$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 0.229753 & -0.973249 \\ 0.973249 & 0.229753 \end{pmatrix} \begin{pmatrix} 5.23607 & 0 \\ 0. & 0.763932 \end{pmatrix} \begin{pmatrix} 0.229753 & 0.973249 \\ -0.973249 & 0.229753 \end{pmatrix}. \quad (3.168)$$

We also have

$$\boldsymbol{\alpha}' = \mathsf{Q}^T \cdot \boldsymbol{\alpha}, \tag{3.169}$$

$$= \begin{pmatrix} 0.229753 & 0.973249 \\ -0.973249 & 0.229753 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \tag{3.170}$$

$$= \begin{pmatrix} 0.229753\alpha_1 + 0.973249\alpha_2 \\ -0.973249\alpha_1 + 0.229753\alpha_2 \end{pmatrix}. \tag{3.171}$$

Our equation for $\mathcal{D}$ in rotated and then unrotated coordinates becomes

$$\mathcal{D} = \boldsymbol{\alpha}'^T \cdot \Lambda \cdot \boldsymbol{\alpha}', \tag{3.172}$$

$$= \lambda^{(1)}\alpha_1'^2 + \lambda^{(2)}\alpha_2'^2, \tag{3.173}$$

$$= 5.23607\alpha_1'^2 + 0.763932\alpha_2'^2, \tag{3.174}$$

$$= 5.23607(0.229753\alpha_1 + 0.973249\alpha_2)^2 + 0.763932(-0.973249\alpha_1 + 0.763932\alpha_2)^2. \tag{3.175}$$

Full expansion recovers our original

$$\mathcal{D} = \alpha_1^2 + 2\alpha_1\alpha_2 + 5\alpha_2^2. \tag{3.176}$$

One should prefer Eq. (3.175) over the original, equivalent, and easy to obtain form of Eq. (3.176). That is because Eq. (3.175) is obviously an ellipse because of the positive coefficients on the quadratic terms. Moreover, the form gives the alignment of the major and minor axes via the rotation matrix Q^T. Simple trigonometric analysis shows the principal axes are rotated clockwise by $\theta = 13°$. This can be found by considering one of the orthonormal eigenvectors in Q and computing

$$\theta = \arctan \frac{0.229753}{-0.973249} = -13.2825°. \tag{3.177}$$

3.8 Expansion rate

Consider a small material region of fluid, also called a particle of fluid. As introduced in Ch. 2.4.6.5.1, we define a material region as a region enclosed by a surface across which there is no flux of mass. We shall later see in Ch. 4.1 by invoking the mass conservation axiom for a non-relativistic system, that the implication is that the mass of a material region is constant, but we need not yet consider this. In general the volume containing this particle can increase or decrease. It is useful to quantify the rate of this increase or decrease. Additionally, this will give a flavor of the analysis to come for the conservation axioms.

Taking $MR(t)$ to denote the time-dependent finite material region in space, we must have

$$V_{MR} = \int_{MR(t)} dV. \tag{3.178}$$

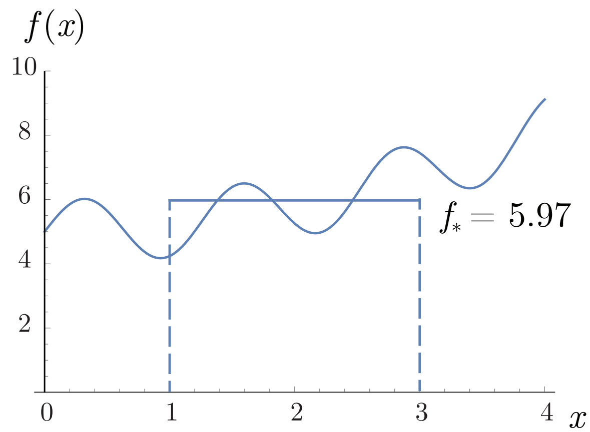

Figure 3.5: Plot illustrating the mean value theorem applied to the function $f(x) = 5+x^2/5+\sin 5x$ within the domain $x \in [1, 3]$. The mean value theorem holds that the mean value of the function is the area under the curve, i.e. the integral, scaled by the domain length, here $3 - 1 = 2$. So the mean value theorem gives $f_* = (\int_1^3 (5 + x^2/5 + \sin 5x)\ dx)/(3 - 1) = (88 + 3(\sin 5)(\sin 10))/15 = 5.97$.

Using the Reynolds transport theorem, Eq. (2.271), we take the time derivative of both sides and obtain

$$\frac{dV_{MR}}{dt} = \int_{MR(t)} \underbrace{\partial_o(1)}_{=0}\ dV + \int_{MS(t)} n_i v_i\ dS, \tag{3.179}$$

$$= \int_{MS(t)} n_i v_i\ dS, \tag{3.180}$$

$$= \int_{MR(t)} \partial_i v_i\ dV, \qquad \text{by Gauss's theorem, Eq. (2.250)}, \tag{3.181}$$

$$= (\partial_i v_i)_* V_{MR}, \qquad \text{by the mean value theorem.} \tag{3.182}$$

We recall from calculus the mean value theorem that states that for any integral, a mean value can be defined, denoted by a $*$, as for example $\int_a^b f(x)\ dx = f_*(b - a)$. We give a simple example illustrating the mean value theorem in Fig. 3.5. As we shrink the size of the material volume to zero, the mean value approaches the local value, so we get

$$\frac{1}{V_{MR}} \frac{dV_{MR}}{dt} = (\partial_i v_i)_*, \tag{3.183}$$

$$\lim_{V_{MR}\to 0} \frac{1}{V_{MR}} \frac{dV_{MR}}{dt} = \partial_i v_i = \nabla^T \cdot \mathbf{v} = \text{div}\ \mathbf{v} = \text{tr}\ \mathsf{D}. \tag{3.184}$$

Equation (3.184) describes the *relative expansion rate* also known as the *dilatation rate* of a material fluid particle. A fluid particle for which $\partial_i v_i = 0$ must have a relative expansion rate of zero, and satisfies conditions to be an *incompressible fluid.* The velocity field for an incompressible fluid is solenoidal.

3.9 Invariants of the strain rate tensor

The tensor associated with straining (also called the deformation rate tensor or strain rate tensor) $\partial_{(i}v_{j)}$ is symmetric. Consequently, it has three real eigenvalues, $\lambda_{\dot{\epsilon}}^{(i)}$, and an orientation for which the strain rate is aligned with the eigenvectors. As with stress, there are also three principal invariants of strain rate, the analog to Eqs. (2.156-2.158):

$$I_{\dot{\epsilon}}^{(1)} = \partial_{(i}v_{i)} = \partial_i v_i = \lambda_{\dot{\epsilon}}^{(1)} + \lambda_{\dot{\epsilon}}^{(2)} + \lambda_{\dot{\epsilon}}^{(3)}, \tag{3.185}$$

$$I_{\dot{\epsilon}}^{(2)} = \frac{1}{2}(\partial_{(i}v_{i)}\partial_{(j}v_{j)} - \partial_{(i}v_{j)}\partial_{(j}v_{i)}) = \lambda_{\dot{\epsilon}}^{(1)}\lambda_{\dot{\epsilon}}^{(2)} + \lambda_{\dot{\epsilon}}^{(2)}\lambda_{\dot{\epsilon}}^{(3)} + \lambda_{\dot{\epsilon}}^{(3)}\lambda_{\dot{\epsilon}}^{(1)}, \tag{3.186}$$

$$I_{\dot{\epsilon}}^{(3)} = \epsilon_{ijk}\partial_{(1}v_{i)}\partial_{(2}v_{j)}\partial_{(3}v_{k)} = \lambda_{\dot{\epsilon}}^{(1)}\lambda_{\dot{\epsilon}}^{(2)}\lambda_{\dot{\epsilon}}^{(3)}. \tag{3.187}$$

The physical interpretation for $I_{\dot{\epsilon}}^{(1)}$ is obvious in that it is equal to the relative rate of volume change for a material element, $(1/V)dV/dt$. Aris (1962) discusses how $I_{\dot{\epsilon}}^{(2)}$ is related to $(1/V)d^2V/dt^2$ and $I_{\dot{\epsilon}}^{(3)}$ is related to $(1/V)d^3V/dt^3$.

3.10 Invariants of the velocity gradient tensor

For completeness, the invariants of the more general velocity gradient tensor are included. They are also analogous to Eqs. (2.156-2.158):

$$I_{\nabla\mathbf{v}}^{(1)} = \partial_i v_i = \lambda_{\nabla\mathbf{v}}^{(1)} + \lambda_{\nabla\mathbf{v}}^{(2)} + \lambda_{\nabla\mathbf{v}}^{(3)}, \tag{3.188}$$

$$I_{\nabla\mathbf{v}}^{(2)} = \frac{1}{2}((\partial_i v_i)(\partial_j v_j) - (\partial_i v_j)(\partial_j v_i)) = \lambda_{\nabla\mathbf{v}}^{(1)}\lambda_{\nabla\mathbf{v}}^{(2)} + \lambda_{\nabla\mathbf{v}}^{(2)}\lambda_{\nabla\mathbf{v}}^{(3)} + \lambda_{\nabla\mathbf{v}}^{(3)}\lambda_{\nabla\mathbf{v}}^{(1)}, \tag{3.189}$$

$$= \frac{1}{2}\left((\partial_i v_i)(\partial_j v_j) + \partial_{[i}v_{j]}\partial_{[i}v_{j]} - \partial_{(i}v_{j)}\partial_{(i}v_{j)}\right), \tag{3.190}$$

$$= \frac{1}{2}\left((\partial_i v_i)(\partial_j v_j) + \frac{1}{2}\omega_i\omega_i - \partial_{(i}v_{j)}\partial_{(i}v_{j)}\right), \tag{3.191}$$

$$I_{\nabla\mathbf{v}}^{(3)} = \epsilon_{ijk}\partial_1 v_i\partial_2 v_j\partial_3 v_k = \lambda_{\nabla\mathbf{v}}^{(1)}\lambda_{\nabla\mathbf{v}}^{(2)}\lambda_{\nabla\mathbf{v}}^{(3)}. \tag{3.192}$$

3.11 Two-dimensional kinematics

Next, consider some important two-dimensional cases, first for general two-dimensional flows, and then for specific examples.

3.11.1 General two-dimensional flows

For two-dimensional motion, we have the velocity vector as $(v_1, v_2, v_3 = 0)$, and for the unit tangent of the vector separating two nearby particles $(\alpha_1, \alpha_2, \alpha_3 = 0)$.

3.11.1.1 Rotation

Recalling that $dx_i = \alpha_i \; d\mathsf{s}$, for rotation, we have from Eq. (3.106)

$$
\begin{aligned}
dv_j^{(r)} &= \partial_{[i}v_{j]} \; dx_i = \alpha_i \partial_{[i}v_{j]} \; d\mathsf{s}, && (3.193)\\
&= \left(\alpha_1 \partial_{[1}v_{j]} + \alpha_2 \partial_{[2}v_{j]}\right) d\mathsf{s}, && (3.194)\\
dv_1^{(r)} &= \left(\alpha_1 \underbrace{\partial_{[1}v_{1]}}_{=0} + \alpha_2 \partial_{[2}v_{1]}\right) d\mathsf{s}, && (3.195)\\
&= \alpha_2 \partial_{[2}v_{1]} \; d\mathsf{s}, && (3.196)\\
dv_2^{(r)} &= \left(\alpha_1 \partial_{[1}v_{2]} + \alpha_2 \underbrace{\partial_{[2}v_{2]}}_{=0}\right) d\mathsf{s}, && (3.197)\\
&= \alpha_1 \partial_{[1}v_{2]} \; d\mathsf{s}. && (3.198)
\end{aligned}
$$

Rewriting in terms of the actual derivatives, we get

$$
\begin{aligned}
dv_1^{(r)} &= \frac{1}{2}\alpha_2 \left(\partial_2 v_1 - \partial_1 v_2\right) d\mathsf{s}, && (3.199)\\
dv_2^{(r)} &= \frac{1}{2}\alpha_1 \left(\partial_1 v_2 - \partial_2 v_1\right) d\mathsf{s}. && (3.200)
\end{aligned}
$$

Also for the vorticity vector, we get

$$
\omega_k = \epsilon_{kij} \partial_i v_j. \qquad (3.201)
$$

The only non-zero component is ω_3, that comes to

$$
\begin{aligned}
\omega_3 &= \underbrace{\epsilon_{311}}_{=0} \partial_1 v_1 + \underbrace{\epsilon_{312}}_{=1} \partial_1 v_2 + \underbrace{\epsilon_{321}}_{=-1} \partial_2 v_1 + \underbrace{\epsilon_{322}}_{=0} \partial_2 v_2, && (3.202)\\
&= \partial_1 v_2 - \partial_2 v_1. && (3.203)
\end{aligned}
$$

3.11.1.2 Extension

$$
\begin{aligned}
dv_k^{(es)} &= \alpha_k \alpha_i \alpha_j \partial_{(i}v_{j)} \; d\mathsf{s}, && (3.204)\\
&= \alpha_k \left(\alpha_1\alpha_1 \partial_{(1}v_{1)} + \alpha_1\alpha_2 \partial_{(1}v_{2)} + \alpha_2\alpha_1 \partial_{(2}v_{1)} + \alpha_2\alpha_2 \partial_{(2}v_{2)}\right) d\mathsf{s} && (3.205)\\
&= \alpha_k \left(\alpha_1^2 \partial_1 v_1 + \alpha_1\alpha_2 \left(\partial_1 v_2 + \partial_2 v_1\right) + \alpha_2^2 \partial_2 v_2\right) d\mathsf{s}, && (3.206)\\
dv_1^{(es)} &= \alpha_1 \left(\alpha_1^2 \partial_1 v_1 + \alpha_1\alpha_2 \left(\partial_1 v_2 + \partial_2 v_1\right) + \alpha_2^2 \partial_2 v_2\right) d\mathsf{s}, && (3.207)\\
dv_2^{(es)} &= \alpha_2 \left(\alpha_1^2 \partial_1 v_1 + \alpha_1\alpha_2 \left(\partial_1 v_2 + \partial_2 v_1\right) + \alpha_2^2 \partial_2 v_2\right) d\mathsf{s}. && (3.208)
\end{aligned}
$$

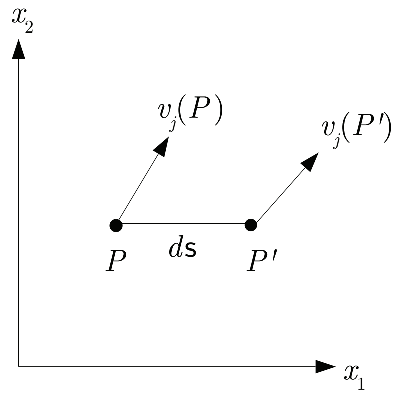

Figure 3.6: Sketch of fluid particle at P and P' in motion.

3.11.1.3 Shear

$$dv_j^{(ss)} = dv_j^{(s)} - dv_j^{(es)}, \tag{3.209}$$

$$= \left(\alpha_i \partial_{(i} v_{j)} - \alpha_j \alpha_i \alpha_k \partial_{(i} v_{k)}\right) d\mathsf{s}, \tag{3.210}$$

$$dv_1^{(ss)} = \left(\alpha_1 \partial_1 v_1 + \alpha_2 \left(\frac{\partial_2 v_1 + \partial_1 v_2}{2}\right) - \alpha_1 \left(\alpha_1^2 \partial_1 v_1 + \alpha_1 \alpha_2 \left(\partial_1 v_2 + \partial_2 v_1\right) + \alpha_2^2 \partial_2 v_2\right)\right) d\mathsf{s}, \tag{3.211}$$

$$dv_2^{(ss)} = \left(\alpha_2 \partial_2 v_2 + \alpha_1 \left(\frac{\partial_1 v_2 + \partial_2 v_1}{2}\right) - \alpha_2 \left(\alpha_1^2 \partial_1 v_1 + \alpha_1 \alpha_2 \left(\partial_1 v_2 + \partial_2 v_1\right) + \alpha_2^2 \partial_2 v_2\right)\right) d\mathsf{s}. \tag{3.212}$$

3.11.1.4 Expansion

$$\frac{1}{V}\frac{dV}{dt} = \partial_1 v_1 + \partial_2 v_2. \tag{3.213}$$

3.11.2 Relative motion along 1 axis

Let us consider in detail the configuration shown in Fig. 3.6 in which the particle separation is along the 1 axis. Hence $\alpha_1 = 1$, $\alpha_2 = 0$, and $\alpha_3 = 0$.

- *Rotation*

$$dv_1^{(r)} = 0, \tag{3.214}$$

$$dv_2^{(r)} = \frac{1}{2}\left(\partial_1 v_2 - \partial_2 v_1\right) \, d\mathsf{s} = \frac{\omega_3}{2} \, d\mathsf{s}. \tag{3.215}$$

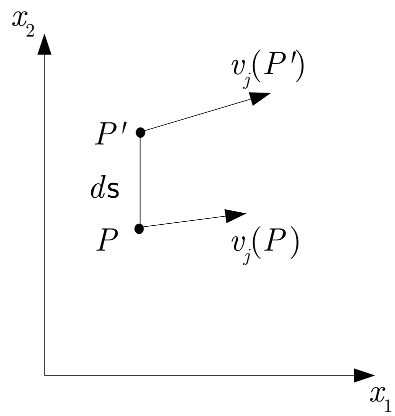

Figure 3.7: Sketch of fluid particle at P and P' in motion.

- *Extension*

$$dv_1^{(es)} = \partial_1 v_1 \; d\mathsf{s}, \tag{3.216}$$
$$dv_2^{(es)} = 0. \tag{3.217}$$

- *Shear*

$$dv_1^{(ss)} = 0, \tag{3.218}$$
$$dv_2^{(ss)} = \frac{1}{2}\left(\partial_1 v_2 + \partial_2 v_1\right) \; d\mathsf{s} = \partial_{(1} v_{2)} \; d\mathsf{s}. \tag{3.219}$$

- *Expansion*:

$$\frac{1}{V}\frac{dV}{dt} = \partial_1 v_1 + \partial_2 v_2. \tag{3.220}$$

3.11.3 Relative motion along 2 axis

Let us consider in detail the configuration shown in Fig. 3.7 in which the particle separation is aligned with the 2 axis. Hence $\alpha_1 = 0$, $\alpha_2 = 1$, and $\alpha_3 = 0$.

- *Rotation*

$$dv_1^{(r)} = \frac{1}{2}\left(\partial_2 v_1 - \partial_1 v_2\right) d\mathsf{s} = -\frac{\omega_3}{2} \; d\mathsf{s}, \tag{3.221}$$
$$dv_2^{(r)} = 0. \tag{3.222}$$

- *Extension*

$$dv_1^{(es)} = 0, \tag{3.223}$$
$$dv_2^{(es)} = \partial_2 v_2 \; d\mathsf{s}. \tag{3.224}$$

- *Shear*

$$dv_1^{(ss)} = \frac{1}{2}\left(\partial_2 v_1 + \partial_1 v_2\right) d\mathsf{s} = \partial_{(1} v_{2)} \; d\mathsf{s}, \tag{3.225}$$
$$dv_2^{(ss)} = 0. \tag{3.226}$$

- *Expansion*:

$$\frac{1}{V}\frac{dV}{dt} = \partial_1 v_1 + \partial_2 v_2. \tag{3.227}$$

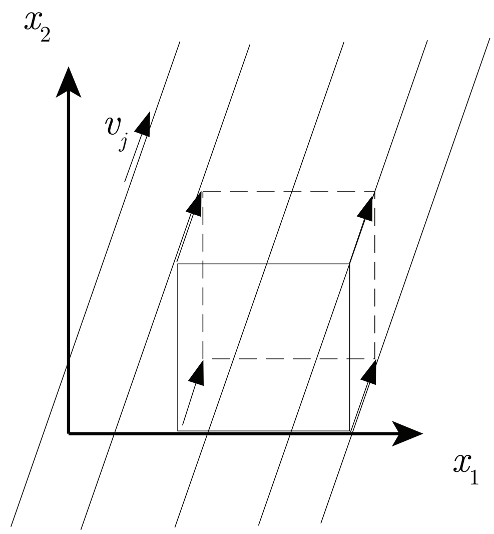

Figure 3.8: Sketch of uniform flow.

3.11.4 Uniform flow

Consider the kinematics of a uniform two-dimensional flow in which

$$v_1 = k_1, \qquad v_2 = k_2, \qquad v_3 = 0, \tag{3.228}$$

as sketched in Fig. 3.8.

- *Streamlines*: $\frac{dx_1}{v_1} = \frac{dx_2}{v_2}$, $\quad \frac{dx_1}{k_1} = \frac{dx_2}{k_2}$, $\quad x_1 = \left(\frac{k_1}{k_2}\right) x_2 + C$.
- *Rotation*: $\omega_3 = \partial_1 v_2 - \partial_2 v_1 = \partial_1(k_1) - \partial_2(k_2) = 0$.
- *Extension*
  - *on 1-axis*: $\partial_1 v_1 = 0$.
  - *on 2-axis*: $\partial_2 v_2 = 0$.
- *Shear for unrotated element*: $\frac{1}{2}(\partial_1 v_2 + \partial_2 v_1) = 0$.
- *Expansion*: $\partial_1 v_1 + \partial_2 v_2 = 0$.
- *Acceleration*:

$$\frac{dv_1}{dt} = \partial_o v_1 + v_1 \partial_1 v_1 + v_2 \partial_2 v_1 = 0 + k_1 \partial_1(k_1) + k_2 \partial_2(k_1) = 0,$$
$$\frac{dv_2}{dt} = \partial_o v_2 + v_1 \partial_1 v_2 + v_2 \partial_2 v_2 = 0 + k_1 \partial_1(k_2) + k_2 \partial_2(k_2) = 0.$$

For this simple flow, the streamlines are straight lines, there is no rotation, no extension, no shear, no expansion, and no acceleration.

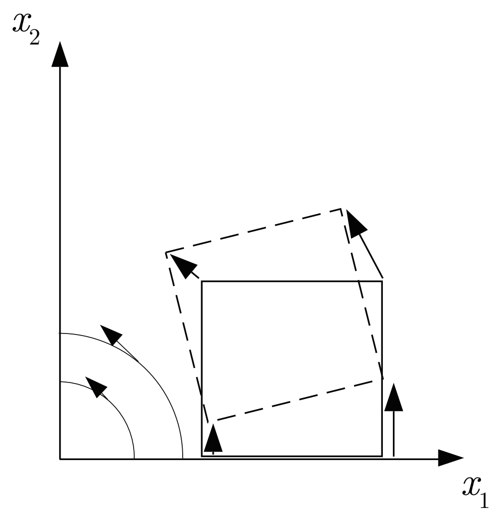

Figure 3.9: Sketch of pure rigid body rotation.

3.11.5 Pure rigid body rotation

Consider the kinematics of a two-dimensional flow in which

$$v_1 = -kx_2, \qquad v_2 = kx_1, \qquad v_3 = 0, \tag{3.229}$$

as sketched in Fig. 3.9.

- *Streamlines*: $\frac{dx_1}{v_1} = \frac{dx_2}{v_2}, \qquad \frac{dx_1}{-kx_2} = \frac{dx_2}{kx_1}, \qquad x_1\ dx_1 = -x_2\ dx_2, \qquad x_1^2 + x_2^2 = C.$
- *Rotation*: $\omega_3 = \partial_1 v_2 - \partial_2 v_1 = \partial_1(kx_1) - \partial_2(-kx_2) = 2k.$
- *Extension*
  - *on 1-axis*: $\partial_1 v_1 = 0,$
  - *on 2-axis*: $\partial_2 v_2 = 0.$
- *Shear for unrotated element*: $\frac{1}{2}\left(\partial_1(kx_1) + \partial_2(-kx_2)\right) = k - k = 0.$
- *Expansion*: $\partial_1 v_1 + \partial_2 v_2 = 0 + 0 = 0.$
- *Acceleration*:

  $\frac{dv_1}{dt} = \partial_o v_1 + v_1\partial_1 v_1 + v_2\partial_2 v_1 = 0 - kx_2\partial_1(-kx_2) + kx_1\partial_2(-kx_2) = -k^2 x_1,$
  $\frac{dv_2}{dt} = \partial_o v_2 + v_1\partial_1 v_2 + v_2\partial_2 v_2 = 0 - kx_2\partial_1(kx_1) + kx_1\partial_2(kx_1) = -k^2 x_2.$

In this flow, the velocity magnitude grows linearly with distance from the origin. This is precisely how a rotating rigid body behaves. The streamlines are circles. The rotation is positive for positive k, hence counterclockwise, there is no deformation in extension or shear, and there is no expansion. The acceleration is pointed towards the origin.

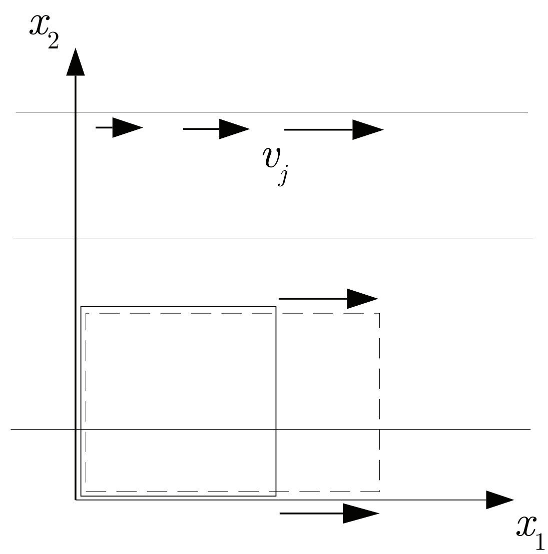

Figure 3.10: Sketch of extensional flow (1-D compressible).

3.11.6 Pure extensional motion (a compressible flow)

Consider the kinematics of a two-dimensional flow in which

$$v_1 = kx_1, \qquad v_2 = 0, \qquad v_3 = 0, \tag{3.230}$$

as sketched in Fig. 3.10.

- *Streamlines*: $\frac{dx_1}{v_1} = \frac{dx_2}{v_2}, \qquad v_2\ dx_1 = v_1\ dx_2, \qquad 0 = kx_1\ dx_2, \qquad x_2 = C.$
- *Rotation*: $\omega_3 = \partial_1 v_2 - \partial_2 v_1 = \partial_1(0) - \partial_2(kx_1) = 0.$
- *Extension*
  - *on 1-axis*: $\partial_1 v_1 = k,$
  - *on 2-axis*: $\partial_2 v_2 = 0.$
- *Shear for unrotated element*: $\frac{1}{2}(\partial_1 v_2 + \partial_2 v_1) = \frac{1}{2}(\partial_1(0) + \partial_2(kx_1)) = 0.$
- *Expansion*: $\partial_1 v_1 + \partial_2 v_2 = k.$
- *Acceleration*:

  $\frac{dv_1}{dt} = \partial_o v_1 + v_1\partial_1 v_1 + v_2\partial_2 v_1 = 0 + kx_1\partial_1(kx_1) + 0\partial_2(kx_1) = k^2x_1,$
  $\frac{dv_2}{dt} = \partial_o v_2 + v_1\partial_1 v_2 + v_2\partial_2 v_2 = 0 + kx_1\partial_1(0) + 0\partial_2(0) = 0.$

In this flow, the streamlines are straight lines; there is no fluid rotation; there is extension (stretching) deformation along the 1-axis, but no shear deformation along this axis. The relative expansion rate is positive for positive k, indicating a compressible flow. The acceleration is confined to the x_1 direction.

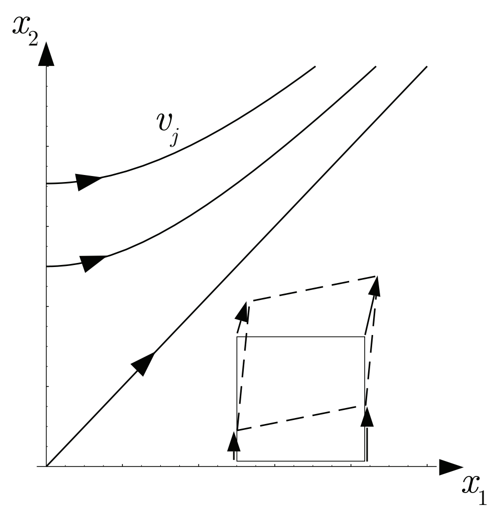

Figure 3.11: Sketch of pure shearing flow.

3.11.7 Pure shear straining

Consider the kinematics of a two-dimensional flow in which

$$v_1 = kx_2, \qquad v_2 = kx_1, \qquad v_3 = 0, \tag{3.231}$$

as sketched in Fig. 3.11.

- *Streamlines*: $\frac{dx_1}{v_1} = \frac{dx_2}{v_2}, \qquad \frac{dx_1}{kx_2} = \frac{dx_2}{kx_1}, \qquad x_1 \; dx_1 = x_2 \; dx_2, \qquad x_1^2 = x_2^2 + C.$
- *Rotation*: $\omega_3 = \partial_1 v_2 - \partial_2 v_1 = \partial_1(kx_1) - \partial_2(kx_2) = k - k = 0.$
- *Extension*
  - *on 1-axis*: $\partial_1 v_1 = \partial_1(kx_2) = 0,$
  - *on 2-axis*: $\partial_2 v_2 = \partial_2(kx_1) = 0.$
- *Shear for unrotated element*: $\frac{1}{2}(\partial_1 v_2 + \partial_2 v_1) = \frac{1}{2}(\partial_1(kx_1) + \partial_2(kx_2)) = k.$
- *Expansion*: $\partial_1 v_1 + \partial_2 v_2 = 0.$
- *Acceleration*:

  $\frac{dv_1}{dt} = \partial_o v_1 + v_1 \partial_1 v_1 + v_2 \partial_2 v_1 = 0 + kx_2 \partial_1(kx_2) + kx_1 \partial_2(kx_2) = k^2 x_1,$
  $\frac{dv_2}{dt} = \partial_o v_2 + v_1 \partial_1 v_2 + v_2 \partial_2 v_2 = 0 + kx_2 \partial_1(kx_1) + kx_1 \partial_2(kx_1) = k^2 x_2.$

In this flow, the streamlines are hyperbolas; there is no rotation or axial extension along the coordinate axes; there is positive shear deformation for an element aligned with the coordinate axes, and no expansion. So, the pure shear deformation preserves volume. The fluid is accelerating away from the origin.

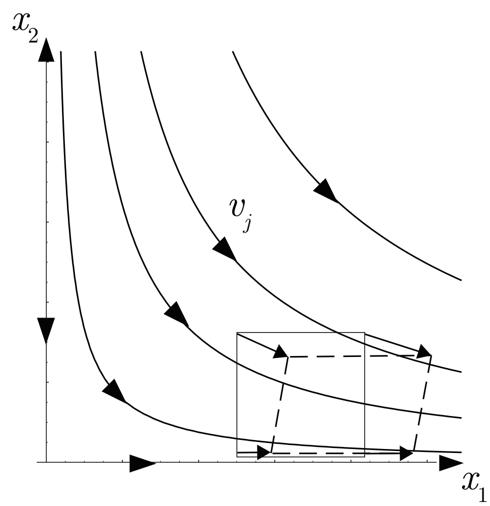

Figure 3.12: Sketch of an ideal corner flow.

3.11.8 Ideal corner flow

Consider the kinematics of a two-dimensional flow in which

$$v_1 = kx_1, \qquad v_2 = -kx_2, \qquad v_3 = 0, \tag{3.232}$$

as sketched in Fig. 3.12.

- *Streamlines*: $\frac{dx_1}{v_1} = \frac{dx_2}{v_2}$, $\quad \frac{dx_1}{kx_1} = \frac{dx_2}{-kx_2}$, $\quad \ln x_1 = -\ln x_2 + C'$, $\quad x_1 x_2 = C$.
- *Rotation*: $\omega_3 = \partial_1 v_2 - \partial_2 v_1 = \partial_1(-kx_2) - \partial_2(kx_1) = 0$.
- *Extension*
  - *on 1-axis*: $\partial_1 v_1 = \partial_1(kx_1) = k$,
  - *on 2-axis*: $\partial_2 v_2 = \partial_2(-kx_2) = -k$.
- *Shear for unrotated element*: $\frac{1}{2}(\partial_1 v_2 + \partial_2 v_1) = \frac{1}{2}(\partial_1(-kx_2) + \partial_2(kx_1)) = 0$.
- *Expansion*: $\partial_1 v_1 + \partial_2 v_2 = k - k = 0$.
- *Acceleration*:

  $\frac{dv_1}{dt} = \partial_o v_1 + v_1 \partial_1 v_1 + v_2 \partial_2 v_1 = 0 + kx_1 \partial_1(kx_1) - kx_2 \partial_2(kx_1) = k^2 x_1$,
  $\frac{dv_2}{dt} = \partial_o v_2 + v_1 \partial_1 v_2 + v_2 \partial_2 v_2 = 0 + kx_1 \partial_1(-kx_2) - kx_2 \partial_2(-kx_2) = k^2 x_2$.

In this flow, the streamlines are hyperbolas; there is no rotation or shear along the coordinate axes; there is extensional strain for an element aligned with the coordinate axes, but no net expansion. So, the ideal corner flow preserves volume. The fluid is accelerating away from the origin.

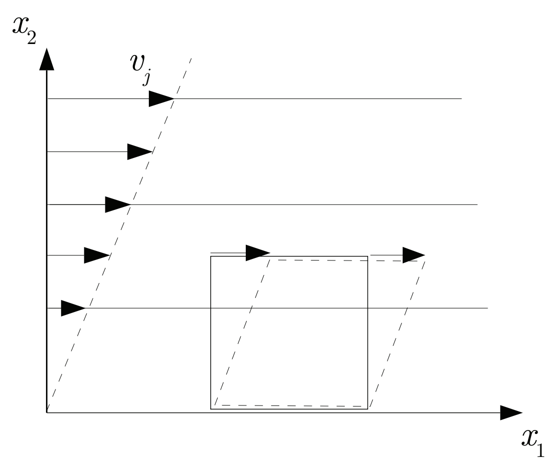

Figure 3.13: Sketch of Couette flow.

3.11.9 Couette flow: shear + rotation

Consider the kinematics of a two-dimensional flow in which

$$v_1 = kx_2, \qquad v_2 = 0, \qquad v_3 = 0, \tag{3.233}$$

as sketched in Fig. 3.13. This is known as a Couette[3] flow.

- *Streamlines*: $\frac{dx_1}{v_1} = \frac{dx_2}{v_2}, \qquad \frac{dx_1}{kx_2} = \frac{dx_2}{0}, \qquad 0 = kx_2\, dx_2, \qquad x_2 = C.$
- *Rotation*: $\omega_3 = \partial_1 v_2 - \partial_2 v_1 = \partial_1(0) - \partial_2(kx_2) = -k.$
- *Extension*
  - *on 1-axis*: $\partial_1 v_1 = \partial_1(kx_2) = 0,$
  - *on 2-axis*: $\partial_2 v_2 = \partial_2(0) = 0.$
- *Shear for unrotated element*: $\frac{1}{2}(\partial_1 v_2 + \partial_2 v_1) = \frac{1}{2}(\partial_1(0) + \partial_2(kx_2)) = \frac{k}{2}.$
- *Expansion*: $\partial_1 v_1 + \partial_2 v_2 = 0.$
- *Acceleration*:

  $\frac{dv_1}{dt} = \partial_o v_1 + v_1\partial_1 v_1 + v_2\partial_2 v_1 = 0 + kx_2\partial_1(kx_2) + 0\partial_2(kx_2) = 0,$
  $\frac{dv_2}{dt} = \partial_o v_2 + v_1\partial_1 v_2 + v_2\partial_2 v_2 = 0 + kx_2\partial_1(0) + 0\partial_2(0) = 0.$

Here the streamlines are straight lines, and the flow is rotational (clockwise because $\omega < 0$ for $k > 0$)! The constant volume rotation is combined with a constant volume shear deformation for the element aligned with the coordinate axes. The fluid is not accelerating.

[3]Maurice Marie Alfred Couette, 1858-1943, French fluid mechanician, rheologist and teacher; student of Joseph Valentin Boussinesq, and faculty member at Catholic University of Angers.

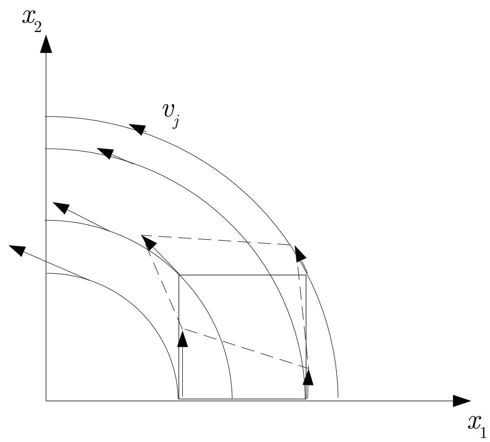

Figure 3.14: Sketch of ideal irrotational vortex.

3.11.10 Ideal irrotational vortex: extension + shear

Consider the kinematics of a two-dimensional flow sketched in Fig. 3.14.

$$v_1 = -k\frac{x_2}{x_1^2 + x_2^2}, \qquad v_2 = k\frac{x_1}{x_1^2 + x_2^2}, \qquad v_3 = 0. \tag{3.234}$$

- *Streamlines*: $\frac{dx_1}{v_1} = \frac{dx_2}{v_2}$, $\quad \frac{dx_1}{-k\frac{x_2}{x_1^2+x_2^2}} = \frac{dx_2}{k\frac{x_1}{x_1^2+x_2^2}}$, $\quad -\frac{dx_1}{x_2} = \frac{dx_2}{x_1}$, $\quad x_1^2 + x_2^2 = C$.
- *Rotation*: $\omega_3 = \partial_1 v_2 - \partial_2 v_1 = \partial_1 \left(k\frac{x_1}{x_1^2+x_2^2}\right) - \partial_2 \left(-k\frac{x_2}{x_1^2+x_2^2}\right) = 0$.
- *Extension*
  - *on 1-axis*: $\partial_1 v_1 = \partial_1 \left(-k\frac{x_2}{x_1^2+x_2^2}\right) = 2k\frac{x_1 x_2}{(x_1^2+x_2^2)^2}$,
  - *on 2-axis*: $\partial_2 v_2 = \partial_2 \left(k\frac{x_1}{x_1^2+x_2^2}\right) = -2k\frac{x_1 x_2}{(x_1^2+x_2^2)^2}$.
- *Shear for unrotated element*: $\frac{1}{2}\left(\partial_1 v_2 + \partial_2 v_1\right) = k\frac{x_2^2 - x_1^2}{(x_1^2+x_2^2)^2}$.
- *Expansion*: $\partial_1 v_1 + \partial_2 v_2 = 0$.
- *Acceleration*:
  $\frac{dv_1}{dt} = \partial_o v_1 + v_1\partial_1 v_1 + v_2 \partial_2 v_1 = -\frac{k^2 x_1}{(x_1^2+x_2^2)^2}$,
  $\frac{dv_2}{dt} = \partial_o v_2 + v_1\partial_1 v_2 + v_2 \partial_2 v_2 = -\frac{k^2 x_2}{(x_1^2+x_2^2)^2}$.

The streamlines are circles, and the fluid element does not rotate about its own axis! It does rotate about the origin. It deforms by extension and shear in such a way that overall the volume is constant.

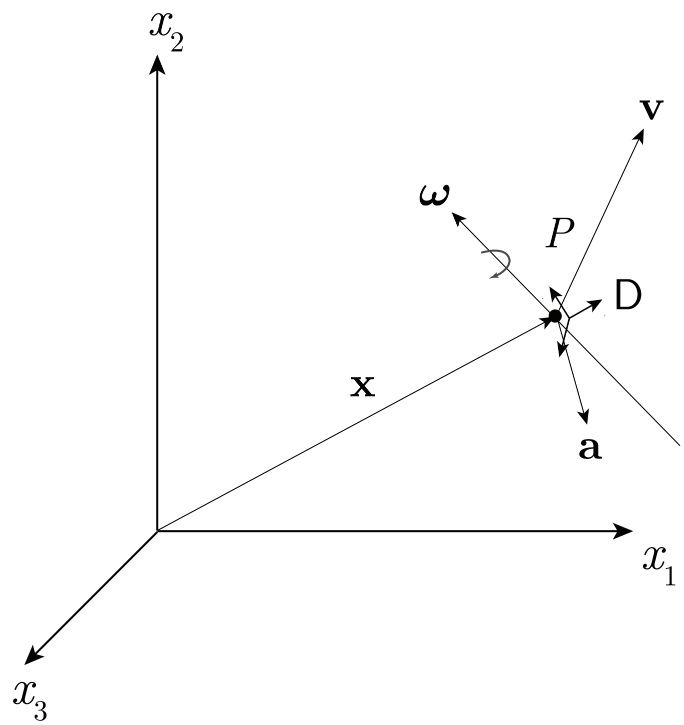

Figure 3.15: Sketch of three-dimensional kinematics.

3.12 Three-dimensional kinematics: summary

We graphically summarize three-dimensional kinematics in Fig. 3.15. Here we depict the motion of a fluid particle P, that has Eulerian position vector $\mathbf{x}$, velocity vector $\mathbf{v}$, and acceleration vector $\mathbf{a}$. The particle P is also shown undergoing a solid-body rotation about an axis aligned with the vorticity vector $\boldsymbol{\omega}$. It also undergoes an extensional deformation aligned with the principal axes of the deformation tensor D. As the position $\mathbf{x}$ is varied continuously, all field quantities, $\mathbf{v}$, $\mathbf{a}$, $\boldsymbol{\omega}$, and D, vary continuously as well.

3.13 Kinematics as a dynamical system

Let us apply some standard notions from dynamical systems theory (see Powers and Sen (2015), Ch. 9.6; Powers (2016), Ch. 6.1; or Paolucci (2016), Ch. 3.2.6) to fluid kinematics. Let us imagine that we are given a time-independent flow field, where the fluid velocity is known and is a function of position only. Then the motion of an individual fluid particle is governed by the following autonomous system of non-linear ordinary differential equations:

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}(t)), \qquad \mathbf{x}(0) = \mathbf{x}^o. \tag{3.235}$$

Here, the initial position of the fluid particle is given by the constant vector $\mathbf{x}^o$. The solution of Eq. (3.235) can be expressed in general form

$$\mathbf{x} = \mathbf{x}(t; \mathbf{x}^o), \tag{3.236}$$

a function of time parameterized by the initial condition of the fluid particle. Such a solution is certainly a pathline, streamline, and streakline. It is also known as a trajectory in the dynamical systems literature.

Let us analyze Eq. (3.235) in some more detail. From the definition of the total derivative, see Eq. (3.94), we have

$$d\mathbf{v} = \underbrace{(\nabla\mathbf{v}^T)^T}_{\mathsf{L}^T} \cdot d\mathbf{x}, \tag{3.237}$$

$$d\mathbf{v} = \mathsf{L}^T \cdot d\mathbf{x}. \tag{3.238}$$

This gives the acceleration vector as

$$\frac{d\mathbf{v}}{dt} = \mathsf{L}^T \cdot \frac{d\mathbf{x}}{dt}, \tag{3.239}$$

$$= \mathsf{L}^T \cdot \mathbf{v}. \tag{3.240}$$

Example 3.5

Study the following non-linear autonomous system, that could describe the steady three-dimensional kinematics of a fluid:

$$\frac{dx_1}{dt} = v_1(x_1, x_2, x_3) = 1 + x_1 x_2 x_3, \qquad x_1^o = 0, \tag{3.241}$$

$$\frac{dx_2}{dt} = v_2(x_1, x_2, x_3) = x_1 + x_2^2 + x_1 x_2^3, \qquad x_2^o = 0, \tag{3.242}$$

$$\frac{dx_3}{dt} = v_3(x_1, x_2, x_3) = 2 - x_1 + x_2 x_3, \qquad x_3^o = 0. \tag{3.243}$$

When considering dynamic systems, one should always consider equilibrium points. Such points exist when $v_1 = v_2 = v_3 = 0$; in fluid mechanics, they are known as *stagnation points.* They are found by solving the nonlinear algebra problem:

$$0 = v_1(x_1, x_2, x_3) = 1 + x_1 x_2 x_3, \tag{3.244}$$
$$0 = v_2(x_1, x_2, x_3) = x_1 + x_2^2 + x_1 x_2^3, \tag{3.245}$$
$$0 = v_3(x_1, x_2, x_3) = 2 - x_1 + x_2 x_3. \tag{3.246}$$

Numerical solution reveals three roots. Two of them are complex, and a third real. As we are generally only concerned with real solutions, we focus only on that root, which is

$$x_{1o} = 1, \qquad x_{2o} = -1.46557, \qquad x_{3o} = 0.682328. \tag{3.247}$$

The "o" as a subscript denotes a stagnation condition, in contrast to "o" as a superscript, that denotes an initial condition. Direct substitution into the equations for velocity confirm this is a stagnation point. Flow in the neighborhood of a stagnation point can be understood by considering the locally linear behavior. Taylor[4] series of the velocity in the neighborhood of the stagnation point reveals the local kinematics are given by the locally linear system

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{pmatrix}}_{=\mathsf{L}^T}\Bigg|_{x_{io}} \begin{pmatrix} x_1 - x_{1o} \\ x_2 - x_{2o} \\ x_3 - x_{3o} \end{pmatrix}, \tag{3.248}$$

$$= \underbrace{\begin{pmatrix} x_2 x_3 & x_1 x_3 & x_1 x_2 \\ 1 + x_2^3 & 2x_2 + 3x_1 x_2^2 & 0 \\ -1 & x_3 & x_2 \end{pmatrix}}_{=\mathsf{L}^T}\Bigg|_{x_{io}} \begin{pmatrix} x_1 - x_{1o} \\ x_2 - x_{2o} \\ x_3 - x_{3o} \end{pmatrix}, \tag{3.249}$$

$$= \underbrace{\begin{pmatrix} -1. & 0.682328 & -1.46557 \\ -2.1479 & 3.51255 & 0 \\ -1 & 0.682328 & -1.46557 \end{pmatrix}}_{=\mathsf{L}^T}\Bigg|_{x_{io}} \begin{pmatrix} x_1 - x_{1o} \\ x_2 - x_{2o} \\ x_3 - x_{3o} \end{pmatrix}. \tag{3.250}$$

As discussed in standard texts on applied mathematics, e.g. Powers and Sen (2015), Ch. 9, the local dynamics in the neighborhood of the stagnation point are dictated by the eigenvalues of the coefficient matrix. Those are easily numerically evaluated as $\lambda = 3.25643, -2.20944$, and 0. The positive, negative, and zero eigenvalues are associated with unstable, stable, and neutrally stable modes, respectively. Because of the presence of both stable and unstable modes, this stagnation point is a so-called *saddle node.* Had all eigenvalues been real and positive, it would have been an unstable *source node.* Had all eigenvalues been real and negative, it would have been a stable *sink node.* Had any eigenvalues contained an imaginary component, the solution could take on a locally oscillatory behavior.

[4]Brook Taylor, 1685-1731, English mathematician and artist, educated at Cambridge, published on capillary action, magnetism, and thermometers, adjudicated the dispute between Newton and Leibniz over priority in developing calculus, contributed to the method of finite differences, invented integration by parts, has his name ascribed to Taylor series of which variants were earlier discovered by Gregory, Newton, Leibniz, Johann Bernoulli, and de Moivre.

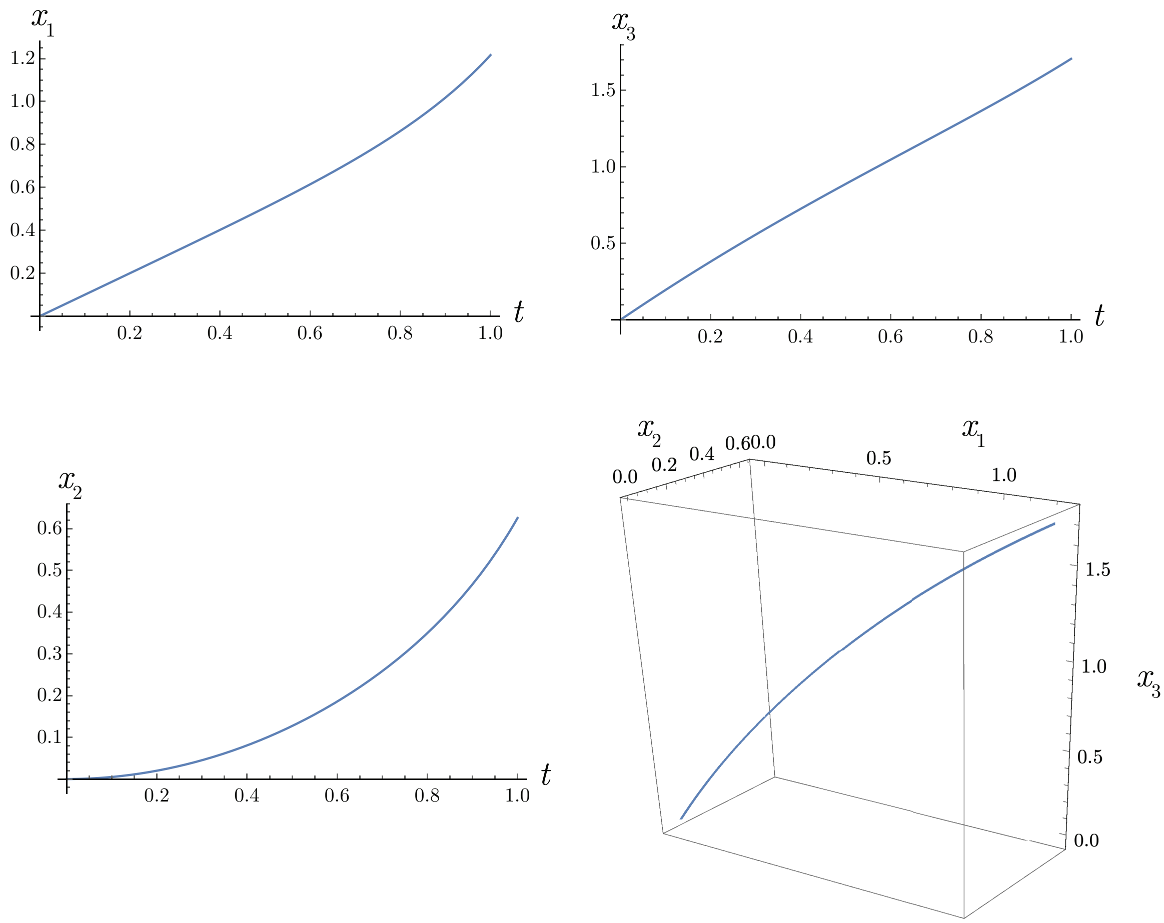

Figure 3.16: Plot of $x_1(t)$, $x_2(t)$, $x_3(t)$, along with the coincident pathline, streamline, and streakline for a steady three-dimensional fluid particle that commences at the origin.

Numerical solution of this nonlinear system of ordinary differential equations yields $x_1(t)$, $x_2(t)$, $x_3(t)$, that for this time-independent velocity field induces the particle pathlines, streamlines, and streaklines. All are plotted in Fig. 3.16. We could also apply the complete mathematical theory of dynamic systems to understand the system better.

We can use Eq. (3.240) to calculate the acceleration vector field:

$$\begin{pmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \\ \frac{dv_3}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{pmatrix} \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix}, \tag{3.251}$$

$$= \begin{pmatrix} x_2 x_3 & x_1 x_3 & x_1 x_2 \\ 1 + x_2^3 & 2x_2 + 3x_1 x_2^2 & 0 \\ -1 & x_3 & x_2 \end{pmatrix} \begin{pmatrix} 1 + x_1 x_2 x_3 \\ x_1 + x_2^2 + x_1 x_2^3 \\ 2 - x_1 + x_2 x_3 \end{pmatrix}, \tag{3.252}$$

$$= \begin{pmatrix} 2x_1x_2 - x_1^2x_2 + x_1^2x_3 + x_2x_3 + 2x_1x_2^2x_3 + x_1^2x_2^3x_3 + x_1x_2^2x_3^2 \\ 1 + 2x_1x_2 + 3x_1^2x_2^2 + 3x_2^3 + 5x_1x_2^4 + 3x_1^2x_2^5 + x_1x_2x_3 + x_1x_2^4x_3 \\ -1 + 2x_2 - x_1x_2 + x_1x_3 - x_1x_2x_3 + 2x_2^2x_3 + x_1x_2^3x_3 \end{pmatrix}. \quad (3.253)$$

If we know the kinematics of a fluid particle, we know everything about its motion, including its acceleration. We shall soon, Ch. 4.2, discuss things like Newton's second law of motion that relates acceleration to forces. If we know the acceleration, it is possible to induce what the force was that generated it by simply multiplying the acceleration by the mass. Rarely is this the case however. It is more common to know something about the forces and to use this to deduce what the motion is.

Now, we seek to analyze a particular pathline. Note that the velocity vector is tangent to the fluid particle trajectory. Let us study a unit vector that happens to be tangent to the velocity field:

$$\boldsymbol{\alpha}_t = \frac{\mathbf{v}}{|\mathbf{v}|}. \quad (3.254)$$

Next, use the quotient rule to examine how the unit tangent vector evolves with time:

$$\frac{d\boldsymbol{\alpha}_t}{dt} = \frac{1}{|\mathbf{v}|}\frac{d\mathbf{v}}{dt} - \frac{\mathbf{v}}{|\mathbf{v}|^2}\frac{d|\mathbf{v}|}{dt}. \quad (3.255)$$

We can scale Eq. (3.240) by $|\mathbf{v}|$ to get $(1/|\mathbf{v}|)d\mathbf{v}/dt = \mathsf{L}^T \cdot \mathbf{v}/|\mathbf{v}| = \mathsf{L}^T \cdot \boldsymbol{\alpha}_t$. Thus Eq. (3.255) can be rewritten as

$$\frac{d\boldsymbol{\alpha}_t}{dt} = \mathsf{L}^T \cdot \boldsymbol{\alpha}_t - \frac{\mathbf{v}}{|\mathbf{v}|^2}\frac{d|\mathbf{v}|}{dt}, \quad (3.256)$$

$$= \mathsf{L}^T \cdot \boldsymbol{\alpha}_t - \boldsymbol{\alpha}_t\frac{1}{|\mathbf{v}|}\frac{d|\mathbf{v}|}{dt}. \quad (3.257)$$

Next consider the following series of operations starting with Eq. (3.240):

$$\frac{d\mathbf{v}}{dt} = \mathsf{L}^T \cdot \mathbf{v}, \quad (3.258)$$

$$\mathbf{v}^T \cdot \frac{d\mathbf{v}}{dt} = \mathbf{v}^T \cdot \mathsf{L}^T \cdot \mathbf{v}, \quad (3.259)$$

$$\frac{d}{dt}\left(\frac{\mathbf{v}^T \cdot \mathbf{v}}{2}\right) = \mathbf{v}^T \cdot \mathsf{L}^T \cdot \mathbf{v}, \quad (3.260)$$

$$\frac{d}{dt}\left(\frac{|\mathbf{v}|^2}{2}\right) = \mathbf{v}^T \cdot \mathsf{L}^T \cdot \mathbf{v}, \quad (3.261)$$

$$|\mathbf{v}|\frac{d}{dt}\left(|\mathbf{v}|\right) = \mathbf{v}^T \cdot \mathsf{L}^T \cdot \mathbf{v}, \quad (3.262)$$

$$\frac{1}{|\mathbf{v}|}\frac{d}{dt}\left(|\mathbf{v}|\right) = \frac{\mathbf{v}^T}{|\mathbf{v}|} \cdot \mathsf{L}^T \cdot \frac{\mathbf{v}}{|\mathbf{v}|}, \quad (3.263)$$

$$\frac{1}{|\mathbf{v}|}\frac{d}{dt}\left(|\mathbf{v}|\right) = \boldsymbol{\alpha}_t^T \cdot \mathsf{L}^T \cdot \boldsymbol{\alpha}_t. \quad (3.264)$$

Now substitute Eq. (3.264) into Eq. (3.257) to get

$$\frac{d\boldsymbol{\alpha}_t}{dt} = \mathsf{L}^T \cdot \boldsymbol{\alpha}_t - \left(\boldsymbol{\alpha}_t^T \cdot \mathsf{L}^T \cdot \boldsymbol{\alpha}_t\right) \boldsymbol{\alpha}_t. \tag{3.265}$$

As an aside, take the dot product of Eq. (3.265) with $\boldsymbol{\alpha}_t$ to get

$$\begin{aligned}
\boldsymbol{\alpha}_t^T \cdot \frac{d\boldsymbol{\alpha}_t}{dt} &= \boldsymbol{\alpha}_t^T \cdot \mathsf{L}^T \cdot \boldsymbol{\alpha}_t - \left(\boldsymbol{\alpha}_t^T \cdot \mathsf{L}^T \cdot \boldsymbol{\alpha}_t\right) \underbrace{\boldsymbol{\alpha}_t^T \cdot \boldsymbol{\alpha}_t}_{=1}, && (3.266)\\
&= \boldsymbol{\alpha}_t^T \cdot \mathsf{L}^T \cdot \boldsymbol{\alpha}_t - \boldsymbol{\alpha}_t^T \cdot \mathsf{L}^T \cdot \boldsymbol{\alpha}_t, && (3.267)\\
&= 0. && (3.268)
\end{aligned}$$

This must be an identity, because $\boldsymbol{\alpha}_t^T \cdot \boldsymbol{\alpha}_t = 1$, and its time derivative gives $\boldsymbol{\alpha}_t^T \cdot d\boldsymbol{\alpha}/dt = 0$. Now recalling Eq. (3.99), and employing $\boldsymbol{\alpha}_t^T \cdot \mathsf{R}^T \cdot \boldsymbol{\alpha}_t = 0$, because of the anti-symmetry of R, and $\mathsf{D}^T = \mathsf{D}$, because of the symmetry of D, Eq. (3.265) can be rewritten as

$$\frac{d\boldsymbol{\alpha}_t}{dt} = \mathsf{L}^T \cdot \boldsymbol{\alpha}_t - \left(\boldsymbol{\alpha}_t^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_t\right) \boldsymbol{\alpha}_t. \tag{3.269}$$

Let us consider how a volume stretches in a direction aligned with the velocity vector. We first specialize the general differential arc length to that found along the particle path: $d\mathsf{s} = ds$. Now, recall from geometry that the square of the differential arc length must be

$$ds^2 = d\mathbf{x}^T \cdot d\mathbf{x}, \tag{3.270}$$

where $d\mathbf{x}$ is also confined to the particle path. Consider now how this quantity changes with time when we move with the particle:

$$\begin{aligned}
\frac{d}{dt}(ds)^2 &= \frac{d}{dt}\left(d\mathbf{x}^T \cdot d\mathbf{x}\right), && (3.271)\\
&= d\mathbf{x}^T \cdot \frac{d}{dt}(d\mathbf{x}) + \left(\frac{d}{dt}(d\mathbf{x})\right)^T \cdot d\mathbf{x}, && (3.272)\\
&= d\mathbf{x}^T \cdot d\left(\frac{d\mathbf{x}}{dt}\right) + \left(d\left(\frac{d\mathbf{x}}{dt}\right)\right)^T \cdot d\mathbf{x}, && (3.273)\\
&= d\mathbf{x}^T \cdot d\mathbf{v} + d\mathbf{v}^T \cdot d\mathbf{x}, && (3.274)\\
&= 2\ d\mathbf{x}^T \cdot d\mathbf{v}, && (3.275)\\
&= 2\ d\mathbf{x}^T \cdot \mathsf{L}^T \cdot d\mathbf{x}, && (3.276)\\
2\ ds\frac{d}{dt}(ds) &= 2\ d\mathbf{x}^T \cdot \mathsf{L}^T \cdot d\mathbf{x}, && (3.277)\\
\frac{1}{ds}\frac{d}{dt}(ds) &= \frac{d\mathbf{x}}{ds}^T \cdot \mathsf{L}^T \cdot \frac{d\mathbf{x}}{ds}. && (3.278)
\end{aligned}$$

Recall now that

$$\boldsymbol{\alpha}_t = \frac{\mathbf{v}}{|\mathbf{v}|}, \tag{3.279}$$

$$= \frac{\frac{d\mathbf{x}}{dt}}{\frac{ds}{dt}}, \tag{3.280}$$

$$= \frac{d\mathbf{x}}{ds}. \tag{3.281}$$

So, Eq. (3.278) can be rewritten as

$$\frac{1}{ds}\frac{d}{dt}(ds) = \boldsymbol{\alpha}_t^T \cdot \mathsf{L}^T \cdot \boldsymbol{\alpha}_t, \tag{3.282}$$

$$\frac{d}{dt}(\ln ds) = \boldsymbol{\alpha}_t^T \cdot (\mathsf{D} + \mathsf{R})^T \cdot \boldsymbol{\alpha}_t, \tag{3.283}$$

$$= \boldsymbol{\alpha}_t^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_t, \tag{3.284}$$

$$= \mathsf{D} : \boldsymbol{\alpha}_t \boldsymbol{\alpha}_t^T. \tag{3.285}$$

This relative tangential stretching rate is closely related to the result of Eq. (3.125) for extensional strain rate. Specializing Eq. (3.125) for a particle pathline, and combining, we can say

$$d\mathbf{v}^{(es)} = (\boldsymbol{\alpha}_t^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_t)\boldsymbol{\alpha}_t \; ds, \tag{3.286}$$

$$\frac{d\mathbf{v}^{(es)}}{ds} = (\boldsymbol{\alpha}_t^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_t)\boldsymbol{\alpha}_t, \tag{3.287}$$

$$\boldsymbol{\alpha}_t^T \cdot \frac{d\mathbf{v}^{(es)}}{ds} = (\boldsymbol{\alpha}_t^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_t) \underbrace{\boldsymbol{\alpha}_t^T \cdot \boldsymbol{\alpha}_t}_{=1}, \tag{3.288}$$

$$= \boldsymbol{\alpha}_t^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_t = \frac{1}{ds}\frac{d}{dt}(ds), \tag{3.289}$$

$$= \boldsymbol{\alpha}_t^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_t = \frac{1}{ds} d\left(\frac{ds}{dt}\right), \tag{3.290}$$

$$= \boldsymbol{\alpha}_t^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_t = \frac{d|\mathbf{v}|}{ds}, \tag{3.291}$$

$$= \mathsf{D} : \boldsymbol{\alpha}_t \boldsymbol{\alpha}_t^T = \frac{d|\mathbf{v}|}{ds}. \tag{3.292}$$

Here, we invoked Eq. (3.284) to obtain Eq. (3.290). The quantity $\boldsymbol{\alpha}_t^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_t = \mathsf{D} : \boldsymbol{\alpha}_t \boldsymbol{\alpha}_t^T$ is a measure of how the magnitude of the velocity changes with respect to arc length along the particle path.

We can gain further insight into how velocity magnitude changes by a diagonal decomposition of $\mathsf{D} = \mathsf{Q} \cdot \Lambda \cdot \mathsf{Q}^T$, where Q is an orthogonal rotation matrix with the normalized eigenvectors of D in its columns, and Λ is the diagonal matrix with the eigenvalues of D in its diagonal. Thus

$$\frac{d|\mathbf{v}|}{ds} = \boldsymbol{\alpha}_t^T \cdot \underbrace{\mathsf{Q} \cdot \Lambda \cdot \mathsf{Q}^T}_{\mathsf{D}} \cdot \boldsymbol{\alpha}_t, \tag{3.293}$$

$$= (\mathsf{Q}^T \cdot \boldsymbol{\alpha}_t)^T \cdot \Lambda \cdot (\mathsf{Q}^T \cdot \boldsymbol{\alpha}_t). \tag{3.294}$$

The operation $\mathsf{Q}^T \cdot \boldsymbol{\alpha}_t \equiv \boldsymbol{\alpha}_s$ generates a new rotated unit vector $\boldsymbol{\alpha}_s = (\alpha_{s1}, \alpha_{s2}, \alpha_{s3})^T$. Thus we can state

$$\frac{d|\mathbf{v}|}{ds} = \alpha_{s1}^2 \lambda^{(1)} + \alpha_{s2}^2 \lambda^{(2)} + \alpha_{s3}^2 \lambda^{(3)}, \tag{3.295}$$

$$1 = \alpha_{s1}^2 + \alpha_{s2}^2 + \alpha_{s3}^2. \tag{3.296}$$

The rate of change of the velocity magnitude along a particle pathline can be understood to be a weighted average of the eigenvalues of the deformation tensor D. In the special case in which $\boldsymbol{\alpha}_t$ is the i^{th} eigenvector of D, we simply get $d|\mathbf{v}|/ds = \lambda^{(i)}$, where $\lambda^{(i)}$ is the corresponding eigenvalue.

If we extend Eq. (3.184) to differential material volumes, we could say the relative expansion rate is

$$\frac{1}{dV}\frac{d}{dt}(dV) = \text{tr } \mathsf{D}, \tag{3.297}$$

$$\frac{d}{dt}(\ln dV) = \text{tr } \mathsf{D}. \tag{3.298}$$

Now our differential volume can be formed by

$$dV = dA\ ds, \tag{3.299}$$

where dA is the cross-sectional area normal to the flow direction. Thus

$$\ln dV = \ln dA + \ln ds, \tag{3.300}$$

$$\ln dA = \ln dV - \ln ds, \tag{3.301}$$

$$\frac{d}{dt}(\ln dA) = \frac{d}{dt}(\ln dV) - \frac{d}{dt}(\ln ds). \tag{3.302}$$

Substitute from Eqs. (3.284,3.298) to get the relative rate of change of the differential area normal to the flow direction:

$$\frac{d}{dt}(\ln dA) = \text{tr } \mathsf{D} - \boldsymbol{\alpha}_t^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_t. \tag{3.303}$$

This relation, while not identical, is similar to the expression for shear strain rate, Eq. (3.135). We can also use Eq. (2.98) to rewrite Eq. (3.303) as

$$\frac{d}{dt}(\ln dA) = \mathsf{D} : \mathsf{I} - \mathsf{D} : \boldsymbol{\alpha}_t \boldsymbol{\alpha}_t^T, \tag{3.304}$$

$$= \mathsf{D} : \left(\mathsf{I} - \boldsymbol{\alpha}_t \boldsymbol{\alpha}_t^T\right). \tag{3.305}$$

Now the matrix $\mathsf{I} - \boldsymbol{\alpha}_t \boldsymbol{\alpha}_t^T$ has some surprising properties. It is singular and has rank two. Because it is symmetric, it has a set of three orthogonal eigenvectors that can be normalized to form an orthonormal set. Its three eigenvalues are 1, 1, and 0. Remarkably, the eigenvector associated with the zero eigenvalue must be parallel to and can be selected as $\boldsymbol{\alpha}_t$, the unit tangent to the curve. Thus the other two eigenvectors can be thought of as unit normals to the curve, that we label $\boldsymbol{\alpha}_{n1}$ and $\boldsymbol{\alpha}_{n2}$. These eigenvectors are not unique; however, a set can always be found. We can summarize the decomposition in the following steps:

$$\mathsf{I} - \boldsymbol{\alpha}_t \boldsymbol{\alpha}_t^T = \mathsf{Q} \cdot \Lambda \cdot \mathsf{Q}^T, \tag{3.306}$$

$$= \begin{pmatrix} \vdots & \vdots & \vdots \\ \boldsymbol{\alpha}_{n1} & \boldsymbol{\alpha}_{n2} & \boldsymbol{\alpha}_t \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cdots & \boldsymbol{\alpha}_{n1}^T & \cdots \\ \cdots & \boldsymbol{\alpha}_{n2}^T & \cdots \\ \cdots & \boldsymbol{\alpha}_t^T & \cdots \end{pmatrix}, \tag{3.307}$$

$$= \boldsymbol{\alpha}_{n1} \boldsymbol{\alpha}_{n1}^T + \boldsymbol{\alpha}_{n2} \boldsymbol{\alpha}_{n2}^T. \tag{3.308}$$

The two unit normals are orthogonal to each other, $\boldsymbol{\alpha}_{n1}^T \cdot \boldsymbol{\alpha}_{n2} = 0$. Thus, we have

$$\frac{d}{dt}(\ln dA) = \mathsf{D} : \left(\boldsymbol{\alpha}_{n1} \boldsymbol{\alpha}_{n1}^T + \boldsymbol{\alpha}_{n2} \boldsymbol{\alpha}_{n2}^T \right), \tag{3.309}$$

$$= \mathsf{D} : \boldsymbol{\alpha}_{n1} \boldsymbol{\alpha}_{n1}^T + \mathsf{D} : \boldsymbol{\alpha}_{n2} \boldsymbol{\alpha}_{n2}^T, \tag{3.310}$$

$$= \boldsymbol{\alpha}_{n1}^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_{n1} + \boldsymbol{\alpha}_{n2}^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_{n2}. \tag{3.311}$$

Comparing to Eq. (3.284) that has one mode associated with $\boldsymbol{\alpha}_t$ available for stretching of the one-dimensional arc length in the streamwise direction, there are two modes associated with $\boldsymbol{\alpha}_{n1}$, $\boldsymbol{\alpha}_{n2}$ available for stretching the two-dimensional area.

The form $\boldsymbol{\alpha}_{n1}^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_{n1}$ suggests it determines the relative normal stretching rate in the direction of $\boldsymbol{\alpha}_{n1}$; a similar rate exists for the other normal direction. One might imagine that there exists a normal direction that yields extreme values for relative normal stretching rates. It is easily shown this achieved by the following. First, define a rectangular matrix, $\widehat{\mathsf{Q}}$, whose columns are populated by $\boldsymbol{\alpha}_{n1}$ and $\boldsymbol{\alpha}_{n2}$:

$$\widehat{\mathsf{Q}} = \begin{pmatrix} \vdots & \vdots \\ \boldsymbol{\alpha}_{n1} & \boldsymbol{\alpha}_{n2} \\ \vdots & \vdots \end{pmatrix}. \tag{3.312}$$

Then project the 3×3 matrix D onto this basis to form the 2×2 matrix $\widehat{\mathsf{D}}$ associated with stretching in the directions normal to the motion:

$$\widehat{\mathsf{D}} = \widehat{\mathsf{Q}}^T \cdot \mathsf{D} \cdot \widehat{\mathsf{Q}}. \tag{3.313}$$

The eigenvalues of $\widehat{\mathsf{D}}$ give the maximum and minimum values of the relative normal stretching rates, and the eigenvectors give the associated directions of extremal normal stretching.

Looked at another way and motivated by standard results from differential geometry, we can make special choices, $\boldsymbol{\alpha}_{n1} = \boldsymbol{\alpha}_{np}$, $\boldsymbol{\alpha}_{n2} = \boldsymbol{\alpha}_{nb}$, where $\boldsymbol{\alpha}_{np}$ is the so-called "principal normal unit vector" and $\boldsymbol{\alpha}_{nb}$ is the so-called "bi-normal unit vector." The following results are described in more detail in many sources, e.g. Powers and Sen (2015), p. 89. We have the so-called "Frenet-Serret"[5] relations:

$$\frac{d\boldsymbol{\alpha}_t}{ds} = \kappa\boldsymbol{\alpha}_{np}, \tag{3.314}$$

$$\frac{d\boldsymbol{\alpha}_{np}}{ds} = -\kappa\boldsymbol{\alpha}_t - \tau\boldsymbol{\alpha}_{nb}, \tag{3.315}$$

$$\frac{d\boldsymbol{\alpha}_{nb}}{ds} = \tau\boldsymbol{\alpha}_{np}. \tag{3.316}$$

Here κ is the so-called "curvature," of the curve and τ is the so-called "torsion" of the curve. One can, with effort show that κ and τ are given by

$$\kappa = \frac{\sqrt{\left|\frac{d^2\mathbf{x}}{dt^2}\right|^2\left|\frac{d\mathbf{x}}{dt}\right|^2 - \left(\frac{d\mathbf{x}}{dt}^T\cdot\frac{d^2\mathbf{x}}{dt^2}\right)^2}}{\left|\frac{d\mathbf{x}}{dt}\right|^3} = \frac{\left|\frac{d\mathbf{x}}{dt}\times\frac{d^2\mathbf{x}}{dt^2}\right|}{\left|\frac{d\mathbf{x}}{dt}\right|^3}, \tag{3.317}$$

$$\tau = \frac{-\left(\frac{d\mathbf{x}}{dt}\times\frac{d^2\mathbf{x}}{dt^2}\right)^T\cdot\frac{d^3\mathbf{x}}{dt^3}}{\left|\frac{d^2\mathbf{x}}{dt^2}\right|^2\left|\frac{d\mathbf{x}}{dt}\right|^2 - \left(\frac{d\mathbf{x}}{dt}^T\cdot\frac{d^2\mathbf{x}}{dt^2}\right)^2}. \tag{3.318}$$

Note κ and τ are expressed here as functions of time. This is certainly the case for a particle moving along a path in time. But just as the intrinsic curvature of a mountain road is independent of the speed of the vehicle traveling on the road, despite the traveling vehicle experiencing a time-dependency of curvature, the curvature and torsion can be considered more fundamentally to be functions of position only, given that the velocity field is known as a function of position. Analysis reveals in fact that

$$\kappa = \frac{\sqrt{\left(\mathbf{v}^T\cdot\mathsf{L}\cdot\mathsf{L}^T\cdot\mathbf{v}\right)\left(\mathbf{v}^T\cdot\mathbf{v}\right) - \left(\mathbf{v}^T\cdot\mathsf{L}^T\cdot\mathbf{v}\right)^2}}{\left(\mathbf{v}^T\cdot\mathbf{v}\right)^{3/2}}. \tag{3.319}$$

One could also develop an expression for torsion that is explicitly dependent on position. The expression is complicated and requires the use of third order tensors to capture the higher order spatial variations.

We can also use this intrinsic orthonormal basis to get

$$\frac{d}{dt}(\ln dA) = \mathsf{D} : \left(\boldsymbol{\alpha}_{np}\boldsymbol{\alpha}_{np}^T + \boldsymbol{\alpha}_{nb}\boldsymbol{\alpha}_{nb}^T\right), \tag{3.320}$$

$$= \mathsf{D} : \boldsymbol{\alpha}_{np}\boldsymbol{\alpha}_{np}^T + \mathsf{D} : \boldsymbol{\alpha}_{nb}\boldsymbol{\alpha}_{nb}^T, \tag{3.321}$$

$$= \boldsymbol{\alpha}_{np}^T\cdot\mathsf{D}\cdot\boldsymbol{\alpha}_{np} + \boldsymbol{\alpha}_{nb}^T\cdot\mathsf{D}\cdot\boldsymbol{\alpha}_{nb}. \tag{3.322}$$

The following example, adapted from Powers and Sen (2015), illustrates how kinematics illuminates the general field of nonlinear dynamical systems.

[5] Jean Frédeŕic Frenet, 1816-1900, and Joseph Alfred Serret, 1819-1885, French mathematicians.

Example 3.6

Consider the example of Mengers[6]

$$\frac{dx_1}{dt} = \frac{1}{20}(1 - x_1^2), \tag{3.323}$$

$$\frac{dx_2}{dt} = -2x_2 - \frac{35}{16}x_2 + 2(1 - x_1^2)x_3, \tag{3.324}$$

$$\frac{dx_3}{dt} = x_2 + x_3, \tag{3.325}$$

and identify so-called heteroclinic trajectories and their attractiveness.

There are only two finite equilibria for this system, a saddle at $(-1, 0, 0)^T$ and a sink at $(1, 0, 0)$. Because the first equation is uncoupled from the second two and is sufficiently simple, it can be integrated exactly to form $x_1 = \tanh(t/20)$. This, coupled with $x_2 = 0$ and $x_3 = 0$, satisfies all differential equations and connects the equilibria, so the x_1 axis for $x_1 \in [-1, 1]$ is what is known as the heteroclinic trajectory. One then asks if nearby trajectories are attracted to it. This can be answered by a local geometry-based analysis. Our system is of the form $d\mathbf{x}/dt = \mathbf{v}(\mathbf{x})$. Let us consider its behavior in the neighborhood of a generic point $\mathbf{x}_0$ that is on the heteroclinic trajectory, but is far from equilibrium. We then locally linearize our system as

$$\frac{d}{dt}(\mathbf{x} - \mathbf{x}_0) = \underbrace{\mathbf{v}(\mathbf{x}_0)}_{\text{translation}} + \underbrace{\mathsf{L}|_{\mathbf{x}_0} \cdot (\mathbf{x} - \mathbf{x}_0)}_{\text{deformation+rotation}} + \dots, \tag{3.326}$$

$$= \underbrace{\mathbf{v}(\mathbf{x}_0)}_{\text{translation}} + \underbrace{\mathsf{D}|_{\mathbf{x}_0} \cdot (\mathbf{x} - \mathbf{x}_0)}_{\text{deformation}} + \underbrace{\mathsf{R}|_{\mathbf{x}_0} \cdot (\mathbf{x} - \mathbf{x}_0)}_{\text{rotation}} + \dots. \tag{3.327}$$

Here, we have employed the local velocity gradient L as well as its symmetric (D) and anti-symmetric (R) parts:

$$\mathsf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \mathsf{D} + \mathsf{R}, \qquad \mathsf{D} = \frac{\mathsf{L} + \mathsf{L}^T}{2}, \qquad \mathsf{R} = \frac{\mathsf{L} - \mathsf{L}^T}{2}. \tag{3.328}$$

The symmetry of D allows definition of a real orthonormal basis. For this three-dimensional system, the dual vector $\boldsymbol{\omega}$ of the anti-symmetric R defines the axis of rotation, and its magnitude ω describes the rotation rate. Now the relative volumetric stretching rate is given by $\operatorname{tr} \mathsf{L} = \operatorname{tr} \mathsf{D} = \operatorname{div} \mathbf{v}$. And it is not difficult to show that the linear stretching rate $\mathcal{D}$ associated with any direction with unit normal $\boldsymbol{\alpha}$ is $\mathcal{D} = \boldsymbol{\alpha}^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}$.

For our system, we have

$$\mathsf{L} = \begin{pmatrix} -\frac{x_1}{10} & 0 & 0 \\ -4x_1x_3 & -2 & -\frac{35}{16} + 2(1 - x_1^2) \\ 0 & 1 & 1 \end{pmatrix}. \tag{3.329}$$

We see that the relative volumetric expansion rate is

$$\operatorname{tr} \mathsf{L} = -1 - \frac{x_1}{10}. \tag{3.330}$$

[6] J. D. Mengers, 2012, "Slow invariant manifolds for reaction-diffusion systems," Ph.D. Dissertation, University of Notre Dame, Notre Dame, Indiana.

Because on the heteroclinic trajectory $x_1 \in [-1,1]$, we always have a locally shrinking volume on that trajectory. Now by inspection the unit tangent vector to the heteroclinic trajectory is $\boldsymbol{\alpha}_t = (1,0,0)^T$. So the tangential stretching rate on the heteroclinic trajectory is

$$\mathcal{D}_t = \boldsymbol{\alpha}_t^T \cdot \mathsf{L} \cdot \boldsymbol{\alpha}_t = -\frac{x_1}{10}. \tag{3.331}$$

So near the saddle we have $\mathcal{D}_t = 1/10$, and near the sink we have $\mathcal{D}_t = -1/10$. Now we are concerned with stretching in directions normal to the heteroclinic trajectory. Certainly two unit normal vectors are $\boldsymbol{\alpha}_{n1} = (0,1,0)^T$ and $\boldsymbol{\alpha}_{n2} = (0,0,1)^T$. But there are also infinitely many other unit normals. A detailed optimization calculation reveals however that if we 1) form the 3×2 matrix Q_n with $\boldsymbol{\alpha}_{n1}$ and $\boldsymbol{\alpha}_{n2}$ in its columns:

$$\mathsf{Q}_n = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.332}$$

and 2) form the 2×2 matrices D_n and R_n associated with the plane normal to the heteroclinic trajectory

$$\mathsf{D}_n = \mathsf{Q}_n^T \cdot \mathsf{D} \cdot \mathsf{Q}_n, \qquad \mathsf{R}_n = \mathsf{Q}_n^T \cdot \mathsf{R} \cdot \mathsf{Q}_n, \tag{3.333}$$

that a) the eigenvalues of D_n give the extreme values of the normal stretching rates $\mathcal{D}_{n1}$ and $\mathcal{D}_{n2}$, and the normalized eigenvectors give the associated directions for extreme normal stretching and b) the magnitude of extremal rotation in the hyperplane normal to $\boldsymbol{\alpha}_t$ is given by $\omega = ||\mathsf{R}_n||_2$. On the heteroclinic trajectory, we find

$$\mathsf{D} = \begin{pmatrix} -\frac{x_1}{10} & 0 & 0 \\ 0 & -2 & -\frac{19}{32} + 1 - x_1^2 \\ 0 & -\frac{19}{32} + 1 - x_1^2 & 1 \end{pmatrix}. \tag{3.334}$$

The reduced deformation tensor associated with motion in the normal plane is

$$\mathsf{D}_n = \mathsf{Q}_n^T \cdot \mathsf{D} \cdot \mathsf{Q}_n = \begin{pmatrix} -2 & -\frac{19}{32} + 1 - x_1^2 \\ -\frac{19}{32} + 1 - x_1^2 & 1 \end{pmatrix}. \tag{3.335}$$

Its eigenvalues give the extremal normal stretching rates that are

$$\mathcal{D}_{n,1,2} = -\frac{1}{2} \pm \frac{\sqrt{2473 - 832x_1^2 + 1024x_1^4}}{32}. \tag{3.336}$$

For $x_1 \in [-1,1]$, we have $\mathcal{D}_{n,1} \approx 1$ and $\mathcal{D}_{n,2} \approx -2$. *Because of the presence of a positive normal stretching rate, one cannot guarantee trajectories are attracted to the heteroclinic trajectory, even though volume of nearby points is shrinking.* Positive normal stretching does not guarantee divergence from the heteroclinic trajectory; it permits it. Rotation can orient a collection of nearby points into regions where there is either positive or negative normal stretching. There are two possibilities for the heteroclinic trajectory to be attracting: either 1) all normal stretching rates are negative, or 2) the rotation rate is sufficiently fast and the overall system is volume-decreasing,[7] so that the integrated effect is relaxation to the heteroclinic trajectory. For the heteroclinic trajectory to have the additional property of being restricted to the slow dynamics, we must additionally require that the smallest normal stretching rate be larger than the tangential stretching rate.

We illustrate these notions in the sketch of Fig. 3.17. Here we imagine a sphere of points as initial

[7]Such systems have $\nabla^T \cdot \mathbf{v} < 0$. In the dynamic systems literature, this is known as a dissipative system; however, in fluid mechanics we reserve the word "dissipative" for systems that have thermodynamic irreversibilities.

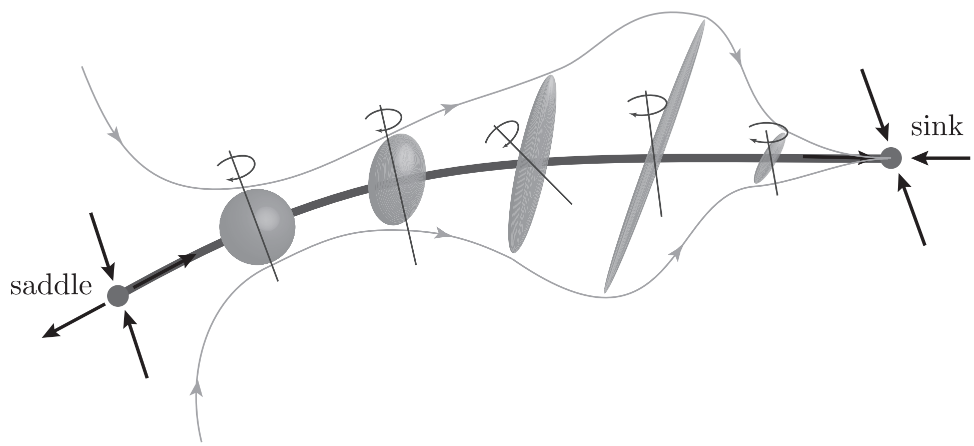

Figure 3.17: Sketch of a phase volume with $\nabla^T \cdot \mathbf{v} < 0$ showing heteroclinic connection between a saddle and sink equilibria along with the evolution of a set of points initially configured as a sphere as they move into regions with some positive normal stretching rates.

conditions near the saddle. We imagine that the system is such that the volume shrinks as the sphere moves. While the overall volume shrinks, one of the normal stretching rates is positive, admitting divergence of nearby trajectories from the heteroclinic trajectory. Rotation orients the volume into a region where negative normal stretching brings all points ultimately to the sink.

For our system, families of trajectories are shown in Fig. 3.18a, and it is seen that there is divergence from the heteroclinic trajectory. This must be attributed to some points experiencing positive normal stretching away from the heteroclinic trajectory. For this case, the rotation rate is $\omega = -51/32 + 1 - x_1^2$. Thus, the local rotation has a magnitude of near unity near the heteroclinic orbit, and the time scales of rotation are close to the time scales of normal stretching.

We can modify the system to include more rotation. For instance, replacing Eq. (3.325) by $dx_3/dt = 10x_2 + x_3$ introduces a sufficient amount of rotation to render the heteroclinic trajectory to be attractive to nearby trajectories. Detailed analysis reveals that this small change 1) does not change the location of the two equilibria, 2) does not change the heteroclinic trajectory connecting the two equilibria, 3) modifies the dynamics near each equilibrium such that both have two stable oscillatory modes, with the equilibrium at $(-1,0,0)^T$ also containing a third unstable mode and that at $(1,0,0)^T$ containing a third stable mode, 4) does not change that the system has a negative volumetric stretch rate on the heteroclinic trajectory, 5) does not change that a positive normal stretching mode exists on the heteroclinic trajectory, and 6) enhances the rotation such that the heteroclinic trajectory is locally attractive. This is illustrated in Fig. 3.18b.

Had the local velocity gradient been purely symmetric, interpretation would be much easier. It is the effect of a non-zero anti-symmetric part of L that induces the geometrical complexities of rotation. Such systems are often known as *non-normal dynamical systems.*

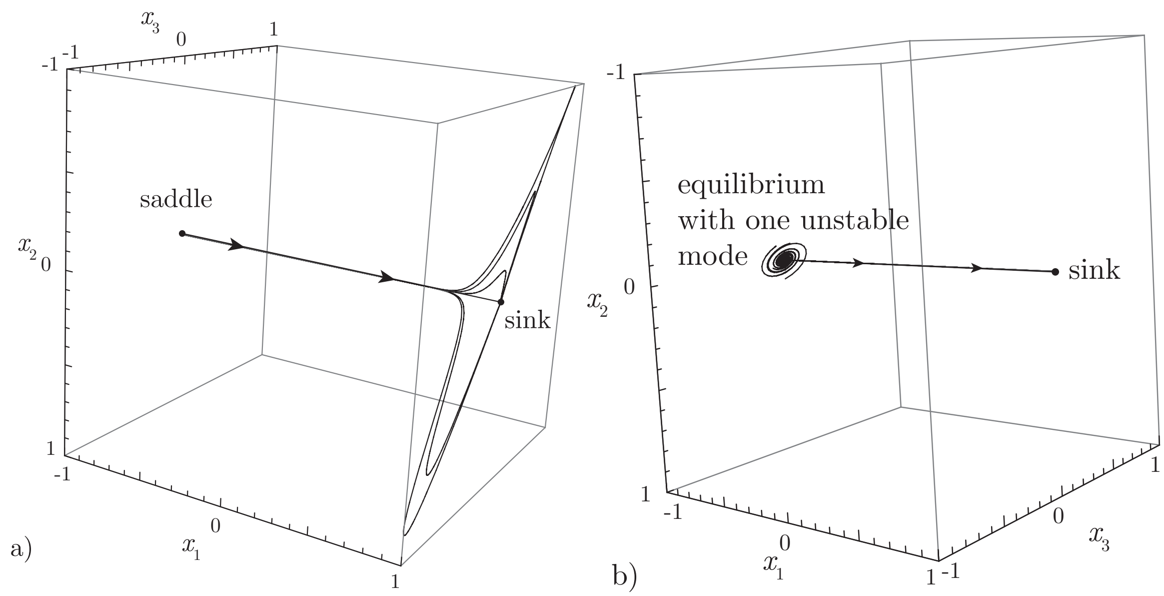

Figure 3.18: Plots of trajectories near the heteroclinic connection between equilibria with one unstable mode and a sink illustrating a) divergence of nearby trajectories due to positive normal stretching with insufficiently rapid rotation and b) convergence of nearby trajectories in the presence of positive normal stretching with sufficiently rapid rotation.

Chapter 4

Conservation axioms

see Panton, Chapter 4,
see Hughes and Gaylord, Chapter 1,
see Yih, Chapters 1 and 2,
see Whitaker, Chapters 4 and 5,
see Aris, Chapters 5 and 6.

A fundamental goal of this chapter is to convert the verbal notions that embody the basic axioms of non-relativistic continuum mechanics into usable mathematical expressions. First, we must list those axioms. The axioms themselves are simply principles that have been observed to have wide validity as long as the particle velocity is small relative to the speed of light and length scales are sufficiently large to contain many molecules. Many of these axioms can be applied to molecules as well. The axioms cannot be proven. They are simply statements that have been useful in describing the universe.

A summary of the axioms in words is as follows:

- *Mass conservation principle*: The time rate of change of mass of a material region is zero.

- *Linear momenta principle*: The time rate of change of the linear momenta of a material region is equal to the sum of forces acting on the region. This is Euler's generalization of Newton's second law of motion.

- *Angular momenta principle*: The time rate of change of the angular momenta of a material region is equal to the sum of the torques acting on the region. This was first formulated by Euler.

- *Energy conservation principle*: The time rate of change of energy within a material region is equal to the rate that energy is received by heat and work interactions. This is the first law of thermodynamics.

- *Entropy inequality*: The time rate of change of entropy within a material region is greater than or equal to the ratio of the rate of heat transferred to the region and the absolute temperature of the region. This is the second law of thermodynamics.

Some secondary concepts related to these axioms are as follows:

- The local stress on one side of a surface is identically opposite that stress on the opposite side.
- Stress can be separated into *thermodynamic* and *viscous* stress.
- Forces can be separated into *surface* and *body* forces.
- In the absence of body couples, the angular momenta principle reduces to a nearly trivial statement.
- The energy equation can be separated into *mechanical* and *thermal components*. The mechanical energy is associated with ordered kinetic energy at the macroscale, and the thermal energy is associated with random kinetic energy of molecular motion at the microscale.

Next we shall systematically convert these words into mathematical form.

4.1 Mass

The mass conservation axiom is simple to state mathematically. It is

$$\frac{d}{dt}\left(m_{MR(t)}\right) = 0. \tag{4.1}$$

As introduced in Ch. 2.4.6.5.1, $MR(t)$ stands for a material region that can evolve in time, and $m_{MR(t)}$ is the mass in the material region. A relevant material region is sketched in Fig. 4.1. We can define the mass of the material region based upon the local value of density:

$$m_{MR(t)} = \int_{MR(t)} \rho \; dV. \tag{4.2}$$

So, the mass conservation axiom is

$$\frac{d}{dt}\int_{MR(t)} \rho \; dV = 0. \tag{4.3}$$

Recalling Leibniz's rule, Eq. (2.268), $\frac{d}{dt}\int_{AR(t)}[\;]\; dV = \int_{AR(t)} \partial_o[\;]\; dV + \int_{AS(t)} n_i w_i [\;]\; dS$, we specialize the arbitrary velocity to the fluid velocity so that $w_i = v_i$. This is because we are

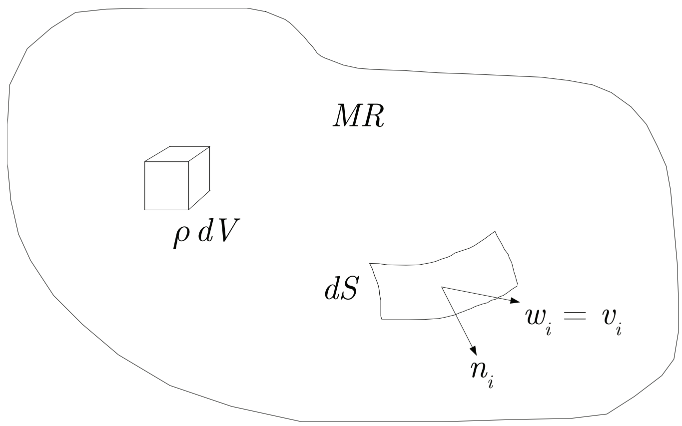

Figure 4.1: Sketch of finite material region MR, infinitesimal mass element $\rho\ dV$, and infinitesimal surface element dS with unit normal n_i, and general velocity w_i equal to fluid velocity v_i.

considering a material region, and thus the Reynolds transport theorem, Eq. (2.271). So we get

$$\frac{d}{dt}\int_{MR(t)} \rho\ dV = \int_{MR(t)} \partial_o \rho\ dV + \int_{MS(t)} n_i v_i \rho\ dS = 0. \tag{4.4}$$

Now, Eq. (4.4) is in fact the most fundamental representation of the mass conservation principle. It applies for both continuous flows as well as for flows with embedded discontinuities such as the shock waves we will study in Ch. 8.4.1.

For this chapter, we will assume that there are no embedded discontinuities, and proceed forward. Now we invoke Gauss's theorem, Eq. (2.250) $\int_{MR(t)} \partial_i[\ ]\ dV = \int_{MS(t)} n_i[\ ]\ dS$, to convert a surface integral to a volume integral to get the mass conservation axiom to read as

$$\int_{MR(t)} \partial_o \rho\ dV + \int_{MR(t)} \partial_i(\rho v_i)\ dV = 0, \tag{4.5}$$

$$\int_{MR(t)} \left(\partial_o \rho + \partial_i(\rho v_i)\right)\ dV = 0. \tag{4.6}$$

Now, in an important step, we realize that the only way for this integral, that has arbitrary limits of integration, to always be zero, is for the integrand itself to always be zero. Hence, we have

$$\partial_o \rho + \partial_i(\rho v_i) = 0. \tag{4.7}$$

This step requires all state variables be continuous, and so cannot be done if discontinuities, such as shock waves, are embedded within $MR(t)$, as will be discussed in Ch. 8.4.1. We write this in expanded Cartesian and Gibbs notation as

$$\partial_o \rho + \partial_1(\rho v_1) + \partial_2(\rho v_2) + \partial_3(\rho v_3) = 0, \tag{4.8}$$

$$\frac{\partial \rho}{\partial t} + \nabla^T \cdot (\rho \mathbf{v}) = 0. \tag{4.9}$$

These equations, along with Eq. (4.7), are all in what is known as *conservative or divergence form.* The conservative form shows mass (equivalently ρ) is conserved when mass fluxes, $\rho\mathbf{v}$, are in balance. There are several alternative forms for this axiom. Using the product rule, we can say also

$$\underbrace{\partial_o \rho + v_i \partial_i \rho}_{\text{material derivative of density}} + \rho \partial_i v_i = 0, \tag{4.10}$$

or, writing in what is called the *non-conservative* form,

$$\frac{d\rho}{dt} + \rho\partial_i v_i = 0, \tag{4.11}$$

$$\frac{d\rho}{dt} + \rho\nabla^T \cdot \mathbf{v} = 0, \tag{4.12}$$

$$(\partial_o\rho + v_1\partial_1\rho + v_2\partial_2\rho + v_3\partial_3\rho) + \rho\left(\partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3\right) = 0. \tag{4.13}$$

For flows with no embedded discontinuities, the conservative and non-conservative forms give identical information. So, we can also say

$$\underbrace{\frac{1}{\rho}\frac{d\rho}{dt}}_{\text{relative rate of density increase}} = - \underbrace{\partial_i v_i}_{\text{relative rate of particle volume expansion}}. \tag{4.14}$$

Recalling Eq. (3.184), we see the relative rate of density increase of a fluid particle is the negative of its relative rate of expansion, as expected. So, we also have

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\frac{1}{V_{MR}}\frac{dV_{MR}}{dt}, \tag{4.15}$$

$$\rho\frac{dV_{MR}}{dt} + V_{MR}\frac{d\rho}{dt} = 0, \tag{4.16}$$

$$\frac{d}{dt}(\rho V_{MR}) = 0, \tag{4.17}$$

$$\frac{d}{dt}(m_{MR}) = 0. \tag{4.18}$$

This returns us to our original mass conservation statement, Eq. (4.1). We note that in a relativistic system, in which mass-energy is conserved, but not mass, that we can have a material region, that is a region bounded by a surface across which there is no flux of mass, for which the mass can indeed change, thus violating our non-relativistic mass conservation axiom.

Let us consider a special case of the Reynolds transport theorem, Eq. (2.271) for a fluid that obeys mass conservation. The general tensor in Eq. (2.271) can be recast as

$$T_{jk....} = \rho\mathcal{T}_{jk....} \tag{4.19}$$

This is useful when $T_{jk...}$ as some intensive property that has units of some quantity per unit mass. Then $\mathcal{T}_{jl...}$ is the same quantity per unit volume. Then the Reynolds transport

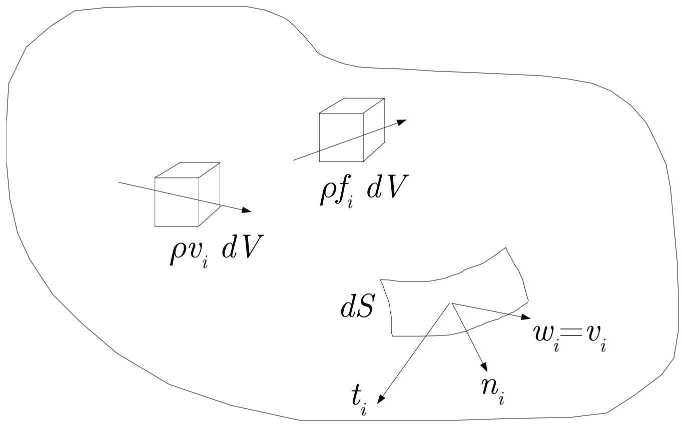

Figure 4.2: Sketch of finite material region MR, infinitesimal linear momenta element $\rho v_i\ dV$, infinitesimal body force element $\rho f_i\ dV$, and infinitesimal surface element dS with unit normal n_i, surface traction t_i and general velocity w_i equal to fluid velocity v_i.

theorem becomes

$$\frac{d}{dt}\int_{MR(t)} \rho \mathcal{T}_{jk....}(x_i,t)\ dV = \int_{MR(t)} \partial_o\left(\rho \mathcal{T}_{jk....}\right)\ dV + \int_{MS(t)} n_l \rho v_l \mathcal{T}_{jk....}\ dS, \tag{4.20}$$

$$= \int_{MR(t)} \left(\partial_o\left(\rho \mathcal{T}_{jk....}\right) + \partial_l\left(\rho v_l \mathcal{T}_{jk....}\right)\right)\ dV, \tag{4.21}$$

$$= \int_{MR(t)} \left(\mathcal{T}_{jk...}\left(\underbrace{\partial_o \rho + \partial_l\left(\rho v_l\right)}_{=0}\right) + \rho\left(\underbrace{\partial_o \mathcal{T}_{jk....} + v_l \partial_l \mathcal{T}_{jk....}}_{=d\mathcal{T}_{jk...}/dt}\right)\right)\ dV, \tag{4.22}$$

$$= \int_{MR(t)} \rho \frac{d\mathcal{T}_{jk...}}{dt}\ dV. \tag{4.23}$$

4.2 Linear momenta

4.2.1 Statement of the principle

The linear momenta conservation axiom is simple to state mathematically. It is

$$\underbrace{\frac{d}{dt}\int_{MR(t)} \rho v_i\ dV}_{\text{rate of change of linear momenta}} = \underbrace{\int_{MR(t)} \rho f_i\ dV}_{\text{body forces}} + \underbrace{\int_{MS(t)} t_i\ dS}_{\text{surface forces}}. \tag{4.24}$$

Again $MR(t)$ stands for a material region that can evolve in time. A relevant material region is sketched in Fig. 4.2. The term f_i represents a body force per unit mass. An example of such a force would be the gravitational force acting on a body, that when scaled by mass, yields g_i. The term t_i is a traction, that is a vector representing force per unit area. A major challenge of this section will be to express the traction vector in terms of what is known as the stress tensor.

Consider first the left hand side, LHS, of the linear momenta principle

$$\begin{aligned} LHS &= \int_{MR(t)} \partial_o(\rho v_i)\ dV + \int_{MS(t)} n_j \rho v_i v_j\ dS, \qquad \text{from Reynolds,} \qquad (4.25) \\ &= \int_{MR(t)} \left(\partial_o(\rho v_i) + \partial_j(\rho v_j v_i)\right)\ dV, \qquad \text{from Gauss.} \qquad (4.26) \end{aligned}$$

So, the linear momenta principle is

$$\int_{MR(t)} \left(\partial_o(\rho v_i) + \partial_j(\rho v_j v_i)\right)\ dV = \int_{MR(t)} \rho f_i\ dV + \int_{MS(t)} t_i\ dS. \qquad (4.27)$$

These are all expressed in terms of volume integrals except for the term involving surface forces.

4.2.2 Surface forces

The surface force per unit area is a vector we call the traction t_j. It has the units of stress, but it is not formally a stress, which is a tensor. The traction is a function of both position x_i and surface orientation n_k: $t_j = t_j(x_i, n_k)$.

We intend to demonstrate the following: The traction can be stated in terms of a *stress tensor* T_{ij} as written next:

$$\begin{aligned} t_j &= n_i T_{ij}, \\ \mathbf{t}^T &= \mathbf{n}^T \cdot \mathsf{T}, \\ \mathbf{t} &= \mathsf{T}^T \cdot \mathbf{n}. \qquad (4.28) \end{aligned}$$

The following excursions are necessary to show this.

- *Show force on one side of surface equal and opposite to that on the opposite side*

  Let us apply the principle of linear momenta to the material region as sketched in Fig. 4.3. Here we indicate the dependency of the traction on orientation by notation such as $t_i\left(n_i^{II}\right)$. This does not indicate multiplication, nor that i is a dummy index here. In Fig. 4.3, the thin pillbox has width Δl, circumference s, and a surface area for the circular region of ΔS. Surface I is a circular region; surface II is the opposite circular region, and surface III is the cylindrical side.

  We apply the mean value theorem to the linear momenta principle for this region and get

$$\left(\partial_o(\rho v_i) + \partial_j(\rho v_j v_i)\right)^* (\Delta S)(\Delta l) = (\rho f_i)^* (\Delta S)(\Delta l)$$

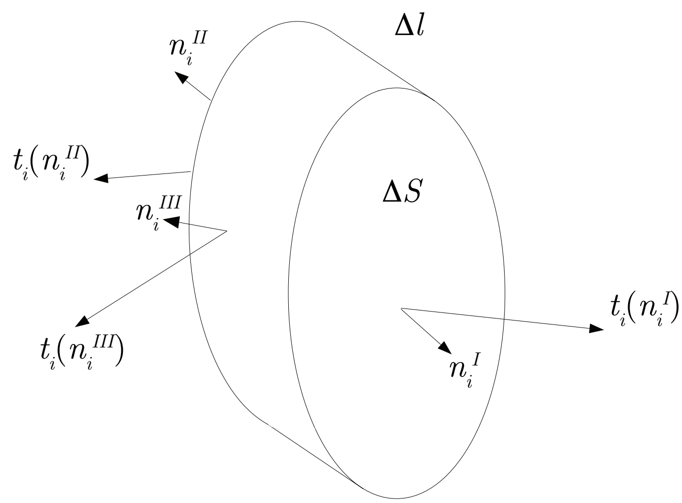

Figure 4.3: Sketch of pillbox element for stress analysis.

$$+t_i^*(n_i^I)\Delta S$$
$$+t_i^*(n_i^{II})\Delta S$$
$$+t_i^*(n_i^{III})s(\Delta l). \qquad (4.29)$$

Now we let $\Delta l \to 0$, holding for now s and ΔS fixed to obtain

$$0 = \left(t_i^*(n_i^I) + t_i^*(n_i^{II})\right) \Delta S. \qquad (4.30)$$

Now letting $\Delta S \to 0$, so that the mean value approaches the local value, and taking $n_i^I = -n_i^{II} \equiv n_i$, we get a useful result

$$t_i(n_i) = -t_i(-n_i). \qquad (4.31)$$

At an infinitesimal length scale, the traction on one side of a surface is equal an opposite that on the other. That is, there is a local force balance. This applies even if there is velocity and acceleration of the material on a macroscale. On the microscale, surface forces dominate inertia and body forces. This is a useful general principle to remember. It the fundamental reason why microorganisms have different propulsion systems that macroorganisms: they are fighting different forces.

- *Study stress on arbitrary plane and relate to stress on coordinate planes*

  Now let us consider a rectangular parallelepiped aligned with the Cartesian axes that has been sliced at an oblique angle to form a tetrahedron. We will apply the linear momenta principle to this geometry and make a statement about the existence of a stress tensor. The described material region is sketched in Fig. 4.4. Let ΔL be a characteristic length scale of the tetrahedron. Also let four unit normals n_j exist, one for each surface. They will be $-n_1$, $-n_2$, $-n_3$ for the surfaces associated with each

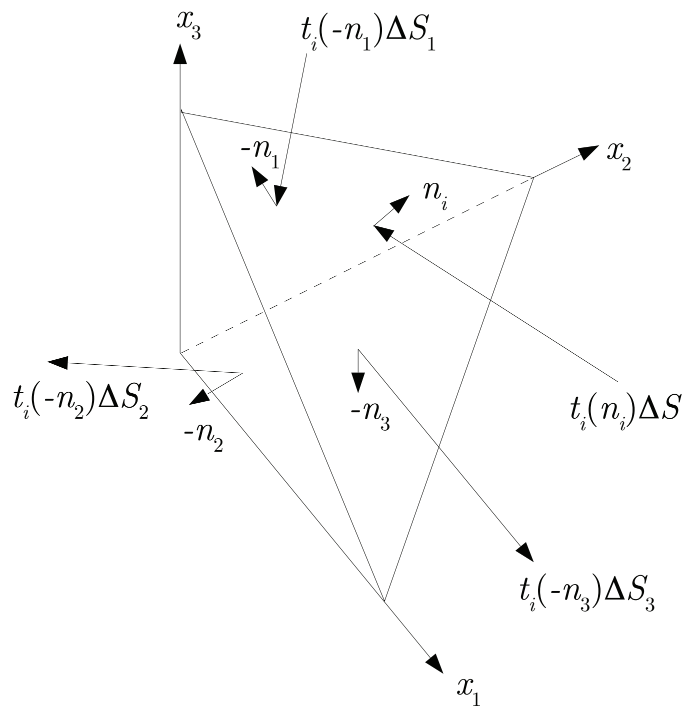

Figure 4.4: Sketch of tetrahedral element for stress analysis on an arbitrary plane.

coordinate direction. They are negative because the outer normal points opposite to the direction of the axes. Let n_i be the normal associated with the oblique face. Let ΔS denote the surface area of each face.

Now the volume of the tetrahedron must be of order L^3 and the surface area of order L^2. Thus applying the mean value theorem to the linear momenta principle, we obtain the form

$$(\text{inertia}) \times (\Delta L)^3 = (\text{body forces}) \times (\Delta L)^3 + (\text{surface forces}) \times (\Delta L)^2. \tag{4.32}$$

As before, for small volumes, $\Delta L \to 0$, and the linear momenta principle reduces to

$$\sum \text{surface forces} = 0. \tag{4.33}$$

Applying this to the configuration of Fig. 4.4, we get

$$0 = t_i^*(n_i)\Delta S + t_i^*(-n_1)\Delta S_1 + t_i^*(-n_2)\Delta S_2 + t_i^*(-n_3)\Delta S_3. \tag{4.34}$$

But we know that $t_j(n_j) = -t_j(-n_j)$, so

$$t_i^*(n_i)\Delta S = t_i^*(n_1)\Delta S_1 + t_i^*(n_2)\Delta S_2 + t_i^*(n_3)\Delta S_3. \tag{4.35}$$

Now it is not a difficult geometry problem to show that $n_i \Delta S = \Delta S_i$, so we get

$$t_i^*(n_i)\Delta S = n_1 t_i^*(n_1)\Delta S + n_2 t_i^*(n_2)\Delta S + n_3 t_i^*(n_3)\Delta S, \tag{4.36}$$
$$t_i^*(n_i) = n_1 t_i^*(n_1) + n_2 t_i^*(n_2) + n_3 t_i^*(n_3). \tag{4.37}$$

Now we can consider terms like t_i to obviously be a vector, and the indicator, for example (n_1), tells us with which surface the vector is associated. This is precisely what a tensor does, and in fact we can say

$$t_i(n_i) = n_1 T_{1i} + n_2 T_{2i} + n_3 T_{3i}. \tag{4.38}$$

In shorthand, we can say the same thing with

$$t_i = n_j T_{ji}, \qquad \text{or equivalently} \qquad t_j = n_i T_{ij}, \qquad \text{QED.} \tag{4.39}$$

Here T_{ij} is the component of stress in the j direction associated with the surface whose normal is in the i direction.

- *Consider pressure and the viscous stress tensor*

Pressure is a familiar concept from thermodynamics and fluid statics. It is often tempting and sometimes correct to think of the pressure as the force per unit area normal to a surface and the force tangential to a surface being somehow related to frictional forces. We shall see that in general, this view is too simplistic.

First recall from thermodynamics that what we will call p, the *thermodynamic pressure*, is for a simple compressible substance a function of at most two intensive thermodynamic variables, say $p = f(\rho, e)$, where e is the specific internal energy. Also recall that the thermodynamic pressure must be a normal stress, as thermodynamics considers formally only materials at rest, and viscous stresses are associated with moving fluids.

To distinguish between thermodynamic stresses and other stresses, let us define the *viscous stress tensor* τ_{ij} as follows

$$\tau_{ij} = T_{ij} + p\delta_{ij}. \tag{4.40}$$

Recall that T_{ij} is the *total stress tensor*. We obviously also have

$$T_{ij} = -p\delta_{ij} + \tau_{ij}. \tag{4.41}$$

With this definition, pressure is positive in compression, while T_{ij} and τ_{ij} are positive in tension. Let us also define the *mechanical pressure*, $p^{(m)}$, as the negative of the average normal surface stress

$$p^{(m)} \equiv -\frac{1}{3} T_{ii} = -\frac{1}{3}(T_{11} + T_{22} + T_{33}). \tag{4.42}$$

The often invoked *Stokes' assumption*, that remains a subject of widespread misunderstanding since it was first made in 1845,[1] is often adopted for lack of a good alternative in answer to a question that will be addressed later in Ch. 5.4.3. It asserts that the thermodynamic pressure is equal to the mechanical pressure:

$$p = p^{(m)} = -\frac{1}{3}T_{ii}. \tag{4.43}$$

Presumably a pressure measuring device in a moving flow field would actually measure the mechanical pressure, and not necessarily the thermodynamic pressure, so it is important to have this issue clarified for proper reconciliation of theory and measurement. It will be seen that Stokes' assumption gives some minor æsthetic pleasure in certain limits, but it is not well-established, and is more a convenience than a requirement for most materials. It is the case that various incarnations of more fundamental kinetic theory under the assumption of a dilute gas composed of inert hard spheres give rise to the conclusion that Stokes' assumption is valid. At moderate densities, these hard sphere kinetic theory models predict that Stokes' assumption is invalid. However, none of the common kinetic theory models is able to predict results from experiments, that nevertheless also give indication, albeit indirect, that Stokes' assumption is invalid. Kinetic theories and experiments that consider polyatomic molecules, that can suffer vibrational and rotational effects as well, show further deviation from Stokes' assumption. It is often plausibly argued that these so-called non-equilibrium effects, that is molecular vibration and rotation, that are only important in high speed flow applications in which the flow velocity is on the order of the fluid sound speed, are the mechanisms that cause Stokes' assumption to be violated. Because they only are important in high speed applications, they are difficult to measure, though measurement of the decay of acoustic waves has provided some data. For liquids, there is little to no theory, and the limited data indicates that Stokes' assumption is invalid.

Now contracting Eq. (4.41), we get

$$T_{ii} = -p\delta_{ii} + \tau_{ii}. \tag{4.44}$$

Using the fact that $\delta_{ii} = 3$ and inserting Eq. (4.43) in Eq. (4.44), we find *for a fluid that obeys Stokes' assumption* that

$$T_{ii} = \frac{1}{3}T_{ii}(3) + \tau_{ii}, \tag{4.45}$$

$$0 = \tau_{ii}. \tag{4.46}$$

That is to say, the trace of the viscous stress tensor is zero. Moreover, for a fluid that obeys Stokes' assumption, we can interpret the viscous stress as the deviation from the

[1]Stokes, G. G., 1845, "On the theories of internal friction of fluids in motion," *Transactions of the Cambridge Philosophical Society*, 8: 287-305.

mean stress; that is, the viscous stress is a deviatoric stress:

$$\underbrace{T_{ij}}_{\text{total stress}} = \underbrace{\frac{1}{3}T_{kk}\delta_{ij}}_{\text{mean stress}} + \underbrace{\tau_{ij}}_{\text{deviatoric stress}}, \qquad \text{valid only if Stokes' assumption holds.} \tag{4.47}$$

If Stokes' assumption does not hold, then a portion of τ_{ij} will also contribute to the mean stress; that is, the viscous stress is not then entirely deviatoric.

Finally, let us note what the traction vector is when the fluid is static. For a static fluid, there is no viscous stress, so $\tau_{ij} = 0$, and we have

$$T_{ij} = -p\delta_{ij}, \qquad \text{static fluid.} \tag{4.48}$$

We get the traction vector for a static fluid on any surface with normal n_i by

$$t_j = n_i T_{ij} = -pn_i\delta_{ij} = -pn_j, \qquad \text{static fluid.} \tag{4.49}$$

Changing indices, we see $t_i = -pn_i$, that is the traction vector must be oriented in the same direction as the surface normal for a static fluid; all stresses are normal to any arbitrarily oriented surface.

4.2.3 Final form of linear momenta equation

We are now prepared to write the linear momenta equation in final form. Substituting our expression for the traction vector, Eq. (4.39) into the linear momenta expression, Eq. (4.27), we get

$$\int_{MR(t)} (\partial_o(\rho v_i) + \partial_j(\rho v_j v_i))\ dV = \int_{MR(t)} \rho f_i\ dV + \int_{MS(t)} n_j T_{ji}\ dS. \tag{4.50}$$

Using Gauss's theorem, Eq. (2.250), to convert the surface integral into a volume integral, and combining all under one integral sign, we get

$$\int_{MR(t)} (\partial_o(\rho v_i) + \partial_j(\rho v_j v_i) - \rho f_i - \partial_j T_{ji})\ dV = 0. \tag{4.51}$$

Making the same argument as before regarding arbitrary material volumes, this must then require that the integrand be zero (we actually must require all variables be continuous to make this work), so we obtain

$$\partial_o(\rho v_i) + \partial_j(\rho v_j v_i) - \rho f_i - \partial_j T_{ji} = 0. \tag{4.52}$$

Using then $T_{ij} = -p\delta_{ij} + \tau_{ij}$, we get in Cartesian index, Gibbs[2], and full notation

$$\partial_o(\rho v_i) + \partial_j(\rho v_j v_i) = \rho f_i - \partial_i p + \partial_j \tau_{ji}, \tag{4.53}$$

[2]Here the transpose notation is particularly cumbersome and unfamiliar, though necessary for full consistency. One will more commonly see this equation written simply as $\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \rho \mathbf{f} - \nabla p + \nabla \cdot \boldsymbol{\tau}$.

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \left(\nabla^T \cdot (\rho \mathbf{v}\mathbf{v}^T)\right)^T = \rho \mathbf{f} - \nabla p + \left(\nabla^T \cdot \boldsymbol{\tau}\right)^T, \tag{4.54}$$
$$\partial_o(\rho v_1) + \partial_1(\rho v_1 v_1) + \partial_2(\rho v_2 v_1) + \partial_3(\rho v_3 v_1) = \rho f_1 - \partial_1 p + \partial_1 \tau_{11} + \partial_2 \tau_{21} + \partial_3 \tau_{31}, \tag{4.55}$$
$$\partial_o(\rho v_2) + \partial_1(\rho v_1 v_2) + \partial_2(\rho v_2 v_2) + \partial_3(\rho v_3 v_2) = \rho f_2 - \partial_2 p + \partial_1 \tau_{12} + \partial_2 \tau_{22} + \partial_3 \tau_{32}, \tag{4.56}$$
$$\partial_o(\rho v_3) + \partial_1(\rho v_1 v_3) + \partial_2(\rho v_2 v_3) + \partial_3(\rho v_3 v_3) = \rho f_3 - \partial_3 p + \partial_1 \tau_{13} + \partial_2 \tau_{23} + \partial_3 \tau_{33}. \tag{4.57}$$

The form is known as the linear momenta principle cast in conservative or divergence form. It is the first choice of forms for many numerical simulations, as discretizations of this form of the equation naturally preserve the correct values of global linear momenta, up to roundoff error.

However, there is a commonly used reduced, non-conservative form that makes some analysis and physical interpretation easier. Let us use the product rule to expand the linear momenta principle, then rearrange it, and use mass conservation, Eq. (4.7), and the definition of material derivative to rewrite the expression:

$$\rho \partial_o v_i + v_i \partial_o \rho + v_i \partial_j(\rho v_j) + \rho v_j \partial_j v_i = \rho f_i - \partial_i p + \partial_j \tau_{ji}, \tag{4.58}$$
$$\rho(\partial_o v_i + v_j \partial_j v_i) + v_i(\underbrace{\partial_o \rho + \partial_j(\rho v_j)}_{=0 \text{ by mass conservation}}) = \rho f_i - \partial_i p + \partial_j \tau_{ji}, \tag{4.59}$$
$$\rho \underbrace{(\partial_o v_i + v_j \partial_j v_i)}_{=\frac{dv_i}{dt}} = \rho f_i - \partial_i p + \partial_j \tau_{ji}, \tag{4.60}$$
$$\rho \frac{dv_i}{dt} = \rho f_i - \partial_i p + \partial_j \tau_{ji}, \tag{4.61}$$
$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f} - \nabla p + \left(\nabla^T \cdot \boldsymbol{\tau}\right)^T. \tag{4.62}$$

Written in full, this becomes

$$\rho(\partial_o v_1 + v_1 \partial_1 v_1 + v_2 \partial_2 v_1 + v_3 \partial_3 v_1) = \rho f_1 - \partial_1 p + \partial_1 \tau_{11} + \partial_2 \tau_{21} + \partial_3 \tau_{31}, \tag{4.63}$$
$$\rho(\partial_o v_2 + v_1 \partial_1 v_2 + v_2 \partial_2 v_2 + v_3 \partial_3 v_2) = \rho f_2 - \partial_2 p + \partial_1 \tau_{12} + \partial_2 \tau_{22} + \partial_3 \tau_{32}, \tag{4.64}$$
$$\rho(\partial_o v_3 + v_1 \partial_1 v_3 + v_2 \partial_2 v_3 + v_3 \partial_3 v_3) = \rho f_3 - \partial_3 p + \partial_1 \tau_{13} + \partial_2 \tau_{23} + \partial_3 \tau_{33}. \tag{4.65}$$

So, we see that particles accelerate due to body forces and unbalanced surface forces. If the surface forces are non-zero but uniform, they will have no gradient or divergence, and hence not contribute to accelerating a particle.

Example 4.1

Show Newton's linear momenta principle in the limit of no viscous stress or body force,

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla p, \tag{4.66}$$

is invariant under the Galilean transformation of Ch. 1.4, written here as

$$x_1' = x_1 - v_{1o} t, \tag{4.67}$$
$$x_2' = x_2 - v_{2o} t, \tag{4.68}$$
$$x_3' = x_3 - v_{3o} t, \tag{4.69}$$
$$t' = t. \tag{4.70}$$

We see $dt' = dt$. The spatial coordinates of fluid particles have time derivatives, with respect to the equivalent t or t', of

$$\frac{dx_1'}{dt'} = \frac{dx_1}{dt} - v_{1o}, \tag{4.71}$$

$$\frac{dx_2'}{dt'} = \frac{dx_2}{dt} - v_{2o}, \tag{4.72}$$

$$\frac{dx_3'}{dt'} = \frac{dx_3}{dt} - v_{3o}. \tag{4.73}$$

Defining the fluid particle velocities as usual, $dx_i/dt = v_i$, $dx_i'/dt' = v_i'$, we see

$$v_1' = v_1 - v_{1o}, \tag{4.74}$$

$$v_2' = v_2 - v_{2o}, \tag{4.75}$$

$$v_3' = v_3 - v_{3o}. \tag{4.76}$$

Let us expand the equation set to be considered for transformation:

$$\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1}, \tag{4.77}$$

$$\frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} + v_3 \frac{\partial v_2}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_2}, \tag{4.78}$$

$$\frac{\partial v_3}{\partial t} + v_1 \frac{\partial v_3}{\partial x_1} + v_2 \frac{\partial v_3}{\partial x_2} + v_3 \frac{\partial v_3}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_3}. \tag{4.79}$$

We need representations of the partial derivatives in the transformed coordinate system. Here it is advantageous to consider a so-called *space-time* formulation. Our original Cartesian system is obviously found by inverting the given transformation:

$$x_1 = x_1' + v_{1o} t', \tag{4.80}$$

$$x_2 = x_2' + v_{2o} t', \tag{4.81}$$

$$x_3 = x_3' + v_{3o} t', \tag{4.82}$$

$$t = t'. \tag{4.83}$$

For a general space-time transformation, we have

$$\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \\ dt \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial x_1}{\partial x_1'} & \frac{\partial x_1}{\partial x_2'} & \frac{\partial x_1}{\partial x_3'} & \frac{\partial x_1}{\partial t'} \\ \frac{\partial x_2}{\partial x_1'} & \frac{\partial x_2}{\partial x_2'} & \frac{\partial x_2}{\partial x_3'} & \frac{\partial x_2}{\partial t'} \\ \frac{\partial x_3}{\partial x_1'} & \frac{\partial 3_1}{\partial x_2'} & \frac{\partial 3_1}{\partial x_3'} & \frac{\partial 3_1}{\partial t'} \\ \frac{\partial t}{\partial x_1'} & \frac{\partial t}{\partial x_2'} & \frac{\partial t}{\partial x_3'} & \frac{\partial t}{\partial t'} \end{pmatrix}}_{\mathsf{J}} \begin{pmatrix} dx_1' \\ dx_2' \\ dx_3' \\ dt' \end{pmatrix}. \tag{4.84}$$

We have the Jacobian matrix J, specialized for our Galilean transformation, as

$$\mathsf{J} = \begin{pmatrix} \frac{\partial x_1}{\partial x_1'} & \frac{\partial x_1}{\partial x_2'} & \frac{\partial x_1}{\partial x_3'} & \frac{\partial x_1}{\partial t'} \\ \frac{\partial x_2}{\partial x_1'} & \frac{\partial x_2}{\partial x_2'} & \frac{\partial x_2}{\partial x_3'} & \frac{\partial x_2}{\partial t'} \\ \frac{\partial x_3}{\partial x_1'} & \frac{\partial 3_1}{\partial x_2'} & \frac{\partial 3_1}{\partial x_3'} & \frac{\partial 3_1}{\partial t'} \\ \frac{\partial t}{\partial x_1'} & \frac{\partial t}{\partial x_2'} & \frac{\partial t}{\partial x_3'} & \frac{\partial t}{\partial t'} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & v_{1o} \\ 0 & 1 & 0 & v_{2o} \\ 0 & 0 & 1 & v_{3o} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.85}$$

Here $J = \det \mathsf{J} = 1$, so the transformation is "volume"- and "orientation"-preserving, in the sense of space-time, and never singular. We also can easily show

$$\left(\mathsf{J}^T\right)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v_{1o} & -v_{2o} & -v_{3o} & 1 \end{pmatrix}, \tag{4.86}$$

and thus have from Eq. (2.284) that

$$\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial t} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v_{1o} & -v_{2o} & -v_{3o} & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1'} \\ \frac{\partial}{\partial x_2'} \\ \frac{\partial}{\partial x_3'} \\ \frac{\partial}{\partial t'} \end{pmatrix}. \tag{4.87}$$

This gives then the simple

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_1'}, \qquad \frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_2'}, \qquad \frac{\partial}{\partial x_3} = \frac{\partial}{\partial x_3'}, \tag{4.88}$$

and the slightly more complicated

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - v_{1o}\frac{\partial}{\partial x_1'} - v_{2o}\frac{\partial}{\partial x_2'} - v_{3o}\frac{\partial}{\partial x_3'}. \tag{4.89}$$

Let us apply the full Galilean transformation to one of the linear momenta equations, Eq. (4.77).

$$\frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial x_1} + v_2\frac{\partial v_1}{\partial x_2} + v_3\frac{\partial v_1}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1}, \tag{4.90}$$

First, we replace all the velocities with their respective relative velocities:

$$\begin{aligned} \frac{\partial}{\partial t}(v_1' + v_{1o}) + (v_1' + v_{1o})\frac{\partial}{\partial x_1}(v_1' + v_{1o}) + (v_2' + v_{2o})\frac{\partial}{\partial x_2}(v_1' + v_{1o}) + (v_3' + v_{3o})\frac{\partial}{\partial x_3}(v_1' + v_{1o}) \\ = -\frac{1}{\rho}\frac{\partial p}{\partial x_1}. \end{aligned} \tag{4.91}$$

Because v_{io} are all constant, the equation simplifies to

$$\frac{\partial v_1'}{\partial t} + (v_1' + v_{1o})\frac{\partial v_1'}{\partial x_1} + (v_2' + v_{2o})\frac{\partial v_1'}{\partial x_2} + (v_3' + v_{3o})\frac{\partial v_1'}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1}. \tag{4.92}$$

Because of how spatial derivatives transform, Eq. (4.88), we can say

$$\frac{\partial v_1'}{\partial t} + (v_1' + v_{1o})\frac{\partial v_1'}{\partial x_1'} + (v_2' + v_{2o})\frac{\partial v_1'}{\partial x_2'} + (v_3' + v_{3o})\frac{\partial v_1'}{\partial x_3'} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1'}. \tag{4.93}$$

We next transform the time derivative via Eq. (4.89) to get

$$\begin{aligned} \frac{\partial v_1'}{\partial t'} - v_{1o}\frac{\partial v_1'}{\partial x_1'} - v_{2o}\frac{\partial v_1'}{\partial x_2'} - v_{3o}\frac{\partial v_1'}{\partial x_3'} + (v_1' + v_{1o})\frac{\partial v_1'}{\partial x_1'} + (v_2' + v_{2o})\frac{\partial v_1'}{\partial x_2'} + (v_3' + v_{3o})\frac{\partial v_1'}{\partial x_3'} \\ = -\frac{1}{\rho}\frac{\partial p}{\partial x_1'}. \end{aligned} \tag{4.94}$$

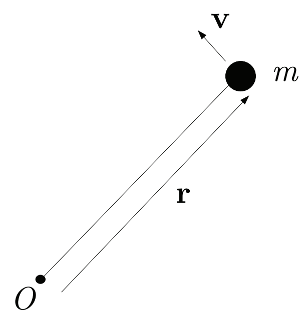

Figure 4.5: Sketch of particle of mass m velocity $\mathbf{v}$ rotating about an axis centered at point O, with radial distance vector $\mathbf{r}$.

This simplifies significantly to yield an equation that is invariant in form from the untransformed Eq. (4.77):

$$\frac{\partial v_1'}{\partial t'} + v_1' \frac{\partial v_1'}{\partial x_1'} + v_2' \frac{\partial v_1'}{\partial x_2'} + v_3' \frac{\partial v_1'}{\partial x_3'} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1'}. \tag{4.95}$$

This extends to the 2 and 3 linear momentum equations, yielding the general transformed linear momenta equation to be represented as

$$\frac{\partial v_i'}{\partial t'} + v_j' \frac{\partial v_i'}{\partial x_j'} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i'}. \tag{4.96}$$

In terms of the material derivative, we could say

$$\frac{dv_i'}{dt'} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i'}. \tag{4.97}$$

The invariance of the linear momemta principle under Galilean transformation is the linchpin of Newtonian mechanics.

4.3 Angular momenta

It is often easy to overlook the angular momenta principle, and its consequence is so simple that, it is often just asserted without proof. In fact in classical rigid body mechanics, it is redundant with the linear momenta principle. It is, however, an independent axiom for continuous deformable media.

Let us first recall some notions from classical rigid body mechanics, while referring to the sketch of Fig. 4.5. We have the angular momenta vector $\mathbf{L}$ for the particle of Fig. 4.5

$$\mathbf{L} = \mathbf{r} \times (m\mathbf{v}). \tag{4.98}$$

Any force $\mathbf{F}$ that acts on m with lever arm $\mathbf{r}$ induces a torque $\hat{\mathbf{T}}$ that is

$$\hat{\mathbf{T}} = \mathbf{r} \times \mathbf{F}. \tag{4.99}$$

Now let us apply these notions for an infinitesimal fluid particle with differential mass $\rho\ dV$.

$$\begin{aligned}
\text{Angular momenta} &= \mathbf{r} \times (\rho\ dV)\mathbf{v} = \rho\epsilon_{ijk} r_j v_k\ dV, && (4.100)\\
\text{Torque of body force} &= \mathbf{r} \times \mathbf{f}(\rho\ dV) = \rho\epsilon_{ijk} r_j f_k\ dV, && (4.101)\\
\text{Torque of surface force} &= \mathbf{r} \times \mathbf{t}\ dS = \epsilon_{ijk} r_j t_k\ dS, &&\\
&= \mathbf{r} \times (\mathbf{n}^T \cdot \mathsf{T})^T\ dS = \epsilon_{ijk} r_j n_p T_{pk}\ dS, && (4.102)\\
\text{Angular momenta from surface couples} &= \mathbf{n}^T \cdot \mathsf{H}\ dS = n_k H_{ki}\ dS. && (4.103)
\end{aligned}$$

Now the principle, that in words says the time rate of change of angular momenta of a material region is equal to the sum of external couples (or torques) on the system becomes mathematically,

$$\underbrace{\frac{d}{dt}\int_{MR(t)} \rho\epsilon_{ijk} r_j v_k\ dV}_{\text{Apply Reynolds then Gauss}} = \int_{MR(t)} \rho\epsilon_{ijk} r_j f_k\ dV + \underbrace{\int_{MS(t)} \left(\epsilon_{ijk} r_j n_p T_{pk} + n_k H_{ki}\right)\ dS}_{\text{apply Gauss}}. \tag{4.104}$$

We apply Reynolds transport theorem and Gauss's theorem to the indicated terms and let the volume of the material region shrink to zero now. First with Reynolds, we get

$$\begin{aligned}
&\int_{MR(t)} \partial_o \rho\epsilon_{ijk} r_j v_k\ dV + \int_{MS(t)} \epsilon_{ijk}\rho r_j v_k n_p v_p\ dS = \\
&\int_{MR(t)} \rho\epsilon_{ijk} r_j f_k\ dV + \int_{MS(t)} \left(\epsilon_{ijk} r_j n_p T_{pk} + n_k H_{ki}\right)\ dS.
\end{aligned} \tag{4.105}$$

Next with Gauss we get

$$\begin{aligned}
&\int_{MR(t)} \partial_o \rho\epsilon_{ijk} r_j v_k\ dV + \int_{MR(t)} \epsilon_{ijk}\partial_p(\rho r_j v_k v_p)\ dV = \\
&\int_{MR(t)} \rho\epsilon_{ijk} r_j f_k\ dV + \int_{MR(t)} \epsilon_{ijk}\partial_p(r_j T_{pk})\ dV + \int_{MR(t)} \partial_k H_{ki}\ dV.
\end{aligned} \tag{4.106}$$

As the region is arbitrary, the integrand formed by placing all terms under the same integral must be zero, that yields

$$\epsilon_{ijk}\left(\partial_o(\rho r_j v_k) + \partial_p(\rho r_j v_p v_k) - \rho r_j f_k - \partial_p(r_j T_{pk})\right) = \partial_k H_{ki}. \tag{4.107}$$

Using the product rule to expand some of the derivatives, we get

$$\epsilon_{ijk}\left(r_j\partial_o(\rho v_k) + \rho v_k \underbrace{\partial_o r_j}_{=0} + r_j\partial_p(\rho v_p v_k) + \rho v_p v_k \underbrace{\partial_p r_j}_{\delta_{pj}} - r_j\rho f_k - r_j\partial_p T_{pk} - T_{pk}\underbrace{\partial_p r_j}_{\delta_{pj}}\right) = \partial_k H_{ki}. \tag{4.108}$$

Applying the simplifications indicated and rearranging, we get

$$\epsilon_{ijk} r_j \underbrace{\left(\partial_o(\rho v_k) + \partial_p(\rho v_p v_k) - \rho f_k - \partial_p T_{pk}\right)}_{=0 \text{ by linear momenta}} = \partial_k H_{ki} - \rho \epsilon_{ijk} v_j v_k + \epsilon_{ijk} T_{jk}. \tag{4.109}$$

So, we can say,

$$\partial_k H_{ki} = \epsilon_{ijk}(\rho v_j v_k - T_{jk}) = \underbrace{\epsilon_{ijk}}_{\text{anti-sym.}} \left(\underbrace{\rho v_j v_k}_{\text{sym.}} - \underbrace{T_{(jk)}}_{\text{sym.}} - \underbrace{T_{[jk]}}_{\text{anti-sym.}} \right), \tag{4.110}$$

$$= -\epsilon_{ijk} T_{[jk]}. \tag{4.111}$$

We have utilized the fact that the tensor inner product of any anti-symmetric tensor with any symmetric tensor must be zero. Now, if we have the case where there are no externally imposed angular momenta fields, such as could be the case when electromagnetic forces are important, we have the common condition of $H_{ki} = 0$, and the angular momenta principle reduces to the simple statement that

$$T_{[ij]} = 0. \tag{4.112}$$

That is, the anti-symmetric part of the stress tensor must be zero. Hence, the stress tensor, absent any surface couples, must be symmetric, and we get in Cartesian index and Gibbs notation:

$$T_{ij} = T_{ji}, \tag{4.113}$$

$$\mathsf{T} = \mathsf{T}^T. \tag{4.114}$$

4.4 Energy

We recall the first law of thermodynamics, that states the time rate of change of a material region's internal and kinetic energy is equal to the rate of heat transferred to the material region less the rate of work done by the material region. Here we have adopted the common engineering sign convention for heat and work, motivated by steam engine analysis, for which thermal energy came "in" and work came "out." Mathematically, this is stated as

$$\frac{d\mathcal{E}}{dt} = \frac{dQ}{dt} - \frac{dW}{dt}. \tag{4.115}$$

In this case (though this is not uniformly enforced in these notes), the upper case letters denote extensive thermodynamic properties. For example, $\mathcal{E}$ is extensive *total energy*, inclusive of internal and kinetic,

$$\mathcal{E} = \rho V \left(e + \frac{1}{2} v_j v_j \right), \tag{4.116}$$

with SI units of J.[3] We could have included potential energy in $\mathcal{E}$, but will instead absorb it into the work term W. The corresponding intensive total energy with SI units of J/kg is

$$\varepsilon = e + v_j v_j/2. \tag{4.117}$$

Let us consider each term in the first law of thermodynamics in detail and then write the equation in final form.

4.4.1 Total energy term

For a fluid particle, the differential amount of extensive total energy is

$$d\mathcal{E} = \rho\left(e + \frac{1}{2}v_j v_j\right)\, dV, \tag{4.118}$$

$$= \underbrace{\rho\, dV}_{\text{mass}} \qquad \underbrace{\left(e + \frac{1}{2}v_j v_j\right)}_{\text{specific internal +kinetic energy}} \qquad . \tag{4.119}$$

4.4.2 Work term

Recall that work is done when a force acts through a distance, and a work rate arises when a force acts through a distance at a particular rate in time (hence, a velocity is involved). Recall also that work is the dot product (inner product) of the force vector with the position or velocity that gives the true work or work rate. In shorthand, we could say

$$dW = d\mathbf{x}^T \cdot \mathbf{F}, \tag{4.120}$$

$$\frac{dW}{dt} = \frac{d\mathbf{x}^T}{dt} \cdot \mathbf{F} = \mathbf{v}^T \cdot \mathbf{F}. \tag{4.121}$$

Here W has the SI units of J, and $\mathbf{F}$ has the SI units of N. We contrast this with our expression for body force per unit mass $\mathbf{f}$, that has SI units of N/kg = m/s^2. Now for the materials we consider, we must describe work done by two types of forces: 1) body, and 2) surface.

- *Work rate done by a body force*

$$\text{Work rate done } by \text{ force on fluid} = (\rho\, dV)(f_i)v_i, \tag{4.122}$$

$$\text{Work rate done } by \text{ fluid} = -\rho v_i f_i\, dV. \tag{4.123}$$

[3]The computational fluid dynamics literature often makes the unfortunate choice of defining the "total energy" as $E = e + v_j v_j/2$ with units of J/kg. Thus, it is really a specific energy and violates the thermodynamics convention that lower case variables are used for intensive properties. We will not use this nomenclature, and will generally reserve upper case variables for extensive properties. We will consider "total" to imply the sum of internal and kinetic, that could either be extensive or intensive. We take the extensive internal energy to be $E = \rho V e$.

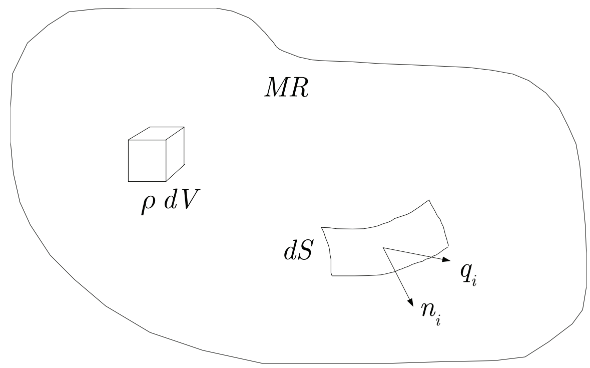

Figure 4.6: Sketch of finite material region MR, infinitesimal mass element $\rho\ dV$, and infinitesimal surface element dS with unit normal n_i, and heat flux vector q_i.

- *Work rate done by a surface force*

$$\text{Work rate done } by \text{ force on fluid} = (t_i\ dS)v_i = ((n_j T_{ji})\ dS)v_i, \qquad (4.124)$$

$$\text{Work rate done } by \text{ fluid} = -n_j T_{ji} v_i\ dS. \qquad (4.125)$$

4.4.3 Heat transfer term

The only thing confusing about the heat transfer rate is the sign convention. We recall that heat transfer *to* a body is associated with an increase in that body's energy. Now following the scenario sketched in the material region of Fig. 4.6, we define the heat flux vector q_i as a vector that points in the direction of thermal energy flow that has units of energy per area per time; in SI this would be W/m^2. So, we have

$$\text{heat transfer rate } from \text{ body through } dS = n_i q_i\ dS, \qquad (4.126)$$

$$\text{heat transfer rate } to \text{ body through } dS = -n_i q_i\ dS. \qquad (4.127)$$

4.4.4 Conservative form of the energy equation

Putting the words of the first law into equation form, we get

$$\underbrace{\frac{d}{dt}\int_{MR(t)} \rho\left(e + \frac{1}{2}v_j v_j\right) dV}_{\frac{d\mathcal{E}}{dt}} = \underbrace{\int_{MS(t)} (-n_i q_i)\ dS}_{\frac{dQ}{dt}} - \underbrace{\left(\int_{MS(t)} (-n_i T_{ij} v_j)\ dS + \int_{MR(t)} (-\rho f_i v_i)\ dV\right)}_{\frac{dW}{dt}}, \qquad (4.128)$$

$$\frac{d}{dt}\int_{MR(t)} \rho\left(e + \frac{1}{2}v_j v_j\right) dV = -\int_{MS(t)} n_i q_i\ dS + \int_{MS(t)} n_i T_{ij} v_j\ dS + \int_{MR(t)} \rho f_i v_i\ dV. \qquad (4.129)$$

Skipping the details of an identical application of the Reynolds transport theorem and Gauss's theorem, and shrinking the volume to approach zero, we obtain the differential equation of energy in *conservative* or divergence form (in first Cartesian index then Gibbs notation):

$$\underbrace{\partial_o\left(\rho\left(e+\frac{1}{2}v_jv_j\right)\right)}_{\text{rate of change of total energy}} + \underbrace{\partial_i\left(\rho v_i\left(e+\frac{1}{2}v_jv_j\right)\right)}_{\text{advection of total energy}} = - \underbrace{\partial_i q_i}_{\text{diffusive heat flux}} + \underbrace{\partial_i(T_{ij}v_j)}_{\text{surface force work rate}} + \underbrace{\rho v_i f_i}_{\text{body force work rate}}, \tag{4.130}$$

$$\frac{\partial}{\partial t}\left(\rho\left(e+\frac{1}{2}\mathbf{v}^T\cdot\mathbf{v}\right)\right) + \nabla^T\cdot\left(\rho\mathbf{v}\left(e+\frac{1}{2}\mathbf{v}^T\cdot\mathbf{v}\right)\right) = -\nabla^T\cdot\mathbf{q} + \nabla^T\cdot(\mathsf{T}\cdot\mathbf{v}) + \rho\mathbf{v}^T\cdot\mathbf{f}. \tag{4.131}$$

This is a scalar equation as there are no free indices.

We can segregate the work done by the surface forces into that done by pressure forces and that done by viscous forces by rewriting this in terms of p and τ_{ij} as follows

$$\partial_o\left(\rho\left(e+\frac{1}{2}v_jv_j\right)\right) + \partial_i\left(\rho v_i\left(e+\frac{1}{2}v_jv_j\right)\right) = -\partial_i q_i - \partial_i(pv_i) + \partial_i(\tau_{ij}v_j) + \rho v_i f_i, \tag{4.132}$$

$$\frac{\partial}{\partial t}\left(\rho\left(e+\frac{1}{2}\mathbf{v}^T\cdot\mathbf{v}\right)\right) + \nabla^T\cdot\left(\rho\mathbf{v}\left(e+\frac{1}{2}\mathbf{v}^T\cdot\mathbf{v}\right)\right) = -\nabla^T\cdot\mathbf{q} - \nabla^T\cdot(p\mathbf{v}) + \nabla^T\cdot(\boldsymbol{\tau}\cdot\mathbf{v}) + \rho\mathbf{v}^T\cdot\mathbf{f}. \tag{4.133}$$

4.4.5 Secondary forms of the energy equation

While the energy equation just derived is perfectly valid for all continuous materials, it is common to see other forms. Many will be described here.

4.4.5.1 Enthalpy-based conservative formulation

It is common to bring the pressure-volume work term to the left side to rewrite the conservative energy equation, Eq. (4.132), as

$$\partial_o\left(\rho\left(e+\frac{1}{2}v_jv_j\right)\right) + \partial_i\left(\rho v_i\left(e+\frac{1}{2}v_jv_j+\frac{p}{\rho}\right)\right) = -\partial_i q_i + \partial_i(\tau_{ij}v_j) + \rho v_i f_i, \tag{4.134}$$

$$\frac{\partial}{\partial t}\left(\rho\left(e+\frac{1}{2}\mathbf{v}^T\cdot\mathbf{v}\right)\right) + \nabla^T\cdot\left(\rho\mathbf{v}\left(e+\frac{1}{2}\mathbf{v}^T\cdot\mathbf{v}+\frac{p}{\rho}\right)\right) = -\nabla^T\cdot\mathbf{q} + \nabla^T\cdot(\boldsymbol{\tau}\cdot\mathbf{v}) + \rho\mathbf{v}^T\cdot\mathbf{f}. \tag{4.135}$$

Recall from elementary thermodynamics the specific *enthalpy* h is defined as

$$h = e + \frac{p}{\rho}. \tag{4.136}$$

Using this definition, the first law in conservative form can be rewritten as

$$\partial_o \left(\rho \left(e + \frac{1}{2} v_j v_j \right) \right) + \partial_i \left(\rho v_i \left(h + \frac{1}{2} v_j v_j \right) \right) = -\partial_i q_i + \partial_i (\tau_{ij} v_j) + \rho v_i f_i, \tag{4.137}$$

$$\frac{\partial}{\partial t} \left(\rho \left(e + \frac{1}{2} \mathbf{v}^T \cdot \mathbf{v} \right) \right) + \nabla^T \cdot \left(\rho \mathbf{v} \left(h + \frac{1}{2} \mathbf{v}^T \cdot \mathbf{v} \right) \right) = -\nabla^T \cdot \mathbf{q} + \nabla^T \cdot (\boldsymbol{\tau} \cdot \mathbf{v}) + \rho \mathbf{v}^T \cdot \mathbf{f}. \tag{4.138}$$

Note that both e and h are present in this form.

4.4.5.2 Mechanical energy equation

The mechanical energy equation has no foundation in the first law of thermodynamics; instead, it is entirely a consequence of the linear momenta principle. It is the type of energy that is often considered in classical Newtonian particle mechanics, a world in which energy is either potential or kinetic but not thermal. We include it here because one needs to be able to distinguish mechanical from thermal energy and it will be useful in later analyses.

The mechanical energy equation, a pure consequence of the linear momenta principle, is obtained by taking the dot product (inner product) of the velocity vector with the linear momenta principle:

$$\mathbf{v}^T \cdot \text{linear momenta.}$$

In detail, we get

$$v_j \left(\rho \partial_o v_j + \rho v_i \partial_i v_j \right) = \rho v_j f_j - v_j \partial_j p + (\partial_i \tau_{ij}) v_j, \tag{4.139}$$

$$\rho \partial_o \left(\frac{v_j v_j}{2} \right) + \rho v_i \partial_i \left(\frac{v_j v_j}{2} \right) = \rho v_j f_j - v_j \partial_j p + (\partial_i \tau_{ij}) v_j, \tag{4.140}$$

$$\tfrac{v_j v_j}{2} \text{ mass} : \frac{v_j v_j}{2} \partial_o \rho + \frac{v_j v_j}{2} \partial_i (\rho v_i) = 0. \tag{4.141}$$

We add Eqs. (4.140) and (4.141) and use the product rule to get

$$\partial_o \left(\rho \frac{v_j v_j}{2} \right) + \partial_i \left(\rho v_i \frac{v_j v_j}{2} \right) = \rho v_j f_j - v_j \partial_j p + (\partial_i \tau_{ij}) v_j. \tag{4.142}$$

$$\frac{\partial}{\partial t} \left(\rho \frac{\mathbf{v}^T \cdot \mathbf{v}}{2} \right) + \nabla^T \cdot \left(\rho \mathbf{v} \frac{\mathbf{v}^T \cdot \mathbf{v}}{2} \right) = \rho \mathbf{v}^T \cdot \mathbf{f} - \mathbf{v}^T \cdot \nabla p + \left(\nabla^T \cdot \boldsymbol{\tau} \right) \cdot \mathbf{v}. \tag{4.143}$$

The term $\rho v_j v_j / 2$ represents the volume-averaged kinetic energy, with SI units J/m^3. The mechanical energy equation, Eq. (4.142), predicts the kinetic energy increases due to three effects:

- fluid motion in the direction of a body force,
- fluid motion in the direction of *decreasing* pressure, or
- fluid motion in the direction of *increasing* viscous stress.

Body forces themselves affect mechanical energy, while it is imbalances in surface forces that affect mechanical energy.

We could also summarize the non-conservative form of the mechanical energy equation, Eq. (4.140), as

$$\rho \frac{d}{dt}\left(\frac{v_j v_j}{2}\right) = \rho v_j f_j - v_j \partial_j p + (\partial_i \tau_{ij}) v_j, \tag{4.144}$$

$$\rho \frac{d}{dt}\left(\frac{\mathbf{v}^T \cdot \mathbf{v}}{2}\right) = \rho \mathbf{v}^T \cdot \mathbf{f} - \mathbf{v}^T \cdot \nabla p + \left(\nabla^T \cdot \boldsymbol{\tau}\right) \cdot \mathbf{v}. \tag{4.145}$$

4.4.5.3 Thermal energy equation

If we take the conservative form of the energy equation (4.132) and subtract from it the mechanical energy equation (4.142), we get an equation for the evolution of thermal energy:

$$\partial_o(\rho e) + \partial_i(\rho v_i e) = -\partial_i q_i - p\partial_i v_i + \tau_{ij}\partial_i v_j, \tag{4.146}$$

$$\frac{\partial}{\partial t}(\rho e) + \nabla^T \cdot (\rho \mathbf{v} e) = -\nabla^T \cdot \mathbf{q} - p\nabla^T \cdot \mathbf{v} + \boldsymbol{\tau} : \nabla \mathbf{v}^T. \tag{4.147}$$

Here ρe is the volume-averaged internal energy with SI units J/m^3. The thermal energy equation (4.146) predicts thermal energy (or internal energy) increases due to

- negative gradients in heat flux (more heat enters than leaves),
- pressure force accompanied by a mean negative volumetric deformation (that is, a uniform compression; note that $\partial_i v_i$ is the relative expansion rate), or
- viscous force associated with a deformation[4] (we will worry about the sign later).

In contrast to mechanical energy, thermal energy changes do not require surface force imbalances; instead they require kinematic deformation. Moreover, body forces have no influence on thermal energy. The work done by a body force is partitioned entirely to the mechanical energy of a body.

[4]For a general fluid, this includes a mean volumetric deformation as well as a deviatoric deformation. If the fluid satisfies Stokes' assumption, it is only the deviatoric deformation that induces a change in internal energy in the presence of viscous stress.

4.4.5.4 Non-conservative energy equation

We can obtain the commonly used non-conservative form of the energy equation, also known as the energy equation following a fluid particle, by the following operations. First expand the thermal energy equation (4.146):

$$\rho\partial_o e + e\partial_o\rho + \rho v_i\partial_i e + e\partial_i(\rho v_i) = -\partial_i q_i - p\partial_i v_i + \tau_{ij}\partial_i v_j. \tag{4.148}$$

Then regroup and notice terms common from mass conservation, Eq. (4.7):

$$\rho\underbrace{(\partial_o e + v_i\partial_i e)}_{\frac{de}{dt}} + e\underbrace{(\partial_o\rho + \partial_i(\rho v_i))}_{=0 \text{ by mass}} = -\partial_i q_i - p\partial_i v_i + \tau_{ij}\partial_i v_j, \tag{4.149}$$

so we get

$$\rho\frac{de}{dt} = -\partial_i q_i - p\partial_i v_i + \tau_{ij}\partial_i v_j, \tag{4.150}$$

$$\rho\frac{de}{dt} = -\nabla^T\cdot\mathbf{q} - p\nabla^T\cdot\mathbf{v} + \boldsymbol{\tau} : \nabla\mathbf{v}^T. \tag{4.151}$$

We can get an equation that is reminiscent of elementary thermodynamics, valid for small volumes V by multiplying Eq. (4.150) by V and using Eq. (3.184) to replace $\partial_i v_i$ by its known value in terms of the relative expansion rate to obtain

$$\rho V\frac{de}{dt} = -V\partial_i q_i - p\frac{dV}{dt} + V\tau_{ij}\partial_i v_j. \tag{4.152}$$

The only term not usually found in elementary thermodynamics texts is the third on the right hand side, which is a viscous work term.

4.4.5.5 Energy equation in terms of enthalpy

Often the energy equation is fully cast in terms of enthalpy. This is generally valid, but especially useful in constant pressure environments. Now starting with the energy equation following a particle (4.150), we can use one form of the mass equation, Eq. (4.14), to eliminate the relative expansion rate $\partial_i v_i$ in favor of the material derivative of density to get

$$\rho\frac{de}{dt} = -\partial_i q_i + \frac{p}{\rho}\frac{d\rho}{dt} + \tau_{ij}\partial_i v_j. \tag{4.153}$$

Rearranging, we get

$$\rho\left(\frac{de}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt}\right) = -\partial_i q_i + \tau_{ij}\partial_i v_j. \tag{4.154}$$

Now differentiating Eq. (4.136), $h = e + p/\rho$, we find

$$\begin{aligned} dh &= de - \frac{p}{\rho^2}\, d\rho + \frac{1}{\rho}\, dp, && (4.155) \\ \frac{dh}{dt} &= \frac{de}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt} + \frac{1}{\rho}\frac{dp}{dt}, && (4.156) \\ \frac{de}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt} &= \frac{dh}{dt} - \frac{1}{\rho}\frac{dp}{dt}, && (4.157) \\ \rho\frac{de}{dt} - \frac{p}{\rho}\frac{d\rho}{dt} &= \rho\frac{dh}{dt} - \frac{dp}{dt}. && (4.158) \end{aligned}$$

So, using Eq. (4.158) to eliminate de/dt in Eq. (4.154) in favor of dh/dt, the energy equation in terms of enthalpy becomes

$$\begin{aligned} \rho\frac{dh}{dt} &= \frac{dp}{dt} - \partial_i q_i + \tau_{ij}\partial_i v_j, && (4.159) \\ \rho\frac{dh}{dt} &= \frac{dp}{dt} - \nabla^T \cdot \mathbf{q} + \boldsymbol{\tau} : \nabla \mathbf{v}^T. && (4.160) \end{aligned}$$

4.4.5.6 Energy equation in terms of entropy

By using standard relations from thermodynamics, we can write the energy equation in terms of entropy. It is important to note that this is just an algebraic substitution. The physical principle that this equation will represent is still *energy* conservation.

Recall the Gibbs equation from thermodynamics, that serves to define entropy s:

$$T\ ds = de + p\ d\hat{v}. \tag{4.161}$$

Here T is the absolute temperature, and $\hat{v}$ is the specific volume, $\hat{v} = V/m = 1/\rho$. In terms of ρ, the Gibbs equation is

$$T\ ds = de - \frac{p}{\rho^2}\ d\rho. \tag{4.162}$$

Taking the material derivative of Eq. (4.162) , that is operationally equivalent to dividing by dt, and solving for de/dt,we get

$$\frac{de}{dt} = T\frac{ds}{dt} + \frac{p}{\rho^2}\frac{d\rho}{dt}. \tag{4.163}$$

This is still essentially a thermodynamic definition of s. Now use Eq. (4.163) in the non-conservative energy equation (4.150) to get an alternate expression for the first law:

$$\rho T\frac{ds}{dt} + \frac{p}{\rho}\frac{d\rho}{dt} = -\partial_i q_i - p\partial_i v_i + \tau_{ij}\partial_i v_j. \tag{4.164}$$

Recalling Eq. (4.14), $-\partial_i v_i = (1/\rho)(d\rho/dt)$, we have

$$\rho T \frac{ds}{dt} = -\partial_i q_i + \tau_{ij}\partial_i v_j, \tag{4.165}$$

$$\rho \frac{ds}{dt} = -\frac{1}{T}\partial_i q_i + \frac{1}{T}\tau_{ij}\partial_i v_j. \tag{4.166}$$

Using the fact that from the quotient rule we have $\partial_i(q_i/T) = (1/T)\partial_i q_i - (q_i/T^2)\partial_i T$, we can then say

$$\rho \frac{ds}{dt} = -\partial_i\left(\frac{q_i}{T}\right) - \frac{1}{T^2}q_i\partial_i T + \frac{1}{T}\tau_{ij}\partial_i v_j, \tag{4.167}$$

$$\rho \frac{ds}{dt} = -\nabla^T\cdot\left(\frac{\mathbf{q}}{T}\right) - \frac{1}{T^2}\mathbf{q}^T\cdot\nabla T + \frac{1}{T}\boldsymbol{\tau} : \nabla\mathbf{v}^T. \tag{4.168}$$

From this statement, we can conclude from the *first law* of thermodynamics that the entropy of a fluid particle changes due to heat transfer and to deformation in the presence of viscous stress. We will make a more precise statement about entropy changes after we introduce the second law of thermodynamics.

The energy equation in terms of entropy can be written in conservative or divergence form by adding the product of s and the mass equation, $s\partial_o\rho + s\partial_i(\rho v_i) = 0$, to Eq. (4.167) to obtain

$$\partial_o(\rho s) + \partial_i(\rho v_i s) = -\partial_i\left(\frac{q_i}{T}\right) - \frac{1}{T^2}q_i\partial_i T + \frac{1}{T}\tau_{ij}\partial_i v_j, \tag{4.169}$$

$$\frac{\partial}{\partial t}(\rho s) + \nabla^T\cdot(\rho\mathbf{v}s) = -\nabla^T\cdot\left(\frac{\mathbf{q}}{T}\right) - \frac{1}{T^2}\mathbf{q}^T\cdot\nabla T + \frac{1}{T}\boldsymbol{\tau} : \nabla\mathbf{v}^T. \tag{4.170}$$

4.5 Entropy inequality

Let us use a non-rigorous method to suggest a form of the entropy inequality that is consistent with classical thermodynamics. Recall the mathematical statement of the entropy inequality from classical thermodynamics:

$$dS \geq \frac{dQ}{T}. \tag{4.171}$$

Here S is the extensive entropy, with SI units J/K, and Q is the heat energy into a system with SI units of J. Notice that entropy can go up or down in a process, depending on the heat transferred. If the process is adiabatic, $dQ = 0$, and the entropy can either remain fixed or rise. Now for our continuous material we have

$$dS = \rho s\ dV, \tag{4.172}$$

$$dQ = -q_i n_i\ dA\ dt. \tag{4.173}$$

Here we have used s for the specific entropy, that has SI units J/kg/K. We have also changed, for obvious reasons, the notation for our element of surface area, now dA, rather than the previous dS. Notice we must be careful with our sign convention. When the heat flux vector is aligned with the outward normal, heat leaves the system. Because we want positive dQ to represent heat into a system, we need the negative sign.

The second law becomes then

$$\rho s \; dV \geq -\frac{q_i}{T} n_i \; dA \; dt. \tag{4.174}$$

Now integrate over the finite geometry: on the left side this is a volume integral and the right side this is an area integral.

$$\int_{MR(t)} \rho s \; dV \quad \geq \quad \left(\int_{MS(t)} -\frac{q_i}{T} n_i \; dA \right) dt. \tag{4.175}$$

Differentiating with respect to time and then applying our typical machinery to the second law gives rise to

$$\frac{d}{dt} \int_{MR(t)} \rho s \; dV \quad \geq \quad \int_{MS(t)} -\frac{q_i}{T} n_i \; dA, \tag{4.176}$$

$$\int_{MR(t)} \partial_o(\rho s) \; dV + \int_{MS(t)} \rho s v_i n_i \; dA \quad \geq \quad \int_{MS(t)} -\frac{q_i}{T} n_i \; dA, \tag{4.177}$$

$$\int_{MR(t)} \left(\partial_o(\rho s) + \partial_i(\rho s v_i)\right) \; dV \quad \geq \quad \int_{MR(t)} -\partial_i \left(\frac{q_i}{T}\right) \; dV, \tag{4.178}$$

$$\int_{MR(t)} \left(\partial_o(\rho s) + \partial_i(\rho s v_i)\right) \; dV \quad = \quad \int_{MR(t)} -\partial_i \left(\frac{q_i}{T}\right) \; dV + \int_{MR(t)} I \; dV. \tag{4.179}$$

Here we have defined the *irreversibility*, $I \geq 0$, as a positive semi-definite scalar. It is simply a convenience to replace the inequality with an equality. Then we have the conservative form

$$\partial_o(\rho s) + \partial_i(\rho s v_i) = -\partial_i \left(\frac{q_i}{T}\right) + I. \tag{4.180}$$

Invoking mass conservation, Eq. (4.7), we easily get the non-conservative form

$$\rho \frac{ds}{dt} = -\partial_i \left(\frac{q_i}{T}\right) + I. \tag{4.181}$$

This is the second law. Now if we subtract from this the first law written in terms of entropy, Eq. (4.167), we get the result

$$I = -\frac{1}{T^2} q_i \partial_i T + \frac{1}{T} \underbrace{\tau_{ij} \partial_i v_j}_{\Phi}. \tag{4.182}$$

As an aside, we have defined the commonly used *viscous dissipation function* Φ as

$$\Phi \equiv \tau_{ij} \partial_i v_j. \tag{4.183}$$

For symmetric stress tensors, we also have $\Phi = \tau_{ij}\partial_{(i}v_{j)}$. Now because $I \geq 0$, we can view the entirety of the second law as the following constraint, sometimes called the *weak form* of the *Clausius-Duhem*[5][6] inequality:

$$-\frac{1}{T^2}q_i\partial_i T + \frac{1}{T}\tau_{ij}\partial_i v_j \geq 0, \tag{4.184}$$

$$-\frac{1}{T^2}\mathbf{q}^T \cdot \nabla T + \frac{1}{T}\boldsymbol{\tau} : \nabla \mathbf{v}^T \geq 0, \tag{4.185}$$

$$-\frac{1}{T^2}\mathbf{q}^T \cdot \nabla T + \frac{1}{T}\boldsymbol{\tau} : \mathsf{L} \geq 0. \tag{4.186}$$

Recalling that τ_{ij} is symmetric by the angular momenta principle for no external couples, and, consequently, that its tensor inner product with the velocity gradient only has a contribution from the symmetric part of the velocity gradient (that is, the deformation rate or strain rate tensor), the entropy inequality reduces slightly to

$$-\frac{1}{T^2}q_i\partial_i T + \frac{1}{T}\tau_{ij}\partial_{(i}v_{j)} \geq 0, \tag{4.187}$$

$$-\frac{1}{T^2}\mathbf{q}^T \cdot \nabla T + \frac{1}{T}\boldsymbol{\tau} : \left(\frac{\nabla \mathbf{v}^T + \left(\nabla \mathbf{v}^T\right)^T}{2}\right) \geq 0, \tag{4.188}$$

$$-\frac{1}{T^2}\mathbf{q}^T \cdot \nabla T + \frac{1}{T}\boldsymbol{\tau} : \mathsf{D} \geq 0. \tag{4.189}$$

We shall see in upcoming sections that we will be able to specify q_i and τ_{ij} in such a fashion that is both consistent with experiment and satisfies the entropy inequality.

The more restrictive (and in some cases, *overly* restrictive) *strong form* of the Clausius-Duhem inequality requires each term to be greater than or equal to zero. For our system the strong form, realizing that the absolute temperature $T > 0$, is

$$-q_i\partial_i T \geq 0, \qquad \underbrace{\tau_{ij}\partial_{(i}v_{j)}}_{\Phi} \geq 0, \tag{4.190}$$

$$-\mathbf{q}^T \cdot \nabla T \geq 0, \qquad \boldsymbol{\tau} : \left(\frac{\nabla \mathbf{v}^T + \left(\nabla \mathbf{v}^T\right)^T}{2}\right) \geq 0. \tag{4.191}$$

It is straightforward to show that terms that generate entropy due to viscous work also dissipate mechanical energy. This can be cleanly demonstrated by considering the mechanisms that cause mechanical energy the change within a finite fixed control volume V. First consider the non-conservative form of the mechanical energy equation, Eq. (4.144):

$$\rho\frac{d}{dt}\left(\frac{v_j v_j}{2}\right) = \rho v_j f_j - v_j\partial_j p + v_j\partial_i\tau_{ij}. \tag{4.192}$$

[5]Rudolf Clausius, 1822-1888, Prussian-born German mathematical physicist, key figure in making thermodynamics a science, author of well-known statement of the second law of thermodynamics, taught at Zürich Polytechnikum, University of Würzburg, and University of Bonn.

[6]Pierre Maurice Marie Duhem, 1861-1916, French physicist, mathematician, and philosopher, taught at Lille, Rennes, and the University of Bordeaux.

Now use the product rule to restate the pressure and viscous work terms so as to achieve

$$\rho \frac{d}{dt}\left(\frac{v_j v_j}{2}\right) = \rho v_j f_j - \partial_j(v_j p) + p\partial_j v_j + \partial_i(\tau_{ij} v_j) - \underbrace{\tau_{ij}\partial_i v_j}_{=\Phi \geq 0}. \tag{4.193}$$

So, here we see what induces *local* changes in mechanical energy. We see that body forces, pressure forces and viscous forces in general can induce the mechanical energy to rise or fall. However that part of the viscous stresses that is associated with the viscous dissipation, Φ, is guaranteed to induce a *local decrease* in mechanical energy. It is sometimes said that this is a transformation in which mechanical energy dissipates into thermal energy.

To study global changes in mechanical energy, we consider the conservative form of the mechanical energy equation, Eq. (4.142), here written in the same way that takes advantage of application of the product rule to the pressure and viscous terms:

$$\partial_o\left(\rho \frac{v_j v_j}{2}\right) + \partial_i\left(\rho v_i \frac{v_j v_j}{2}\right) = \rho v_j f_j - \partial_j(v_j p) + p\partial_j v_j + \partial_i(\tau_{ij} v_j) - \tau_{ij}\partial_i v_j. \tag{4.194}$$

Now integrate over a fixed control volume with closed boundaries, so that

$$\begin{aligned} \int_V \partial_o\left(\rho \frac{v_j v_j}{2}\right)\, dV + \int_V \partial_i\left(\rho v_i \frac{v_j v_j}{2}\right)\, dV &= \int_V \rho v_j f_j\, dV - \int_V \partial_j(v_j p)\, dV + \int_V p\partial_j v_j\, dV \\ &\quad + \int_V \partial_i(\tau_{ij} v_j)\, dV - \int_V \tau_{ij}\partial_i v_j\, dV. \end{aligned} \tag{4.195}$$

Applying Leibniz's rule, Eq. (2.268), and Gauss's law, Eq. (2.250), we get

$$\begin{aligned} \frac{d}{dt}\int_V \rho \frac{v_j v_j}{2}\, dV + \int_S n_i \rho v_i \frac{v_j v_j}{2}\, dS &= \int_V \rho v_j f_j\, dV - \int_S n_j v_j p\, dS + \int_V p\partial_j v_j\, dV \\ &\quad + \int_S n_i(\tau_{ij} v_j)\, dS - \int_V \tau_{ij}\partial_i v_j\, dV. \end{aligned} \tag{4.196}$$

Now on the surface of the closed fixed volume, the velocity is zero, so we get

$$\frac{d}{dt}\int_V \rho \frac{v_j v_j}{2}\, dV = \int_V \rho v_j f_j\, dV + \int_V p\partial_j v_j\, dV - \int_V \underbrace{\tau_{ij}\partial_i v_j}_{\text{positive}}\, dV. \tag{4.197}$$

Now the strong form of the second law requires that $\tau_{ij}\partial_i v_j = \tau_{ij}\partial_{(i} v_{j)} \geq 0$. So, we see for a finite fixed closed volume of fluid that a body force and pressure force in conjunction with local volume changes can cause the global mechanical energy to either grow or decay, the viscous stress always induces a decay of global mechanical energy; in other words it is a dissipative effect.

4.6 Integral forms

- Our governing equations are formulated based upon laws that apply to a *material element*.
- We are not often interested in an actual material element but in some other fixed or moving region in space.
- Rules for such systems can by formulated with Leibniz's rule in conjunction with the differential forms of our axioms.

Let us first recall Leibniz's rule (2.273) for an arbitrary scalar function f (that has no relation to our body force term) over a time-dependent arbitrary region $AR(t)$:

$$\frac{d}{dt}\int_{AR(t)} f \; dV = \int_{AR(t)} \partial_o f \; dV + \int_{AS(t)} n_i w_i f \; dS. \tag{4.198}$$

Recall that w_i is the velocity of the arbitrary surface, not necessarily the particle velocity.

4.6.1 Mass

We rewrite the mass conservation, Eq. (4.7), as

$$\partial_o \rho = -\partial_i(\rho v_i). \tag{4.199}$$

Now let us use this, and let $f = \rho$ in Leibniz's rule, Eq. (4.198), to get

$$\frac{d}{dt}\int_{AR(t)} \rho \; dV = \int_{AR(t)} \partial_o \rho \; dV + \int_{AS(t)} n_i w_i \rho \; dS, \tag{4.200}$$

$$\frac{d}{dt}\int_{AR(t)} \rho \; dV = \int_{AR(t)} (-\partial_i(\rho v_i)) \; dV + \int_{AS(t)} n_i w_i \rho \; dS. \tag{4.201}$$

Invoking Gauss, Eq. (2.250), we get

$$\frac{d}{dt}\int_{AR(t)} \rho \; dV = \int_{AS(t)} n_i \rho (w_i - v_i) \; dS. \tag{4.202}$$

Now consider three special cases.

4.6.1.1 Fixed region

We take $w_i = 0$. So the arbitrary region that is a function of time, $AR(t)$, becomes a fixed region, FR. It is bounded by a fixed surface FS.

$$\frac{d}{dt}\int_{FR} \rho \; dV = -\int_{FS} n_i \rho v_i \; dS, \tag{4.203}$$

$$\frac{d}{dt}\int_{FR} \rho \; dV + \int_{FS} n_i \rho v_i \; dS = 0. \tag{4.204}$$

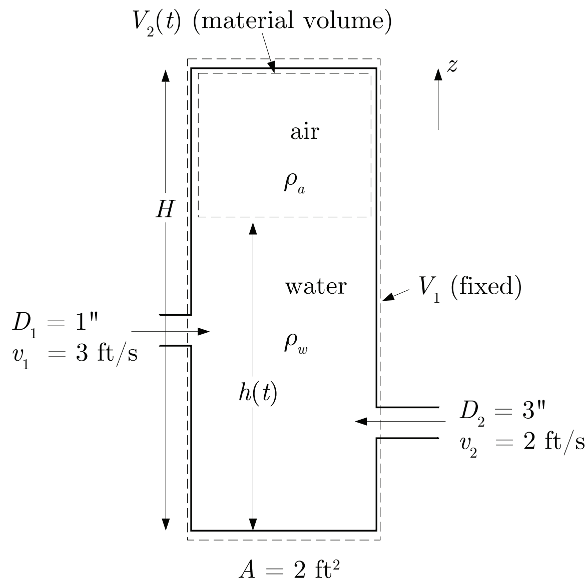

Figure 4.7: Sketch of volume with water and air being filled with water.

4.6.1.2 Material region

Here we take $w_i = v_i$.

$$\frac{d}{dt}\int_{MR(t)} \rho \, dV = 0. \tag{4.205}$$

4.6.1.3 Moving solid enclosure with holes

Say the region considered is a solid enclosure with holes through with fluid can enter and exit. The our arbitrary surface $AS(t)$ can be specified as

$$AS(t) = A_e(t) \quad \text{area of entrances and exits} \tag{4.206}$$
$$+A_s(t) \quad \text{solid moving surface with } w_i = v_i \tag{4.207}$$
$$+A_s \quad \text{fixed solid surface with } w_i = v_i = 0. \tag{4.208}$$

Then we get

$$\frac{d}{dt}\int_{AR(t)} \rho \, dV + \int_{A_e(t)} \rho n_i (v_i - w_i) \, dS = 0. \tag{4.209}$$

Example 4.2

Consider the volume sketched in Fig. 4.7. Water enters a circular hole of diameter $D_1 = 1$" with velocity $v_1 = 3$ ft/s. Water enters another circular hole of diameter $D_2 = 3$" with velocity $v_2 = 2$ ft/s. The cross sectional area of the cylindrical tank is $A = 2$ ft^2. The tank has height H. Water at density ρ_w exists in the tank at height $h(t)$. Air at density ρ_a fills the remainder of the tank. Find the rate of rise of the water dh/dt.

Consider two control volumes

- V_1: the fixed region enclosing the entire tank, and
- $V_2(t)$: the material region attached to the air.

First, let us write mass conservation for the material region 2:

$$\frac{d}{dt}\int_{V_2} \rho_a \; dV = 0, \tag{4.210}$$

$$\frac{d}{dt}\int_{h(t)}^{H} \rho_a A \; dz = 0. \tag{4.211}$$

Mass conservation for V_1 is

$$\frac{d}{dt}\int_{V_1} \rho \; dV + \int_{A_e} \rho v_i n_i \; dS = 0. \tag{4.212}$$

Now break up V_1 and write A_e explicitly

$$\frac{d}{dt}\int_0^{h(t)} \rho_w A \; dz + \underbrace{\frac{d}{dt}\int_{h(t)}^{H} \rho_a A \; dz}_{=0} = -\int_{A_1} \rho_w v_i n_i \; dS - \int_{A_2} \rho_w v_i n_i \; dS, \tag{4.213}$$

$$\frac{d}{dt}\int_0^{h(t)} \rho_w A \; dz = \rho_w v_1 A_1 + \rho_w v_2 A_2, \tag{4.214}$$

$$= \frac{\rho_w \pi}{4}(v_1 D_1^2 + v_2 D_2^2), \tag{4.215}$$

$$\rho_w A \frac{d}{dt}\int_0^{h(t)} dz = \frac{\rho_w \pi}{4}(v_1 D_1^2 + v_2 D_2^2), \tag{4.216}$$

$$\frac{dh}{dt} = \frac{\pi}{4A}(v_1 D_1^2 + v_2 D_2^2), \tag{4.217}$$

$$= \frac{\pi}{4(2 \text{ ft}^2)}\left(\left(3 \; \frac{\text{ft}}{\text{s}}\right)\left(\frac{1}{12} \text{ ft}\right)^2 + \left(2 \; \frac{\text{ft}}{\text{s}}\right)\left(\frac{3}{12} \text{ ft}\right)^2\right), \tag{4.218}$$

$$= \frac{7\pi}{384} \; \frac{\text{ft}}{\text{s}} = 0.057 \; \frac{\text{ft}}{\text{s}}. \tag{4.219}$$

4.6.2 Linear momenta

Let us perform the same exercise for the linear momenta equation. First, in a strictly mathematical step, apply Leibniz's rule, Eq. (4.198), to linear momenta, $f = \rho v_i$:

$$\frac{d}{dt}\int_{AR(t)} \rho v_i \; dV = \int_{AR(t)} \partial_o(\rho v_i) \; dV + \int_{AS(t)} n_j w_j \rho v_i \; dS. \tag{4.220}$$

Now invoke the physical linear momenta axiom, Eq. (4.52). Here the axiom gives us an expression for $\partial_o(\rho v_i)$. We will also convert volume integrals to surface integrals via Gauss's theorem, Eq. (2.250), to get

$$\frac{d}{dt}\int_{AR(t)} \rho v_i \ dV = -\int_{AS(t)} (\rho n_j(v_j - w_j)v_i + n_i p - n_j \tau_{ij}) \ dS + \int_{AR(t)} \rho f_i \ dV. \tag{4.221}$$

Now momenta flux terms only have values at entrances and exits (at solid surfaces, we get $v_i = w_i$), so we can say

$$\frac{d}{dt}\int_{AR(t)} \rho v_i \ dV + \int_{A_e(t)} \rho n_j(v_j - w_j)v_i \ dS = -\int_{AS(t)} n_i p \ dS + \int_{AS(t)} n_j \tau_{ij} \ dS + \int_{AR(t)} \rho f_i \ dV. \tag{4.222}$$

The surface forces are evaluated along all surfaces, not just entrances and exits.

4.6.3 Energy

Applying the same analysis to the energy equation, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{AR(t)} \rho\left(e + \frac{1}{2}v_j v_j\right) \ dV \ &= \ -\int_{AS(t)} \rho n_i(v_i - w_i)\left(e + \frac{1}{2}v_j v_j\right) \ dS \\ &\quad -\int_{AR(t)} n_i q_i \ dS \\ &\quad -\int_{AS(t)} (n_i v_i p - n_i \tau_{ij} v_j) \ dS \\ &\quad +\int_{AR(t)} \rho v_i f_i \ dV. \end{aligned} \tag{4.223}$$

4.6.4 General expression

If we have a governing equation from a physical principle that is of the form

$$\partial_o f_j + \partial_i(v_i f_j) = \partial_i g_j + h_j, \tag{4.224}$$

then we can say for an arbitrary volume that

$$\underbrace{\frac{d}{dt}\int_{AR(t)} f_j \ dV}_{\text{change of } f_j} = \underbrace{-\int_{AS(t)} n_i f_j(v_i - w_i) \ dS}_{\text{flux of } f_j} + \underbrace{\int_{AS(t)} n_i g_j \ dS}_{\text{effect of } g_j} + \underbrace{\int_{AR(t)} h_j \ dV}_{\text{effect of } h_j}. \tag{4.225}$$

4.7 Summary of axioms in differential form

Here we pause to summarize the mathematical form of our axioms. We give the Cartesian index, Gibbs, and the full non-orthogonal index notation. All details of development of the non-orthogonal index notation are omitted, and the reader is referred to Aris (1962) for a full development. We will first present the conservative form and then the non-conservative form.

4.7.1 Conservative form

4.7.1.1 Cartesian index form

$$\partial_o \rho + \partial_i(\rho v_i) = 0, \tag{4.226}$$

$$\partial_o(\rho v_i) + \partial_j(\rho v_j v_i) = \rho f_i - \partial_i p + \partial_j \tau_{ji}, \tag{4.227}$$

$$\tau_{ij} = \tau_{ji}, \tag{4.228}$$

$$\partial_o\left(\rho\left(e + \frac{1}{2}v_j v_j\right)\right) + \partial_i\left(\rho v_i\left(e + \frac{1}{2}v_j v_j\right)\right) = -\partial_i q_i - \partial_i(p v_i) + \partial_i(\tau_{ij} v_j) + \rho v_i f_i, \tag{4.229}$$

$$\partial_o(\rho s) + \partial_i(\rho s v_i) \geq -\partial_i\left(\frac{q_i}{T}\right). \tag{4.230}$$

4.7.1.2 Gibbs form

$$\frac{\partial \rho}{\partial t} + \nabla^T \cdot (\rho \mathbf{v}) = 0, \tag{4.231}$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \left(\nabla^T \cdot (\rho \mathbf{v}\mathbf{v}^T)\right)^T = \rho \mathbf{f} - \nabla p + \left(\nabla^T \cdot \boldsymbol{\tau}\right)^T, \tag{4.232}$$

$$\boldsymbol{\tau} = \boldsymbol{\tau}^T, \tag{4.233}$$

$$\frac{\partial}{\partial t}\left(\rho\left(e + \frac{1}{2}\mathbf{v}^T \cdot \mathbf{v}\right)\right) + \nabla^T \cdot \left(\rho \mathbf{v}\left(e + \frac{1}{2}\mathbf{v}^T \cdot \mathbf{v}\right)\right) = -\nabla^T \cdot \mathbf{q} - \nabla^T \cdot (p\mathbf{v}) + \nabla^T \cdot (\boldsymbol{\tau} \cdot \mathbf{v}) + \rho \mathbf{v}^T \cdot \mathbf{f}, \tag{4.234}$$

$$\frac{\partial}{\partial t}(\rho s) + \nabla^T \cdot (\rho s \mathbf{v}) \geq -\nabla^T \cdot \left(\frac{\mathbf{q}}{T}\right). \tag{4.235}$$

4.7.1.3 Non-orthogonal index form

Here we present the governing equations for general non-orthogonal coordinate systems. Few texts give a proper exposition of the conservative form of the equations in non-orthogonal coordinates. Here we have extended the development of Vinokur[7] to include the effects of

[7]Vinokur, M., 1974, "Conservation equations of gasdynamics," *Journal of Computational Physics*, 14(2): 105-125.

momenta and energy diffusion. This extension has been guided by general notions found in standard works such as Aris (1962) as well as Liseikin (2010).

$$\frac{\partial}{\partial t}\left(\sqrt{g}\ \rho\right) + \frac{\partial}{\partial x^k}\left(\sqrt{g}\ \rho v^k\right) = 0, \tag{4.236}$$

$$\frac{\partial}{\partial t}\left(\sqrt{g}\ \rho v^j \frac{\partial \xi^i}{\partial x^j}\right) + \frac{\partial}{\partial x^k}\left(\sqrt{g}\ \rho v^j v^k \frac{\partial \xi^i}{\partial x^j}\right) = \sqrt{g}\ \rho f^j \frac{\partial \xi^i}{\partial x^j} - \frac{\partial}{\partial x^k}\left(\sqrt{g}\ p g^{jk} \frac{\partial \xi^i}{\partial x^j}\right) + \frac{\partial}{\partial x^k}\left(\sqrt{g}\ \tau^{jk} \frac{\partial \xi^i}{\partial x^j}\right), \tag{4.237}$$

$$\frac{\partial}{\partial t}\left(\sqrt{g}\ \rho\left(e + \frac{1}{2} g_{ij} v^i v^j\right)\right) + \frac{\partial}{\partial x^k}\left(\sqrt{g}\ \rho v^k\left(e + \frac{1}{2} g_{ij} v^i v^j\right)\right) = -\frac{\partial}{\partial x^k}\left(\sqrt{g}\ q^k\right) - \frac{\partial}{\partial x^k}\left(\sqrt{g}\ p v^k\right) + \frac{\partial}{\partial x^k}\left(\sqrt{g}\ g_{ij} v^j \tau^{ik}\right) + \sqrt{g}\ \rho g_{ij} v^j f^i, \tag{4.238}$$

$$\frac{\partial}{\partial t}\left(\sqrt{g}\ \rho s\right) + \frac{\partial}{\partial x^k}\left(\sqrt{g}\ \rho s v^k\right) \geq -\frac{\partial}{\partial x^k}\left(\sqrt{g}\ \frac{q^k}{T}\right). \tag{4.239}$$

4.7.2 Non-conservative form

4.7.2.1 Cartesian index form

$$\frac{d\rho}{dt} = -\rho \partial_i v_i, \tag{4.240}$$

$$\rho \frac{dv_i}{dt} = \rho f_i - \partial_i p + \partial_j \tau_{ji}, \tag{4.241}$$

$$\tau_{ij} = \tau_{ji}, \tag{4.242}$$

$$\rho \frac{de}{dt} = -\partial_i q_i - p \partial_i v_i + \tau_{ij} \partial_i v_j, \tag{4.243}$$

$$\rho \frac{ds}{dt} \geq -\partial_i \left(\frac{q_i}{T}\right). \tag{4.244}$$

4.7.2.2 Gibbs form

$$\frac{d\rho}{dt} = -\rho \nabla^T \cdot \mathbf{v}, \tag{4.245}$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f} - \nabla p + \left(\nabla^T \cdot \boldsymbol{\tau} \right)^T, \tag{4.246}$$

$$\boldsymbol{\tau} = \boldsymbol{\tau}^T, \tag{4.247}$$

$$\rho \frac{de}{dt} = -\nabla^T \cdot \mathbf{q} - p \nabla^T \cdot \mathbf{v} + \boldsymbol{\tau} : \nabla \mathbf{v}^T, \tag{4.248}$$

$$\rho \frac{ds}{dt} \geq -\nabla^T \cdot \left(\frac{\mathbf{q}}{T} \right). \tag{4.249}$$

4.7.2.3 Non-orthogonal index form

This version is likely correct but has not been checked carefully!

$$\frac{\partial \rho}{\partial t} + v^i \frac{\partial \rho}{\partial x^i} = -\frac{\rho}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g}\ v^i \right), \tag{4.250}$$

$$\rho \left(\frac{\partial v^i}{\partial t} + v^j \left(\frac{\partial v^i}{\partial x^j} + \Gamma^i_{jl} v^l \right) \right) = \rho f^i - g^{ij} \frac{\partial p}{\partial x^j} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(\sqrt{g}\ \tau^{ij} \right) + \Gamma^i_{jk} \tau^{jk}, \tag{4.251}$$

$$\begin{aligned} \rho \left(\frac{\partial e}{\partial t} + v^i \frac{\partial e}{\partial x^i} \right) &= -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g}\ q^i \right) - \frac{p}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g}\ v^i \right) \\ &\quad + g_{ik} \tau^{kj} \left(\frac{\partial v^i}{\partial x^j} + \Gamma^i_{jl} v^l \right), \end{aligned} \tag{4.252}$$

$$\rho \left(\frac{\partial s}{\partial t} + v^i \frac{\partial s}{\partial x^i} \right) \geq -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g}\ \frac{q^i}{T} \right). \tag{4.253}$$

The term $\Gamma^i_{jl} v^l$ in the linear momenta equation can be shown to represent the effects of non-Cartesian terms such as centripetal and Coriolis accelerations, to be fully explored in Ch. 7.1.1.

4.7.3 Physical interpretations

Each term in the governing axioms represents a physical mechanism. This approach is emphasized in the classical text by Bird, Stewart, and Lightfoot (2007) on transport processes. In general, the equations which are partial differential equations can be represented in the following form:

$$\text{local change} = \text{advection} + \text{diffusion} + \text{source}. \tag{4.254}$$

Here we consider advection and diffusion to be types of transport phenomena. If we have a fixed volume of material, a property of that material, such as its thermal energy, can change because an outside flow sweeps energy in from outside due to bulk fluid motion. That is advection.[8] The thermal energy can also change because random molecular motions allow slow leakage to the outside or leakage in from the outside. That is diffusion. Or the material

[8]Often the term "convection" is used in a similar fashion as "advection." While there is not a universal consensus on the distinctions between these two common terms, "advection" does seem to always be associated with bulk fluid motion that has $v_i \neq 0$. Such motion is macroscopic and identifiable. It is thus

can undergo intrinsic changes inside, such as viscous work, that converts kinetic energy into thermal energy.

Let us write the Gibbs form of the non-conservative equations of mass, linear momenta, and energy in a slightly different way to illustrate these mechanisms:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= \quad \text{local change in mass} \\
& -\mathbf{v}^T \cdot \nabla \rho \quad \text{advection of mass} \\
& +0 \quad \text{diffusion of mass} \\
& -\rho \nabla^T \cdot \mathbf{v}, \quad \text{volume expansion source,} \qquad (4.255) \\
\rho \frac{\partial \mathbf{v}}{\partial t} &= \quad \text{local change in linear momenta} \\
& -\rho \left(\mathbf{v}^T \cdot \nabla\right) \mathbf{v} \quad \text{advection of linear momenta} \\
& + \left(\nabla^T \cdot \boldsymbol{\tau}\right)^T, \quad \text{diffusion of linear momenta} \qquad (4.256) \\
& +\rho \mathbf{f} \quad \text{body force source of linear momenta} \\
& -\nabla p \quad \text{pressure gradient source of linear momenta,} \\
\rho \frac{\partial e}{\partial t} &= \quad \text{local change in thermal energy} \\
& -\rho \mathbf{v}^T \cdot \nabla e \quad \text{advection of thermal energy} \\
& -\nabla^T \cdot \mathbf{q} \quad \text{diffusion of thermal energy} \\
& -p \nabla^T \cdot \mathbf{v} \quad \text{pressure work thermal energy source} \\
& +\boldsymbol{\tau} : \nabla \mathbf{v}^T \quad \text{viscous work thermal energy source.} \qquad (4.257)
\end{aligned}$$

Briefly considering the second law, we note that the irreversibility I is solely associated with diffusion of linear momenta and diffusion of energy. This makes sense in that diffusion is associated with random molecular motions and thus disorder. Advection is associated with an ordered motion of matter in that we retain knowledge of the position of the matter. Pressure-volume work is a reversible work and does not contribute to entropy changes. A portion of the heat transfer can be considered to be reversible. All of the work done by the viscous forces is irreversible work.

4.8 Incompleteness of the axioms

The beauty of these axioms is that they are valid for any material that can be modeled as a continuum under the influence of the forces we have mentioned. Specifically, they are valid for both solid and fluid mechanics, which is remarkable.

associated with information and thus order. Advection can apply to mass, momentum, or energy. "Convection" is more commonly restricted to energy, and sometimes includes both energy advection and diffusion effects. Diffusion is associated with disordered, random motion of molecules. Diffusion is non-zero, even if there is no bulk fluid motion, $v_i = 0$. That is to say, a fluid at rest at the macroscale is generally in motion at the microscale, where the motion is random.

While the axioms are complete, the equations are not! We have twenty-three unknowns here $\rho(1), v_i(3), f_i(3), p(1), \tau_{ij}(9), e(1), q_i(3), T(1), s(1)$, and only eight equations (one mass, three linear momenta, three independent angular momenta, one energy). We cannot really count the second law as an equation, as it is an inequality. Whatever result we get must be consistent with it. Whatever the case, we are short a number of equations. We will see in Ch. 5 how we use constitutive equations, equations founded in empiricism, that in some sense model sub-continuum effects that we have ignored, to complete our system.

Chapter 5

Constitutive equations

see Panton, Chapter 6,
see Hughes and Gaylord, Chapter 1,
see Aris, Chapters 5 and 6.

In this chapter, we return to the problem of completing our set of equations as introduced in Ch. 4.8. Constitutive equations are additional equations based on experiment that are not as fundamental as the previously developed axioms that can complete our continuum description. They can be rather *ad hoc* relations that in some sense model the sub-continuum nano-structure. In some cases, for example, the sub-continuum kinetic theory of gases, we can show that when the sub-continuum is formally averaged, that we obtain commonly used constitutive equations. In most cases however, constitutive equations simply represent curve fits to basic experimental results, that can vary widely from material to material. As is briefly discussed below, constitutive equations are not completely arbitrary. Whatever is proposed must allow our final equations to be invariant under Galilean transformations and rotations as well as satisfy the entropy inequality.

For example, we might hope to develop a constitutive equation for the heat flux vector q_i. Being naïve, we might in general expect it to be a function of a large number of variables:

$$q_i = q_i(\rho, p, T, v_i, \tau_{ij}, f_i, e, s, \ldots). \tag{5.1}$$

The principles of continuum mechanics will rule out some possibilities, but still allow a broad range of forms.

5.1 Frame and material indifference

Our choice of a constitutive law must be invariant under a Galilean transformation (frame invariance) a rotation (material indifference). Say for example, we propose that the heat flux vector is proportional to the velocity vector

$$q_i = av_i, \qquad \text{trial constitutive relation.} \tag{5.2}$$

If we changed frames such that velocities in the moving frame were $u_i = v_i - V$, we would have $q_i = a(u_i + V)$. With this constitutive law, we find a physical quantity is dependent on the frame velocity, that we observe to be non-physical; hence we rule out this trial constitutive relation.

A commonly used constitutive law for stress in a one-dimensional experiment is

$$\tau_{12} = b(\partial_1 v_2)^a (\partial_1 u_2)^b, \tag{5.3}$$

where u_2 is the displacement of particle. While this may fit one-dimensional data well, it is in no way clear how one could simply extend this to write an expression for τ_{ij}, and many propositions will fail to satisfy material indifference.

5.2 Second law restrictions and Onsager relations

The entropy inequality from the second law of thermodynamics provides additional restrictions on the form of constitutive equations. Recall the second law (equivalently, the weak form of the Clausius-Duhem inequality, Eq. (4.187)) tells us that

$$-\frac{1}{T^2} q_i \partial_i T + \frac{1}{T} \tau_{ij} \partial_{(i} v_{j)} \geq 0. \tag{5.4}$$

We would like to find forms of q_i and τ_{ij} that are consistent with the weak form of the entropy inequality, Eq. (5.4).

5.2.1 Weak form of the Clausius-Duhem inequality

The weak form suggests that we may want to consider both q_i and τ_{ij} to be functions involving the temperature gradient $\partial_i T$ and the deformation tensor $\partial_{(i} v_{j)}$.

5.2.1.1 Non-physical motivating example

To see that this is actually too general of an assumption, it suffices to consider a one-dimensional limit. In the one-dimensional limit, the weak form of the entropy inequality, Eq. (4.187), reduces to

$$-\frac{1}{T^2} q \frac{\partial T}{\partial x} + \frac{1}{T} \tau \frac{\partial u}{\partial x} \geq 0. \tag{5.5}$$

We can write this in a vector form as

$$\left(-\frac{1}{T}\frac{\partial T}{\partial x} \quad \frac{1}{u}\frac{\partial u}{\partial x} \right) \begin{pmatrix} \frac{q}{T} \\ \frac{\tau u}{T} \end{pmatrix} \geq 0. \tag{5.6}$$

A factor of u/u was introduced to the viscous stress term. This allows for a necessary dimensional consistency in that q/T has the same units as $\tau u/T$. Let us then hypothesize a linear relationship exists between the generalized fluxes q/T and $\tau u/T$ and the generalized driving gradients $-(1/T)\partial T/\partial x$ and $(1/u)\partial u/\partial x$:

$$\frac{q}{T} = C_{11}\left(-\frac{1}{T}\frac{\partial T}{\partial x}\right) + C_{12}\frac{1}{u}\frac{\partial u}{\partial x}, \tag{5.7}$$

$$\frac{\tau u}{T} = C_{21}\left(-\frac{1}{T}\frac{\partial T}{\partial x}\right) + C_{22}\frac{1}{u}\frac{\partial u}{\partial x}. \tag{5.8}$$

In matrix form this becomes

$$\begin{pmatrix} \frac{q}{T} \\ \frac{\tau u}{T} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} -\frac{1}{T}\frac{\partial T}{\partial x} \\ \frac{1}{u}\frac{\partial u}{\partial x} \end{pmatrix}. \tag{5.9}$$

We then substitute this hypothesized relationship into the entropy inequality to obtain

$$\left(-\frac{1}{T}\frac{\partial T}{\partial x} \quad \frac{1}{u}\frac{\partial u}{\partial x} \right) \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} -\frac{1}{T}\frac{\partial T}{\partial x} \\ \frac{1}{u}\frac{\partial u}{\partial x} \end{pmatrix} \geq 0. \tag{5.10}$$

We next segregate the matrix C_{ij} into a symmetric and anti-symmetric part to get

$$\left(-\frac{1}{T}\frac{\partial T}{\partial x} \quad \frac{1}{u}\frac{\partial u}{\partial x} \right) \left(\begin{pmatrix} C_{11} & \frac{C_{12}+C_{21}}{2} \\ \frac{C_{21}+C_{12}}{2} & C_{22} \end{pmatrix} + \begin{pmatrix} 0 & \frac{C_{12}-C_{21}}{2} \\ \frac{C_{21}-C_{12}}{2} & 0 \end{pmatrix} \right) \begin{pmatrix} -\frac{1}{T}\frac{\partial T}{\partial x} \\ \frac{1}{u}\frac{\partial u}{\partial x} \end{pmatrix} \geq 0. \tag{5.11}$$

Distributing the multiplication, we find

$$\begin{aligned} &\left(-\frac{1}{T}\frac{\partial T}{\partial x} \quad \frac{1}{u}\frac{\partial u}{\partial x} \right) \begin{pmatrix} C_{11} & \frac{C_{12}+C_{21}}{2} \\ \frac{C_{21}+C_{12}}{2} & C_{22} \end{pmatrix} \begin{pmatrix} -\frac{1}{T}\frac{\partial T}{\partial x} \\ \frac{1}{u}\frac{\partial u}{\partial x} \end{pmatrix} \\ &+ \underbrace{\left(-\frac{1}{T}\frac{\partial T}{\partial x} \quad \frac{1}{u}\frac{\partial u}{\partial x} \right) \begin{pmatrix} 0 & \frac{C_{12}-C_{21}}{2} \\ \frac{C_{21}-C_{12}}{2} & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{T}\frac{\partial T}{\partial x} \\ \frac{1}{u}\frac{\partial u}{\partial x} \end{pmatrix}}_{=0} \geq 0. \end{aligned} \tag{5.12}$$

The second term is identically zero for all values of temperature and velocity gradients. So what remains is the inequality involving only a symmetric matrix:

$$\left(-\frac{1}{T}\frac{\partial T}{\partial x} \quad \frac{1}{u}\frac{\partial u}{\partial x} \right) \begin{pmatrix} C_{11} & \frac{C_{12}+C_{21}}{2} \\ \frac{C_{21}+C_{12}}{2} & C_{22} \end{pmatrix} \begin{pmatrix} -\frac{1}{T}\frac{\partial T}{\partial x} \\ \frac{1}{u}\frac{\partial u}{\partial x} \end{pmatrix} \geq 0. \tag{5.13}$$

Now in a well known result from linear algebra, a necessary and sufficient condition for satisfying this inequality is that the new coefficient matrix be positive semi-definite. Further, the matrix will be positive semi-definite if it has positive semi-definite eigenvalues. The eigenvalues of the new coefficient matrix can be shown to be

$$\lambda = \frac{1}{2}\left((C_{11} + C_{22}) \pm \sqrt{(C_{11} - C_{22})^2 + (C_{12} + C_{21})^2} \right). \tag{5.14}$$

Because the terms inside the radical are positive semi-definite, the eigenvalues must be real. This is a consequence of the parent matrix being symmetric. Now we require two positive semi-definite eigenvalues. First, if $C_{11} + C_{22} < 0$, we obviously have at least one negative eigenvalue, so we demand that $C_{11} + C_{22} \geq 0$. We then must have

$$C_{11} + C_{22} \geq \sqrt{(C_{11} - C_{22})^2 + (C_{12} + C_{21})^2}. \tag{5.15}$$

This gives rise to

$$(C_{11} + C_{22})^2 \geq (C_{11} - C_{22})^2 + (C_{12} + C_{21})^2. \tag{5.16}$$

Expanding and simplifying, one gets

$$C_{11}C_{22} \geq \left(\frac{C_{12} + C_{21}}{2}\right)^2. \tag{5.17}$$

Now the right side is positive semi-definite, so the left side must be also. Thus

$$C_{11}C_{22} \geq 0. \tag{5.18}$$

The only way for the sum and product of C_{11} and C_{22} to be positive semi-definite is to demand that $C_{11} \geq 0$ and $C_{22} \geq 0$. Thus we arrive at the final set of conditions to satisfy the second law:

$$C_{11} \geq 0, \tag{5.19}$$
$$C_{22} \geq 0, \tag{5.20}$$
$$C_{11}C_{22} \geq \left(\frac{C_{12} + C_{21}}{2}\right)^2. \tag{5.21}$$

Now an important school of thought, founded by Onsager[1] in twentieth century thermodynamics takes an extra step and makes the *further* assertion that the original matrix C_{ij} itself must be symmetric. That is $C_{12} = C_{21}$. This remarkable assertion is independent of the second law, and is, for other scenarios, consistent with experimental results. Consequently, the second law in combination with Onsager's independent demand, requires that

$$C_{11} \geq 0, \tag{5.22}$$
$$C_{22} \geq 0, \tag{5.23}$$
$$C_{12} \leq \sqrt{C_{11}C_{22}}. \tag{5.24}$$

All this said, we must dismiss our hypothesis in this specific case on other physical grounds, namely that such a hypothesis results in an infinite shear stress for a fluid at rest! In the special case in which $\partial T/\partial x = 0$, our hypothesis predicts $\tau = C_{22}(T/u^2)(\partial u/\partial x)$. Obviously this is inconsistent with any observation and so we reject this hypothesis. Additionally, this assumed form is not frame invariant because of the velocity dependency. So, why did we go to this trouble? First, we now have confidence that we should not expect to find heat flux to depend on deformation. Second, it illustrates some general techniques in continuum mechanics. Moreover, the techniques we used have actually been applied to other more complex phenomena which are physical, and of great practical importance.

[1]Lars Onsager, 1903-1976, Norwegian-born American physical chemist, earned Ph.D. and taught at Yale, developed a systematic theory for irreversible chemical processes.

5.2.1.2 Real physical effects

That such a matrix such as we studied in the previous section was asserted to be symmetric is a manifestation of what is known as a general *Onsager relation*, developed by Onsager in 1931 with a statistical mechanics basis for more general systems and for which he was awarded a Nobel Prize in chemistry in 1968. These actually describe a surprising variety of physical phenomena, and are described in detail many texts, including Fung and Woods. A well-known example is the Peltier[2] effect in which conduction of both heat and electrical charge is influenced by gradients of charge and temperature. This forms the basis of the operation of a thermocouple. Other relations exist are the Soret[3] effect in which diffusive mass fluxes are induced by temperature gradients, the Dufour effect in which a diffusive energy flux is induced by a species concentration gradient, the Hall[4] effect for coupled electrical and magnetic effects (that explains the operation of an electric motor), the Seeback[5] effect in which electromotive forces are induced by different conducting elements at different temperatures, the Thomson[6] effect in which heat is transferred when electric current flows in a conductor in which there is a temperature gradient, and the principle of detailed balance for multi-species chemical reactions.

5.2.2 Strong form of the Clausius-Duhem inequality

A less general way to satisfy the second law is to take the sufficient (but not necessary!) condition that each individual term in the entropy inequality to be greater than or equal to zero:

$$-\frac{1}{T^2}q_i\partial_i T \geq 0, \qquad \text{and} \tag{5.25}$$

$$\frac{1}{T}\tau_{ij}\partial_{(i}v_{j)} \geq 0. \tag{5.26}$$

Once again, this is called the *strong form* of the entropy inequality (or the strong form of the Clausius-Duhem inequality), and is potentially overly restrictive.

[2]Jean Charles Athanase Peltier, 1785-1845, French clockmaker, retired at 30 to study science.

[3]Charles Soret, 1854-1904, Swiss physicist and chemist.

[4]Edwin Herbert Hall, 1855-1938, Maine-born American physicist, educated at Johns Hopkins University where he discovered the Hall effect while working on his dissertation, taught at Harvard.

[5]Thomas Johann Seebeck, 1770-1831, German medical doctor who studied at Berlin and Göttingen.

[6]William Thomson (Lord Kelvin), 1824-1907, Belfast-born British mathematician and physicist, graduated and taught at Glasgow University, key figure in all of 19th century engineering science including mathematics, thermodynamics, and electrodynamics.

5.3 Fourier's law

Let us examine the restriction on q_i from the strong form of the entropy inequality to infer the common constitutive relation known as *Fourier's law*.[7] The portion of the strong form of the entropy inequality with which we are concerned here is Eq. (5.25):

$$-\frac{1}{T^2} q_i \partial_i T \geq 0. \tag{5.27}$$

Now *one* way to guarantee this inequality is satisfied is to specify the constitutive relation for the heat flux vector as

$$q_i = -k\partial_i T, \qquad \text{with} \qquad k \geq 0. \tag{5.28}$$

This is the well known Fourier's law for an isotropic material, where k is the thermal conductivity. It has the proper behavior under Galilean transformations and rotations; more importantly, it is consistent with macroscale experiments for isotropic materials and can be justified from an underlying microscale theory. Substitution of Fourier's law for an isotropic material into the strong form of entropy inequality, Eq. (5.25), yields

$$\frac{1}{T^2} k(\partial_i T)(\partial_i T) \geq 0, \tag{5.29}$$

that for $k \geq 0$ is a true statement. The second law allows other forms as well. The expression $q_i = -k((\partial_j T)(\partial_j T))\partial_i T$ is consistent with the second law. It does not match experiments well for most materials however.

Following Duhamel,[8] we can also generalize Fourier's law for an anisotropic material. Let us only consider anisotropic materials for which the conductivity in any given direction is a constant. For such materials, the thermal conductivity is a tensor k_{ij}, and Fourier's law generalizes to

$$q_i = -k_{ij}\partial_j T. \tag{5.30}$$

This effectively states that for a fixed temperature gradient, the heat flux depends on the orientation. This is characteristic of anisotropic substances such as layered materials. Substitution of the generalized Fourier's law into the strong form of the entropy inequality, Eq. (5.25), gives now

$$\frac{1}{T^2} k_{ij}(\partial_j T)(\partial_i T) \geq 0, \tag{5.31}$$

$$\frac{1}{T^2} (\partial_i T) k_{ij}(\partial_j T) \geq 0, \tag{5.32}$$

$$\frac{1}{T^2} (\nabla T)^T \cdot \mathsf{K} \cdot \nabla T \geq 0. \tag{5.33}$$

[7] Jean Baptiste Joseph Fourier, 1768-1830, French mathematician and Egyptologist who studied the transfer of heat and the representation of mathematical functions by infinite series summations of other functions. Son of a tailor.

[8] Jean Marie Constant Duhamel, 1797-1872, highly regarded mathematics teacher at École Polytechnique in Paris who applied mathematics to problems in heat transfer, mechanics, and acoustics.

Now $1/T^2 > 0$, so we must have $(\partial_i T) k_{ij} (\partial_j T) \geq 0$ for all possible values of ∇T. Now any possible anti-symmetric portion of k_{ij} cannot contribute to the inequality. We can see this by expanding k_{ij} in the entropy inequality to get

$$\partial_i T \left(\frac{1}{2}(k_{ij} + k_{ji}) + \frac{1}{2}(k_{ij} - k_{ji}) \right) \partial_j T \geq 0, \tag{5.34}$$

$$\partial_i T \left(k_{(ij)} + k_{[ij]} \right) \partial_j T \geq 0, \tag{5.35}$$

$$(\partial_i T) k_{(ij)} (\partial_j T) + \underbrace{(\partial_i T) k_{[ij]} (\partial_j T)}_{=0} \geq 0, \tag{5.36}$$

$$(\partial_i T) k_{(ij)} (\partial_j T) \geq 0. \tag{5.37}$$

The anti-symmetric part of k_{ij} makes no contribution to the entropy generation because it involves the tensor inner product of a symmetric tensor with an anti-symmetric tensor, that is identically zero.

Next, we again use the well-known result from linear algebra that the entropy inequality is satisfied if $k_{(ij)}$ is a positive semi-definite tensor. This will be the case if all the eigenvalues of $k_{(ij)}$ are non-negative. That this is sufficient to satisfy the entropy inequality is made plausible if we consider $\partial_j T$ to be an eigenvector, so that $k_{(ij)} \partial_j T = \lambda \delta_{ij} \partial_j T$ giving rise to an entropy inequality of

$$(\partial_i T) \lambda \delta_{ij} (\partial_j T) \geq 0, \tag{5.38}$$

$$\lambda (\partial_i T)(\partial_i T) \geq 0. \tag{5.39}$$

The inequality holds for all $\partial_i T$ as long as $\lambda \geq 0$.

Further now, when we consider the contribution of the heat flux vector to the energy equation, we see any possible anti-symmetric portion of the conductivity tensor will be inconsequential as well. This is seen by the following analysis, that considers only relevant terms in the energy equation

$$\rho \frac{de}{dt} = -\partial_i q_i + \ldots, \tag{5.40}$$

$$= \partial_i \left(k_{ij} \partial_j T \right) + \ldots, \tag{5.41}$$

$$= k_{ij} \partial_i \partial_j T + \ldots, \tag{5.42}$$

$$= \left(k_{(ij)} + k_{[ij]} \right) \partial_i \partial_j T + \ldots, \tag{5.43}$$

$$= k_{(ij)} \partial_i \partial_j T + \underbrace{k_{[ij]} \partial_i \partial_j T}_{=0} + \ldots, \tag{5.44}$$

$$= k_{(ij)} \partial_i \partial_j T + \ldots. \tag{5.45}$$

So, it seems any possible anti-symmetric portion of k_{ij} will have no consequence as far as the first or second laws are concerned. However, an anti-symmetric portion of k_{ij} would induce a heat flux orthogonal to the direction of the temperature gradient. In a remarkable confirmation of Onsager's principle, experimental measurements on anisotropic crystalline materials demonstrate that there is no component of heat flux orthogonal to the temperature gradient, and thus, the conductivity matrix k_{ij} in fact has zero anti-symmetric part, and thus is symmetric, $k_{ij} = k_{ji}$. For our particular case with a tensorial conductivity, the competing effects are the heat fluxes in three directions, caused by temperature gradients in three directions:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = -\begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix} \begin{pmatrix} \partial_1 T \\ \partial_2 T \\ \partial_3 T \end{pmatrix}. \tag{5.46}$$

The symmetry condition, Onsager's principle, requires that $k_{12} = k_{21}$, $k_{13} = k_{31}$, and $k_{23} = k_{32}$. So, the experimentally verified Onsager's principle further holds that the heat flux for an anisotropic material is given by

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = -\begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{pmatrix} \begin{pmatrix} \partial_1 T \\ \partial_2 T \\ \partial_3 T \end{pmatrix}. \tag{5.47}$$

Now it is well known that the conductivity matrix k_{ij} will be positive semi-definite if all its eigenvalues are non-negative. The eigenvalues will be guaranteed real upon adopting Onsager symmetry. The characteristic polynomial for the eigenvalues is given by

$$\lambda^3 - I_k^{(1)}\lambda^2 + I_k^{(2)}\lambda - I_k^{(3)} = 0, \tag{5.48}$$

where the invariants of the conductivity tensor k_{ij}, are given by the standard

$$I_k^{(1)} = k_{ii} = \text{tr}\,\mathsf{K}, \tag{5.49}$$

$$I_k^{(2)} = \frac{1}{2}(k_{ii}k_{jj} - k_{ij}k_{ji}) = (\det \mathsf{K})\left(\text{tr}\,\mathsf{K}^{-1}\right), \tag{5.50}$$

$$I_k^{(3)} = \epsilon_{ijk}k_{1j}k_{2j}k_{3j} = \det \mathsf{K}. \tag{5.51}$$

In a standard result from linear algebra, one can show that if all three invariants are positive semi-definite, then the eigenvalues are all positive semi-definite, and as a result, the matrix itself is positive semi-definite. Hence, in order for k_{ij} to be positive semi-definite we demand that

$$I_k^{(1)} \geq 0, \tag{5.52}$$

$$I_k^{(2)} \geq 0, \tag{5.53}$$

$$I_k^{(3)} \geq 0, \tag{5.54}$$

that is equivalent to demanding that

$$k_{11} + k_{22} + k_{33} \geq 0, \tag{5.55}$$

$$k_{11}k_{22} + k_{11}k_{33} + k_{22}k_{33} - k_{12}^2 - k_{13}^2 - k_{23}^2 \geq 0, \tag{5.56}$$

$$k_{13}(k_{12}k_{23} - k_{22}k_{13}) + k_{23}(k_{12}k_{13} - k_{11}k_{23}) + k_{33}(k_{11}k_{22} - k_{12}k_{12}) \geq 0. \tag{5.57}$$

If $\det \mathsf{K} \neq 0$, the conditions reduce to

$$\text{tr } \mathsf{K} \geq 0, \tag{5.58}$$
$$\text{tr } \mathsf{K}^{-1} \geq 0, \tag{5.59}$$
$$\det \mathsf{K} > 0. \tag{5.60}$$

Now by considering $\partial_i T = (1,0,0)^T$, and demanding $(\partial_i T) k_{ij} (\partial_j T) \geq 0$, we conclude that $k_{11} \geq 0$. Similarly, by considering $\partial_i T = (0,1,0)^T$ and $\partial_i T = (0,0,1)^T$, we conclude that $k_{22} \geq 0$ and $k_{33} \geq 0$, respectively. Thus $\text{tr } \mathsf{K} \geq 0$ is automatically satisfied. In equation form, we then have

$$k_{11} \geq 0, \tag{5.61}$$
$$k_{22} \geq 0, \tag{5.62}$$
$$k_{33} \geq 0, \tag{5.63}$$
$$k_{11}k_{22} + k_{11}k_{33} + k_{22}k_{33} - k_{12}^2 - k_{13}^2 - k_{23}^2 \geq 0, \tag{5.64}$$
$$k_{13}(k_{12}k_{23} - k_{22}k_{13}) + k_{23}(k_{12}k_{13} - k_{11}k_{23}) + k_{33}(k_{11}k_{22} - k_{12}k_{12}) \geq 0. \tag{5.65}$$

While by no means a proof, numerical experimentation gives strong indication that the remaining conditions can be satisfied if, loosely stated, $k_{11}, k_{22}, k_{33} >> |k_{12}|, |k_{23}|, |k_{13}|$. That is, for positive semi-definiteness,

- *each* diagonal element must be positive semi-definite,
- off-diagonal terms can be positive or negative, and
- diagonal terms must have amplitudes that are, loosely speaking, larger than the amplitudes of off-diagonal terms.

Example 5.1

Let us consider heat conduction in the limit of two dimensions and a constant anisotropic conductivity tensor, without imposing Onsager's conditions.

Let us take then

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = -\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} \partial_1 T \\ \partial_2 T \end{pmatrix}. \tag{5.66}$$

The second law demands that

$$\begin{pmatrix} \partial_1 T & \partial_2 T \end{pmatrix} \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} \partial_1 T \\ \partial_2 T \end{pmatrix} \geq 0. \tag{5.67}$$

This is expanded as

$$\begin{pmatrix} \partial_1 T & \partial_2 T \end{pmatrix} \left(\begin{pmatrix} k_{11} & \frac{k_{12}+k_{21}}{2} \\ \frac{k_{21}+k_{12}}{2} & k_{22} \end{pmatrix} + \begin{pmatrix} 0 & \frac{k_{12}-k_{21}}{2} \\ \frac{k_{21}-k_{12}}{2} & 0 \end{pmatrix} \right) \begin{pmatrix} \partial_1 T \\ \partial_2 T \end{pmatrix} \geq 0. \tag{5.68}$$

As before, the anti-symmetric portion makes no contribution to the left hand side, giving rise to

$$\begin{pmatrix} \partial_1 T & \partial_2 T \end{pmatrix} \begin{pmatrix} k_{11} & \frac{k_{12}+k_{21}}{2} \\ \frac{k_{21}+k_{12}}{2} & k_{22} \end{pmatrix} \begin{pmatrix} \partial_1 T \\ \partial_2 T \end{pmatrix} \geq 0. \tag{5.69}$$

And, demanding that the eigenvalues of the symmetric part of the conductivity tensor be positive gives rise to the conditions, identical to that of an earlier analysis, that

$$k_{11} \geq 0, \tag{5.70}$$

$$k_{22} \geq 0, \tag{5.71}$$

$$k_{11}k_{22} \geq \left(\frac{k_{12}+k_{21}}{2}\right)^2. \tag{5.72}$$

The energy equation becomes

$$\rho \frac{de}{dt} = -\partial_i q_i + \ldots, \tag{5.73}$$

$$= \begin{pmatrix} \partial_1 & \partial_2 \end{pmatrix} \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} \partial_1 T \\ \partial_2 T \end{pmatrix} + \ldots, \tag{5.74}$$

$$= \begin{pmatrix} \partial_1 & \partial_2 \end{pmatrix} \begin{pmatrix} k_{11}\partial_1 T + k_{12}\partial_2 T \\ k_{21}\partial_1 T + k_{22}\partial_2 T \end{pmatrix} + \ldots, \tag{5.75}$$

$$= k_{11}\partial_1\partial_1 T + (k_{12}+k_{21})\partial_1\partial_2 T + k_{22}\partial_2\partial_2 T + \ldots, \tag{5.76}$$

$$= k_{11}\frac{\partial^2 T}{\partial x_1^2} + (k_{12}+k_{21})\frac{\partial^2 T}{\partial x_1 \partial x_2} + k_{22}\frac{\partial^2 T}{\partial x_2^2} + \ldots. \tag{5.77}$$

One sees that the energy evolution depends only on the symmetric part of the conductivity tensor.

Imposition of Onsager's relations gives simply $k_{12} = k_{21}$, giving rise to second law restrictions of

$$k_{11} \geq 0, \tag{5.78}$$

$$k_{22} \geq 0, \tag{5.79}$$

$$k_{11}k_{22} \geq k_{12}^2, \tag{5.80}$$

and an energy equation of

$$\rho \frac{de}{dt} = k_{11}\frac{\partial^2 T}{\partial x_1^2} + 2k_{12}\frac{\partial^2 T}{\partial x_1 \partial x_2} + k_{22}\frac{\partial^2 T}{\partial x_2^2} + \ldots. \tag{5.81}$$

Example 5.2

Consider the ramifications of a heat flux vector in violation of Onsager's principle: flux in which the anisotropic conductivity is purely anti-symmetric. For simplicity consider an incompressible solid with constant specific heat c. For the heat flux, we take

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = -\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix} \begin{pmatrix} \partial_1 T \\ \partial_2 T \end{pmatrix}. \tag{5.82}$$

This holds that heat flux in the 1 direction is induced only by temperature gradients in the 2 direction and heat flux in the 2 direction is induced only by temperature gradients in the 1 direction.

The second law demands that

$$\begin{pmatrix} \partial_1 T & \partial_2 T \end{pmatrix} \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix} \begin{pmatrix} \partial_1 T \\ \partial_2 T \end{pmatrix} \geq 0, \tag{5.83}$$

$$\begin{pmatrix} \partial_1 T & \partial_2 T \end{pmatrix} \begin{pmatrix} -\beta \partial_2 T \\ \beta \partial_1 T \end{pmatrix} \geq 0, \tag{5.84}$$

$$-\beta(\partial_1 T)(\partial_2 T) + \beta(\partial_1 T)(\partial_2 T) \geq 0, \tag{5.85}$$

$$0 \geq 0. \tag{5.86}$$

So, the second law holds.

For the incompressible solid with constant heat capacity, the velocity field is zero, and the energy equation reduces to the simple

$$\rho c \frac{\partial T}{\partial t} = -\partial_i q_i. \tag{5.87}$$

Imposing our unusual expression for heat flux, we get

$$\rho c \frac{\partial T}{\partial t} = \begin{pmatrix} \partial_1 & \partial_2 \end{pmatrix} \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix} \begin{pmatrix} \partial_1 T \\ \partial_2 T \end{pmatrix}, \tag{5.88}$$

$$= \begin{pmatrix} \partial_1 & \partial_2 \end{pmatrix} \begin{pmatrix} -\beta \partial_2 T \\ \beta \partial_1 T \end{pmatrix}, \tag{5.89}$$

$$= -\beta \partial_1 \partial_2 T + \beta \partial_1 \partial_2 T, \tag{5.90}$$

$$= 0. \tag{5.91}$$

So, this unusual heat flux vector is one that induces no change in temperature. In terms of the first law of thermodynamics, a net energy flux into a control volume in the 1 direction is exactly counterbalanced by an net energy flux out of the same control volume in the 2 direction. Thus the first law holds as well.

Let us consider a temperature distribution for this unusual material. And let us consider it to apply to the domain $x \in [0,1]$, $y \in [0,1]$, $t \in [0,\infty]$. Take

$$T(x_1, x_2, t) = x_2. \tag{5.92}$$

Obviously this satisfies the first law as $\partial T/\partial t = 0$. Let us check the heat flux.

$$q_1 = \beta \partial_2 T = \beta, \tag{5.93}$$

$$q_2 = -\beta \partial_1 T = 0. \tag{5.94}$$

Now the lower boundary at $x_2 = 0$ has $T = 0$. The upper boundary has $x_2 = 1$ so $T = 1$. And this constant temperature gradient in the 2 direction is inducing a constant heat flux in the 1 direction, $q_1 = -\beta$. The energy flux that enters at $x_1 = 0$ departs at $x_1 = 1$, maintaining energy conservation.

One can consider an equivalent problem in cylindrical coordinates. Taking

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta, \tag{5.95}$$

and applying the chain rule,

$$\begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial x_1} & \frac{\partial \theta}{\partial x_1} \\ \frac{\partial r}{\partial x_2} & \frac{\partial \theta}{\partial x_2} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix}, \tag{5.96}$$

one finds

$$\begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix}. \tag{5.97}$$

So, transforming $q_1 = \beta\partial_2 T$, and $q_2 = -\beta\partial_1 T$ gives

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \beta \begin{pmatrix} \sin\theta & \frac{\cos\theta}{r} \\ -\cos\theta & \frac{\sin\theta}{r} \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial r} \\ \frac{\partial T}{\partial \theta} \end{pmatrix}. \tag{5.98}$$

Standard trigonometry gives

$$\begin{pmatrix} q_r \\ q_\theta \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{\text{rotation matrix}} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \tag{5.99}$$

Applying the rotation matrix to both sides gives then

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \beta \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sin\theta & \frac{\cos\theta}{r} \\ -\cos\theta & \frac{\sin\theta}{r} \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial r} \\ \frac{\partial T}{\partial \theta} \end{pmatrix}, \tag{5.100}$$

$$\begin{pmatrix} q_r \\ q_\theta \end{pmatrix} = \beta \begin{pmatrix} 0 & \frac{1}{r} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial r} \\ \frac{\partial T}{\partial \theta} \end{pmatrix}, \tag{5.101}$$

or simply

$$q_r = \frac{\beta}{r}\frac{\partial T}{\partial \theta}, \tag{5.102}$$

$$q_\theta = -\beta \frac{\partial T}{\partial r}. \tag{5.103}$$

Now the steady state temperature distribution in the annular region $1/2 < r < 1$, $T = r$, describes a domain with an inner boundary held at $T = 1/2$ and an outer boundary held at $T = 1$. Such a temperature distribution would induce a heat flux in the θ direction only, so that $q_r = 0$ and $q_\theta = -\beta$. That is, the heat goes round and round the domain, but never enters or exits at any boundary.

Now such a flux is counterintuitive precisely because it has never been observed or measured. It is for this reason that we can adopt Onsager's hypothesis and demand that, independent of the first and second laws of thermodynamics,

$$\beta = 0, \tag{5.104}$$

and the conductivity tenser is purely symmetric.

5.4 Stress-strain rate relation for a Newtonian fluid

We now seek to satisfy the second part of the strong form of the entropy inequality, Eq. (5.26). Recalling that $T > 0$, this reduces to

$$\underbrace{\tau_{ij}\partial_{(i}v_{j)}}_{\Phi} \geq 0. \tag{5.105}$$

This form suggests that we seek a constitutive equation for the viscous stress tensor τ_{ij} that is a function of the deformation tensor $\partial_{(i}v_{j)}$. Fortunately, such a form exists, that moreover agrees with macroscale experiments and microscale theories. Here we will focus on the simplest of such theories, for what is known as a *Newtonian fluid*, a fluid that is isotropic and whose viscous stress varies linearly with strain rate. In general, this is a discipline unto itself known as *rheology*.

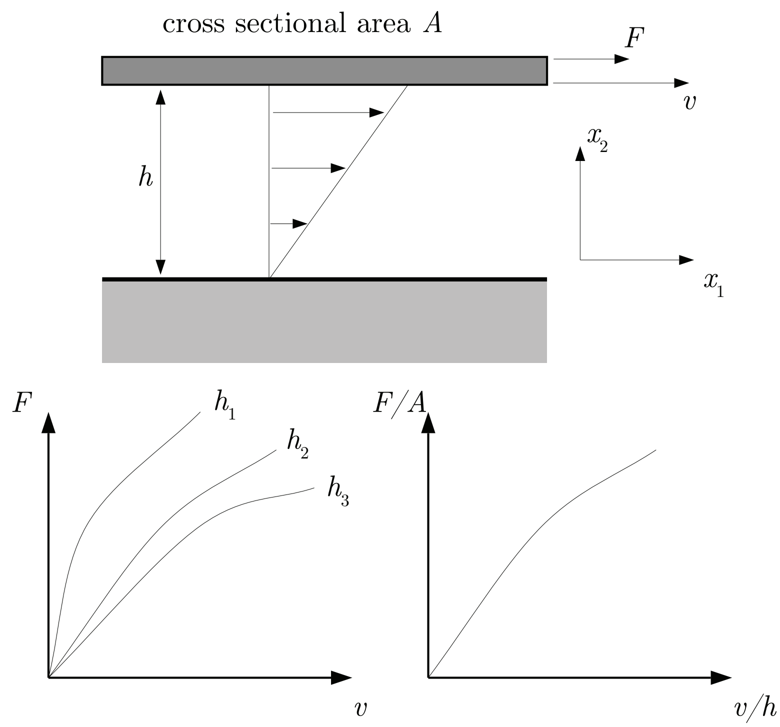

Figure 5.1: Sketch of simple Couette flow experiment with measurements of stress versus strain rate.

5.4.1 Underlying experiments

We can pull a flat plate over a fluid and measure the force necessary to maintain a specified velocity. This situation and some expected results are sketched in Fig. 5.1. We observe that

- At the upper and lower plate surfaces, the fluid has the same velocity of each plate. This is called the *no-slip* condition.
- The faster the velocity v of the upper plate is, the higher the force necessary to pull the plate is. The increase can be linear or non-linear.
- When experiments are carried out with different plate area and different gap width, a single universal curve results when F/A is plotted against v/h.
- The velocity profile is linear with increasing x_2.

In a way similar on a molecular scale to energy diffusion, this experiment is describing a diffusion of momentum from the pulled plate into the fluid below it. The constitutive equation we develop for viscous stress, when combined with the governing axioms, will model momentum diffusion.

We can associate F/A with a shear stress: τ_{21}, recalling stress on the 2 face in the 1 direction. We can associate v/h with a velocity gradient, here $\partial_2 v_1$. We note that considering

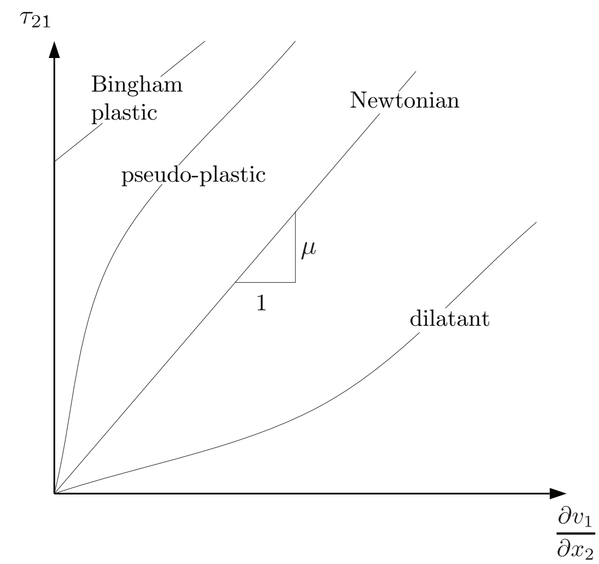

Figure 5.2: Variation of viscous stress with strain rate for typical fluids.

the velocity gradient is essentially equivalent to considering the deformation gradient, as far as the second law is concerned, and so we will be loose here in our use of the term. We loosely define the *coefficient of viscosity* μ for this configuration as

$$\mu = \frac{\tau_{21}}{\partial_2 v_1} = \frac{\text{viscous stress}}{\text{strain rate}}. \tag{5.106}$$

We shall see in the next section that μ is better described as the *first coefficient of viscosity* and that a second coefficient is also required to fully characterize an isotropic fluid. The viscosity is the analog of Young's[9] modulus in solid mechanics, that is the ratio of stress to strain. In general μ is a thermodynamic property of a material. It is often a strong function of temperature, but can vary with pressure as well. A Newtonian fluid has a viscosity that does not depend on strain rate (but could depend on temperature and pressure). A non-Newtonian fluid has a viscosity that is strain rate-dependent (and possible temperature and pressure). Some typical behavior is sketched in Fig. 5.2. We shall focus here on fluids whose viscosity is not a function of strain rate. Much of our development will be valid for temperature- and pressure-dependent viscosity, while most actual examples will consider only constant viscosity.

5.4.2 Analysis for isotropic Newtonian fluid

Here we shall outline the method described by Whitaker (pp. 139-145) to describe the viscous stress as a function of strain rate for an isotropic fluid with constant viscosity. An isotropic

[9]Thomas Young, 1773-1829, English physician and physicist whose experiments in interferometry revived the wave theory of light, Egyptologist who helped decipher the Rosetta stone, worked on surface tension in fluids, gave the word "energy" scientific significance, and developed Young's modulus in elasticity.

fluid has no directional dependencies when subjected to a force. A fluid composed of aligned long chain polymers is an example of a fluid that is most likely not isotropic. Following Whitaker, we

- *postulate* that stress is a function of deformation rate (strain rate) only:[10]

$$\tau_{ij} = f_{ij}(\partial_{(k}v_{l)}). \tag{5.107}$$

Written out in more detail, we have postulated a relationship of the form

$$\begin{aligned}
\tau_{11} &= f_{11}(\partial_{(1}v_{1)}, \partial_{(2}v_{2)}, \partial_{(3}v_{3)}, \partial_{(1}v_{2)}, \partial_{(2}v_{3)}, \partial_{(3}v_{1)}\partial_{(2}v_{1)}, \partial_{(3}v_{2)}, \partial_{(1}v_{3)}), && (5.108)\\
\tau_{12} &= f_{12}(\partial_{(1}v_{1)}, \partial_{(2}v_{2)}, \partial_{(3}v_{3)}, \partial_{(1}v_{2)}, \partial_{(2}v_{3)}, \partial_{(3}v_{1)}\partial_{(2}v_{1)}, \partial_{(3}v_{2)}, \partial_{(1}v_{3)}), && (5.109)\\
&\vdots\\
\tau_{33} &= f_{33}(\partial_{(1}v_{1)}, \partial_{(2}v_{2)}, \partial_{(3}v_{3)}, \partial_{(1}v_{2)}, \partial_{(2}v_{3)}, \partial_{(3}v_{1)}\partial_{(2}v_{1)}, \partial_{(3}v_{2)}, \partial_{(1}v_{3)}). && (5.110)
\end{aligned}$$

- require that $\tau_{ij} = 0$ if $\partial_{(i}v_{j)} = 0$; hence, no strain rate, no stress.

- require that stress is *linearly* related to strain rate:

$$\tau_{ij} = \hat{C}_{ijkl}\partial_{(k}v_{l)}. \tag{5.111}$$

This is the imposition of the assumption of a Newtonian fluid. Here $\hat{C}_{ijkl}$ is a fourth order tensor. Thus we have in matrix form

$$\begin{pmatrix} \tau_{11}\\ \tau_{22}\\ \tau_{33}\\ \tau_{12}\\ \tau_{23}\\ \tau_{31}\\ \tau_{21}\\ \tau_{32}\\ \tau_{13} \end{pmatrix} = \begin{pmatrix}
\hat{C}_{1111} & \hat{C}_{1122} & \hat{C}_{1133} & \hat{C}_{1112} & \hat{C}_{1123} & \hat{C}_{1131} & \hat{C}_{1121} & \hat{C}_{1132} & \hat{C}_{1113}\\
\hat{C}_{2211} & \hat{C}_{2222} & \hat{C}_{2233} & \hat{C}_{2212} & \hat{C}_{2223} & \hat{C}_{2231} & \hat{C}_{2221} & \hat{C}_{2232} & \hat{C}_{2213}\\
\hat{C}_{3311} & \hat{C}_{3322} & \hat{C}_{3333} & \hat{C}_{3312} & \hat{C}_{3323} & \hat{C}_{3331} & \hat{C}_{3321} & \hat{C}_{3332} & \hat{C}_{3313}\\
\hat{C}_{1211} & \hat{C}_{1222} & \hat{C}_{1233} & \hat{C}_{1212} & \hat{C}_{1223} & \hat{C}_{1231} & \hat{C}_{1221} & \hat{C}_{1232} & \hat{C}_{1213}\\
\hat{C}_{2311} & \hat{C}_{2322} & \hat{C}_{2333} & \hat{C}_{2312} & \hat{C}_{2323} & \hat{C}_{2331} & \hat{C}_{2321} & \hat{C}_{2332} & \hat{C}_{2313}\\
\hat{C}_{3111} & \hat{C}_{3122} & \hat{C}_{3133} & \hat{C}_{3112} & \hat{C}_{3123} & \hat{C}_{3131} & \hat{C}_{3121} & \hat{C}_{3132} & \hat{C}_{3113}\\
\hat{C}_{2111} & \hat{C}_{2122} & \hat{C}_{2133} & \hat{C}_{2112} & \hat{C}_{2123} & \hat{C}_{2131} & \hat{C}_{2121} & \hat{C}_{2132} & \hat{C}_{2113}\\
\hat{C}_{3211} & \hat{C}_{3222} & \hat{C}_{3233} & \hat{C}_{3212} & \hat{C}_{3223} & \hat{C}_{3231} & \hat{C}_{3221} & \hat{C}_{3232} & \hat{C}_{3213}\\
\hat{C}_{1311} & \hat{C}_{1322} & \hat{C}_{1333} & \hat{C}_{1312} & \hat{C}_{1323} & \hat{C}_{1331} & \hat{C}_{1321} & \hat{C}_{1332} & \hat{C}_{1313}
\end{pmatrix} \begin{pmatrix} \partial_{(1}v_{1)}\\ \partial_{(2}v_{2)}\\ \partial_{(3}v_{3)}\\ \partial_{(1}v_{2)}\\ \partial_{(2}v_{3)}\\ \partial_{(3}v_{1)}\\ \partial_{(2}v_{1)}\\ \partial_{(3}v_{2)}\\ \partial_{(1}v_{3)} \end{pmatrix}. \tag{5.112}$$

There are $3^4 = 81$ unknown coefficients $\hat{C}_{ijkl}$. We found one of them in our simple experiment in which we found

$$\tau_{21} = \tau_{12} = \mu\partial_2 v_1 = \mu(2\partial_{(1}v_{2)}). \tag{5.113}$$

Hence in this special case $\hat{C}_{1212} = 2\mu$.

[10]Thus, we are not allowing viscous stress to be a function of the rigid body rotation rate. While it seems intuitive that rigid body rotation should not induce viscous stress, Batchelor (2000) mentions that there is no rigorous proof for this; hence, we describe our statement as a postulate.

Now we could do eighty-one separate experiments, or we could take advantage of the assumption that the fluid has no directional dependency. We will take the following approach. Observer A conducts an experiment to measure the stress tensor in reference frame $\mathcal{A}$. The observer begins with the "viscosity matrix" $\hat{C}_{ijkl}$. The experiment is conducted by varying strain rate and measuring stress. With complete knowledge A feels confident this knowledge could be used to predict the stress in rotated frame $\mathcal{A}'$.

Consider observer A' who is oriented to frame $\mathcal{A}'$. Oblivious to observer A, A' conducts the same experiment to measure what for her or him is τ'_{ij} The value that A' measures must be the same that A predicts in order for the system to be isotropic. This places restrictions on the viscosity matrix $\hat{C}_{ijkl}$. We intend to show that if the fluid is isotropic, only *two* of the eighty-one coefficients are distinct and non-zero.

We first use symmetry properties of the stress and strain rate tensor to reduce to thirty-six unknown coefficients. We note that in actuality there are only six independent components of stress and six independent components of deformation because both are symmetric tensors. Consequently, we can write our linear stress-strain rate relation as

$$\begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \\ \tau_{23} \\ \tau_{31} \end{pmatrix} = \begin{pmatrix} \hat{C}_{1111} & \hat{C}_{1122} & \hat{C}_{1133} & \hat{C}_{1112}+\hat{C}_{1121} & \hat{C}_{1123}+\hat{C}_{1132} & \hat{C}_{1131}+\hat{C}_{1113} \\ \hat{C}_{2211} & \hat{C}_{2222} & \hat{C}_{2233} & \hat{C}_{2212}+\hat{C}_{2221} & \hat{C}_{2223}+\hat{C}_{2232} & \hat{C}_{2231}+\hat{C}_{2213} \\ \hat{C}_{3311} & \hat{C}_{3322} & \hat{C}_{3333} & \hat{C}_{3312}+\hat{C}_{3321} & \hat{C}_{3323}+\hat{C}_{3332} & \hat{C}_{3331}+\hat{C}_{3313} \\ \hat{C}_{1211} & \hat{C}_{1222} & \hat{C}_{1233} & \hat{C}_{1212}+\hat{C}_{1221} & \hat{C}_{1223}+\hat{C}_{1232} & \hat{C}_{1231}+\hat{C}_{1213} \\ \hat{C}_{2311} & \hat{C}_{2322} & \hat{C}_{2333} & \hat{C}_{2312}+\hat{C}_{2321} & \hat{C}_{2323}+\hat{C}_{2332} & \hat{C}_{2331}+\hat{C}_{2313} \\ \hat{C}_{3111} & \hat{C}_{3122} & \hat{C}_{3133} & \hat{C}_{3112}+\hat{C}_{3121} & \hat{C}_{3123}+\hat{C}_{3132} & \hat{C}_{3131}+\hat{C}_{3113} \end{pmatrix} \begin{pmatrix} \partial_{(1}v_{1)} \\ \partial_{(2}v_{2)} \\ \partial_{(3}v_{3)} \\ \partial_{(1}v_{2)} \\ \partial_{(2}v_{3)} \\ \partial_{(3}v_{1)} \end{pmatrix}. \tag{5.114}$$

Now adopting Whitaker's notation for simplification, we define this matrix of $\hat{C}$'s as a new matrix of C's. Here, now C itself is not a tensor, while $\hat{C}$ is a tensor. We take equivalently then

$$\begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \\ \tau_{23} \\ \tau_{31} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix} \begin{pmatrix} \partial_{(1}v_{1)} \\ \partial_{(2}v_{2)} \\ \partial_{(3}v_{3)} \\ \partial_{(1}v_{2)} \\ \partial_{(2}v_{3)} \\ \partial_{(3}v_{1)} \end{pmatrix}. \tag{5.115}$$

Next, recalling that for tensorial quantities

$$\tau'_{ij} = \ell_{ki}\ell_{lj}\tau_{kl}, \tag{5.116}$$

$$\partial'_{(i}v'_{j)} = \ell_{ki}\ell_{lj}\partial_{(k}v_{l)}, \tag{5.117}$$

let us subject our fluid to a battery of rotations and see what can be concluded by enforcing material indifference.

- 180° *rotation about* x_3 *axis*

  For this rotation, sketched in Fig. 5.3. we have direction cosines

$$\ell_{ki} = \begin{pmatrix} \ell_{11}=-1 & \ell_{12}=0 & \ell_{13}=0 \\ \ell_{21}=0 & \ell_{22}=-1 & \ell_{23}=0 \\ \ell_{31}=0 & \ell_{32}=0 & \ell_{33}=1 \end{pmatrix}. \tag{5.118}$$

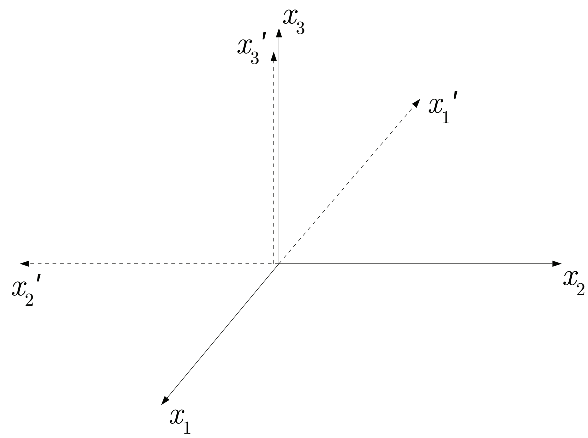

Figure 5.3: Rotation of 180° about x_3 axis.

So, applying Eq. (2.10), we see

$$\begin{pmatrix} x_1' & x_2' & x_3' \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.119}$$

that yields

$$x_1' = -x_1, \qquad x_2' = -x_2, \qquad x_3' = x_3, \tag{5.120}$$

which is consistent with Fig. 5.3. Because $\det \ell_{ki} = 1$, the transformation is a rotation. Applying the transformation rules to each term in the shear stress tensor, we get

$$\tau_{11}' = \ell_{k1}\ell_{l1}\tau_{kl} = (-1)^2\tau_{11} = \tau_{11}, \tag{5.121}$$
$$\tau_{22}' = \ell_{k2}\ell_{l2}\tau_{kl} = (-1)^2\tau_{22} = \tau_{22}, \tag{5.122}$$
$$\tau_{33}' = \ell_{k3}\ell_{l3}\tau_{kl} = (1)^2\tau_{33} = \tau_{33}, \tag{5.123}$$
$$\tau_{12}' = \ell_{k1}\ell_{l2}\tau_{kl} = (-1)^2\tau_{12} = \tau_{12}, \tag{5.124}$$
$$\tau_{23}' = \ell_{k2}\ell_{l3}\tau_{kl} = (-1)(1)\tau_{23} = -\tau_{23}, \tag{5.125}$$
$$\tau_{31}' = \ell_{k3}\ell_{l1}\tau_{kl} = (1)(-1)\tau_{31} = -\tau_{31}. \tag{5.126}$$

Likewise, we find that

$$\partial_{(1}'v_{1)}' = \partial_{(1}v_{1)}, \tag{5.127}$$
$$\partial_{(2}'v_{2)}' = \partial_{(2}v_{2)}, \tag{5.128}$$
$$\partial_{(3}'v_{3)}' = \partial_{(3}v_{3)}, \tag{5.129}$$
$$\partial_{(1}'v_{2)}' = \partial_{(1}v_{2)}, \tag{5.130}$$
$$\partial_{(2}'v_{3)}' = -\partial_{(2}v_{3)}, \tag{5.131}$$
$$\partial_{(3}'v_{1)}' = -\partial_{(3}v_{1)}. \tag{5.132}$$

Now our observer A' who is in the rotated system would say, for instance that

$$\tau'_{11} = C_{11}\partial'_{(1}v'_{1)} + C_{12}\partial'_{(2}v'_{2)} + C_{13}\partial'_{(3}v'_{3)} + C_{14}\partial'_{(1}v'_{2)} + C_{15}\partial'_{(2}v'_{3)} + C_{16}\partial'_{(3}v'_{1)}, \quad (5.133)$$

while our observer A who used tensor algebra to predict τ'_{11} would say

$$\tau'_{11} = C_{11}\partial'_{(1}v'_{1)} + C_{12}\partial'_{(2}v'_{2)} + C_{13}\partial'_{(3}v'_{3)} + C_{14}\partial'_{(1}v'_{2)} - C_{15}\partial'_{(2}v'_{3)} - C_{16}\partial'_{(3}v'_{1)}. \quad (5.134)$$

Because we want both predictions to be the same, we must require that

$$C_{15} = C_{16} = 0. \quad (5.135)$$

In matrix form, our observer A would predict for the rotated frame that

$$\begin{pmatrix} \tau'_{11} \\ \tau'_{22} \\ \tau'_{33} \\ \tau'_{12} \\ -\tau'_{23} \\ -\tau'_{31} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix} \begin{pmatrix} \partial'_{(1}v'_{1)} \\ \partial'_{(2}v'_{2)} \\ \partial'_{(3}v'_{3)} \\ \partial'_{(1}v'_{2)} \\ -\partial'_{(2}v'_{3)} \\ -\partial'_{(3}v'_{1)} \end{pmatrix}. \quad (5.136)$$

To retain material difference between the predictions of our two observers, we thus require that $C_{15} = C_{16} = C_{25} = C_{26} = C_{35} = C_{36} = C_{45} = C_{46} = C_{51} = C_{52} = C_{53} = C_{54} = C_{61} = C_{62} = C_{63} = C_{64} = 0$. This eliminates 16 coefficients and gives our viscosity matrix the form

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{21} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{31} & C_{32} & C_{33} & C_{34} & 0 & 0 \\ C_{41} & C_{42} & C_{43} & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & C_{56} \\ 0 & 0 & 0 & 0 & C_{65} & C_{66} \end{pmatrix}. \quad (5.137)$$

with only 20 independent coefficients.

- 180° *rotation about* x_1 *axis*

This rotation is sketched in Fig. 5.4. Applying Eq. (2.10), we see

$$\begin{pmatrix} x'_1 & x'_2 & x'_3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (5.138)$$

that yields

$$x'_1 = x_1, \qquad x'_2 = -x_2, \qquad x'_3 = -x_3, \quad (5.139)$$

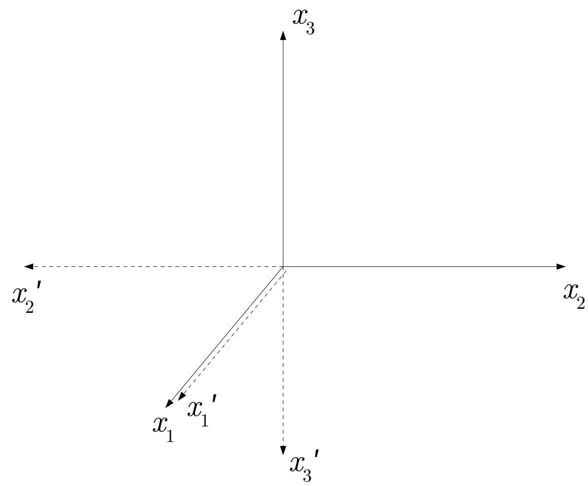

Figure 5.4: Rotation of 180° about x_1 axis.

that is consistent with Fig. 5.4. Leaving out the rest of the details of the previous section, this rotation has a set of direction cosines of

$$\ell_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{5.140}$$

Application of this rotation leads to the conclusion that the viscosity matrix must be of the form

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix}. \tag{5.141}$$

with only 12 independent coefficients.

- 180° *rotation about* x_2 *axis*

  One is tempted to perform this rotation as well, but nothing new is learned from it!

- 90° *rotation about* x_1 *axis*

  Having exhausted 180° rotations, let us turn to 90° rotations. We first rotate about the x_1 axis. This rotation is sketched in Fig. 5.5. This rotation has a set of direction cosines of

$$\ell_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.142}$$

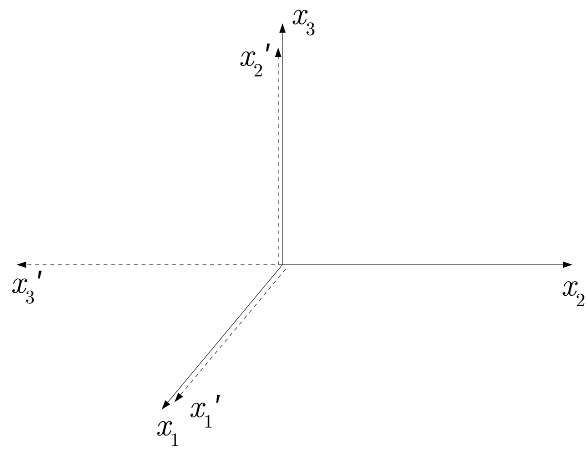

Figure 5.5: Rotation of 90° about x_1 axis.

Because $\det \ell_{ki} = 1$, the transformation is a rotation. Certainly also the length of each column vector is unity, and each column vector dotted into other column vectors has a value of zero, so the column vectors are orthonormal. Application of this rotation leads to the conclusion that the viscosity matrix must be of the form

$$\begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{21} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix}. \tag{5.143}$$

with only 8 independent coefficients.

- 90° *rotation about* x_3 *axis*

This rotation is sketched in Fig. 5.6. This rotation has a set of direction cosines of

$$\ell_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.144}$$

Because $\det \ell_{ki} = 1$, the transformation is a rotation. Application of this rotation leads to the conclusion that the viscosity matrix must be of the form

$$\begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{pmatrix}. \tag{5.145}$$

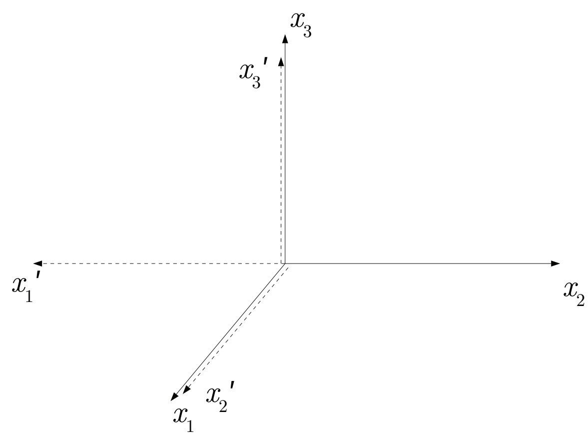

Figure 5.6: Rotation of 90° about x_3 axis.

with only 3 independent coefficients.

- 90° *rotation about* x_2 *axis*

  We learn nothing from this rotation.

- 45° *rotation about* x_3 *axis*

  This rotation is sketched in Fig. 5.7. This rotation has a set of direction cosines of

$$\ell_{ij} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.146}$$

Because $\det \ell_{ki} = 1$, the transformation is a rotation. After a lot of algebra, application of this rotation leads to the conclusion that the viscosity matrix must be of the form

$$\begin{pmatrix} C_{44}+C_{12} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{44}+C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{44}+C_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{pmatrix}. \tag{5.147}$$

with only 2 independent coefficients.

Try as we might, we cannot reduce this any further with more rotations. It can be proved more rigorously, as shown in most books on tensor analysis, that this is the furthest reduction that can be made. So, for an isotropic Newtonian fluid, we can expect two independent coefficients to parameterize the relation between strain rate and viscous stress. The relation

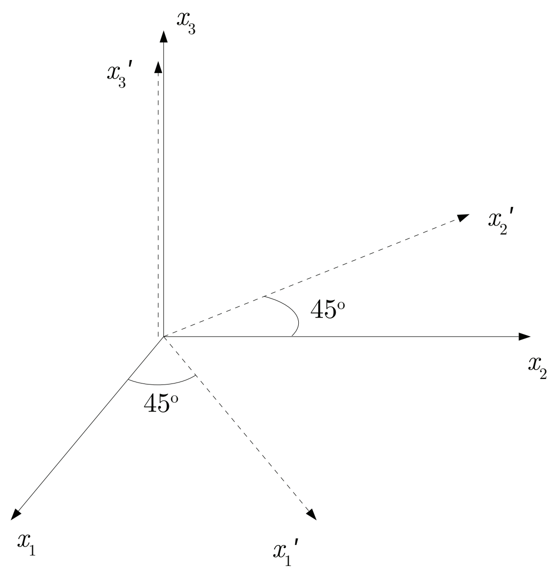

Figure 5.7: Rotation of 45° about x_3 axis.

between stress and strain rate can be expressed in detail as

$$\tau_{11} = C_{44}\partial_{(1}v_{1)} + C_{12}\left(\partial_{(1}v_{1)} + \partial_{(2}v_{2)} + \partial_{(3}v_{3)}\right), \tag{5.148}$$
$$\tau_{22} = C_{44}\partial_{(2}v_{2)} + C_{12}\left(\partial_{(1}v_{1)} + \partial_{(2}v_{2)} + \partial_{(3}v_{3)}\right), \tag{5.149}$$
$$\tau_{33} = C_{44}\partial_{(3}v_{3)} + C_{12}\left(\partial_{(1}v_{1)} + \partial_{(2}v_{2)} + \partial_{(3}v_{3)}\right), \tag{5.150}$$
$$\tau_{12} = C_{44}\partial_{(1}v_{2)}, \tag{5.151}$$
$$\tau_{23} = C_{44}\partial_{(2}v_{3)}, \tag{5.152}$$
$$\tau_{31} = C_{44}\partial_{(3}v_{1)}. \tag{5.153}$$

Using traditional notation, we take

- $C_{44} \equiv 2\mu$, where μ is the *first coefficient of viscosity*, and
- $C_{12} \equiv \lambda$, where λ is the *second coefficient of viscosity*.

There are a variety of other nomenclatures for μ and λ. Following Paolucci (2016), we can also call μ the *shear viscosity* and λ the *dilatational viscosity*. We also can define the *bulk viscosity*, ζ, as

$$\zeta \equiv \lambda + \frac{2}{3}\mu, \tag{5.154}$$

which is a term in common usage. A similar analysis in solid mechanics leads one to conclude for an isotropic material in which the stress tensor is linearly related to the strain (rather than the strain rate) gives rise to two independent coefficients, the elastic modulus and the shear modulus. In solids, these both can be measured, and they are independent. In terms of our original fourth order tensor, we can write the linear relationship $\tau_{ij} = \hat{C}_{ijkl}\partial_{(i}v_{j)}$ as

$$\begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \\ \tau_{23} \\ \tau_{31} \\ \tau_{21} \\ \tau_{32} \\ \tau_{13} \end{pmatrix} = \begin{pmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu \end{pmatrix} \begin{pmatrix} \partial_{(1}v_{1)} \\ \partial_{(2}v_{2)} \\ \partial_{(3}v_{3)} \\ \partial_{(1}v_{2)} \\ \partial_{(2}v_{3)} \\ \partial_{(3}v_{1)} \\ \partial_{(2}v_{1)} \\ \partial_{(3}v_{2)} \\ \partial_{(1}v_{3)} \end{pmatrix}. \tag{5.155}$$

We note that because of the symmetry of $\partial_{(i}v_{j)}$ that this representation is not unique in that the following, as well as other linear combinations, is an identically equivalent statement:

$$\begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \\ \tau_{23} \\ \tau_{31} \\ \tau_{21} \\ \tau_{32} \\ \tau_{13} \end{pmatrix} = \begin{pmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 & \mu \\ 0 & 0 & 0 & \mu & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \partial_{(1}v_{1)} \\ \partial_{(2}v_{2)} \\ \partial_{(3}v_{3)} \\ \partial_{(1}v_{2)} \\ \partial_{(2}v_{3)} \\ \partial_{(3}v_{1)} \\ \partial_{(2}v_{1)} \\ \partial_{(3}v_{2)} \\ \partial_{(1}v_{3)} \end{pmatrix}. \tag{5.156}$$

In shorthand Cartesian index and Gibbs notation, the viscous stress tensor is given by

$$\tau_{ij} = 2\mu\partial_{(i}v_{j)} + \lambda\partial_k v_k \delta_{ij}, \tag{5.157}$$

$$\boldsymbol{\tau} = 2\mu\left(\frac{\nabla\mathbf{v}^T + (\nabla\mathbf{v}^T)^T}{2}\right) + \lambda(\nabla^T\cdot\mathbf{v})\mathsf{I}. \tag{5.158}$$

By performing minor algebraic manipulations, the viscous stress tensor can be cast in a way that elucidates more of the physics of how strain rate influences stress. It is easily verified by direct expansion that the viscous stress tensor can be written as

$$\tau_{ij} = \underbrace{\left((2\mu+3\lambda)\underbrace{\frac{\partial_k v_k}{3}}_{\text{mean strain rate}}\delta_{ij}\right)}_{\text{mean viscous stress}} + \underbrace{2\mu\left(\underbrace{\partial_{(i}v_{j)} - \frac{1}{3}\partial_k v_k\delta_{ij}}_{\text{deviatoric strain rate}}\right)}_{\text{deviatoric viscous stress}}, \tag{5.159}$$

$$\boldsymbol{\tau} = (2\mu+3\lambda)\left(\frac{\nabla^T\cdot\mathbf{v}}{3}\right)\mathsf{I} + 2\mu\left(\frac{\nabla\mathbf{v}^T + (\nabla\mathbf{v}^T)^T}{2} - \frac{1}{3}\left(\nabla^T\cdot\mathbf{v}\right)\mathsf{I}\right). \tag{5.160}$$

Here it is seen that a mean strain rate, really a volumetric change, induces a mean viscous stress, as long as $\lambda \neq -(2/3)\mu$. If either $\lambda = -(2/3)\mu$ or $\partial_k v_k = 0$, all viscous stress is deviatoric. Further, for $\mu \neq 0$, a deviatoric strain rate induces a deviatoric viscous stress. Eliminating λ in favor of the bulk viscosity ζ, we can say

$$\tau_{ij} = \zeta\ \partial_k v_k \delta_{ij} + 2\mu \left(\partial_{(i} v_{j)} - \frac{1}{3} \partial_k v_k \delta_{ij} \right), \tag{5.161}$$

$$\boldsymbol{\tau} = \zeta \left(\nabla^T \cdot \mathbf{v} \right) \mathsf{I} + 2\mu \left(\frac{\nabla \mathbf{v}^T + (\nabla \mathbf{v}^T)^T}{2} - \frac{1}{3} \left(\nabla^T \cdot \mathbf{v} \right) \mathsf{I} \right). \tag{5.162}$$

We can form the mean viscous stress by contracting the viscous stress tensor:

$$\frac{1}{3}\tau_{ii} = \left(\frac{2}{3}\mu + \lambda \right) \partial_k v_k = \zeta\ \partial_k v_k. \tag{5.163}$$

The mean viscous stress is a scalar, and is thus independent of orientation; it is directly proportional to the first invariant of the viscous stress tensor. Obviously the mean viscous stress is zero if $\lambda = -(2/3)\mu$, which occurs if the bulk viscosity is zero. Now the total stress tensor is given by

$$T_{ij} = -p\delta_{ij} + 2\mu \partial_{(i} v_{j)} + \lambda \partial_k v_k \delta_{ij}, \tag{5.164}$$

$$\mathsf{T} = -p\mathsf{I} + 2\mu \left(\frac{\nabla \mathbf{v}^T + (\nabla \mathbf{v}^T)^T}{2} \right) + \lambda (\nabla^T \cdot \mathbf{v}) \mathsf{I}. \tag{5.165}$$

We notice the stress tensor has three components, 1) a uniform diagonal tensor with the hydrostatic pressure, 2) a tensor that is directly proportional to the strain rate tensor, and 3) a uniform diagonal tensor that is proportional to the first invariant of the strain rate tensor: $I_{\dot{\epsilon}}^{(1)} = \text{tr}\ (\partial_{(i} v_{k)}) = \partial_k v_k$. Consequently, the stress tensor can be written as

$$T_{ij} = \underbrace{\left(-p + \lambda I_{\dot{\epsilon}}^{(1)} \right) \delta_{ij}}_{\text{isotropic}} + \underbrace{2\mu \partial_{(i} v_{j)}}_{\text{linear in strain rate}}, \tag{5.166}$$

$$\mathsf{T} = \left(-p + \lambda I_{\dot{\epsilon}}^{(1)} \right) \mathsf{I} + 2\mu \left(\frac{\nabla \mathbf{v}^T + (\nabla \mathbf{v}^T)^T}{2} \right). \tag{5.167}$$

Recalling that $\delta_{ij} = \mathsf{I}$ as well as $I_{\dot{\epsilon}}^{(1)}$ are invariant under a rotation of coordinate axes, we deduce that the stress is related linearly to the strain rate. Moreover when the axes are rotated to be aligned with the principal axes of strain rate, the stress is purely normal stress and takes on its principal value.

Let us next consider two typical elements to aid in interpreting the relation between viscous stress and strain rate for a general Newtonian fluid.

5.4.2.1 Diagonal component

Consider a typical diagonal component of the viscous stress tensor, say τ_{11}:

$$\tau_{11} = \underbrace{\left((2\mu + 3\lambda)\underbrace{\left(\frac{\partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3}{3}\right)}_{\text{mean strain rate}}\right)}_{\text{mean viscous stress}} + \underbrace{2\mu\left(\underbrace{\partial_1 v_1 - \frac{1}{3}(\partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3)}_{\text{deviatoric strain rate}}\right)}_{\text{deviatoric viscous stress}}. \quad (5.168)$$

If we choose our axes to be the principal axes of the strain-rate tensor, then these terms will appear on the diagonal of the stress tensor and there will be no off-diagonal elements. Thus, the fundamental physics of the stress-strain relationship are completely embodied in a natural way in this expression.

5.4.2.2 Off-diagonal component

If we are not aligned with the principal axes, then off-diagonal terms will be non-zero. A typical off-diagonal component of the viscous stress tensor, say τ_{12}, has the following form:

$$\tau_{12} = 2\mu\left(\partial_{(1}v_{2)} + \lambda\partial_k v_k \underbrace{\delta_{12}}_{=0}\right), \quad (5.169)$$

$$= 2\mu\partial_{(1}v_{2)}, \quad (5.170)$$

$$= \mu(\partial_1 v_2 + \partial_2 v_1). \quad (5.171)$$

Note that this is associated with shear deformation for elements aligned with the 1 and 2 axes, and that it is independent of the value of λ, that is only associated with the mean strain rate.

5.4.3 Stokes' assumption

It is a straightforward matter to measure μ. It is not at all straightforward to measure λ. As discussed earlier in Ch. 4.2.2, Stokes in the mid-nineteenth century suggested to require that the mechanical pressure (that is the average normal stress) be equal to the thermodynamic pressure; see Eq. (4.43) and the surrounding discussion. We have seen that the consequence of this is Eq. (4.46): $\tau_{ii} = 0$. If we enforce this on our expression for τ_{ij}, we get

$$\tau_{ii} = 0 = 2\mu\partial_{(i}v_{i)} + \lambda\partial_k v_k \delta_{ii}, \quad (5.172)$$

$$= 2\mu\partial_i v_i + 3\lambda\partial_k v_k, \quad (5.173)$$

$$= 2\mu\partial_i v_i + 3\lambda\partial_i v_i, \quad (5.174)$$

$$= (2\mu + 3\lambda)\partial_i v_i. \quad (5.175)$$

Because in general $\partial_i v_i \neq 0$, Stokes' assumption implies that

$$\lambda = -\frac{2}{3}\mu, \qquad \text{iff Stokes' assumption satisfied.} \tag{5.176}$$

Stated another way, a fluid that satisfies Stokes' assumption has a bulk viscosity of zero:

$$\zeta = 0, \qquad \text{iff Stokes' assumption satisfied.} \tag{5.177}$$

So, a Newtonian fluid satisfying Stokes' assumption has the following constitutive equation for viscous stress

$$\tau_{ij} = \underbrace{2\mu \underbrace{\left(\partial_{(i}v_{j)} - \frac{1}{3}\partial_k v_k \delta_{ij}\right)}_{\text{deviatoric strain rate}}}_{\text{deviatoric viscous stress}}, \tag{5.178}$$

$$\boldsymbol{\tau} = 2\mu\left(\frac{(\nabla \mathbf{v}^T + (\nabla \mathbf{v}^T)^T)}{2} - \frac{1}{3}(\nabla^T \cdot \mathbf{v})\mathbf{I}\right). \tag{5.179}$$

Incompressible flows have $\partial_i v_i = 0$; thus, λ plays no role in determining the viscous stress in such flows. For the fluid that obeys Stokes' assumption, the viscous stress is entirely deviatoric and is induced only by a deviatoric strain rate.

5.4.4 Second law restrictions

Recall that in order that the constitutive equation for viscous stress be consistent with second law of thermodynamics, that it is sufficient (but perhaps overly restrictive) to require that Eq. (5.26) hold:

$$\frac{1}{T}\underbrace{\tau_{ij}\partial_{(i}v_{j)}}_{=\Phi} \geq 0. \tag{5.180}$$

In terms of the viscous dissipation function Φ, this is

$$\frac{1}{T}\Phi \geq 0. \tag{5.181}$$

Invoking our constitutive equation for viscous stress, and realizing that the absolute temperature $T > 0$, we have then that the viscous dissipation function Φ must satisfy

$$\Phi = (2\mu\partial_{(i}v_{j)} + \lambda\partial_k v_k \delta_{ij})(\partial_{(i}v_{j)}) \geq 0. \tag{5.182}$$

This reduces to the sum of two squares:

$$\Phi = 2\mu\partial_{(i}v_{j)}\partial_{(i}v_{j)} + \lambda\partial_k v_k \partial_i v_i \geq 0. \tag{5.183}$$

We then seek restrictions on μ and λ such that this is true. Obviously requiring $\mu \geq 0$ and $\lambda \geq 0$ guarantees satisfaction of the second law. However, Stokes' assumption of $\lambda = -2\mu/3$ does not meet this criterion, and so we are motivated to check more carefully to see if we actually need to be that restrictive.

5.4.4.1 One-dimensional systems

Let us first check the criterion for a strictly one-dimensional system. For such a system, our second law restriction reduces to

$$
\begin{aligned}
2\mu\partial_{(1}v_{1)}\partial_{(1}v_{1)} + \lambda\partial_1 v_1 \partial_1 v_1 &\geq 0, &(5.184)\\
(2\mu+\lambda)\partial_1 v_1 \partial_1 v_1 &\geq 0, &(5.185)\\
2\mu+\lambda &\geq 0, &(5.186)\\
\lambda &\geq -2\mu. &(5.187)
\end{aligned}
$$

Obviously if $\mu > 0$ and $\lambda = -2\mu/3$, the entropy inequality is satisfied. We also could satisfy the inequality for negative μ with sufficiently large positive λ.

5.4.4.2 Two-dimensional systems

Extending this to a two-dimensional system is more complicated. For such systems, expansion of our second law condition gives

$$
\begin{aligned}
2\mu\partial_{(1}v_{1)}\partial_{(1}v_{1)} + 2\mu\partial_{(1}v_{2)}\partial_{(1}v_{2)} + 2\mu\partial_{(2}v_{1)}\partial_{(2}v_{1)} + 2\mu\partial_{(2}v_{2)}\partial_{(2}v_{2)}&\\
+\lambda\left(\partial_{(1}v_{1)} + \partial_{(2}v_{2)}\right)\left(\partial_{(1}v_{1)} + \partial_{(2}v_{2)}\right) \geq 0.& \qquad (5.188)
\end{aligned}
$$

Taking advantage of symmetry of the deformation tensor, we can say

$$
2\mu\partial_{(1}v_{1)}\partial_{(1}v_{1)} + 4\mu\partial_{(1}v_{2)}\partial_{(1}v_{2)} + 2\mu\partial_{(2}v_{2)}\partial_{(2}v_{2)} + \lambda\left(\partial_{(1}v_{1)} + \partial_{(2}v_{2)}\right)\left(\partial_{(1}v_{1)} + \partial_{(2}v_{2)}\right) \geq 0. \qquad (5.189)
$$

Expanding the product and regrouping gives

$$
(2\mu+\lambda)\partial_{(1}v_{1)}\partial_{(1}v_{1)} + 4\mu\partial_{(1}v_{2)}\partial_{(1}v_{2)} + (2\mu+\lambda)\partial_{(2}v_{2)}\partial_{(2}v_{2)} + 2\lambda\partial_{(1}v_{1)}\partial_{(2}v_{2)} \geq 0. \qquad (5.190)
$$

In matrix form, we can write this inequality in the form known from linear algebra as a quadratic form (see Powers and Sen (2015), Ch. 7):

$$
\Phi = \begin{pmatrix} \partial_{(1}v_{1)} & \partial_{(2}v_{2)} & \partial_{(1}v_{2)} \end{pmatrix} \begin{pmatrix} (2\mu+\lambda) & \lambda & 0 \\ \lambda & (2\mu+\lambda) & 0 \\ 0 & 0 & 4\mu \end{pmatrix} \begin{pmatrix} \partial_{(1}v_{1)} \\ \partial_{(2}v_{2)} \\ \partial_{(1}v_{2)} \end{pmatrix} \geq 0. \qquad (5.191)
$$

As we have discussed before, the condition that this hold for all values of the deformation is that the symmetric part of the coefficient matrix have eigenvalues that are greater than or equal to zero. In fact, here the coefficient matrix is purely symmetric. Let us find the eigenvalues κ of the coefficient matrix. The eigenvalues are found by evaluating the following equation

$$
\begin{vmatrix} (2\mu+\lambda)-\kappa & \lambda & 0 \\ \lambda & (2\mu+\lambda)-\kappa & 0 \\ 0 & 0 & 4\mu-\kappa \end{vmatrix} = 0. \qquad (5.192)
$$

We get the characteristic polynomial

$$(4\mu - \kappa)\left((2\mu + \lambda - \kappa)^2 - \lambda^2\right) = 0. \tag{5.193}$$

This has roots

$$\kappa = 4\mu, \tag{5.194}$$
$$\kappa = 2\mu, \tag{5.195}$$
$$\kappa = 2(\mu + \lambda). \tag{5.196}$$

For the two-dimensional system, we see now formally that we must satisfy both

$$\mu \geq 0, \tag{5.197}$$
$$\lambda \geq -\mu. \tag{5.198}$$

This is more restrictive than for the one-dimensional system, but we see that a fluid obeying Stokes' assumption $\lambda = -2\mu/3$ still satisfies this inequality.

5.4.4.3 Three-dimensional systems

For a full three-dimensional variation, the entropy inequality $(2\mu\partial_{(i}v_{j)} + \lambda\partial_k v_k \delta_{ij})(\partial_{(i}v_{j)}) \geq 0$, when expanded, is equivalent to the following quadratic form

$$\Phi = \begin{pmatrix} \partial_{(1}v_{1)} & \partial_{(2}v_{2)} & \partial_{(3}v_{3)} & \partial_{(1}v_{2)} & \partial_{(2}v_{3)} & \partial_{(3}v_{1)} \end{pmatrix} \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 4\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 4\mu \end{pmatrix} \begin{pmatrix} \partial_{(1}v_{1)} \\ \partial_{(2}v_{2)} \\ \partial_{(3}v_{3)} \\ \partial_{(1}v_{2)} \\ \partial_{(2}v_{3)} \\ \partial_{(3}v_{1)} \end{pmatrix} \geq 0. \tag{5.199}$$

Again this must hold for arbitrary values of the deformation, so we must require that the eigenvalues κ of the interior matrix be greater than or equal to zero to satisfy the entropy inequality. It is easy to show that the six eigenvalues for the interior matrix are

$$\kappa = 2\mu, \tag{5.200}$$
$$\kappa = 2\mu, \tag{5.201}$$
$$\kappa = 4\mu, \tag{5.202}$$
$$\kappa = 4\mu, \tag{5.203}$$
$$\kappa = 4\mu, \tag{5.204}$$
$$\kappa = 3\lambda + 2\mu. \tag{5.205}$$

Two of the eigenvalues are degenerate, but this is not a particular problem. We need now that $\kappa \geq 0$, so the entropy inequality requires that

$$\mu \geq 0, \tag{5.206}$$
$$\lambda \geq -\frac{2}{3}\mu. \tag{5.207}$$

Obviously a fluid that satisfies Stokes' assumption does not violate the entropy inequality, but it does give rise to a minimum level of satisfaction. This does not mean the fluid is isentropic! It simply means one of the six eigenvalues is zero.

Now using standard techniques from linear algebra for quadratic forms (see Powers and Sen (2015), Ch. 7), the entropy inequality can, after much effort, be manipulated into the form

$$\begin{aligned}\Phi = \frac{2}{3}\mu\left((\partial_{(1}v_{1)} - \partial_{(2}v_{2)})^2 + (\partial_{(2}v_{2)} - \partial_{(3}v_{3)})^2 + (\partial_{(3}v_{3)} - \partial_{(1}v_{1)})^2\right) \\ + \left(\lambda + \frac{2}{3}\mu\right)(\partial_{(1}v_{1)} + \partial_{(2}v_{2)} + \partial_{(3}v_{3)})^2 \\ +4\mu((\partial_{(1}v_{2)})^2 + (\partial_{(2}v_{3)})^2 + (\partial_{(3}v_{1)})^2) \geq 0. \end{aligned} \tag{5.208}$$

Obviously, this is a sum of perfect squares, and holds for all values of the strain rate tensor. It can be verified by direct expansion that this term is identical to the strong form of the entropy inequality for viscous stress. It can further be verified by direct expansion that the entropy inequality can also be written more compactly as

$$\Phi = 2\mu \underbrace{\left(\partial_{(i}v_{j)} - \frac{1}{3}\partial_k v_k \delta_{ij}\right)\left(\partial_{(i}v_{j)} - \frac{1}{3}\partial_k v_k \delta_{ij}\right)}_{\text{(deviatoric strain rate)}^2} + \left(\lambda + \frac{2}{3}\mu\right)\underbrace{(\partial_i v_i)(\partial_j v_j)}_{\text{(mean strain rate)}^2} \geq 0. \tag{5.209}$$

So, we see that for a Newtonian fluid that the increase in entropy due to viscous dissipation is attributable to two effects: deviatoric strain rate and mean strain rate. The terms involving both are perfect squares, so as long as $\mu \geq 0$ and $\lambda \geq -2\mu/3$, the second law is not violated by viscous effects.

We can also write the strong form of the entropy inequality for a Newtonian fluid $(2\mu\partial_{(i}v_{j)} + \lambda\partial_k v_k \delta_{ij})(\partial_{(i}v_{j)}) \geq 0$, in terms of the principal invariants of strain rate. Leaving out details, that can be verified by direct expansion of all terms, we find the following form

$$\Phi = 2\mu\left(\frac{2}{3}\left(I_{\dot{\epsilon}}^{(1)}\right)^2 - 2I_{\dot{\epsilon}}^{(2)}\right) + \left(\lambda + \frac{2}{3}\mu\right)\left(I_{\dot{\epsilon}}^{(1)}\right)^2 \geq 0. \tag{5.210}$$

Because this is in terms of the invariants, we are assured that it is independent of the orientation of the coordinate system.

It is, however, not obvious that this form is positive semi-definite. We can use the definitions of the invariants of strain rate to rewrite the inequality as

$$\Phi = 2\mu\left(\partial_{(i}v_{j)}\partial_{(j}v_{i)} - \frac{1}{3}(\partial_i v_i)(\partial_j v_j)\right) + \left(\lambda + \frac{2}{3}\mu\right)(\partial_i v_i)(\partial_j v_j) \geq 0. \tag{5.211}$$

In terms of the eigenvalues of the strain rate tensor, κ_1, κ_2, and κ_3, this becomes

$$\Phi = 2\mu\left(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 - \frac{1}{3}(\kappa_1 + \kappa_2 + \kappa_3)^2\right) + \left(\lambda + \frac{2}{3}\mu\right)(\kappa_1 + \kappa_2 + \kappa_3)^2 \geq 0. \tag{5.212}$$

This then reduces to a positive semi-definite form:

$$\Phi = \frac{2}{3}\mu \left((\kappa_1 - \kappa_2)^2 + (\kappa_1 - \kappa_3)^2 + (\kappa_2 - \kappa_3)^2\right) + \left(\lambda + \frac{2}{3}\mu\right)(\kappa_1 + \kappa_2 + \kappa_3)^2 \geq 0. \quad (5.213)$$

Because the eigenvalues are invariant under rotation, this form is invariant.

We summarize by noting relations between mean and deviatoric stress and strain rates for Newtonian fluids. The influence of each on each has been seen or is easily shown to be as follows:

- A mean strain rate will induce a *time rate of change* in the mean thermodynamic stress via traditional thermodynamic relations[11] and will induce an additional mean viscous stress for fluids that do not obey Stokes' assumption.
- A deviatoric strain rate will not directly induce a mean stress.
- A deviatoric strain rate will directly induce a deviatoric stress.
- A mean strain rate will induce entropy production only for a fluid that does not obey Stokes' assumption.
- A deviatoric strain rate will always induce entropy production in a viscous fluid.

5.5 Equations of state

Thermodynamic equations of state provide algebraic relations between variables such as pressure, temperature, energy, and entropy. They do not involve velocity. They are formally valid for materials at rest. As long as the times scales of equilibration of the thermodynamic variables are much faster than the finest time scales of fluid dynamics, it is a valid assumption to use an ordinary equations of state. Such assumptions can be violated in high speed flows in which vibrational and rotational modes of oscillation become excited. They may also be invalid in highly rarefied flows such as might occur in the upper atmosphere.

Typically, we will require two types of equations, a *thermal equation of state* that gives the pressure as a function of two independent thermodynamic variables, e.g.

$$p = p(\rho, T), \quad (5.214)$$

and a *caloric equation of state* that gives the internal energy as a function of two independent thermodynamic variables, e.g.

$$e = e(\rho, T). \quad (5.215)$$

There are additional conditions regarding internal consistency of the equations of state; that is, just any stray functional forms will not do.

[11] e.g. for an isothermal ideal gas $dp/dt = RT(d\rho/dt) = -\rho RT \partial_i v_i$.

We outline here a method for generating equations of state with internal consistency based on satisfying the entropy inequality. First let us define a new thermodynamic variable, a, the Helmholtz[12] free energy:

$$a = e - Ts. \tag{5.216}$$

We can take the material time derivative of Eq. (5.216) to get

$$\frac{da}{dt} = \frac{de}{dt} - T\frac{ds}{dt} - s\frac{dT}{dt}. \tag{5.217}$$

It is shown in thermodynamics texts that there are a set of natural, "canonical," variables for describing a which are T and ρ. That is, we take $a = a(T, \rho)$. Taking the time derivative of this form of a and using the chain rule tells us another form for da/dt:

$$\frac{da}{dt} = \left.\frac{\partial a}{\partial T}\right|_{\rho} \frac{dT}{dt} + \left.\frac{\partial a}{\partial \rho}\right|_{T} \frac{d\rho}{dt}. \tag{5.218}$$

Now we also have the energy equation and entropy inequality:

$$\rho\frac{de}{dt} = -\partial_i q_i - p\partial_i v_i + \tau_{ij}\partial_i v_j, \tag{5.219}$$

$$\rho\frac{ds}{dt} \geq -\partial_i\left(\frac{q_i}{T}\right). \tag{5.220}$$

Using Eq. (5.217) to eliminate de/dt in favor of da/dt in the energy equation, Eq. (5.219), gives a modified energy equation:

$$\rho\left(\frac{da}{dt} + T\frac{ds}{dt} + s\frac{dT}{dt}\right) = -\partial_i q_i - p\partial_i v_i + \tau_{ij}\partial_i v_j. \tag{5.221}$$

Next, we use Eq. (5.218) to eliminate da/dt in Eq. (5.221) to get

$$\rho\left(\left.\frac{\partial a}{\partial T}\right|_{\rho} \frac{dT}{dt} + \left.\frac{\partial a}{\partial \rho}\right|_{T} \frac{d\rho}{dt} + T\frac{ds}{dt} + s\frac{dT}{dt}\right) = -\partial_i q_i - p\partial_i v_i + \tau_{ij}\partial_i v_j. \tag{5.222}$$

Now in this modified energy equation, we solve for $\rho\, ds/dt$ to get

$$\rho\frac{ds}{dt} = -\frac{1}{T}\partial_i q_i - \frac{p}{T}\partial_i v_i + \frac{1}{T}\tau_{ij}\partial_i v_j - \frac{\rho}{T}\left.\frac{\partial a}{\partial T}\right|_{\rho} \frac{dT}{dt} - \frac{\rho}{T}\left.\frac{\partial a}{\partial \rho}\right|_{T} \frac{d\rho}{dt} - \frac{\rho s}{T}\frac{dT}{dt}. \tag{5.223}$$

Substituting this version of the energy conservation equation into the second law, Eq. (5.220), gives

$$-\frac{1}{T}\partial_i q_i - \frac{p}{T}\partial_i v_i + \frac{1}{T}\tau_{ij}\partial_i v_j - \frac{\rho}{T}\left.\frac{\partial a}{\partial T}\right|_{\rho} \frac{dT}{dt} - \frac{\rho}{T}\left.\frac{\partial a}{\partial \rho}\right|_{T} \frac{d\rho}{dt} - \frac{\rho s}{T}\frac{dT}{dt} \geq -\partial_i\left(\frac{q_i}{T}\right). \tag{5.224}$$

[12]Hermann von Helmholtz, 1821-1894, Potsdam-born German physicist and philosopher, descendant of William Penn, the founder of Pennsylvania, empiricist and refuter of the notion that scientific conclusions could be drawn from philosophical ideas, graduated from medical school, wrote convincingly on the science and physiology of music, developed theories of vortex motion as well as thermodynamics and electrodynamics.

Rearranging and using the mass conservation relation to eliminate $\partial_i v_i$, we get

$$-\frac{q_i}{T^2}\partial_i T - \frac{p}{T}\left(-\frac{1}{\rho}\frac{d\rho}{dt}\right) + \frac{1}{T}\tau_{ij}\partial_i v_j - \frac{\rho}{T}\left.\frac{\partial a}{\partial T}\right|_\rho \frac{dT}{dt} - \frac{\rho}{T}\left.\frac{\partial a}{\partial \rho}\right|_T \frac{d\rho}{dt} - \frac{\rho s}{T}\frac{dT}{dt} \geq 0, \quad (5.225)$$

$$-\frac{q_i}{T}\partial_i T + \frac{p}{\rho}\frac{d\rho}{dt} + \tau_{ij}\partial_i v_j - \rho\left.\frac{\partial a}{\partial T}\right|_\rho \frac{dT}{dt} - \rho\left.\frac{\partial a}{\partial \rho}\right|_T \frac{d\rho}{dt} - \rho s\frac{dT}{dt} \geq 0, \quad (5.226)$$

$$-\frac{q_i}{T}\partial_i T + \tau_{ij}\partial_i v_j + \frac{1}{\rho}\frac{d\rho}{dt}\left(p - \rho^2\left.\frac{\partial a}{\partial \rho}\right|_T\right) - \rho\frac{dT}{dt}\left(s + \left.\frac{\partial a}{\partial T}\right|_\rho\right) \geq 0. \quad (5.227)$$

Now in our discussion of the strong form of the energy inequality, we have already found forms for q_i and τ_{ij} for which the terms involving these phenomena are positive semi-definite. We can guarantee the remaining two terms are consistent with the second law, and are associated with reversible processes by requiring that

$$p = \rho^2\left.\frac{\partial a}{\partial \rho}\right|_T, \quad (5.228)$$

$$s = -\left.\frac{\partial a}{\partial T}\right|_\rho. \quad (5.229)$$

For example, if we take the non-obvious, but experimentally defensible choice for a of

$$a = c_v(T - T_o) - c_v T\ln\left(\frac{T}{T_o}\right) + RT\ln\left(\frac{\rho}{\rho_o}\right), \quad (5.230)$$

then we get for pressure

$$p = \rho^2\left.\frac{\partial a}{\partial \rho}\right|_T = \rho^2\left(\frac{RT}{\rho}\right) = \rho RT. \quad (5.231)$$

This equation for pressure is a thermal equation of state for an *ideal gas*, and R is known as the gas constant. It is the ratio of the universal gas constant and the molecular mass of the particular gas.

Solving for entropy s, we get

$$s = -\left.\frac{\partial a}{\partial T}\right|_\rho = c_v\ln\left(\frac{T}{T_o}\right) - R\ln\left(\frac{\rho}{\rho_o}\right). \quad (5.232)$$

Then, we get for e

$$e = a + Ts = c_v(T - T_o). \quad (5.233)$$

We call this equation for energy a caloric equation of state for *calorically perfect* gas. It is calorically perfect because the specific heat at constant volume c_v is assumed a true constant here. In general for ideal gases, it can be shown to be at most a function of temperature.

Chapter 6

Governing equations: summary and special cases

see Panton, Chapters 5 and 8,
see Hughes and Gaylord, Chapter 1,
see Saffman, Chapter 1.

In this chapter, we consider a variety of secondary topics related to the governing equations. We briefly discuss boundary and interface conditions, necessary for a complete system, summarize the partial differential equations in various forms, present some special cases of the governing equations, present the equations in a dimensionless form, and consider a few cases where the linear momenta equation can be integrated once.

6.1 Boundary and interface conditions

At fluid solid interfaces, it is observed in the continuum regime that the fluid sticks to the solid boundary, so that we can safely take the fluid and solid velocities to be identical at the interface. This is called the no-slip condition. As one approaches the molecular level, this breaks down.

At the interface of two distinct, immiscible fluids, one requires that stress and energy flux both be continuous across the interface. Density need not be continuous in the absence of mass diffusion. Were mass diffusion present, the fluids would not be immiscible, and density would be a continuous variable. Additionally the effect of surface tension may need to be accounted for. We shall not consider surface tension in this course, but many texts give a complete treatment.

6.2 Complete set of compressible Navier-Stokes equations

Here we pause once more to write a complete set of equations, the compressible Navier[1] -Stokes equations, written here for a fluid that satisfies Stokes' assumption, but for which the viscosity μ (as well as thermal conductivity k) may be variable. They are an expanded version of those presented in Ch. 4.7.

6.2.0.4 Conservative form

6.2.0.4.1 Cartesian index form

$$\begin{aligned}
\partial_o \rho + \partial_i(\rho v_i) &= 0, && (6.1)\\
\partial_o(\rho v_i) + \partial_j(\rho v_j v_i) &= \rho f_i - \partial_i p \\
&\quad + \partial_j \left(2\mu \left(\partial_{(j} v_{i)} - \frac{1}{3}\partial_k v_k \delta_{ji} \right) \right), && (6.2)\\
\partial_o \left(\rho \left(e + \frac{1}{2} v_j v_j \right) \right) \\
+ \partial_i \left(\rho v_i \left(e + \frac{1}{2} v_j v_j \right) \right) &= \rho v_i f_i - \partial_i (p v_i) + \partial_i (k \partial_i T) \\
&\quad + \partial_i \left(2\mu \left(\partial_{(i} v_{j)} - \frac{1}{3} \partial_k v_k \delta_{ij} \right) v_j \right), && (6.3)\\
p &= p(\rho, T), && (6.4)\\
e &= e(\rho, T), && (6.5)\\
\mu &= \mu(\rho, T), && (6.6)\\
k &= k(\rho, T). && (6.7)
\end{aligned}$$

6.2.0.4.2 Gibbs form

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla^T \cdot (\rho \mathbf{v}) &= 0, && (6.8)\\
\frac{\partial}{\partial t}(\rho \mathbf{v}) + \left(\nabla^T \cdot (\rho \mathbf{v}\mathbf{v}^T) \right)^T &= \rho \mathbf{f} - \nabla p \\
&\quad + \left(\nabla^T \cdot \left(2\mu \left(\frac{\nabla \mathbf{v}^T + (\nabla \mathbf{v}^T)^T}{2} - \frac{1}{3} (\nabla^T \cdot \mathbf{v}) \mathbf{I} \right) \right) \right)^T, && (6.9)\\
\frac{\partial}{\partial t} \left(\rho \left(e + \frac{1}{2} \mathbf{v}^T \cdot \mathbf{v} \right) \right) \\
+ \nabla^T \cdot \left(\rho \mathbf{v} \left(e + \frac{1}{2} \mathbf{v}^T \cdot \mathbf{v} \right) \right) &= \rho \mathbf{v}^T \cdot \mathbf{f} - \nabla^T \cdot (p \mathbf{v}) + \nabla^T \cdot (k \nabla T)
\end{aligned}$$

[1] Claude Louis Marie Henri Navier, 1785-1836, Dijon-born French civil engineer and mathematician, studied under Fourier, taught applied mechanics at École des Ponts et Chaussées, replaced Cauchy as professor at École Polytechnique, specialist in road and bridge building, did not fully understand shear stress in a fluid and used faulty logic in arriving at his equations.

$$+\nabla^T \cdot \left(\left(2\mu \left(\frac{\nabla \mathbf{v}^T + (\nabla \mathbf{v}^T)^T}{2} - \frac{1}{3}(\nabla^T \cdot \mathbf{v})\mathbf{I} \right) \right) \cdot \mathbf{v} \right), \quad (6.10)$$

$$p = p(\rho, T), \quad (6.11)$$

$$e = e(\rho, T), \quad (6.12)$$

$$\mu = \mu(\rho, T), \quad (6.13)$$

$$k = k(\rho, T). \quad (6.14)$$

6.2.0.5 Non-conservative form

6.2.0.5.1 Cartesian index form

$$\frac{d\rho}{dt} = -\rho \partial_i v_i, \quad (6.15)$$

$$\rho \frac{dv_i}{dt} = \rho f_i - \partial_i p + \partial_j \left(2\mu \left(\partial_{(j} v_{i)} - \frac{1}{3} \partial_k v_k \delta_{ji} \right) \right), \quad (6.16)$$

$$\rho \frac{de}{dt} = -p \partial_i v_i + \partial_i (k \partial_i T) + 2\mu \left(\partial_{(i} v_{j)} - \frac{1}{3} \partial_k v_k \delta_{ij} \right) \partial_i v_j, \quad (6.17)$$

$$p = p(\rho, T), \quad (6.18)$$

$$e = e(\rho, T), \quad (6.19)$$

$$\mu = \mu(\rho, T), \quad (6.20)$$

$$k = k(\rho, T). \quad (6.21)$$

6.2.0.5.2 Gibbs form

$$\frac{d\rho}{dt} = -\rho \nabla^T \cdot \mathbf{v}, \quad (6.22)$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f} - \nabla p + \left(\nabla^T \cdot \left(2\mu \left(\frac{\nabla \mathbf{v}^T + (\nabla \mathbf{v})^T}{2} - \frac{1}{3}(\nabla^T \cdot \mathbf{v})\mathbf{I} \right) \right) \right)^T, \quad (6.23)$$

$$\rho \frac{de}{dt} = -p \nabla^T \cdot \mathbf{v} + \nabla^T \cdot (k \nabla T) + 2\mu \left(\frac{\nabla \mathbf{v}^T + (\nabla \mathbf{v}^T)^T}{2} - \frac{1}{3}(\nabla^T \cdot \mathbf{v})\mathbf{I} \right) : \nabla \mathbf{v}^T, \quad (6.24)$$

$$p = p(\rho, T), \quad (6.25)$$

$$e = e(\rho, T), \quad (6.26)$$

$$\mu = \mu(\rho, T), \quad (6.27)$$

$$k = k(\rho, T). \quad (6.28)$$

We take μ, and k to be thermodynamic properties of temperature and density. In practice, both dependencies are often weak, especially the dependency of μ and k on density. We also assume we know the form of the external body force per unit mass f_i. We also no longer formally require the angular momenta principle, as it has been absorbed into our constitutive equation for viscous stress. We also need not write the second law, as we can guarantee its satisfaction as long as $\mu \geq 0, k \geq 0$.

In summary, we have nine unknowns, ρ, $v_i(3)$, p, e, T, μ, and k, and nine equations, mass, linear momenta (3), energy, thermal state, caloric state, and thermodynamic relations for viscosity and thermal conductivity. When coupled with initial, interface, and boundary conditions, all dependent variables can, in principle, be expressed as functions of position x_i and time t, and this knowledge utilized to design devices of practical importance.

6.3 Incompressible Navier-Stokes equations with constant properties

If we make the assumption, that can be justified in the limit when fluid particle velocities are small relative to the velocity of sound waves in the fluid, that density changes following a particle are negligible (that is, $d\rho/dt \to 0$), the Navier-Stokes equations simplify considerably. This does not imply the density is constant everywhere in the flow. Our assumption allows for stratified flows, for which the density of individual particles still can remain constant. We shall also assume viscosity μ, and thermal conductivity k are constants, though this is not necessary.

Let us examine the mass, linear momenta, and energy equations in this limit.

6.3.1 Mass

Expanding the mass equation

$$\partial_o \rho + \partial_i(\rho v_i) = 0, \tag{6.29}$$

we get

$$\underbrace{\partial_o \rho + v_i \partial_i \rho}_{\frac{d\rho}{dt} \to 0} + \rho \partial_i v_i = 0. \tag{6.30}$$

We are assuming the first two terms in this expression, that form $d\rho/dt$, go to zero; hence the mass equation becomes $\rho \partial_i v_i = 0$. Because $\rho > 0$, we can say

$$\partial_i v_i = 0, \tag{6.31}$$

$$\nabla^T \cdot \mathbf{v} = 0. \tag{6.32}$$

So, for an incompressible fluid, the relative expansion rate for a fluid particle is zero, by Eq. (3.184).

6.3.2 Linear momenta

Let us first consider the viscous term:

$$\partial_j \left(2\mu \left(\partial_{(j} v_{i)} - \frac{1}{3} \underbrace{\partial_k v_k}_{=0} \delta_{ij} \right) \right), \tag{6.33}$$

$$\partial_j \left(2\mu \left(\partial_{(j} v_{i)}\right)\right), \tag{6.34}$$
$$\partial_j \left(\mu \left(\partial_i v_j + \partial_j v_i\right)\right). \tag{6.35}$$

Because μ is constant here, we get

$$\mu \left(\partial_j \partial_i v_j + \partial_j \partial_j v_i\right), \tag{6.36}$$
$$\mu \left(\partial_i \underbrace{\partial_j v_j}_{=0} + \partial_j \partial_j v_i\right), \tag{6.37}$$
$$\mu \partial_j \partial_j v_i. \tag{6.38}$$

Everything else in the linear momenta equation is unchanged; hence, we get

$$\rho \partial_o v_i + \rho v_j \partial_j v_i = \rho f_i - \partial_i p + \mu \partial_j \partial_j v_i, \tag{6.39}$$
$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f} - \nabla p + \mu \nabla^2 \mathbf{v}. \tag{6.40}$$

In the incompressible constant viscosity limit, the mass and linear momenta equations form a complete set of four equations in four unknowns: p,v_i. We will see that in this limit the energy equation is coupled to mass, and linear momenta, but it is only a one-way coupling.

6.3.3 Energy

Let us also choose our material to be a liquid, for which the specific heat at constant pressure, c_p is nearly identical to the specific heat at constant volume c_v as long as the ratio $T\alpha_p^2/\kappa_T/\rho/c_p << 1$. Here α_p is the coefficient of isobaric expansion, and κ_T is the coefficient of isothermal compressibility. As long as the liquid is well away from the vaporization point, this is a good assumption for most materials. We will thus take for the liquid $c_p = c_v = c$. For an incompressible gas there are some subtleties to this analysis, involving the low Mach number limit that makes the results not obvious. We will not address that problem in this course; many texts do, but many also shove the problem under the rug! For a compressible gas there are no such problems. For an incompressible liquid whose specific heat is a constant, we have $e = cT + e_o$. The compressible energy equation in full generality is

$$\rho \frac{de}{dt} = -p \partial_i v_i - \partial_i q_i + \tau_{ij} \partial_i v_j. \tag{6.41}$$

Imposing our constitutive equations and assumption of incompressibility onto this, we get

$$\rho \frac{d}{dt}\left(cT + e_o\right) = -p \underbrace{\partial_i v_i}_{=0} - \partial_i(-k\partial_i T) + 2\mu \left(\partial_{(i} v_{j)} - \frac{1}{3} \underbrace{\partial_k v_k}_{=0} \delta_{ij}\right) \partial_i v_j, \tag{6.42}$$
$$\rho c \frac{dT}{dt} = k \partial_i \partial_i T + 2\mu \partial_{(i} v_{j)} \partial_i v_j, \tag{6.43}$$

$$
\begin{aligned}
&= k\partial_i\partial_i T + 2\mu\,\underbrace{\partial_{(i}v_{j)}}_{\text{sym.}}\left(\underbrace{\partial_{(i}v_{j)}}_{\text{sym.}} + \underbrace{\partial_{[i}v_{j]}}_{\text{anti-sym.}}\right), && (6.44)\\
&= k\partial_i\partial_i T + \underbrace{2\mu\partial_{(i}v_{j)}\partial_{(i}v_{j)}}_{\Phi}, && (6.45)\\
\rho c\frac{dT}{dt} &= k\nabla^2 T + \underbrace{2\mu\nabla\mathbf{v}^T : \nabla\mathbf{v}^T}_{\Phi}. && (6.46)
\end{aligned}
$$

For incompressible flows with constant properties, the viscous dissipation function Φ reduces to

$$\Phi = 2\mu\partial_{(i}v_{j)}\partial_{(i}v_{j)}. \tag{6.47}$$

It is a scalar function and obviously positive for $\mu > 0$ because it is a tensor inner product of a tensor with itself.

6.3.4 Summary of incompressible constant property equations

The incompressible constant property equations for a liquid are summarized below in Gibbs notation:

$$
\begin{aligned}
\nabla^T \cdot \mathbf{v} &= 0, && (6.48)\\
\rho\frac{d\mathbf{v}}{dt} &= \rho\mathbf{f} - \nabla p + \mu\nabla^2\mathbf{v}, && (6.49)\\
\rho c\frac{dT}{dt} &= k\nabla^2 T + \Phi. && (6.50)
\end{aligned}
$$

For an ideal gas, it turns out that we should replace c by c_p. The alternative, c_v would seem to be the proper choice, but careful analysis in the limit of low Mach number shows this to be incorrect.

6.3.5 Limits for one-dimensional diffusion

For a static fluid ($v_i = 0$), we have $d/dt = \partial/\partial t$ and $\Phi = 0$; hence the energy equation can be written in a familiar form

$$\frac{\partial T}{\partial t} = \alpha\nabla^2 T. \tag{6.51}$$

Here we take α

$$\alpha = \frac{k}{\rho c}, \tag{6.52}$$

to be defined as the *thermal diffusivity*. In SI, thermal diffusivity has units of m^2/s. For one-dimensional cases where all variation is in the x_2 direction, we get

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x_2^2}. \tag{6.53}$$

Compare this to the momentum equation for a specific form of the velocity field, namely, $v_i(x_i) = v_1(x_2, t)$. When we also have no pressure gradient and no body force, the linear momenta principle reduces to

$$\frac{\partial v_1}{\partial t} = \nu \frac{\partial^2 v_1}{\partial x_2^2}. \tag{6.54}$$

Here we take ν

$$\nu = \frac{\mu}{\rho}, \tag{6.55}$$

to be defined as the *momentum diffusivity*, more commonly known as the *kinematic viscosity*. In SI, momentum diffusivity has units of m^2/s; these are the same as for thermal diffusivity. This equation has an identical form to that for one-dimensional energy diffusion. In fact the physical mechanism governing both, random molecular collisions, is the same.

6.4 Euler equations

The *Euler equations* are best described as the special case of the compressible Navier-Stokes equations in the limit in which diffusion is negligibly small; thus, we consider $\tau_{ij} \to 0$, $q_i \to 0$. For an isotropic fluid that obeys Fourier's law, we could also insist that $\mu \to 0$, $k \to 0$. As Euler equations are typically used for compressible flows for which body forces are often negligible, we also take $f_i \to 0$, though this can be relaxed. A version of these equations was first presented by Euler,[2] although he only considered the mass and linear momenta principles. With the later nineteenth century development of thermodynamics, the Euler equations have been taken to include the first law as well. For a complete set, they must be supplemented by appropriate thermodynamic equations of state; we leave these in a general form here.

Because the Euler equations neglect entropy-generating diffusive mechanisms, for continuous regions of flow, the second law tells us that the entropy is constant. However, we shall see later in Ch. 8.4 that shock discontinuities induce entropy changes. Also, because viscous stress is negligible, the angular momenta principle is irrelevant for the Euler equations. We summarize some of the various forms of the Euler equations next. They represent seven equations for the seven unknowns ρ, v_i, p, e, T.

6.4.1 Conservative form

6.4.1.1 Cartesian index form

$$\partial_o \rho + \partial_i(\rho v_i) = 0, \tag{6.56}$$

$$\partial_o(\rho v_i) + \partial_j(\rho v_j v_i + p\delta_{ji}) = 0, \tag{6.57}$$

[2] Euler, L., 1757, "Principes généraux du mouvement des fluides," *Mémoires de l'Académie des Sciences de Berlin*, 11: 274-315.

$$\partial_o \left(\rho \left(e + \frac{1}{2} v_j v_j \right) \right) + \partial_i \left(\rho v_i \left(e + \frac{1}{2} v_j v_j + \frac{p}{\rho} \right) \right) = 0, \tag{6.58}$$

$$p = p(\rho, T), \tag{6.59}$$

$$e = e(\rho, T). \tag{6.60}$$

6.4.1.2 Gibbs form

$$\frac{\partial \rho}{\partial t} + \nabla^T \cdot (\rho \mathbf{v}) = 0, \tag{6.61}$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \left(\nabla^T \cdot (\rho \mathbf{v} \mathbf{v}^T) + p \mathbf{I} \right)^T = \mathbf{0}, \tag{6.62}$$

$$\frac{\partial}{\partial t} \left(\rho \left(e + \frac{1}{2} \mathbf{v}^T \cdot \mathbf{v} \right) \right) + \nabla^T \cdot \left(\rho \mathbf{v} \left(e + \frac{1}{2} \mathbf{v}^T \cdot \mathbf{v} + \frac{p}{\rho} \right) \right) = 0, \tag{6.63}$$

$$p = p(\rho, T), \tag{6.64}$$

$$e = e(\rho, T). \tag{6.65}$$

6.4.2 Non-conservative form

6.4.2.1 Cartesian index form

$$\frac{d\rho}{dt} = -\rho \partial_i v_i, \tag{6.66}$$

$$\rho \frac{dv_i}{dt} = -\partial_i p, \tag{6.67}$$

$$\rho \frac{de}{dt} = -p \partial_i v_i, \tag{6.68}$$

$$p = p(\rho, T), \tag{6.69}$$

$$e = e(\rho, T). \tag{6.70}$$

6.4.2.2 Gibbs form

$$\frac{d\rho}{dt} = -\rho \nabla^T \cdot \mathbf{v}, \tag{6.71}$$

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p, \tag{6.72}$$

$$\rho \frac{de}{dt} = -p \nabla^T \cdot \mathbf{v}, \tag{6.73}$$

$$p = p(\rho, T), \tag{6.74}$$

$$e = e(\rho, T). \tag{6.75}$$

6.4.3 Alternate forms of the energy equation

The neglect of entropy-generating mechanisms allows the energy equation in the Euler equations to be cast in some simple forms that clearly illuminate the fluid's behavior. One can specialize Eq. (4.153) to cast the energy equation as

$$\rho \frac{de}{dt} = \frac{p}{\rho}\frac{d\rho}{dt}, \tag{6.76}$$

$$\frac{de}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt} = 0. \tag{6.77}$$

In terms of differentials this is simply

$$de - \frac{p}{\rho^2}\, d\rho = 0, \tag{6.78}$$

which, when compared to the Gibbs equation, Eq. (4.162), tells us that this flow is isentropic, $ds = 0$, on a particle pathline. Moreover, using the definition of specific volume, $\hat{v} = 1/\rho$, the non-conservative form of the energy equation in the Euler equations is simply

$$de = -p\ d\hat{v}. \tag{6.79}$$

That is to say the change is energy is attributable solely to the reversible work done by a pressure force acting to change the volume. In terms of the material derivative, one would say

$$\frac{de}{dt} = -p\frac{d\hat{v}}{dt}. \tag{6.80}$$

Equivalently, one could state the energy equation in terms of entropy by considering Eq. (4.167) in the limit of $q_i = 0$, $\tau_{ij} = 0$:

$$\frac{ds}{dt} = 0, \tag{6.81}$$

$$\partial_o s + v_i \partial_i s = 0, \tag{6.82}$$

$$\frac{\partial s}{\partial t} + \mathbf{v}^T \cdot \nabla s = 0. \tag{6.83}$$

Integrating on a particle pathline, we get $s = C$, where the constant C may vary from pathline to pathline. We adopt the following nomenclature:

- *isentropic flow*: the entropy s remains constant on a pathline, but may vary from pathline to pathiline.
- *homeoentropic flow*: the entropy s is the same constant throughout all of the flow field.

For the special case in which the fluid is a CPIG, we have $p = \rho RT$, $e = c_v T + \hat{e}$, and the first law, Eq. (6.78) reduces to

$$c_v\, dT - \frac{p}{\rho^2}\, d\rho = 0, \tag{6.84}$$

$$c_v\, d\left(\frac{p}{\rho R}\right) - \frac{p}{\rho^2}\, d\rho = 0, \tag{6.85}$$

$$\frac{c_v}{R}\left(\frac{1}{\rho}\, dp - \frac{p}{\rho^2}\, d\rho\right) - \frac{p}{\rho^2}\, d\rho = 0, \tag{6.86}$$

$$\frac{c_v}{c_p - c_v}\left(dp - \frac{p}{\rho}\, d\rho\right) - \frac{p}{\rho}\, d\rho = 0, \tag{6.87}$$

$$\frac{1}{\gamma - 1}\left(dp - \frac{p}{\rho}\, d\rho\right) - \frac{p}{\rho}\, d\rho = 0, \tag{6.88}$$

$$\frac{dp}{p} = \gamma\, \frac{d\rho}{\rho}, \tag{6.89}$$

$$\ln \frac{p}{p_o} = \gamma \ln \frac{\rho}{\rho_o}, \tag{6.90}$$

$$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma}, \tag{6.91}$$

$$\frac{p}{\rho^{\gamma}} = C. \tag{6.92}$$

This is the well-known relation for the isentropic behavior of a CPIG, where C is a constant. We really confined ourselves to a particle pathline as we were considering the material time derivative. So the "constant" C actually can take on different values on different pathlines.

Another way to cast this version of the energy equation (for an inviscid CPIG) is

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0, \tag{6.93}$$

$$\partial_o\left(\frac{p}{\rho^{\gamma}}\right) + v_i \partial_i\left(\frac{p}{\rho^{\gamma}}\right) = 0, \tag{6.94}$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^{\gamma}}\right) + \mathbf{v}^T \cdot \nabla\left(\frac{p}{\rho^{\gamma}}\right) = 0. \tag{6.95}$$

6.5 Dimensionless compressible Navier-Stokes equations

Here we discuss how to scale the Navier-Stokes equations into a set of dimensionless equations. Panton (2013) gives a general background for scaling. White (2006) gives a detailed discussion of the dimensionless form of the Navier-Stokes equations.

Consider the Navier-Stokes equations for a CPIG that has Newtonian behavior, satisfies Stokes' assumption, and has constant viscosity, thermal conductivity, and specific heat:

$$\partial_o \rho + \partial_i(\rho v_i) = 0, \tag{6.96}$$

$$\partial_o(\rho v_i) + \partial_j(\rho v_j v_i) = \rho f_i - \partial_i p + \mu \partial_j \left(2 \left(\partial_{(j} v_{i)} - \frac{1}{3} \partial_k v_k \delta_{ji} \right) \right), \tag{6.97}$$

$$\partial_o \left(\rho \left(e + \frac{1}{2} v_j v_j \right) \right) + \partial_i \left(\rho v_i \left(e + \frac{1}{2} v_j v_j \right) \right) = \rho v_i f_i - \partial_i (p v_i) + k \partial_i \partial_i T + \mu \partial_i \left(2 \left(\partial_{(i} v_{j)} - \frac{1}{3} \partial_k v_k \delta_{ij} \right) v_j \right), \tag{6.98}$$

$$p = \rho R T, \tag{6.99}$$

$$e = c_v T + \hat{e}. \tag{6.100}$$

Here R is the gas constant for the particular gas we are considering, which is the ratio of the universal gas constant $\Re$ and the gas's molecular mass $\mathcal{M}$: $R = \Re/\mathcal{M}$. Also $\hat{e}$ is a constant.

Now solutions to these equations, that may be of the form, for example, of $p(x_1, x_2, x_3, t)$, are necessarily parameterized by the constants from constitutive laws such as c_v, R, μ, k, f_i, in addition to parameters from initial and boundary conditions. That is our solutions will really be of the form

$$p(x_1, x_2, x_3, t; c_v, R, \mu, k, f_i, \ldots). \tag{6.101}$$

It is desirable for many reasons to reduce the number of parametric dependencies of these solutions. Some of these reasons include

- identification of groups of terms that truly govern the features of the flow,
- efficiency of presentation of results, and
- efficiency of design of experiments.

The Navier-Stokes equations (and nearly all sets of physically motivated equations) can be reduced in complexity by considering *scaled* versions of the same equations. For a given problem, the proper scales are *non-unique*, though some choices will be more helpful than others. One generally uses the following rules of thumb in choosing scales:

- reduce variables so that their scaled value is near unity,
- demonstrate that certain physical mechanisms may be negligible relative to other physical mechanisms, and
- simplify initial and boundary conditions.

In forming dimensionless equations, one must usually look for

- characteristic length scale L, and
- characteristic time scale t_c.

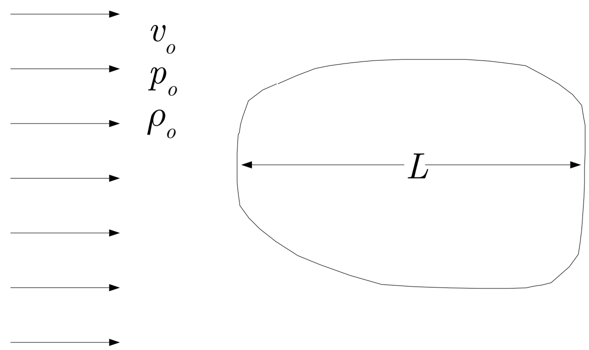

Figure 6.1: Sketch of known flow from infinity approaching body with characteristic length L.

Often an ambient velocity or sound speed exists that can be used to form either a length or time scale, for example

- given $v_o, L \longrightarrow t_c = \frac{L}{v_o}$,
- given $v_o, t_c \longrightarrow L = v_o t_c$.

If for example our physical problem involves the flow over a body of length L (and whose other dimensions are of the same order as L), and free-stream conditions are known to be $p = p_o$, $v_i = (v_o, 0, 0)^T$, $\rho = \rho_o$, as sketched in Fig. 6.1, Knowledge of free-stream pressure and density fixes all other free-stream thermodynamic variables, e.g. e, T, via the thermodynamic relations. For this problem, let the $*$ subscript represent a dimensionless variable. Define the following scaled dependent variables:

$$\rho_* = \frac{\rho}{\rho_o}, \qquad p_* = \frac{p}{p_o}, \qquad v_{*i} = \frac{v_i}{v_o}, \qquad T_* = \frac{\rho_o R}{p_o} T, \qquad e_* = \frac{\rho_o}{p_o} e. \tag{6.102}$$

Define the following scaled independent variables:

$$x_{*i} = \frac{x_i}{L}, \qquad t_* = \frac{v_o}{L} t. \tag{6.103}$$

With these definitions, the operators must also be scaled, that is,

$$\partial_o = \frac{\partial}{\partial t} = \frac{dt_*}{dt}\frac{\partial}{\partial t_*} = \frac{v_o}{L}\frac{\partial}{\partial t_*} = \frac{v_o}{L}\partial_{*o},$$
$$\partial_{*o} = \frac{L}{v_o}\partial_o.$$
$$\partial_i = \frac{\partial}{\partial x_i} = \frac{dx_{*i}}{dx_i}\frac{\partial}{\partial x_{*i}} = \frac{1}{L}\frac{\partial}{\partial x_{*i}} = \frac{1}{L}\partial_{*i},$$
$$\partial_{*i} = L\partial_i. \tag{6.104}$$

6.5.1 Mass

Let us make these substitutions into the mass equation:

$$\partial_o \rho + \partial_i(\rho v_i) = 0, \tag{6.105}$$

$$\frac{v_o}{L}\partial_{*o}(\rho_o\rho_*) + \frac{1}{L}\partial_{*i}(\rho_o\rho_* v_o v_{*i}) = 0, \tag{6.106}$$

$$\frac{\rho_o v_o}{L}\left(\partial_{*o}\rho_* + \partial_{*i}(\rho_* v_{*i})\right) = 0, \tag{6.107}$$

$$\partial_{*o}\rho_* + \partial_{*i}(\rho_* v_{*i}) = 0. \tag{6.108}$$

The mass equation is unchanged in form when we transform to a dimensionless version.

6.5.2 Linear momenta

We have a similar analysis for the linear momenta equation.

$$\begin{aligned}\partial_o(\rho v_i) + \partial_j(\rho v_j v_i) &= \rho f_i - \partial_i p \\ &\quad + \mu \partial_j \left(2\left(\partial_{(j} v_{i)} - \frac{1}{3}\partial_k v_k \delta_{ji}\right)\right),\end{aligned} \tag{6.109}$$

$$\begin{aligned}&\frac{v_o}{L}\partial_{*o}(\rho_o v_o \rho_* v_{*i}) \\ &+\frac{1}{L}\partial_{*j}(\rho_o\rho_* v_o v_{*j} v_o v_{*i}) = \rho_o\rho_* f_i - \frac{1}{L}\partial_{*i}(p_o p_*) \\ &\quad + \frac{\mu}{L}\partial_{*j}\left(\frac{2}{L}\left(\partial_{(*j} v_o v_{*i)} - \frac{1}{3L}\partial_{*k} v_o v_{*k}\delta_{ji}\right)\right),\end{aligned} \tag{6.110}$$

$$\begin{aligned}&\frac{\rho_o v_o^2}{L}\partial_{*o}(\rho_* v_{*i}) \\ &+\frac{\rho_o v_o^2}{L}\partial_{*j}(\rho_* v_{*j} v_{*i}) = \rho_o\rho_* f_i - \frac{p_o}{L}\partial_{*i}(p_*) \\ &\quad + \frac{\mu v_o}{L^2}\partial_{*j}\left(2\left(\partial_{(*j} v_{*i)} - \frac{1}{3}\partial_{*k} v_{*k}\delta_{ji}\right)\right),\end{aligned} \tag{6.111}$$

$$\begin{aligned}\partial_{*o}(\rho_* v_{*i}) + \partial_{*j}(\rho_* v_{*j} v_{*i}) &= \frac{f_i L}{v_o^2}\rho_* - \frac{p_o}{\rho_o v_o^2}\partial_{*i}(p_*) \\ &\quad + \frac{2\mu}{\rho_o v_o L}\partial_{*j}\left(\partial_{(*j} v_{*i)} - \frac{1}{3}\partial_{*k} v_{*k}\delta_{ji}\right).\end{aligned} \tag{6.112}$$

With this scaling, we have generated three distinct dimensionless groups of terms that drive the linear momenta equation:

$$\frac{f_i L}{v_o^2}, \qquad \frac{p_o}{\rho_o v_o^2}, \qquad \text{and} \qquad \frac{\mu}{\rho_o v_o L}. \tag{6.113}$$

These groups are closely related to the following groups of terms, that have the associated interpretations indicated:

- *Froude number Fr*:[3] With the body force per unit mass $f_i = g\hat{g}_i$, where $g > 0$ is the gravitational acceleration magnitude and $\hat{g}_i$ is a unit vector pointing in the direction of gravitational acceleration,

$$Fr^2 \equiv \frac{v_o^2}{gL} = \frac{\text{flow kinetic energy}}{\text{gravitational potential energy}}. \tag{6.114}$$

- *Mach number M_o*:[4] With the Mach number M_o defined as the ratio of the ambient velocity to the ambient sound speed, and recalling that for a CPIG that the square of the ambient sound speed, a_o^2 is $a_o^2 = \gamma p_o/\rho_o$, where γ is the ratio of specific heats $\gamma = c_p/c_v = (1 + R/c_v)$, we have

$$M_o^2 \equiv \frac{v_o^2}{a_o^2} = \frac{v_o^2}{\gamma \frac{p_o}{\rho_o}} = \frac{\rho_o v_o^2}{\gamma p_o} = \frac{v_o^2}{\gamma R T_o} = \frac{\text{flow kinetic energy}}{\text{thermal energy}}. \tag{6.115}$$

  Here we have taken $T_o = p_o/\rho_o/R$.

- *Reynolds number Re*: We have

$$Re \equiv \frac{\rho_o v_o L}{\mu} = \frac{\rho_o v_o^2}{\mu \frac{v_o}{L}} = \frac{\text{dynamic pressure}}{\text{viscous stress}}. \tag{6.116}$$

With these definitions, we get

$$\begin{aligned}\partial_{*o}(\rho_* v_{*i}) + \partial_{*j}(\rho_* v_{*j} v_{*i}) &= \frac{1}{Fr^2}\hat{g}_i \rho_* - \frac{1}{\gamma}\frac{1}{M_o^2}\partial_{*i}(p_*) \\ &+ \frac{2}{Re}\partial_{*j}\left(\partial_{(*j} v_{*i)} - \frac{1}{3}\partial_{*k} v_{*k}\delta_{ji}\right). \end{aligned} \tag{6.117}$$

The relative magnitudes of Fr, M_o, and Re play a crucial role in determining that physical mechanisms are most influential in changing the fluid's linear momenta.

6.5.3 Energy

The analysis is of the exact same form, but more tedious, for the energy equation.

$$\begin{aligned}\partial_o\left(\rho\left(e + \frac{1}{2}v_j v_j\right)\right) + \partial_i\left(\rho v_i\left(e + \frac{1}{2}v_j v_j\right)\right) &= k\partial_i\partial_i T - \partial_i(p v_i) \\ &+ \mu\partial_i\left(2\left(\partial_{(i} v_{j)} - \frac{1}{3}\partial_k v_k \delta_{ij}\right) v_j\right)\end{aligned}$$

[3]William Froude, 1810-1879, English engineer and naval architect, Oxford educated.

[4]Ernst Mach, 1838-1926, Viennese physicist and philosopher who worked in optics, mechanics, and wave dynamics, received doctorate at University of Vienna and taught mathematics at University of Graz and physics at Charles University of Prague, developed fundamental ideas of inertia that influenced Einstein.

$$+\rho v_i f_i, \tag{6.118}$$

$$\begin{aligned}
\frac{v_o}{L}\partial_{*o}\left(\rho_o\rho_*\left(\frac{p_o}{\rho_o}e_* + \frac{1}{2}v_o^2 v_{*j}v_{*j}\right)\right) & \\
+\frac{1}{L}\partial_{i*}\left(\rho_o\rho_* v_o v_{i*}\left(\frac{p_o}{\rho_o}e_* + \frac{1}{2}v_o^2 v_{*j}v_{*j}\right)\right) &= \frac{k}{L^2}\partial_{*i}\partial_{*i}\frac{p_o}{\rho_o R}T_* \\
&-\frac{1}{L}\partial_{*i}(p_o p_* v_o v_{*i}) \\
&+\frac{\mu}{L}\partial_{*i}\left(\frac{2}{L}\left(\partial_{(*i}v_o v_{*j)} - \frac{1}{3}\partial_{*k}v_o v_{*k}\delta_{ij}\right)v_o v_{*j}\right) \\
&+\rho_o\rho_* v_o v_{*i} f_i,
\end{aligned} \tag{6.119}$$

$$\begin{aligned}
\frac{\rho_o v_o}{L}\frac{p_o}{\rho_o}\partial_{*o}\left(\rho_*\left(e_* + \frac{1}{2}\frac{\gamma v_o^2}{\gamma\frac{p_o}{\rho_o}}v_{*j}v_{*j}\right)\right) & \\
+\frac{\rho_o v_o}{L}\frac{p_o}{\rho_o}\partial_{*i}\left(\rho_* v_{*i}\left(e_* + \frac{1}{2}\frac{\gamma v_o^2}{\gamma\frac{p_o}{\rho_o}}v_{*j}v_{*j}\right)\right) &= \frac{k}{L^2}\frac{p_o}{\rho_o}\frac{1}{R}\partial_{*i}\partial_{*i}T_* \\
&-\frac{p_o v_o}{L}\partial_{*i}(p_* v_{*i}) \\
&+\frac{2\mu v_o^2}{L^2}\partial_{*i}\left(\left(\partial_{(*i}v_{*j)} - \frac{1}{3}\partial_{*k}v_{*k}\delta_{ij}\right)v_{*j}\right) \\
&+\rho_o v_o f_i \rho_* v_{*i},
\end{aligned} \tag{6.120}$$

$$\begin{aligned}
\partial_{*o}\left(\rho_*\left(e_* + \frac{1}{2}\frac{\gamma v_o^2}{\gamma\frac{p_o}{\rho_o}}v_{*j}v_{*j}\right)\right) & \\
+\partial_{*i}\left(\rho_* v_{*i}\left(e_* + \frac{1}{2}\frac{\gamma v_o^2}{\gamma\frac{p_o}{\rho_o}}v_{*j}v_{*j}\right)\right) &= \frac{k}{LR\rho_o v_o}\partial_{*i}\partial_{*i}T_* \\
&-\partial_{*i}(p_* v_{*i}) \\
&+\frac{2\mu v_o^2}{L^2}\frac{L}{\rho_o v_o}\frac{1}{\frac{p_o}{\rho_o}}\partial_{*i}\left(\left(\partial_{(*i}v_{*j)} - \frac{1}{3}\partial_{*k}v_{*k}\delta_{ij}\right)v_{*j}\right) \\
&+\frac{f_i L}{\frac{p_o}{\rho_o}}\rho_* v_{*i}.
\end{aligned} \tag{6.121}$$

Now examining the dimensionless groups, we see that

$$\frac{k}{LR\rho_o v_o} = \frac{k}{c_p}\frac{c_p}{R}\frac{1}{L\rho_o v_o} = \frac{k}{\mu c_p}\frac{c_p}{c_p - c_v}\frac{\mu}{\rho_o v_o L} = \frac{1}{Pr}\frac{\gamma}{\gamma-1}\frac{1}{Re}. \tag{6.122}$$

Here we have a new dimensionless group, the Prandtl[5] number, Pr, where

$$Pr \equiv \frac{\mu c_p}{k} = \frac{\frac{\mu}{\rho_o}}{\frac{k}{\rho_o c_p}} = \frac{\text{momentum diffusivity}}{\text{energy diffusivity}} = \frac{\nu}{\alpha}. \tag{6.123}$$

[5]Ludwig Prandtl, 1875-1953, German mechanician and father of aerodynamics, primarily worked at University of Göttingen, discoverer of the boundary layer, pioneer of dirigibles, and advocate of monoplanes.

This has employed definitions of diffusivities given earlier in Eqs. (6.52, 6.55). We also see that

$$\frac{f_i L}{\frac{p_o}{\rho_o}} = \frac{\gamma g L \hat{g}_i}{\gamma \frac{p_o}{\rho_o}} = \frac{v_o^2}{\gamma \frac{p_o}{\rho_o}} \gamma \frac{gL}{v_o^2} \hat{g}_i = \gamma \frac{M_o^2}{Fr^2} \hat{g}_i, \tag{6.124}$$

$$\frac{\gamma v_o^2}{\gamma \frac{p_o}{\rho_o}} = \gamma M_o^2, \tag{6.125}$$

$$\frac{2\mu v_o^2}{L^2} \frac{L}{\rho_o v_o} \frac{1}{\frac{p_o}{\rho_o}} = \frac{2\mu}{\rho_o v_o L} \frac{\gamma v_o^2}{\gamma \frac{p_o}{\rho_o}} = 2 \frac{1}{Re} \gamma M_o^2. \tag{6.126}$$

So, the dimensionless energy equation becomes

$$\begin{aligned} \partial_{*o} \left(\rho_* \left(e_* + \frac{1}{2} \gamma M_o^2 v_{*j} v_{*j} \right) \right) & \\ + \partial_{*i} \left(\rho_* v_{*i} \left(e_* + \frac{1}{2} \gamma M_o^2 v_{*j} v_{*j} \right) \right) &= \frac{\gamma}{\gamma - 1} \frac{1}{Pr} \frac{1}{Re} \partial_{*i} \partial_{*i} T_* \\ & - \partial_{*i} (p_* v_{*i}) \\ & + 2\gamma \frac{M_o^2}{Re} \partial_{*i} \left(\left(\partial_{(*i} v_{*j)} - \frac{1}{3} \partial_{*k} v_{*k} \delta_{ij} \right) v_{*j} \right) \\ & + \frac{\gamma M_o^2}{Fr^2} \hat{g}_i \rho_* v_{*i}. \end{aligned} \tag{6.127}$$

6.5.4 Thermal state equation

$$p_o p_* = \rho_o \rho_* R \left(\frac{p_o}{\rho_o R} \right) T_*, \tag{6.128}$$

$$p_* = \rho_* T_*. \tag{6.129}$$

6.5.5 Caloric state equation

$$\frac{p_o}{\rho_o} e_* = c_v \left(\frac{p_o}{\rho_o R} \right) T_* + \hat{e}, \tag{6.130}$$

$$e_* = \frac{c_v}{R} T_* + \frac{\rho_o \hat{e}}{p_o}, \tag{6.131}$$

$$e_* = \frac{1}{\gamma - 1} T_* + \underbrace{\frac{\rho_o \hat{e}}{p_o}}_{\text{unimportant}}. \tag{6.132}$$

For completeness, we retain the term $\rho_o \hat{e}/p_o$. It actually plays no role in this non-reactive flow because energy only enters via its derivatives. When flows with chemical reactions are modeled, this term may be important.

6.5.6 Upstream conditions

Scaling the upstream conditions, we get

$$p_* = 1, \qquad \rho_* = 1, \qquad v_{*i} = (1,0,0)^T. \tag{6.133}$$

With this we then get secondary relationships

$$T_* = 1, \qquad e_* = \frac{1}{\gamma - 1} + \frac{\rho_o \hat{e}}{p_o}. \tag{6.134}$$

6.5.7 Reduction in parameters

We lastly note that our original system had the following ten independent parameters:

$$\rho_o,\ p_o,\ c_v,\ R,\ L,\ v_o,\ \mu,\ k,\ f_i,\ \hat{e}. \tag{6.135}$$

Our scaled system however has only *six* independent parameters:

$$Re,\ Pr,\ M_o,\ Fr,\ \gamma,\ \frac{\rho_o \hat{e}}{p_o}. \tag{6.136}$$

We have lost no information, nor made any approximations, and we have a system with fewer dependencies.

6.6 First integrals of linear momenta

Under special circumstances, we can integrate the linear momenta principle to obtain a simplified equation. We will consider two cases here, what is known as Bernoulli's[6] equation and Crocco's[7] equation. We will soon consider the Helmholtz vorticity transport equation and Kelvin's circulation theorem, in Secs. 7.4 and 7.5, respectively, that are also first integrals of linear momenta in special cases.

6.6.1 Bernoulli's equation

What we commonly call Bernoulli's equation is really a first integral of the linear momenta principle. Under different assumptions, we can get different flavors of Bernoulli's equation. A first integral of the linear momenta principle exists under the following conditions:

- viscous stresses are negligible relative to other terms, $\tau_{ij} \sim 0$,

[6]Daniel Bernoulli, 1700-1782, Dutch-born Swiss mathematician of the prolific and mathematical Bernoulli family, son of Johann Bernoulli, studied at Heidelberg, Strasbourg, and Basel, receiving M.D. degree, served in St. Petersburg and lectured at the University of Basel, put forth his fluid mechanical principle in the 1738 *Hydrodynamica*, in competition with his father's 1738 *Hydraulica*.

[7]Luigi Crocco, 1909-1986, Sicilian-born, Italian applied mathematician and theoretical aerodynamicist and rocket engineer, taught at University of Rome, Princeton, and Paris.

- the fluid is barotropic, $p = p(\rho)$ or $\rho = \rho(p)$.[8]
- body forces are conservative, so we can write $f_i = -\partial_i \hat{\phi}$, where $\hat{\phi}$ is a known potential function, and
- either
  - the flow is irrotational, $\omega_k = \epsilon_{kij}\partial_i v_j = 0$, or
  - the flow is steady, $\partial_o = 0$.

First consider a version of the general linear momenta equation in non-conservative form, Eq. (4.60) scaled by ρ:

$$\partial_o v_i + v_j \partial_j v_i = -\frac{1}{\rho}\partial_i p + f_i + \frac{1}{\rho}\partial_j \tau_{ji}. \tag{6.137}$$

Now use our vector identity, Eq. (2.261), to rewrite the advective term, and impose our assumptions to arrive at

$$\partial_o v_i + \partial_i \left(\frac{1}{2} v_j v_j\right) - \epsilon_{ijk} v_j \omega_k = -\frac{1}{\rho}\partial_i p - \partial_i \hat{\phi}. \tag{6.138}$$

Now let us define, just for this particular analysis, a new function Υ. We will take Υ to be a function of pressure p, and thus implicitly, a function of x_i and t. For the barotropic fluid, we define Υ as

$$\Upsilon(p(x_i,t)) \equiv \int_{p_o}^{p(x_i,t)} \frac{d\hat{p}}{\rho(\hat{p})}. \tag{6.139}$$

In the special case of incompressible flow with $1/\rho = 1/\rho_o$, we have

$$\Upsilon = \frac{p}{\rho_o} - \frac{p_o}{\rho_o}, \qquad \text{incompressible.} \tag{6.140}$$

In the special case of isothermal flow of an ideal gas with $1/\rho = RT/p$, we have

$$\Upsilon = RT \ln \frac{p}{p_o}, \qquad \text{isothermal ideal gas.} \tag{6.141}$$

In the special case of isentropic flow of CPIG with $1/\rho = (1/\rho_o)(p/p_o)^{1/\gamma}$, see Eq. (6.91), we have

$$\Upsilon = \frac{\gamma}{\gamma+1}\frac{p_o}{\rho_o}\left(\left(\frac{p}{p_o}\right)^{\frac{\gamma+1}{\gamma}} - 1\right), \qquad \text{isentropic CPIG.} \tag{6.142}$$

Recalling Leibniz's rule for one-dimension, Eq. (2.274),

$$\frac{d}{dt}\int_{x=a(t)}^{x=b(t)} f(x,t)\ dx = \int_{x=a(t)}^{x=b(t)} \partial_o f\ dx + \frac{db}{dt} f(b(t),t) - \frac{da}{dt} f(a(t),t), \tag{6.143}$$

[8] Three of the most common barotropic conditions are 1) a constant density fluid, $\rho = C$, 2) an isothermal ideal gas, e.g. $p = \rho(RT)$, where R and T are constant, or 3) an adiabatic CPIG, e.g. $p/p_o = (\rho/\rho_o)^\gamma$, where γ is the ratio of specific heats.

we let $\partial/\partial x_i$ play the role of d/dt to get

$$\frac{\partial}{\partial x_i}\Upsilon = \frac{\partial}{\partial x_i}\int_{p_o}^{p(x_i,t)} \frac{d\hat{p}}{\rho(\hat{p})} = \frac{1}{\rho(p(x_i,t))}\frac{\partial p}{\partial x_i} - \frac{1}{\rho(p_o)}\underbrace{\frac{\partial p_o}{\partial x_i}}_{=0} + \int_{p_o}^{p(x_i,t)} \underbrace{\frac{\partial}{\partial x_i}\left(\frac{1}{\rho(\hat{p})}\right)}_{=0} d\hat{p}. \quad (6.144)$$

As p_o is constant, and the integrand has no *explicit* dependency on x_i, we get

$$\frac{\partial}{\partial x_i}\Upsilon = \frac{1}{\rho(p(x_i,t))}\frac{\partial p}{\partial x_i}. \quad (6.145)$$

So, our linear momenta principle reduces to

$$\partial_o v_i + \partial_i\left(\frac{1}{2}v_j v_j\right) - \epsilon_{ijk}v_j\omega_k = -\partial_i\Upsilon - \partial_i\hat{\phi}, \quad (6.146)$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla^T\cdot\left(\frac{\mathbf{v}^T\cdot\mathbf{v}}{2}\right) - \mathbf{v}\times\boldsymbol{\omega} = -\nabla\Upsilon - \nabla\hat{\phi}. \quad (6.147)$$

Consider now some special cases:

6.6.1.1 Irrotational case

If the fluid is irrotational, we have $\omega_k = \epsilon_{klm}\partial_l v_m = 0$. Consequently, we can write the velocity vector as the gradient of a potential function ϕ, known as the *velocity potential*:

$$\partial_m\phi = v_m. \quad (6.148)$$

If the velocity takes this form, then the vorticity is

$$\omega_k = \epsilon_{klm}\partial_l\partial_m\phi. \quad (6.149)$$

Because ϵ_{klm} is anti-symmetric and $\partial_l\partial_m$ is symmetric, their tensor inner product must be zero; hence, such a flow is irrotational: $\omega_k = \epsilon_{klm}\partial_l\partial_m\phi = 0$. So, the linear momenta principle, Eq. (6.146), reduces to

$$\partial_o\partial_i\phi + \partial_i\left(\frac{1}{2}(\partial_j\phi)(\partial_j\phi)\right) = -\partial_i\Upsilon - \partial_i\hat{\phi}, \quad (6.150)$$

$$\partial_i\left(\partial_o\phi + \frac{1}{2}(\partial_j\phi)(\partial_j\phi) + \Upsilon + \hat{\phi}\right) = 0, \quad (6.151)$$

$$\partial_o\phi + \frac{1}{2}(\partial_j\phi)(\partial_j\phi) + \Upsilon + \hat{\phi} = f(t), \quad (6.152)$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla^T\phi\cdot\nabla\phi + \Upsilon + \hat{\phi} = f(t). \quad (6.153)$$

Here $f(t)$ is an arbitrary function of time, that can be chosen to match conditions in a given problem.

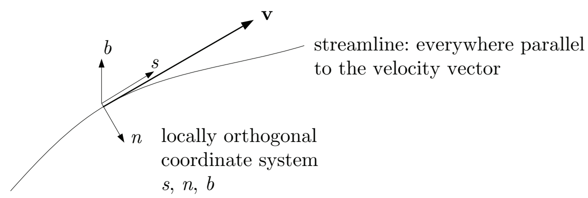

Figure 6.2: Local orthogonal intrinsic coordinate system oriented with local velocity field.

6.6.1.2 Steady case

6.6.1.2.1 Streamline integration Here we take $\partial_o = 0$, but $\omega_k \neq 0$. Rearranging the steady version of the linear momenta equation, Eq. (6.146), we get

$$\partial_i \left(\frac{1}{2} v_j v_j \right) + \partial_i \Upsilon + \partial_i \hat{\phi} = \epsilon_{ijk} v_j \omega_k, \tag{6.154}$$

$$\partial_i \left(\frac{1}{2} v_j v_j + \Upsilon + \hat{\phi} \right) = \epsilon_{ijk} v_j \omega_k. \tag{6.155}$$

Taking the inner product of both sides with v_i, we get

$$v_i \partial_i \left(\frac{1}{2} v_j v_j + \Upsilon + \hat{\phi} \right) = v_i \epsilon_{ijk} v_j \omega_k, \tag{6.156}$$

$$= \underbrace{\epsilon_{ijk} v_i v_j}_{=0} \omega_k, \tag{6.157}$$

$$= 0. \tag{6.158}$$

The term on the right hand side is zero because it is the tensor inner product of a symmetric and anti-symmetric tensor.

For a local coordinate system that has component s aligned with the velocity vector v_i, and the other two directions n, and b, mutually orthogonal, we have $v_i = (v_s, 0, 0)^T$. Such a system is sketched in Fig. 6.2, we will get many simplifications. Our linear momenta principle then reduces to

$$(v_s, 0, 0) \begin{pmatrix} \partial_s [\,] \\ \partial_n [\,] \\ \partial_b [\,] \end{pmatrix} = 0. \tag{6.159}$$

Forming this dot product yields

$$v_s \frac{\partial}{\partial s} \left(\frac{1}{2} v_j v_j + \Upsilon + \hat{\phi} \right) = 0. \tag{6.160}$$

For $v_s \neq 0$, we get that

$$\frac{1}{2} v_j v_j + \Upsilon + \hat{\phi} = C(n, b). \tag{6.161}$$

On a particular streamline, the function $C(n, b)$ will be a constant.

6.6.1.2.2 Lamb surfaces We can extend the idea of integration along a streamline to describe what are known as *Lamb surfaces*[9] by again considering the steady, inviscid linear momenta principle with conservative body forces, Eq. (6.155):

$$\partial_i \left(\frac{1}{2} v_j v_j + \Upsilon + \hat{\phi} \right) = \epsilon_{ijk} v_j \omega_k. \tag{6.162}$$

Now taking the quantity $\mathcal{B}$ to be

$$\mathcal{B} \equiv \frac{1}{2} v_j v_j + \Upsilon + \hat{\phi}, \tag{6.163}$$

the linear momenta principle, Eq. (6.155), becomes

$$\partial_i \mathcal{B} = \epsilon_{ijk} v_j \omega_k. \tag{6.164}$$

Now the vector $\epsilon_{ijk} v_j \omega_k$ is orthogonal to both velocity v_j and vorticity ω_k because of the nature of the cross product. Also the vector $\partial_i \mathcal{B}$ is orthogonal to a surface on which $\mathcal{B}$ is constant. Consequently, the surface on which $\mathcal{B}$ is constant must be tangent to both the velocity and vorticity vectors. Surfaces of constant $\mathcal{B}$ thus are composed of families of streamlines on which the Bernoulli constant has the same value. In addition they contain families of vortex lines. These are the Lamb surfaces of the flow.

6.6.1.3 Irrotational, steady, incompressible case

In this case, we recover the form most commonly used (and misused) of Bernoulli's equation, namely,

$$\frac{1}{2} v_j v_j + \Upsilon + \hat{\phi} = C. \tag{6.165}$$

The constant is truly constant throughout the flow field. With $\Upsilon = p/\rho_o - p_o/\rho_o$ here and $\hat{\phi} = g_z z + \hat{\phi}_o$ (with $g_z > 0$, and rising z corresponding to rising distance from the earth's surface, we get $\mathbf{f} = -\nabla \hat{\phi} = -g_z \mathbf{k}$) for a constant gravitational field, and v the magnitude of the velocity vector, we get

$$\frac{1}{2} v^2 + \frac{p}{\rho} + g_z z = C. \tag{6.166}$$

Here we have absorbed constants p_o/ρ_o and $\hat{\phi}_o$ into C.

6.6.2 Crocco's theorem

It is common, especially in texts on compressible flow, to present what is known as *Crocco's theorem*. The many different versions presented in many standard texts are non-uniform

[9] Sir Horace Lamb, 1849-1934, English fluid mechanician, first studied at Owens College, Manchester followed by mathematics at Cambridge, taught at Adelaide, Australia, then returned to the University of Manchester; prolific writer of textbooks, including *Hydrodynamics* (1993).

and often of unclear validity. Its utility is confined mainly to providing an alternative way of expressing the linear momenta principle that provides some insight into the factors that influence fluid motion. In special cases, it can be integrated to form a more useful relationship, similar to Bernoulli's equation, between fundamental fluid variables. The heredity of this theorem is not always clear, though, as we shall see it is nothing more than a combination of the linear momenta principle coupled with some definitions from thermodynamics. Its derivation is often confined to inviscid flows. Here we will first present a result valid for general viscous flows for the evolution of *stagnation enthalpy*, that is closely related to Crocco's theorem. As introduced earlier on p. 101, a stagnation property, such as stagnation enthalpy, is the value the property acquires when the fluid is brought to rest at a stagnation point. Next we will show how one of the restrictions can be relaxed so as to obtain what we call the *extended Crocco's theorem*. We then show how this reduces to a form that is similar to a form presented in many texts.

6.6.2.1 Stagnation enthalpy variation

First, again consider the general linear momenta equation, Eq. (6.137):

$$\partial_o v_i + v_j \partial_j v_i = -\frac{1}{\rho}\partial_i p + f_i + \frac{1}{\rho}\partial_j \tau_{ji}. \tag{6.167}$$

Now, as before in the development of Bernoulli's equation, use our vector identity, Eq. (2.261), to rewrite the advective term, but retain the viscous terms to get

$$\partial_o v_i + \partial_i \left(\frac{1}{2} v_j v_j\right) - \epsilon_{ijk} v_j \omega_k = -\frac{1}{\rho}\partial_i p + f_i + \frac{1}{\rho}\partial_j \tau_{ji}. \tag{6.168}$$

Taking the dot product with v_i, and rearranging, we get

$$\partial_o \left(\frac{1}{2} v_i v_i\right) + v_i \partial_i \left(\frac{1}{2} v_j v_j\right) = \underbrace{\epsilon_{ijk} v_i v_j}_{=0} \omega_k - \frac{1}{\rho} v_i \partial_i p + v_i f_i + \frac{1}{\rho} v_i \partial_j \tau_{ji}. \tag{6.169}$$

Again, because ϵ_{ijk} is anti-symmetric and $v_i v_j$ is symmetric, their tensor inner product is zero, so we get

$$\partial_o \left(\frac{1}{2} v_i v_i\right) + v_i \partial_i \left(\frac{1}{2} v_j v_j\right) = -\frac{1}{\rho} v_i \partial_i p + v_i f_i + \frac{1}{\rho} v_i \partial_j \tau_{ji}. \tag{6.170}$$

Now recall the Gibbs relation from thermodynamics, Eq. (4.162):

$$T\ ds = de - \frac{p}{\rho^2}\ d\rho. \tag{6.171}$$

Also recall the definition of enthalpy h, Eq. (4.136):

$$h = e + \frac{p}{\rho}. \tag{6.172}$$

Differentiating the equation for enthalpy, we recover Eq. (4.155):

$$dh = de + \frac{1}{\rho}\ dp - \frac{p}{\rho^2}\ d\rho. \tag{6.173}$$

Eliminating de in favor of dh in the Gibbs equation gives

$$T\ ds = dh - \frac{1}{\rho}\ dp. \tag{6.174}$$

If we choose to apply this relation to the motion following a fluid particle, we can say then that

$$T\frac{ds}{dt} = \frac{dh}{dt} - \frac{1}{\rho}\frac{dp}{dt}. \tag{6.175}$$

Expanding, we get

$$T(\partial_o s + v_i \partial_i s) = \partial_o h + v_i \partial_i h - \frac{1}{\rho}(\partial_o p + v_i \partial_i p). \tag{6.176}$$

Rearranging, we get

$$T(\partial_o s + v_i \partial_i s) - (\partial_o h + v_i \partial_i h) + \frac{1}{\rho}\partial_o p = -\frac{1}{\rho} v_i \partial_i p. \tag{6.177}$$

We then use this identity to eliminate the pressure gradient term from the linear momenta equation in favor of enthalpy, entropy, and unsteady pressure terms:

$$\partial_o \left(\frac{1}{2} v_i v_i\right) + v_i \partial_i \left(\frac{1}{2} v_j v_j\right) = T(\partial_o s + v_i \partial_i s) - (\partial_o h + v_i \partial_i h) + \frac{1}{\rho}\partial_o p + v_i f_i + \frac{1}{\rho} v_i \partial_j \tau_{ji}. \tag{6.178}$$

Rearranging slightly, noting that $v_i v_i = v_j v_j$, and assuming the body force is conservative so that $f_i = -\partial_i \hat{\phi}$, we get

$$\partial_o \left(h + \frac{1}{2} v_j v_j + \hat{\phi}\right) + v_i \partial_i \left(h + \frac{1}{2} v_j v_j + \hat{\phi}\right) = T\left(\partial_o s + v_i \partial_i s\right) + \frac{1}{\rho}\partial_o p + \frac{1}{\rho} v_i \partial_j \tau_{ji}. \tag{6.179}$$

Here we have made the common assumption that the body force potential $\hat{\phi}$ is independent of time, that allows us to absorb it within the time derivative. If we define, as is common, the stagnation enthalpy h_o as

$$h_o = h + \frac{1}{2} v_j v_j + \hat{\phi}, \tag{6.180}$$

we can then state

$$\partial_o h_o + v_i \partial_i h_o = T\left(\partial_o s + v_i \partial_i s\right) + \frac{1}{\rho}\partial_o p + \frac{1}{\rho} v_i \partial_j \tau_{ji}, \tag{6.181}$$

$$\frac{dh_o}{dt} = T\frac{ds}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial t} + \frac{1}{\rho}\mathbf{v}^T \cdot \left(\nabla^T \cdot \boldsymbol{\tau}\right)^T \tag{6.182}$$

The stagnation enthalpy is sometimes known as the *total enthalpy*. We can use the first law of thermodynamics written in terms of entropy, Eq. (4.166), $\rho(ds/dt) = -(1/T)\partial_i q_i + (1/T)\tau_{ij}\partial_i v_j$, to eliminate the entropy derivative in favor of those terms that generate entropy to arrive at

$$\rho \frac{dh_o}{dt} = \partial_i(\tau_{ij} v_j - q_i) + \partial_o p, \tag{6.183}$$

$$\rho \frac{dh_o}{dt} = \nabla^T \cdot (\boldsymbol{\tau} \cdot \mathbf{v} - \mathbf{q}) + \frac{\partial p}{\partial t}. \tag{6.184}$$

Thus, we see that the total enthalpy of a fluid particle is influenced by energy and momentum diffusion as well as an unsteady pressure field.

6.6.2.2 Extended Crocco's theorem

With a slight modification of the preceding analysis, we can arrive at the *extended Crocco's theorem*. Begin once more with an earlier version of the linear momenta principle, Eq. (6.168):

$$\partial_o v_i + \partial_i \left(\frac{1}{2} v_j v_j\right) - \epsilon_{ijk} v_j \omega_k = -\frac{1}{\rho}\partial_i p + f_i + \frac{1}{\rho}\partial_j \tau_{ji}. \tag{6.185}$$

Now assume we have a functional representation of enthalpy in the form

$$h = h(s, p). \tag{6.186}$$

Then we get

$$dh = \left.\frac{\partial h}{\partial s}\right|_p ds + \left.\frac{\partial h}{\partial p}\right|_s dp. \tag{6.187}$$

We also thus deduce from the Gibbs relation $dh = T ds + (1/\rho)\ dp$ that

$$\left.\frac{\partial h}{\partial s}\right|_p = T, \qquad \left.\frac{\partial h}{\partial p}\right|_s = \frac{1}{\rho}. \tag{6.188}$$

Now, because we have $h = h(s, p)$, we can take its derivative with respect to each and all of the coordinate directions to obtain

$$\frac{\partial h}{\partial x_i} = \left.\frac{\partial h}{\partial s}\right|_p \frac{\partial s}{\partial x_i} + \left.\frac{\partial h}{\partial p}\right|_s \frac{\partial p}{\partial x_i}. \tag{6.189}$$

or

$$\partial_i h = \left.\frac{\partial h}{\partial s}\right|_p \partial_i s + \left.\frac{\partial h}{\partial p}\right|_s \partial_i p. \tag{6.190}$$

Substituting known values for the thermodynamic derivatives, we get

$$\partial_i h = T\ \partial_i s + \frac{1}{\rho}\ \partial_i p. \tag{6.191}$$

We can use this to eliminate directly the pressure gradient term from the linear momenta equation to obtain then

$$\partial_o v_i + \partial_i \left(\frac{1}{2} v_j v_j\right) - \epsilon_{ijk} v_j \omega_k = T \partial_i s - \partial_i h + f_i + \frac{1}{\rho} \partial_j \tau_{ji}. \tag{6.192}$$

Rearranging slightly, and again assuming the body force is conservative so that $f_i = -\partial_i \hat{\phi}$, we get the extended Crocco's theorem:

$$\partial_o v_i + \partial_i \left(h + \frac{1}{2} v_j v_j + \hat{\phi}\right) = T \partial_i s + \epsilon_{ijk} v_j \omega_k + \frac{1}{\rho} \partial_j \tau_{ji}. \tag{6.193}$$

Again, employing the total enthalpy, $h_o = h + \frac{1}{2} v_j v_j + \hat{\phi}$, we write the extended Crocco's theorem as

$$\partial_o v_i + \partial_i h_o = T \partial_i s + \epsilon_{ijk} v_j \omega_k + \frac{1}{\rho} \partial_j \tau_{ji}, \tag{6.194}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla h_o = T \nabla s + \mathbf{v} \times \boldsymbol{\omega} + \frac{1}{\rho} \nabla^T \cdot \boldsymbol{\tau}. \tag{6.195}$$

6.6.2.3 Traditional Crocco's theorem

For a steady, inviscid flow, the extended Crocco's theorem reduces to what is usually called Crocco's theorem:

$$\partial_i h_o = T \partial_i s + \epsilon_{ijk} v_j \omega_k, \tag{6.196}$$

$$\nabla h_o = T \nabla s + \mathbf{v} \times \boldsymbol{\omega}. \tag{6.197}$$

If the flow is further required to be homeoentropic, we get

$$\partial_i h_o = \epsilon_{ijk} v_j \omega_k, \tag{6.198}$$

$$\nabla h_o = \mathbf{v} \times \boldsymbol{\omega}. \tag{6.199}$$

Similar to Lamb surfaces, we find that surfaces on which h_o is constant are parallel to both the velocity and vorticity vector fields. Taking the dot product with v_i, we get

$$v_i \partial_i h_o = v_i \epsilon_{ijk} v_j \omega_k, \tag{6.200}$$

$$= \epsilon_{ijk} v_i v_j \omega_k, \tag{6.201}$$

$$= 0. \tag{6.202}$$

Integrating this along a streamline, as for Bernoulli's equation, we find

$$h_o = C(n, b), \tag{6.203}$$

$$h + \frac{1}{2} v_j v_j + \hat{\phi} = C(n, b), \tag{6.204}$$

so we see that the stagnation enthalpy is constant along a streamline and varies from streamline to streamline. If the flow is steady, homeoentropic, and irrotational, the total enthalpy will be constant throughout the flow-field:

$$h + \frac{1}{2} v_j v_j + \hat{\phi} = C. \tag{6.205}$$

In terms of internal energy, we can rewrite this as

$$e + \frac{1}{2} v_j v_j + \frac{p}{\rho} + \hat{\phi} = C. \tag{6.206}$$

This is in a remarkably similar form to the Bernoulli equation for a steady, incompressible, irrotational fluid, Eq. (6.166). However, the assumptions for each are very different. Bernoulli made no appeal to the first law of thermodynamics, while Crocco did. This version of the Bernoulli equation is restricted to incompressible flows, while this version of the Crocco equation is fully compressible.

Part II

Governing equations: solutions

Chapter 7

Vortex dynamics

see Panton, Chapter 13,
see Yih, Chapter 2,
see Kuthe and Chow, Chapter 5,
see Lamb, Chapter 7,
see Saffman.

In this chapter we will consider in detail the kinematics and dynamics of rotating fluids, sometimes called vortex dynamics. The two most common quantities that are used to characterize rotating fluids are the vorticity vector, Eq. (3.110):

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}, \tag{7.1}$$

and a new scaler quantity we define as the *circulation*, Γ:

$$\Gamma = \oint_C \mathbf{v}^T \cdot d\mathbf{r}. \tag{7.2}$$

Here $\oint_C$ is the integral about a closed contour C. Both concepts will be important in this chapter.

Although it is possible to use Cartesian index notation to describe a rotating fluid, some of the ideas are better conveyed in a non-Cartesian system, such as the cylindrical coordinate system. For that reason, and for the sake of giving the student more experience with the other common notation, the Gibbs notation will often be used in this chapter.

7.1 Transformations to cylindrical coordinates

The rotation of a fluid about an axis induces an acceleration in that a fluid particle's velocity vector is certainly changing with respect to time. Such a motion is most easily described with a set of cylindrical coordinates. The transformation and inverse transformation to and

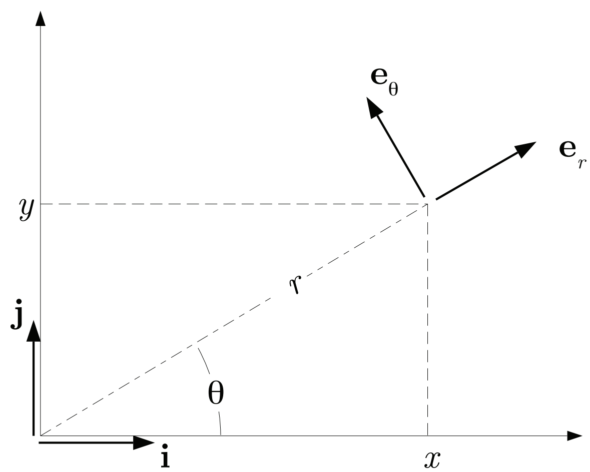

Figure 7.1: Representation of a point in Cartesian and cylindrical coordinates along with unit vectors for both systems.

from cylindrical $(r, \theta, \hat{z})$ coordinates to Cartesian (x, y, z) is given by the familiar

$$x = r\cos\theta, \qquad r = \sqrt{x^2 + y^2}, \tag{7.3}$$

$$y = r\sin\theta, \qquad \theta = \arctan\left(\frac{y}{x}\right), \tag{7.4}$$

$$z = \hat{z}, \qquad \hat{z} = z. \tag{7.5}$$

Most of the basic distinctions between the two systems can be understood by considering two-dimensional geometries. The representation of an arbitrary point in both two-dimensional (x, y) Cartesian and two-dimensional (r, θ) cylindrical coordinate systems along with the unit basis vectors for both systems, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{e}_r$, $\mathbf{e}_\theta$, is sketched in Fig. 7.1. Often a pure two-dimensional representation is called "polar;" while "cylindrical" is reserved for the three-dimensional extension. We will typically use "cylindrical" for either two- or three-dimensional systems.

7.1.1 Centripetal and Coriolis accelerations

The fact that a point in motion is accompanied by changes in the basis vectors with respect to time in the cylindrical representation, but not for Cartesian basis vectors, accounts for the most striking differences in the formulations of the governing equations, namely the appearance of so-called

- centripetal acceleration, and

- Coriolis acceleration,

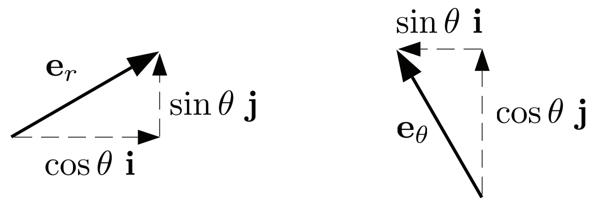

Figure 7.2: Geometrical representation of cylindrical unit vectors in terms of Cartesian unit vectors.

in the cylindrical representation. These were briefly mentioned earlier in Ch. 4.7.2.3. Consider the representations of the velocity vector $\mathbf{v}$ in both coordinate systems:

$$\mathbf{v} = u\mathbf{i} + v\mathbf{j}, \qquad \text{or} \tag{7.6}$$

$$\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta. \tag{7.7}$$

Now the unsteady (as opposed to the advective) part of the acceleration vector of a particle is simply the partial derivative of the velocity vector with respect to time. Now formally, we must allow for variations of the unit basis vectors as well as the components themselves so that

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial u}{\partial t}\mathbf{i} + u\underbrace{\frac{\partial \mathbf{i}}{\partial t}}_{=0} + \frac{\partial v}{\partial t}\mathbf{j} + v\underbrace{\frac{\partial \mathbf{j}}{\partial t}}_{=0}, \tag{7.8}$$

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial v_r}{\partial t}\mathbf{e}_r + v_r\frac{\partial \mathbf{e}_r}{\partial t} + \frac{\partial v_\theta}{\partial t}\mathbf{e}_\theta + v_\theta\frac{\partial \mathbf{e}_\theta}{\partial t}. \tag{7.9}$$

Now the time derivatives of the Cartesian basis vectors are zero, as they are defined not to change with the position of the particle. Hence for a Cartesian representation, we have for the unsteady component of acceleration the familiar:

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial u}{\partial t}\mathbf{i} + \frac{\partial v}{\partial t}\mathbf{j}. \tag{7.10}$$

However the time derivative of the cylindrical basis vectors does change with time for particles in motion! To see this, let us first relate $\mathbf{e}_r$ and $\mathbf{e}_\theta$ to $\mathbf{i}$ and $\mathbf{j}$. From the sketch of Fig. 7.2, it is clear that

$$\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \tag{7.11}$$

$$\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}. \tag{7.12}$$

This is a linear system of equations. We can use Cramer's rule to invert to find

$$\mathbf{i} = \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta, \tag{7.13}$$

$$\mathbf{j} = \sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta. \tag{7.14}$$

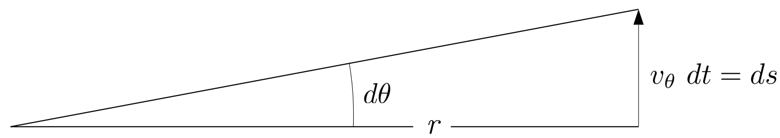

Figure 7.3: Sketch of relation of differential distance ds to velocity in angular direction v_θ.

Now, examining time derivatives of the unit vectors, we see that

$$\frac{\partial \mathbf{e}_r}{\partial t} = -\sin\theta \frac{\partial \theta}{\partial t}\mathbf{i} + \cos\theta \frac{\partial \theta}{\partial t}\mathbf{j}, \tag{7.15}$$

$$= \frac{\partial \theta}{\partial t}\mathbf{e}_\theta, \tag{7.16}$$

and

$$\frac{\partial \mathbf{e}_\theta}{\partial t} = -\cos\theta \frac{\partial \theta}{\partial t}\mathbf{i} - \sin\theta \frac{\partial \theta}{\partial t}\mathbf{j}, \tag{7.17}$$

$$= -\frac{\partial \theta}{\partial t}\mathbf{e}_r. \tag{7.18}$$

so there is a formal variation of the unit vectors with respect to time as long as the angular velocity $\partial\theta/\partial t \neq 0$. So the acceleration vector is

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial v_r}{\partial t}\mathbf{e}_r + v_r \frac{\partial \theta}{\partial t}\mathbf{e}_\theta + \frac{\partial v_\theta}{\partial t}\mathbf{e}_\theta - v_\theta \frac{\partial \theta}{\partial t}\mathbf{e}_r, \tag{7.19}$$

$$= \left(\frac{\partial v_r}{\partial t} - v_\theta \frac{\partial \theta}{\partial t}\right)\mathbf{e}_r + \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial \theta}{\partial t}\right)\mathbf{e}_\theta. \tag{7.20}$$

Now from basic geometry, as sketched in Fig. 7.3, we have

$$ds = r\ d\theta, \tag{7.21}$$

$$v_\theta\ dt = r\ d\theta, \tag{7.22}$$

$$\frac{v_\theta}{r} = \frac{\partial \theta}{\partial t}. \tag{7.23}$$

Consequently, we can write the unsteady component of acceleration as

$$\frac{\partial \mathbf{v}}{\partial t} = \left(\frac{\partial v_r}{\partial t} - \underbrace{\frac{v_\theta^2}{r}}_{\text{centripetal}}\right)\mathbf{e}_r + \left(\frac{\partial v_\theta}{\partial t} + \underbrace{\frac{v_r v_\theta}{r}}_{\text{Coriolis}}\right)\mathbf{e}_\theta. \tag{7.24}$$

Two, apparently *new*, accelerations have appeared as a consequence of the transformation: centripetal acceleration, v_θ^2/r, directed towards the center, and Coriolis acceleration, $v_r v_\theta/r$, directed in the direction of increasing θ. These terms do not have explicit dependency on time derivatives of velocity. And yet when the equations are constructed in this coordinate system, they represent real accelerations, and are consequences of forces. As can be seen by considering the general theory of non-orthogonal coordinate transformations, terms like the centripetal and Coriolis acceleration are associated with the Christoffel symbols of the transformation; see Powers and Sen (2015), Ch. 1.6.

Such terms perhaps contributed to the development of Einstein's theory of general relativity as well. Refusing to accept that our typical expression of a body force, mg, was fundamental, Einstein instead postulated that it was a term that was a relic of a coordinate transformation. He held that we in fact exist in a more complex geometry than classically considered. He constructed his theory of general relativity such that no gravitational force exists, but when coordinate transformations are employed to give us a classical view of the non-relativistic universe, the term mg appears in much the same way as centripetal and Coriolis accelerations appear when we transform to cylindrical coordinates.

7.1.2 Grad and div for cylindrical systems

We can use the chain rule to develop expressions for grad and div in cylindrical coordinate systems. Consider the Cartesian

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}. \tag{7.25}$$

The chain rule gives us

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \hat{z}}{\partial x}\frac{\partial}{\partial \hat{z}}, \tag{7.26}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \hat{z}}{\partial y}\frac{\partial}{\partial \hat{z}}, \tag{7.27}$$

$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \hat{z}}{\partial z}\frac{\partial}{\partial \hat{z}}. \tag{7.28}$$

Now, we have

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\theta, \tag{7.29}$$

$$\frac{\partial r}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{r} = \sin\theta, \tag{7.30}$$

$$\frac{\partial r}{\partial z} = 0, \tag{7.31}$$

and

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}, \tag{7.32}$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}, \quad (7.33)$$

$$\frac{\partial \theta}{\partial z} = 0, \quad (7.34)$$

and

$$\frac{\partial \hat{z}}{\partial x} = 0, \quad (7.35)$$

$$\frac{\partial \hat{z}}{\partial y} = 0, \quad (7.36)$$

$$\frac{\partial \hat{z}}{\partial z} = 1, \quad (7.37)$$

so

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}, \quad (7.38)$$

$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}, \quad (7.39)$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \hat{z}}. \quad (7.40)$$

7.1.2.1 Grad

So now we are prepared to write an explicit form for ∇ in cylindrical coordinates:

$$\nabla = \underbrace{\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)}_{\frac{\partial}{\partial x}}\underbrace{(\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta)}_{\mathbf{i}} + \underbrace{\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)}_{\frac{\partial}{\partial y}}\underbrace{(\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta)}_{\mathbf{j}} + \frac{\partial}{\partial\hat{z}}\mathbf{e}_{\hat{z}}, \quad (7.41)$$

$$\nabla = \left(\left(\cos^2\theta + \sin^2\theta\right)\frac{\partial}{\partial r} + \left(-\frac{\sin\theta\cos\theta}{r} + \frac{\sin\theta\cos\theta}{r}\right)\frac{\partial}{\partial\theta}\right)\mathbf{e}_r + \left(\left(-\sin\theta\cos\theta + \sin\theta\cos\theta\right)\frac{\partial}{\partial r} + \left(\frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r}\right)\frac{\partial}{\partial\theta}\right)\mathbf{e}_\theta + \frac{\partial}{\partial\hat{z}}\mathbf{e}_{\hat{z}}, \quad (7.42)$$

$$\nabla = \frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e}_\theta + \frac{\partial}{\partial\hat{z}}\mathbf{e}_{\hat{z}}. \quad (7.43)$$

We can now write a simple expression for the advective component, $\mathbf{v}^T\cdot\nabla$, of the acceleration vector:

$$\mathbf{v}^T\cdot\nabla = v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + v_{\hat{z}}\frac{\partial}{\partial\hat{z}}. \quad (7.44)$$

7.1.2.2 Div

The divergence is straightforward. In Cartesian coordinates we have

$$\nabla^T \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \tag{7.45}$$

In cylindrical, we replace derivatives with respect to x, y, z with those with respect to $r, \theta, \hat{z}$, so

$$\nabla^T \cdot \mathbf{v} = \cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} + \sin\theta \frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial \hat{z}}. \tag{7.46}$$

Now u, v and w transform in the same way as x, y, and z, so

$$u = v_r \cos\theta - v_\theta \sin\theta, \tag{7.47}$$

$$v = v_r \sin\theta + v_\theta \cos\theta, \tag{7.48}$$

$$w = v_{\hat{z}}. \tag{7.49}$$

Substituting and taking partials, we find that

$$\begin{aligned}
\nabla^T \cdot \mathbf{v} = {} & \cos\theta \left(\cos\theta \frac{\partial v_r}{\partial r} - \underbrace{\sin\theta \frac{\partial v_\theta}{\partial r}}_{\text{A}} \right) - \frac{\sin\theta}{r} \left(\underbrace{\cos\theta \frac{\partial v_r}{\partial \theta}}_{\text{B}} - \sin\theta v_r - \sin\theta \frac{\partial v_\theta}{\partial \theta} - \underbrace{\cos\theta v_\theta}_{\text{C}} \right) \\
& + \sin\theta \left(\sin\theta \frac{\partial v_r}{\partial r} + \underbrace{\cos\theta \frac{\partial v_\theta}{\partial r}}_{\text{A}} \right) + \frac{\cos\theta}{r} \left(\underbrace{\sin\theta \frac{\partial v_r}{\partial \theta}}_{\text{B}} + \cos\theta v_r + \cos\theta \frac{\partial v_\theta}{\partial \theta} - \underbrace{\sin\theta v_\theta}_{\text{C}} \right) \\
& + \frac{\partial v_{\hat{z}}}{\partial \hat{z}}.
\end{aligned} \tag{7.50}$$

When expanded, the terms labeled A, B, and C cancel in this expression. Then using the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, we arrive at the simple form

$$\nabla^T \cdot \mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_{\hat{z}}}{\partial \hat{z}}, \tag{7.51}$$

that is often rewritten as

$$\nabla^T \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_{\hat{z}}}{\partial \hat{z}}. \tag{7.52}$$

Using the same procedure, we can show that the Laplacian operator transforms to

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \hat{z}^2}. \tag{7.53}$$

7.1.2.3 Alternate derivations

Presented here are brief details of an alternate, more formal mathematical derivation of the gradient and Laplacian operators in cylindrical coordinates. General background was presented in Ch. 2.5 and is given in more detail by Powers and Sen (2015), Ch. 1.6. As before, one can transform from the Cartesian system with (x, y, z) as coordinates to the cylindrical system with $(r, \theta, \hat{z})$ as coordinates via $x = r\cos\theta$, $y = r\sin\theta$, $z = \hat{z}$. We will consider the domain $r \in [0, \infty)$, $\theta \in [0, 2\pi]$, $\hat{z} \in (-\infty, \infty)$. Then, with the exception of the origin $(x, y, z) = (0, 0, 0)$, every (x, y, z) will map to a unique $(r, \theta, \hat{z})$.

The Jacobian matrix of the transformation is

$$\mathsf{J} = \frac{\partial(x, y, z)}{\partial(r, \theta, \hat{z})} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \hat{z}} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \hat{z}} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \hat{z}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.54}$$

We have $J = |\mathsf{J}| = r$; this can be shown to tell us that the transformation is singular and thus non-unique when $r = 0$. It is orientation-preserving for $r > 0$, and it is volume-preserving only for $r = 1$; thus, in general it does not preserve volume.

If we take $d\mathbf{x} = (dx, dy, dz)^T$, and $d\mathbf{r} = (dr, d\theta, d\hat{z})^T$, we have

$$d\mathbf{x} = \mathsf{J} \cdot d\mathbf{r}, \tag{7.55}$$

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \hat{z}} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \hat{z}} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \hat{z}} \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\hat{z} \end{pmatrix}. \tag{7.56}$$

Now we want distance to be invariant in either coordinate system. We have the standard result for Cartesian systems that

$$ds^2 = d\mathbf{x}^T \cdot d\mathbf{x} = dx^2 + dy^2 + dz^2. \tag{7.57}$$

For invariance of distance in the transformed system, we thus require

$$ds^2 = d\mathbf{x}^T \cdot d\mathbf{x} = (\mathsf{J} \cdot d\mathbf{r})^T \cdot (\mathsf{J} \cdot d\mathbf{r}), \tag{7.58}$$

$$= d\mathbf{r}^T \cdot \underbrace{\mathsf{J}^T \cdot \mathsf{J}}_{\mathsf{G}} \cdot d\mathbf{r}, \tag{7.59}$$

$$= d\mathbf{r}^T \cdot \mathsf{G} \cdot d\mathbf{r}. \tag{7.60}$$

Recall the metric tensor G is defined by Eq. (2.287) as

$$\mathsf{G} = \mathsf{J}^T \cdot \mathsf{J}, \tag{7.61}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.62}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.63}$$

Because G is diagonal, the implication can be shown to be that new coordinates axes are also orthogonal. So for our system

$$\begin{aligned} ds^2 &= \begin{pmatrix} dr & d\theta & d\hat{z} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\hat{z} \end{pmatrix}, && (7.64) \\ &= dr^2 + (r\ d\theta)^2 + d\hat{z}^2. && (7.65) \end{aligned}$$

Now the gradient operator in the Cartesian system is related to that of the cylindrical system via the same analysis used to obtain Eq. (2.284):

$$\begin{aligned} \nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} &= (\mathsf{J}^T)^{-1} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \hat{z}} \end{pmatrix}, && (7.66) \\ &= \begin{pmatrix} \cos\theta & -\frac{\sin\theta}{r} & 0 \\ \sin\theta & \frac{\cos\theta}{r} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \hat{z}} \end{pmatrix}, && (7.67) \\ &= \begin{pmatrix} \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial\theta} \\ \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial \hat{z}} \end{pmatrix}. && (7.68) \end{aligned}$$

Then we find

$$\begin{aligned} \nabla^T \cdot \mathbf{v} &= \begin{pmatrix} \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial\theta} & \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial\theta} & \frac{\partial}{\partial \hat{z}} \end{pmatrix} \begin{pmatrix} v_r \cos\theta - v_\theta \sin\theta \\ v_r \sin\theta + v_\theta \cos\theta \\ v_{\hat{z}} \end{pmatrix}, && (7.69) \\ &= \frac{1}{r} \frac{\partial}{\partial r}(r v_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_{\hat{z}}}{\partial \hat{z}}. && (7.70) \end{aligned}$$

Consider next the Laplacian operator, $\nabla^2 = \nabla^T \cdot \nabla$, which is

$$\begin{aligned} \nabla^2 &= \nabla^T \cdot \nabla, && (7.71) \\ &= \begin{pmatrix} \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial\theta} & \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial\theta} & \frac{\partial}{\partial \hat{z}} \end{pmatrix} \begin{pmatrix} \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial\theta} \\ \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial \hat{z}} \end{pmatrix}. && (7.72) \end{aligned}$$

Detailed expansion followed by extensive use of trigonometric identities reveals that this reduces to

$$\nabla^T \cdot \nabla = \nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \hat{z}^2}. \qquad (7.73)$$

7.1.3 Incompressible Navier-Stokes equations in cylindrical coordinates

Leaving out some additional details of the transformations, we find that the incompressible Navier-Stokes equations for a Newtonian fluid with constant viscosity and body force confined to the $-\hat{z}$ direction are

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_{\hat{z}}}{\partial \hat{z}}, \tag{7.74}$$

$$\left(\frac{\partial v_r}{\partial t} - \frac{v_\theta^2}{r}\right) + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_{\hat{z}}\frac{\partial v_r}{\partial \hat{z}} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 v_r - \frac{v_r^2}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right), \tag{7.75}$$

$$\left(\frac{\partial v_\theta}{\partial t} + \frac{v_r v_\theta}{r}\right) + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_{\hat{z}}\frac{\partial v_\theta}{\partial \hat{z}} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}\right), \tag{7.76}$$

$$\frac{\partial v_{\hat{z}}}{\partial t} + v_r\frac{\partial v_{\hat{z}}}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_{\hat{z}}}{\partial \theta} + v_{\hat{z}}\frac{\partial v_{\hat{z}}}{\partial \hat{z}} = -\frac{1}{\rho}\frac{\partial p}{\partial \hat{z}} + \nu\nabla^2 v_{\hat{z}} - g_{\hat{z}}. \tag{7.77}$$

Within the acceleration terms, strictly unsteady terms, advective terms, as well as centripetal and Coriolis terms appear. The viscous terms have additional complications that we have not considered in detail but arise because we must transform $\nabla^2\mathbf{v}$, and there are many non-intuitive terms that arise here when expanded in full.

7.2 Ideal rotational vortex

Let us consider the kinematics and dynamics of an ideal rotational vortex, that we define to be a fluid rotating as a solid body. Let us assume incompressible flow, so $\nabla^T \cdot \mathbf{v} = 0$, assume a simple velocity field, and ask what forces could have given rise to that velocity field. We will simply use z for the azimuthal coordinate instead of $\hat{z}$ here. Take

$$v_r = 0, \qquad v_\theta = \frac{\omega_o r}{2}, \qquad v_z = 0. \tag{7.78}$$

This velocity field was also considered in a Cartesian representation in Ch. 3.11.5. The kinematics of this flow are simple and sketched in Fig. 7.4. Here ω_o is a constant. The velocity is zero at the origin and grows in amplitude with linear distance from the origin. The flow is steady, and the streamlines are circles centered about the origin. Obviously, as $r \to \infty$, the theory of relativity would suggest that such a flow would break down as the velocity approached the speed of light. In fact, one would find as well that as the velocities approached the sound speed that compressibility effects would become important far before relativistic effects.

Whatever the case, does this assumed velocity field satisfy incompressible mass conservation?

$$\frac{1}{r}\frac{\partial}{\partial r}(r(0)) + \frac{1}{r}\underbrace{\frac{\partial}{\partial \theta}\left(\frac{\omega_o r}{2}\right)}_{=0} + \frac{\partial}{\partial z}(0) \overset{?}{=} 0. \tag{7.79}$$

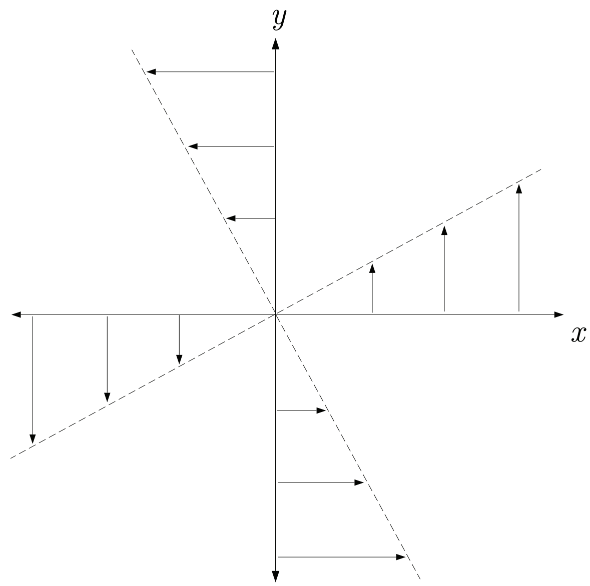

Figure 7.4: Sketch of a fluid rotating as a pure solid body.

Obviously it does.

Next let us consider the acceleration of an element of fluid and the forces that could give rise to that acceleration. First consider that portion of the acceleration that is neither centripetal nor Coriolis for this flow:

$$\underbrace{\frac{\partial}{\partial t}}_{=0} + \underbrace{v_r}_{=0} \frac{\partial}{\partial r} + \frac{v_\theta}{r} \underbrace{\frac{\partial}{\partial \theta}}_{=0} + \underbrace{v_z}_{=0} \frac{\partial}{\partial z} = 0. \tag{7.80}$$

As the only non-zero component of velocity, v_θ, has no dependency on θ, the unsteady and advective portions of the acceleration are zero for this flow. And because $v_r = 0$, there is no Coriolis acceleration. So the only acceleration is centripetal and is $-v_\theta^2/r = -\omega_o^2 r/4$.

Consider now the viscous terms for this flow. We recall for an incompressible Newtonian fluid that

$$\tau_{ij} = 2\mu\partial_{(i}v_{j)} + \lambda \underbrace{\partial_k v_k}_{=0} \delta_{ij}, \tag{7.81}$$

$$= \mu\left(\partial_i v_j + \partial_j v_i\right), \tag{7.82}$$

$$\partial_j \tau_{ij} = \mu\left(\partial_j \partial_i v_j + \partial_j \partial_j v_i\right), \tag{7.83}$$

$$= \mu\left(\partial_i \underbrace{\partial_j v_j}_{=0} + \partial_j \partial_j v_i\right), \tag{7.84}$$

$$= \mu \nabla^2 \mathbf{v}. \tag{7.85}$$

We also note that

$$\nabla \times \boldsymbol{\omega} = \epsilon_{ijk}\partial_j\omega_k = \epsilon_{ijk}\partial_j\epsilon_{kmn}\partial_m v_n, \tag{7.86}$$
$$= \epsilon_{kij}\epsilon_{kmn}\partial_j\partial_m v_n, \tag{7.87}$$
$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\,\partial_j\partial_m v_n, \tag{7.88}$$
$$= \partial_j\partial_i v_j - \partial_j\partial_j v_i, \tag{7.89}$$
$$= \partial_i \underbrace{\partial_j v_j}_{=0} - \partial_j\partial_j v_i, \tag{7.90}$$
$$= -\partial_j\partial_j v_i. \tag{7.91}$$

Comparing, we see that for this incompressible flow,

$$\left(\nabla^T \cdot \boldsymbol{\tau}^T\right)^T = -\mu(\nabla \times \boldsymbol{\omega}). \tag{7.92}$$

Now, using relations that can be developed for the curl in cylindrical coordinates, we have for this flow that

$$\omega_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} = 0, \tag{7.93}$$
$$\omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = 0, \tag{7.94}$$
$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}, \tag{7.95}$$
$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\omega_o r}{2}\right), \tag{7.96}$$
$$= \omega_o. \tag{7.97}$$

So the flow has a constant rotation rate, ω_o. Because it is constant, its curl is zero, and we have for this flow that $\left(\nabla^T \cdot \boldsymbol{\tau}^T\right)^T = \mathbf{0}$. We could just as well show for this flow that $\boldsymbol{\tau} = \mathbf{0}$. That is because the kinematics are those of pure rotation as a solid body with no deformation. No deformation implies no viscous stress.

Hence, the three linear momenta equations in the cylindrical coordinate system reduce to the following:

$$-\frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{7.98}$$
$$0 = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta}, \tag{7.99}$$
$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g_z. \tag{7.100}$$

The r momentum equation strikes a balance between centripetal inertia and radial pressure gradients. The θ momentum equation shows that as there is no acceleration in this direction, there can be no net pressure force to induce it. The z momentum equation enforces a balance between pressure forces and gravitational body forces.

If we take $p = p(r, \theta, z)$ and $p(r_o, \theta, z_o) = p_o$, then

$$\begin{aligned} dp &= \frac{\partial p}{\partial r}\, dr + \frac{\partial p}{\partial \theta}\, d\theta + \frac{\partial p}{\partial z}\, dz, && (7.101)\\ &= \frac{\rho v_\theta^2}{r}\, dr + 0\ d\theta - \rho g_z\ dz, && (7.102)\\ &= \frac{\rho \omega_o^2 r^2}{4r}\, dr - \rho g_z\ dz, && (7.103)\\ &= \frac{\rho \omega_o^2 r}{4}\, dr - \rho g_z\ dz, && (7.104)\\ p - p_o &= \frac{\rho \omega_o^2}{8}(r^2 - r_o^2) - \rho g_z (z - z_o), && (7.105)\\ p(r, z) &= p_o + \frac{\rho \omega_o^2}{8}(r^2 - r_o^2) - \rho g_z (z - z_o). && (7.106) \end{aligned}$$

Now on a surface of constant pressure we have $p(r, z) = \hat{p}$. So

$$\begin{aligned} \hat{p} &= p_o + \frac{\rho \omega_o^2}{8}(r^2 - r_o^2) - \rho g_z (z - z_o), && (7.107)\\ \rho g_z (z - z_o) &= p_o - \hat{p} + \frac{\rho \omega_o^2}{8}(r^2 - r_o^2), && (7.108)\\ z &= z_o + \frac{p_o - \hat{p}}{\rho g_z} + \frac{\omega_o^2}{8 g_z}(r^2 - r_o^2). && (7.109) \end{aligned}$$

So a surface of constant pressure is a parabola in r with a minimum at $r = 0$. This is consistent with what one observes upon spinning a bucket of water.

Now let us rearrange our general equation for the pressure field and eliminate ω using $v_\theta = \omega_o r/2$ and defining $v_{\theta o} = \omega_o r_o/2$:

$$p - \frac{1}{2}\rho v_\theta^2 + \rho g_z z = p_o - \frac{1}{2}\rho v_{\theta o}^2 + \rho g_z z_o = C. \qquad (7.110)$$

This looks similar to the steady *irrotational* incompressible Bernoulli equation in which $p + \frac{1}{2}\rho v^2 + \rho g_z z = K$. But there is a difference in the sign on one of the terms. Now add ρv_θ^2 to both sides of the equation to get

$$p + \frac{1}{2}\rho v_\theta^2 + \rho g_z z = C + \rho v_\theta^2. \qquad (7.111)$$

Now because $v_\theta = \omega_o r/2$, $v_r = 0$, we have lines of constant r as streamlines, and v_θ is constant on those streamlines, so that we get

$$p + \frac{1}{2}\rho v_\theta^2 + \rho g_z z = C', \qquad \text{on a streamline.} \qquad (7.112)$$

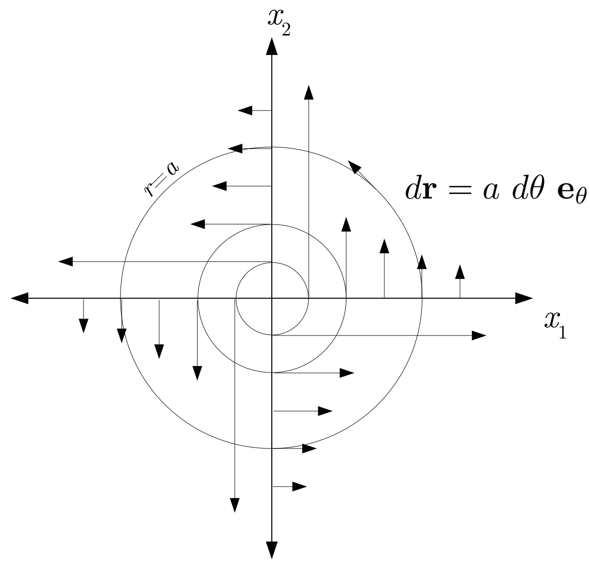

Figure 7.5: Sketch of an ideal irrotational point vortex and a circular contour of $r = a$.

Here C' varies from streamline to streamline.

We lastly note that the circulation for this system depends on position. If we choose our contour integral to be a circle of radius a about the origin we find

$$\Gamma = \oint_C \mathbf{v}^T \cdot d\mathbf{r}, \tag{7.113}$$

$$= \oint_c v_\theta \mathbf{e}_\theta^T \cdot (a\ d\theta\ \mathbf{e}_\theta), \tag{7.114}$$

$$= \int_0^{2\pi} \left(\frac{1}{2}\omega_o a\right)(a\ d\theta), \tag{7.115}$$

$$= \pi a^2 \omega_o. \tag{7.116}$$

7.3 Ideal irrotational vortex

Now let us perform a similar analysis for the following velocity field:

$$v_r = 0, \qquad v_\theta = \frac{\Gamma_o}{2\pi r}, \qquad v_z = 0. \tag{7.117}$$

We have considered the same velocity field as represented in Cartesian coordinates, in Ch. 3.11.10. The kinematics of this flow are also simple and sketched in Fig. 7.5. We see once again that the streamlines are circles about the origin. But here, as opposed to the ideal rotational vortex, $v_\theta \to 0$ as $r \to \infty$ and $v_\theta \to \infty$ as $r \to 0$. The vorticity vector of this flow is

$$\omega_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} = 0, \tag{7.118}$$

$$\omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = 0, \quad (7.119)$$

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}, \quad (7.120)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\Gamma_o}{2\pi r}\right), \quad (7.121)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\Gamma_o}{2\pi}\right) = 0. \quad (7.122)$$

This flow field, that seems to be the epitome of a rotating flow, is formally irrotational as it has zero vorticity! What is happening is that a fluid element not at the origin is actually undergoing severe deformation as it rotates about the origin; however, it does not rotate about its own center of mass. Therefore, the vorticity vector is zero, except at the origin, where it is undefined.

The circulation for this flow about a circle of radius a is

$$\Gamma = \oint_C \mathbf{v}^T \cdot d\mathbf{r}, \quad (7.123)$$

$$= \oint_c v_\theta \mathbf{e}_\theta^T \cdot (a\ d\theta\ \mathbf{e}_\theta), \quad (7.124)$$

$$= \int_0^{2\pi} v_\theta (a\ d\theta), \quad (7.125)$$

$$= \int_0^{2\pi} \frac{\Gamma_o}{2\pi a} a\ d\theta, \quad (7.126)$$

$$= \Gamma_o. \quad (7.127)$$

So the circulation is independent of the radius of the closed contour. In fact it can be shown that as long as the closed contour includes the origin in its interior that any closed contour will have this same circulation. We call Γ_o the ideal irrotational vortex strength, in that it is proportional to the magnitude of the velocity at any radius.

Let us once again consider the forces that could induce the motion of this vortex if the flow happens to be incompressible with constant properties and in a potential field where the gravitational body force per unit mass is $-g_z\mathbf{k}$. Recall again that $\left(\nabla^T \cdot \boldsymbol{\tau}\right)^T = -\mu(\nabla \times \boldsymbol{\omega})$, and that because $\boldsymbol{\omega} = \mathbf{0}$ that $\left(\nabla^T \cdot \boldsymbol{\tau}\right)^T = \mathbf{0}$ for this flow. Because there is deformation here, $\boldsymbol{\tau}$ itself is not zero, but its divergence is zero. For example, if we consider one component of viscous stress $\tau_{r\theta}$ and use standard relations that can be derived for incompressible Newtonian fluids, we find that

$$\tau_{r\theta} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) = \mu r \frac{\partial}{\partial r}\left(\frac{\Gamma_o}{2\pi r^2}\right) = -\frac{\mu \Gamma_o}{\pi r^2}. \quad (7.128)$$

The equations of motion reduce to the same ones as for the ideal rotational vortex:

$$-\frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \quad (7.129)$$

$$0 = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta}, \tag{7.130}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g_z. \tag{7.131}$$

Once more we can deduce a pressure field that is consistent with these and the same set of conditions at $r = r_o$, $z = z_o$, with $p = p_o$:

$$\begin{aligned}
dp &= \frac{\partial p}{\partial r}\,dr + \underbrace{\frac{\partial p}{\partial \theta}}_{=0}\,d\theta + \frac{\partial p}{\partial z}\,dz, &(7.132)\\
&= \frac{\rho v_\theta^2}{r}\,dr - \rho g_z\,dz, &(7.133)\\
&= \frac{\rho \Gamma_o^2}{4\pi^2}\,\frac{dr}{r^3} - \rho g_z\,dz, &(7.134)\\
p - p_o &= -\frac{\rho\Gamma_o^2}{8\pi^2}\left(\frac{1}{r^2} - \frac{1}{r_o^2}\right) - \rho g_z(z - z_o), &(7.135)\\
p + \frac{\rho\Gamma_o^2}{8\pi^2}\frac{1}{r^2} + \rho g_z z &= p_o + \frac{\rho\Gamma_o^2}{8\pi^2}\frac{1}{r_o^2} + \rho g_z z_o, &(7.136)\\
p + \frac{1}{2}\rho v_\theta^2 + \rho g_z z &= p_o + \frac{1}{2}\rho v_{\theta o}^2 + \rho g_z z_o = C. &(7.137)
\end{aligned}$$

This is once again Bernoulli's equation. Here it is for an irrotational flow field that is also time-independent, so the Bernoulli constant C is truly constant for the entire flow field and not just along a streamline.

On isobars we have $p = \hat{p}$ that gives us

$$\begin{aligned}
\hat{p} - p_o &= -\frac{\rho\Gamma_o^2}{8\pi^2}\left(\frac{1}{r^2} - \frac{1}{r_o^2}\right) - \rho g_z(z - z_o), &(7.138)\\
z &= z_o + \frac{p_o - \hat{p}}{\rho g_z} + \frac{\Gamma_o^2}{8\pi^2 g_z}\left(\frac{1}{r^2} - \frac{1}{r_o^2}\right). &(7.139)
\end{aligned}$$

The pressure goes to negative infinity at the origin. One can show that actual forces, obtained by integrating pressure over area, are in fact bounded.

7.4 Helmholtz vorticity transport equation

Here we will take the curl of the linear momenta principle to obtain a relationship, the Helmholtz vorticity transport equation, that shows how the vorticity field evolves in a general fluid.

7.4.1 General development

First, we recall some useful vector identities:

$$(\mathbf{v}^T\cdot\nabla)\mathbf{v} = \nabla\left(\frac{\mathbf{v}^T\cdot\mathbf{v}}{2}\right)+\boldsymbol{\omega}\times\mathbf{v}, \tag{7.140}$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = (\mathbf{b}^T\cdot\nabla)\mathbf{a}-(\mathbf{a}^T\cdot\nabla)\mathbf{b}+\mathbf{a}(\nabla^T\cdot\mathbf{b})-\mathbf{b}(\nabla^T\cdot\mathbf{a}), \tag{7.141}$$

$$\nabla\times(\nabla\phi) = \mathbf{0}, \tag{7.142}$$

$$\nabla^T\cdot(\nabla\times\mathbf{v}) = \nabla^T\cdot\boldsymbol{\omega}=0. \tag{7.143}$$

The first is equivalent to Eq. (2.261); the others are easily proved.

We start now with the linear momenta principle for a general fluid; we recast Eq. (4.246) and write

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}^T\cdot\nabla)\mathbf{v}=\mathbf{f}-\frac{1}{\rho}\nabla p+\frac{1}{\rho}\left(\nabla^T\cdot\boldsymbol{\tau}\right)^T. \tag{7.144}$$

We expand the term $(\mathbf{v}^T\cdot\nabla)\mathbf{v}$ and then apply the curl operator to both sides to get

$$\nabla\times\left(\frac{\partial\mathbf{v}}{\partial t}+\nabla\left(\frac{\mathbf{v}^T\cdot\mathbf{v}}{2}\right)+\boldsymbol{\omega}\times\mathbf{v}\right)=\nabla\times\left(\mathbf{f}-\frac{1}{\rho}\nabla p+\frac{1}{\rho}\left(\nabla^T\cdot\boldsymbol{\tau}\right)^T\right). \tag{7.145}$$

This becomes, via the linearity of the various operators,

$$\frac{\partial}{\partial t}(\underbrace{\nabla\times\mathbf{v}}_{\boldsymbol{\omega}})+\underbrace{\nabla\times\left(\nabla\left(\frac{\mathbf{v}^T\cdot\mathbf{v}}{2}\right)\right)}_{=\mathbf{0}}+\nabla\times\boldsymbol{\omega}\times\mathbf{v}=\nabla\times\mathbf{f}-\nabla\times\left(\frac{1}{\rho}\nabla p\right)+\nabla\times\left(\frac{1}{\rho}\left(\nabla^T\cdot\boldsymbol{\tau}\right)^T\right). \tag{7.146}$$

Using our vector identity for the term with two cross products, Eq. (7.141), we get

$$\underbrace{\frac{\partial\boldsymbol{\omega}}{\partial t}+(\mathbf{v}^T\cdot\nabla)\boldsymbol{\omega}}_{=\frac{d\boldsymbol{\omega}}{dt}}-(\boldsymbol{\omega}^T\cdot\nabla)\mathbf{v}+\boldsymbol{\omega}(\underbrace{\nabla^T\cdot\mathbf{v}}_{=-\frac{1}{\rho}\frac{d\rho}{dt}})-\mathbf{v}(\underbrace{\nabla^T\cdot\boldsymbol{\omega}}_{=0})=\nabla\times\mathbf{f}-\nabla\times\left(\frac{1}{\rho}\nabla p\right)+\nabla\times\left(\frac{1}{\rho}\left(\nabla^T\cdot\boldsymbol{\tau}\right)^T\right). \tag{7.147}$$

Rearranging, we have

$$\frac{d\boldsymbol{\omega}}{dt}-\frac{\boldsymbol{\omega}}{\rho}\frac{d\rho}{dt} = (\boldsymbol{\omega}^T\cdot\nabla)\mathbf{v}+\nabla\times\mathbf{f}-\nabla\times\left(\frac{1}{\rho}\nabla p\right)+\nabla\times\left(\frac{1}{\rho}\left(\nabla^T\cdot\boldsymbol{\tau}\right)^T\right), \tag{7.148}$$

$$\frac{1}{\rho}\frac{d\boldsymbol{\omega}}{dt}-\frac{\boldsymbol{\omega}}{\rho^2}\frac{d\rho}{dt} = \left(\frac{\boldsymbol{\omega}^T}{\rho}\cdot\nabla\right)\mathbf{v}+\frac{1}{\rho}\nabla\times\mathbf{f}-\frac{1}{\rho}\nabla\times\left(\frac{1}{\rho}\nabla p\right)+\frac{1}{\rho}\nabla\times\left(\frac{1}{\rho}\left(\nabla^T\cdot\boldsymbol{\tau}\right)^T\right), \tag{7.149}$$

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}^T}{\rho}\cdot\nabla\right)\mathbf{v}+\frac{1}{\rho}\nabla\times\mathbf{f}-\frac{1}{\rho}\nabla\times\left(\frac{1}{\rho}\nabla p\right)+\frac{1}{\rho}\nabla\times\left(\frac{1}{\rho}\left(\nabla^T\cdot\boldsymbol{\tau}\right)^T\right), \tag{7.150}$$

$$\rho\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\boldsymbol{\omega}^T\cdot\nabla\right)\mathbf{v}+\nabla\times\mathbf{f}-\nabla\times\left(\frac{1}{\rho}\nabla p\right)+\nabla\times\left(\frac{1}{\rho}\left(\nabla^T\cdot\boldsymbol{\tau}\right)^T\right). \tag{7.151}$$

Now consider the term $-\nabla \times ((1/\rho)\nabla p)$. In Einstein notation, we have

$$
\begin{aligned}
-\epsilon_{ijk}\partial_j\left(\frac{1}{\rho}\partial_k p\right) &= -\epsilon_{ijk}\left(\frac{1}{\rho}\partial_j\partial_k p - \frac{1}{\rho^2}(\partial_j\rho)(\partial_k p)\right), && (7.152)\\
&= -\frac{1}{\rho}\underbrace{\epsilon_{ijk}\partial_j\partial_k p}_{=0} + \frac{1}{\rho^2}\epsilon_{ijk}(\partial_j\rho)(\partial_k p), && (7.153)\\
&= \frac{1}{\rho^2}\nabla\rho\times\nabla p. && (7.154)
\end{aligned}
$$

We write the final general form of the vorticity transport equation as

$$
\rho\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \underbrace{\left(\boldsymbol{\omega}^T\cdot\nabla\right)\mathbf{v}}_{\text{A}} + \underbrace{\nabla\times\mathbf{f}}_{\text{B}} + \underbrace{\frac{1}{\rho^2}\nabla\rho\times\nabla p}_{\text{C}} + \underbrace{\nabla\times\left(\frac{1}{\rho}\left(\nabla^T\cdot\boldsymbol{\tau}\right)^T\right)}_{\text{D}}. \qquad (7.155)
$$

Here we see the evolution of the vorticity scaled by the density is affected by four physical processes, that we describe in greater detail directly, namely

- A: bending and stretching of vortex tubes,
- B: non-conservative body forces (if $\mathbf{f} = -\nabla\hat{\phi}$, then $\mathbf{f}$ is conservative, and $\nabla\times\mathbf{f} = -\nabla\times\nabla\hat{\phi} = \mathbf{0}$. For example $\mathbf{f} = -g_z\mathbf{k}$ gives $\hat{\phi} = g_z z$),
- C: non-barotropic, also known as *baroclinic*, effects (if a fluid is barotropic, then $p = p(\rho)$ and $\nabla p = (dp/d\rho)\nabla\rho$ thus $\nabla\rho\times\nabla p = \nabla\rho\times(dp/d\rho)\,\nabla\rho = 0$), and
- D: viscous effects.

7.4.2 Incompressible conservative body force limit

The Helmholtz vorticity transport equation (7.155) reduces significantly in special limiting cases involving incompressible flow in the limit of a conservative body force. In this limit Eq. (7.155) reduces to the following

$$
\frac{d\boldsymbol{\omega}}{dt} = (\boldsymbol{\omega}^T\cdot\nabla)\mathbf{v} + \frac{1}{\rho}\nabla\times(\nabla^T\cdot\boldsymbol{\tau})^T. \qquad (7.156)
$$

7.4.2.1 Isotropic, Newtonian, constant viscosity

Now if we further require that the fluid be isotropic and Newtonian with constant viscosity, the viscous term can be written as

$$
\begin{aligned}
\nabla\times(\nabla^T\cdot\boldsymbol{\tau})^T &= \epsilon_{ijk}\partial_j\partial_m(2\mu(\partial_{(m}v_{k)} - (1/3)\underbrace{\partial_l v_l}_{=0}\delta_{mk})), && (7.157)\\
&= \mu\epsilon_{ijk}\partial_j\partial_m(\partial_m v_k + \partial_k v_m), && (7.158)
\end{aligned}
$$

$$= \mu\epsilon_{ijk}\partial_j(\partial_m\partial_m v_k + \partial_m\partial_k v_m), \tag{7.159}$$

$$= \mu\epsilon_{ijk}\partial_j(\partial_m\partial_m v_k + \partial_k \underbrace{\partial_m v_m}_{=0}), \tag{7.160}$$

$$= \mu\partial_m\partial_m \underbrace{\epsilon_{ijk}\partial_j v_k}_{\boldsymbol{\omega}}, \tag{7.161}$$

$$= \mu\nabla^2\boldsymbol{\omega}. \tag{7.162}$$

So we get, recalling that kinematic viscosity $\nu = \mu/\rho$,

$$\frac{d\boldsymbol{\omega}}{dt} = (\boldsymbol{\omega}^T \cdot \nabla)\mathbf{v} + \nu\nabla^2\boldsymbol{\omega}. \tag{7.163}$$

7.4.2.2 Two-dimensional, isotropic, Newtonian, constant viscosity

If we further require two-dimensionality, then we have $\boldsymbol{\omega} = (0, 0, \omega_3(x_1, x_2))^T$, and $\nabla = (\partial_1, \partial_2, 0)^T$, so $\boldsymbol{\omega}^T \cdot \nabla = 0$. Thus, we get the simple

$$\frac{d\omega_3}{dt} = \nu\nabla^2\omega_3 = \nu\left(\frac{\partial^2\omega_3}{\partial x_1^2} + \frac{\partial^2\omega_3}{\partial x_2^2}\right). \tag{7.164}$$

If the flow is further inviscid $\nu = 0$, we get

$$\frac{d\omega_3}{dt} = 0, \tag{7.165}$$

and we find that there is no tendency for vorticity to change along a streamline. If we further have an initially irrotational state, then we get $\boldsymbol{\omega} = \mathbf{0}$ for all space and time.

7.4.3 Physical interpretations

Let us consider how two of the terms in Eq. (7.155) contribute to the generation of vorticity.

7.4.3.1 Baroclinic (non-barotropic) effects

If a fluid is barotropic then we can write $p = p(\rho)$, or $\rho = \rho(p)$. As an example, an isentropic CPIG has $p/p_o = (\rho/\rho_o)^\gamma$, where γ is the ratio of specific heats, and the o subscript indicates a constant value. Such a gas is barotropic. For such a fluid, we must have by the chain rule that $\partial_i p = (dp/d\rho)\partial_i\rho$. Hence ∇p and $\nabla\rho$ are vectors that point in the same direction. Moreover, isobars (lines of constant pressure) must be parallel to isochores (lines of constant density). If, as sketched in Fig. 7.6, we calculate the resultant vector from the net pressure force, as well as the center of mass for a finite fluid volume, we would see that the resultant force had no lever arm with the center of gravity. Hence it would generate no torque, and no tendency for the fluid element to rotate about its center of mass, hence no vorticity would be generated by this force.

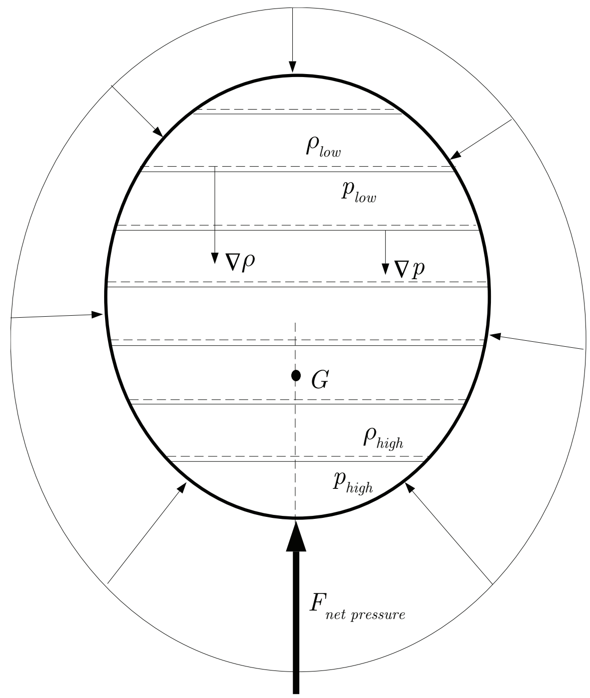

Figure 7.6: Isobars and isochores, center of mass G, and center of pressure for a barotropic fluid.

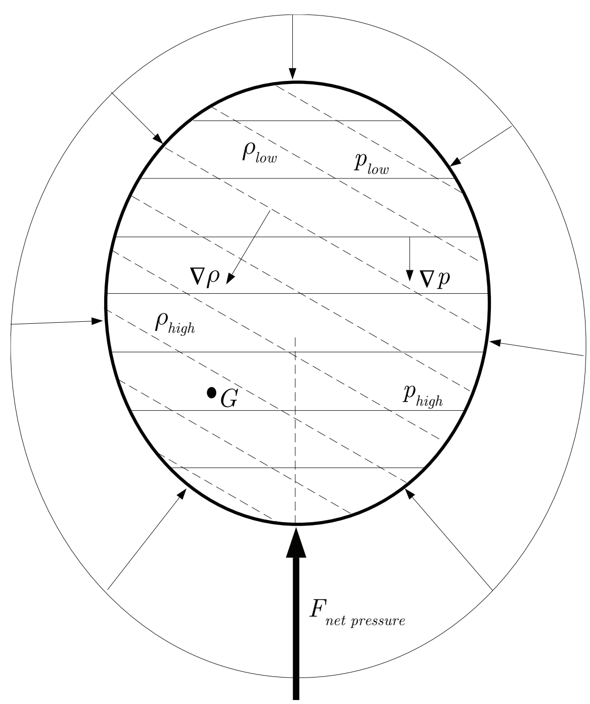

Figure 7.7: Isobars and isochores, center of mass G, and center of pressure for a baroclinic fluid.

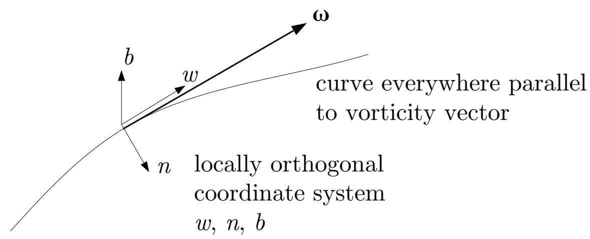

Figure 7.8: Local orthogonal intrinsic coordinate system oriented with local vorticity field.

For a baroclinic fluid, we do not have $p = p(\rho)$; hence, we must expect that ∇p points in a different direction than $\nabla \rho$. If we examine this scenario, as sketched in Fig. 7.7, we discover that the resultant force from the pressure has a non-zero lever arm with the center of mass of the fluid element. Hence, it generates a torque, a tendency to rotate the fluid about G, and vorticity.

7.4.3.2 Bending and stretching of vortex tubes

Now let us consider generation of vorticity by three-dimensional effects. Such effects are commonly characterized as the bending and stretching of what is known as vortex tubes. Here we focus on just the following inviscid equation:

$$\frac{d\boldsymbol{\omega}}{dt} = (\boldsymbol{\omega}^T \cdot \nabla)\mathbf{v}. \tag{7.166}$$

If we consider a coordinate system that is oriented with the vorticity field as sketched in Fig. 7.8, we will get many simplifications. We take the following directions

- w: the direction parallel to the vorticity vector,
- n: the principal normal direction, pointing towards the center of curvature,
- b: the biorthogonal direction, orthogonal to w and n.

With this system, we can say that

$$(\boldsymbol{\omega}^T \cdot \nabla)\mathbf{v} = \left(\begin{pmatrix} \omega_w & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial w} \\ \frac{\partial}{\partial n} \\ \frac{\partial}{\partial b} \end{pmatrix} \right) \mathbf{v}, \tag{7.167}$$

$$= \omega_w \frac{\partial \mathbf{v}}{\partial w}. \tag{7.168}$$

So for the inviscid flow we have

$$\frac{d\boldsymbol{\omega}}{dt} = \omega_w \frac{\partial \mathbf{v}}{\partial w}. \tag{7.169}$$

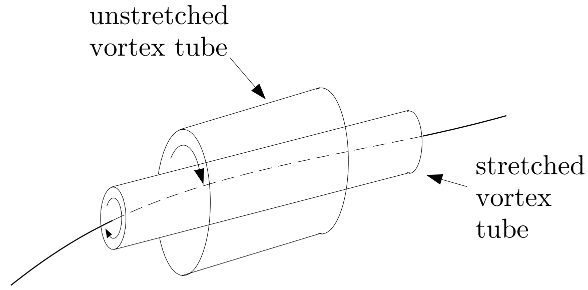

Figure 7.9: Increase in vorticity due to stretching of a vortex tube.

We have in terms of components

$$\frac{d\omega_w}{dt} = \omega_w \frac{\partial v_w}{\partial w}, \tag{7.170}$$

$$\frac{d\omega_n}{dt} = \omega_w \frac{\partial v_n}{\partial w}, \tag{7.171}$$

$$\frac{d\omega_b}{dt} = \omega_w \frac{\partial v_b}{\partial w}. \tag{7.172}$$

The term $\partial v_w/\partial w$ we know from kinematics represents a local stretching or extension. Just as a rotating figure skater increases his or her angular velocity by concentrating his or her mass about a vertical axis, so does a rotating fluid. The first of these expressions says that the component of rotation aligned with the present increases if there is stretching in that direction. This is sketched in Fig. 7.9.

The second and third terms enforce that if v_n or v_b are changing in the w direction, when accompanied by non-zero ω_w, that changes in the non-aligned components of $\boldsymbol{\omega}$ are induced. Hence the previously zero components ω_n, ω_b acquire non-zero values, and the lines parallel to the vorticity vector bend. Hence, we have the term, bending of vortex tubes. It is generally accepted that the bending and stretching of vortex tubes is an important mechanism in the transition from laminar to turbulent flow.

7.5 Kelvin's circulation theorem

Kelvin's circulation theorem describes how the circulation of a material region in a fluid changes with time. We first recall from Eq. (7.2) the definition of circulation Γ:

$$\Gamma = \oint_C \mathbf{v}^T \cdot d\mathbf{x}, \tag{7.173}$$

where C is a closed contour. We next take the material derivative of Γ to get

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\oint_C \mathbf{v}^T \cdot d\mathbf{x}, \tag{7.174}$$

$$= \oint_C \frac{d\mathbf{v}^T}{dt} \cdot d\mathbf{x} + \oint_C \mathbf{v}^T \cdot \frac{d}{dt} d\mathbf{x}, \tag{7.175}$$

$$= \oint_C \frac{d\mathbf{v}^T}{dt} \cdot d\mathbf{x} + \oint_C \mathbf{v}^T \cdot d\left(\frac{d\mathbf{x}}{dt}\right), \tag{7.176}$$

$$= \oint_C \frac{d\mathbf{v}^T}{dt} \cdot d\mathbf{x} + \oint_C \mathbf{v}^T \cdot d\mathbf{v}, \tag{7.177}$$

$$= \oint_C \frac{d\mathbf{v}^T}{dt} \cdot d\mathbf{x} + \underbrace{\oint_C d\left(\frac{1}{2}\mathbf{v}^T \cdot \mathbf{v}\right)}_{=0}, \tag{7.178}$$

$$= \oint_C \left(\frac{d\mathbf{v}}{dt}\right)^T \cdot d\mathbf{x}. \tag{7.179}$$

Here we note that because we have chosen a material region for our closed contour that $d\mathbf{x}/dt$ must be the fluid particle velocity. This then allows us to write the second term as a perfect differential, that integrates over the closed contour to be zero. We continue now by using the linear momenta principle to replace the particle acceleration with density-scaled forces to arrive at

$$\frac{d\Gamma}{dt} = \oint_C \left(\mathbf{f} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\left(\nabla^T \cdot \boldsymbol{\tau}\right)^T\right)^T \cdot d\mathbf{x}. \tag{7.180}$$

If now the fluid is inviscid ($\boldsymbol{\tau} = \mathbf{0}$), the body force is conservative ($\mathbf{f} = -\nabla\hat{\phi}$), and the fluid is barotropic ($(1/\rho)\nabla p = \nabla\Upsilon$), we then have

$$\frac{d\Gamma}{dt} = \oint_C \left(-\nabla\hat{\phi} - \nabla\Upsilon\right)^T \cdot d\mathbf{x}, \tag{7.181}$$

$$= -\oint_C \nabla^T \left(\hat{\phi} + \Upsilon\right) \cdot d\mathbf{x}, \tag{7.182}$$

$$= -\underbrace{\oint_C d\left(\hat{\phi} + \Upsilon\right)}_{=0}. \tag{7.183}$$

The integral on the right hand side is zero because the contour is closed; hence, the integral is path-independent. Consequently, we arrive at the common version of Kelvin's circulation theorem that holds that for a fluid that is inviscid, barotropic, and subjected to conservative body forces, the circulation following a material region does not change with time:

$$\frac{d\Gamma}{dt} = 0. \tag{7.184}$$

This is similar to the Helmholtz equation, that, when we make the additional stipulation of two-dimensionality and incompressibility, gives $d\boldsymbol{\omega}/dt = \mathbf{0}$. This is not surprising as the vorticity is closely linked to the circulation via Stokes' theorem, Eq. (2.260), that states

$$\Gamma = \oint_C \mathbf{v}^T \cdot d\mathbf{x} = \int_S (\nabla \times \mathbf{v})^T \cdot \mathbf{n}\ dS = \int_S \boldsymbol{\omega}^T \cdot \mathbf{n}\ dS. \tag{7.185}$$

7.6 Potential flow of ideal point vortices

Consider the fluid motion induced by the simultaneous interaction of a family of ideal *irrotational* point vortices in an incompressible flow field. Because the flow is irrotational and incompressible, we have the following useful results:

- Because $\nabla \times \mathbf{v} = \mathbf{0}$, we can write the velocity vector as the gradient of a scalar potential ϕ:
$$\mathbf{v} = \nabla \phi, \qquad \text{if irrotational.} \tag{7.186}$$
We call ϕ the *velocity potential*.

- Because $\nabla^T \cdot \mathbf{v} = 0$, we have
$$\nabla^T \cdot \nabla \phi = \nabla^2 \phi = 0, \tag{7.187}$$
or expanding, we have
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \tag{7.188}$$

- We notice that the equation for ϕ is linear; hence the method of superposition is valid here for the velocity potential. That is, we can add an arbitrary number of velocity potentials together and get a viable flow field.

- The irrotational unsteady Bernoulli equation, Eq. (6.153), gives us the time- and space-dependent pressure field. This equation is not linear, so we do not expect pressures from elementary solutions to add to form total pressures.

Recalling that the incompressible, three-dimensional constant viscosity Helmholtz equation can be written via Eq. (7.163) as

$$\frac{d\boldsymbol{\omega}}{dt} = (\boldsymbol{\omega}^T \cdot \nabla)\mathbf{v} + \nu \nabla^2 \boldsymbol{\omega}, \tag{7.189}$$

we see that a flow that is initially irrotational everywhere in an unbounded fluid will always be irrotational, as $d\boldsymbol{\omega}/dt = \mathbf{0}$. There is no mechanism to change the vorticity from its uniform initial value of zero. This even holds for a viscous flow. However, in a bounded medium, the no-slip boundary condition almost always tends to diffuse vorticity into the flow as we shall see. Further for inviscid, barotropic flow, from Kelvin's circulation theorem, Eq. (7.184), the circulation Γ has no tendency to change following a particle; that is, Γ advects unchanged along particle pathlines.

7.6.1 Two interacting ideal vortices

Let us apply this notion to two ideal counterrotating vortices 1 and 2, with respective strengths, Γ_1 and Γ_2, as shown in Fig. 7.10. Were it isolated, vortex 1 would have no

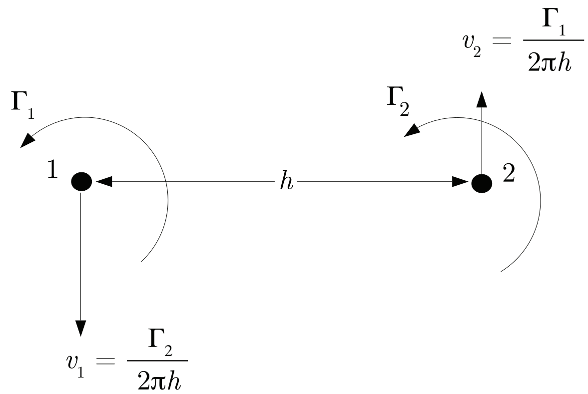

Figure 7.10: Sketch of the mutual influence of two ideal point vortices on each other.

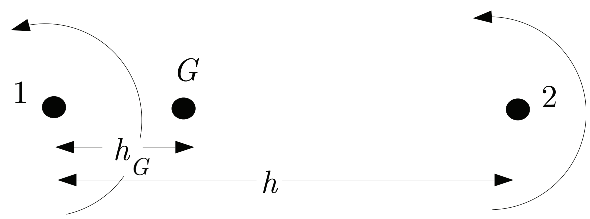

Figure 7.11: Sketch showing the center of rotation G.

tendency to move itself, but would induce a velocity at a distance h away from its center of $\Gamma_1/(2\pi h)$. This induced velocity in fact advects vortex 2, to satisfy Kelvin's circulation theorem. Similarly, vortex 2 induces a velocity of vortex 1 of $\Gamma_2/(2\pi h)$.

The center of rotation G is the point along the 1-2 axis for which the induced velocity is zero, as is illustrated in Fig. 7.11. To calculate it we equate the induced velocities of each vortex

$$\frac{\Gamma_1}{2\pi h_G} = \frac{\Gamma_2}{2\pi(h - h_G)}, \tag{7.190}$$

$$(h - h_G)\Gamma_1 = h_G\Gamma_2, \tag{7.191}$$

$$h\Gamma_1 = h_G(\Gamma_1 + \Gamma_2), \tag{7.192}$$

$$h_G = h\frac{\Gamma_1}{\Gamma_1 + \Gamma_2}. \tag{7.193}$$

A pair of equal strength counterrotating vortices is illustrated in Fig. 7.12. Such vortices induce the same velocity in each other, so they will propagate as a pair at a fixed distance from one another.

7.6.2 Image vortex

If we choose to model the fluid as inviscid, then there is no viscous stress, and we can no longer enforce the no-slip condition at a wall. However at a slip wall, we must require that the

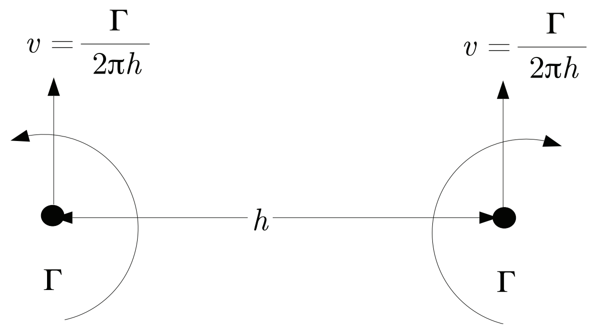

Figure 7.12: Sketch showing a pair of counterrotating vortices of equal strength.

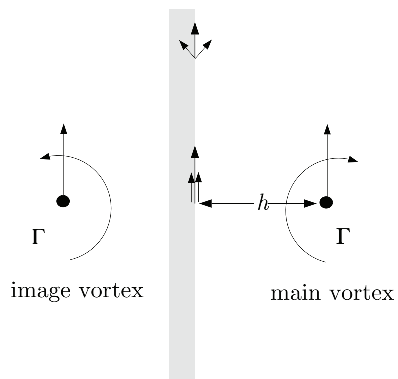

Figure 7.13: Sketch showing a vortex and its image to simulate an inviscid wall.

velocity vector be parallel to the wall. This can be described as a *no-penetration* condition, which is looser than a no-slip condition. No-penetration through a surface with outer normal $\mathbf{n}$ simply requires $\mathbf{v}^T \cdot \mathbf{n} = 0$, and allows no mass to penetrate such a surface, while allowing slip at the surface because $\mathbf{v}^T \cdot \mathbf{t} \neq 0$, with $\mathbf{t}$ as the unit tangent vector to the surface. We can model the motion of an ideal vortex separated by a distance h from an inviscid slip wall by placing a so-called *image vortex* on the other side of the wall. The image vortex will induce a velocity that when superposed with the original vortex, renders the resultant velocity to be parallel to the wall. A vortex and its image vortex, that generates a straight streamline at a wall, is sketched in Fig. 7.13.

7.6.3 Vortex sheets

We can model the slip line between two inviscid fluids moving at different velocities by what is known as a *vortex sheet*. A vortex sheet is sketched in Fig. 7.14. Here we have a distribution of small vortices, each of strength $d\Gamma$, on the x axis. Each of these vortices induces a small velocity $d\mathbf{v}$ at an arbitrary point $(\tilde{x}, \tilde{y})$. The influence of the point vortex at

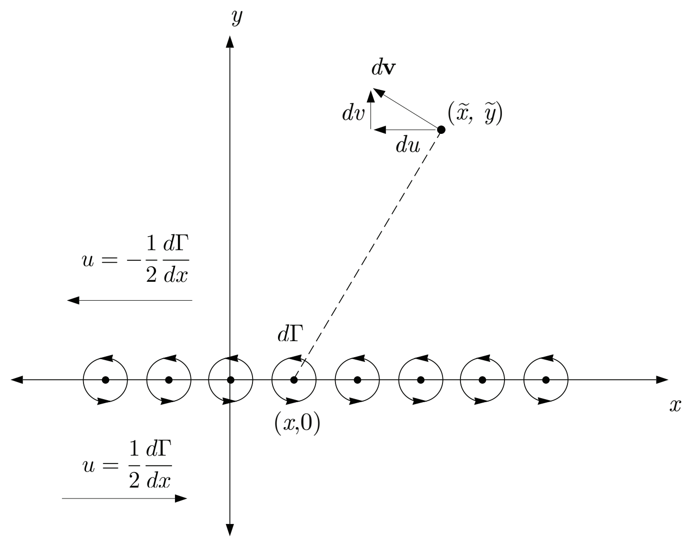

Figure 7.14: Sketch showing schematic of vortex sheet.

$(x, 0)$ is sketched in the figure. It generates a small velocity with magnitude

$$d|\mathbf{v}| = \frac{d\Gamma}{2\pi h} = \frac{d\Gamma}{2\pi\sqrt{(\tilde{x}-x)^2+\tilde{y}^2}}. \tag{7.194}$$

Using basic trigonometry, we can deduce that the influence of the single vortex of differential strength on each velocity component is

$$du = \frac{-d\Gamma\tilde{y}}{2\pi\left((\tilde{x}-x)^2+\tilde{y}^2\right)} = \frac{-\frac{d\Gamma}{dx}\tilde{y}}{2\pi\left((\tilde{x}-x)^2+\tilde{y}^2\right)}\,dx, \tag{7.195}$$

$$dv = \frac{d\Gamma(\tilde{x}-x)}{2\pi\left((\tilde{x}-x)^2+\tilde{y}^2\right)} = \frac{\frac{d\Gamma}{dx}(\tilde{x}-x)}{2\pi\left((\tilde{x}-x)^2+\tilde{y}^2\right)}\,dx. \tag{7.196}$$

Here $d\Gamma/dx$ is a measure of the strength of the vortex sheet. Let us account for the effects of *all* of the differential vortices by integrating from $x = -L$ to $x = L$ and then letting $L \to \infty$. We obtain then the total velocity components u and v at each point to be

$$u = \lim_{L\to\infty} -\frac{\frac{d\Gamma}{dx}}{2\pi}\left(\underbrace{\arctan\left(\frac{L-\tilde{x}}{\tilde{y}}\right)}_{\to\pm\frac{\pi}{2}} + \underbrace{\arctan\left(\frac{L+\tilde{x}}{\tilde{y}}\right)}_{\to\pm\frac{\pi}{2}}\right), \tag{7.197}$$

$$= \begin{cases} -\frac{1}{2}\frac{d\Gamma}{dx}, & \text{if} \quad \tilde{y} > 0, \\ \frac{1}{2}\frac{d\Gamma}{dx}, & \text{if} \quad \tilde{y} < 0, \end{cases} \tag{7.198}$$

$$v = \lim_{L\to\infty} \frac{\frac{d\Gamma}{dx}}{4\pi}\ln\frac{(L-\tilde{x})^2+\tilde{y}^2}{(L+\tilde{x})^2+\tilde{y}^2} = 0. \tag{7.199}$$

So the vortex sheet generates no y component of velocity anywhere in the flow field and two uniform x components of velocity of opposite sign above and below the x axis.

7.6.4 Potential of an ideal irrotational vortex

Let us calculate the velocity potential function ϕ associated with a single ideal irrotational vortex. Consider an ideal irrotational vortex centered at the origin, and represent the velocity field here in cylindrical coordinates:

$$v_r = 0, \qquad v_\theta = \frac{\Gamma_o}{2\pi r}, \qquad v_z = 0. \tag{7.200}$$

We have considered the same velocity field in Sec. 7.3 and, as represented in Cartesian coordinates, in Ch. 3.11.10. Now in cylindrical coordinates the gradient operating on a scalar function gives

$$\nabla\phi = \mathbf{v}, \tag{7.201}$$
$$\frac{\partial\phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\mathbf{e}_\theta + \frac{\partial\phi}{\partial z}\mathbf{e}_z = 0\mathbf{e}_r + \frac{\Gamma_o}{2\pi r}\mathbf{e}_\theta + 0\mathbf{e}_z, \tag{7.202}$$
$$\frac{\partial\phi}{\partial r} = 0, \tag{7.203}$$
$$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\Gamma_o}{2\pi r}, \quad \text{so} \quad \phi = \frac{\Gamma_o}{2\pi}\theta + C(r,z), \tag{7.204}$$
$$\frac{\partial\phi}{\partial z} = 0. \tag{7.205}$$

But because the partials of ϕ with respect to r and z are zero, $C(r,z)$ is at most a constant, that we can set to zero without losing any information regarding the velocity itself

$$\phi = \frac{\Gamma_o}{2\pi}\theta. \tag{7.206}$$

In Cartesian coordinates, we have

$$\phi = \frac{\Gamma_o}{2\pi}\arctan\left(\frac{y}{x}\right). \tag{7.207}$$

Lines of constant potential for the ideal vortex centered at the origin are sketched in Fig. 7.15.

7.6.5 Interaction of multiple vortices

Here we will consider the interactions of a large number of vortices by using the method of superposition for the velocity potentials.

If we have two vortices with strengths Γ_1 and Γ_2 centered at arbitrary locations (x_1, y_1) and (x_2, y_2), as sketched in Fig. 7.16, the potential for each is given by

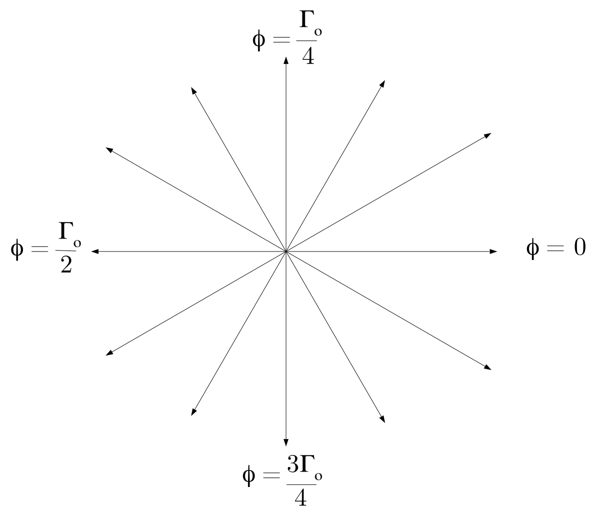

Figure 7.15: Lines of constant potential for ideal irrotational vortex.

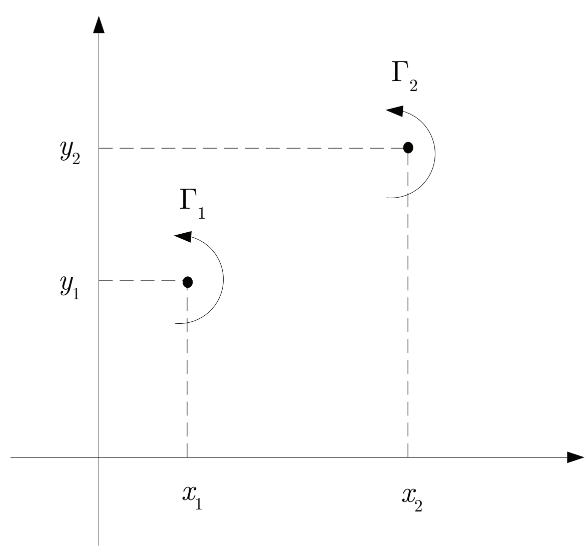

Figure 7.16: Two vortices at arbitrary locations.

$$\phi_1 = \frac{\Gamma_1}{2\pi}\arctan\left(\frac{y-y_1}{x-x_1}\right), \qquad \phi_2 = \frac{\Gamma_2}{2\pi}\arctan\left(\frac{y-y_2}{x-x_2}\right). \tag{7.208}$$

Because the equation governing the velocity potential, $\nabla^2\phi = 0$, is linear we can add the two potentials and still satisfy the overall equation so that

$$\phi = \frac{\Gamma_1}{2\pi}\arctan\left(\frac{y-y_1}{x-x_1}\right) + \frac{\Gamma_2}{2\pi}\arctan\left(\frac{y-y_2}{x-x_2}\right), \tag{7.209}$$

is a legitimate solution. Taking the gradient of ϕ,

$$\begin{aligned}\nabla\phi &= \left(-\left(\frac{\Gamma_1}{2\pi}\right)\frac{y-y_1}{(x-x_1)^2+(y-y_1)^2} - \left(\frac{\Gamma_2}{2\pi}\right)\frac{y-y_2}{(x-x_2)^2+(y-y_2)^2}\right)\mathbf{i}\\ &\quad + \left(\left(\frac{\Gamma_1}{2\pi}\right)\frac{x-x_1}{(x-x_1)^2+(y-y_1)^2} + \left(\frac{\Gamma_2}{2\pi}\right)\frac{x-x_2}{(x-x_2)^2+(y-y_2)^2}\right)\mathbf{j}, \end{aligned}\tag{7.210}$$

so that

$$u(x,y) = -\left(\frac{\Gamma_1}{2\pi}\right)\frac{y-y_1}{(x-x_1)^2+(y-y_1)^2} - \left(\frac{\Gamma_2}{2\pi}\right)\frac{y-y_2}{(x-x_2)^2+(y-y_2)^2}, \tag{7.211}$$

$$v(x,y) = \left(\frac{\Gamma_1}{2\pi}\right)\frac{x-x_1}{(x-x_1)^2+(y-y_1)^2} + \left(\frac{\Gamma_2}{2\pi}\right)\frac{x-x_2}{(x-x_2)^2+(y-y_2)^2}. \tag{7.212}$$

Extending this to a collection of N vortices located at (x_i, y_i) at a given time, we have the following for the velocity field:

$$u(x,y) = -\sum_{i=1}^{N}\left(\frac{\Gamma_i}{2\pi}\right)\frac{y-y_i}{(x-x_i)^2+(y-y_i)^2}, \tag{7.213}$$

$$v(x,y) = \sum_{i=1}^{N}\left(\frac{\Gamma_i}{2\pi}\right)\frac{x-x_i}{(x-x_i)^2+(y-y_i)^2}. \tag{7.214}$$

Now to advect (that is, to move) the kth vortex, we move it with the velocity induced by the other vortices, because vortices advect with the flow. Recalling that the velocity is the time derivative of the position $u_k = dx_k/dt, v_k = dy_k/dt$, we then get the following $2N$ non-linear ordinary differential equations for the $2N$ unknowns, the x and y positions of each of the N vortices:

$$\frac{dx_k}{dt} = \sum_{i=1, i\neq k}^{N} -\left(\frac{\Gamma_i}{2\pi}\right)\frac{y_k-y_i}{(x_k-x_i)^2+(y_k-y_i)^2}, \quad x_k(0) = x_k^o, \quad k = 1,\ldots,N, \tag{7.215}$$

$$\frac{dy_k}{dt} = \sum_{i=1, i\neq k}^{N} \left(\frac{\Gamma_i}{2\pi}\right)\frac{x_k-x_i}{(x_k-x_i)^2+(y_k-y_i)^2}, \quad y_k(0) = y_k^o, \quad k = 1,\ldots,N. \tag{7.216}$$

This set of equations, except for three or fewer point vortices, must be integrated numerically. These equations form what is commonly termed a Biot-Savart[1,2] law. These ordinary differential equations are highly non-linear and typically give rise to chaotic motion of the point vortices. It is a similar calculation to the motion of point masses in a Newtonian gravitational field, except that the essential variation goes as $1/r$ for vortices and $1/r^2$ for Newtonian gravitational fields. Thus, the dynamics are different. Nevertheless just as calculations for large numbers of celestial bodies can give rise to solar systems, clusters of planets, and galaxies, similar "galaxies" of vortices can be predicted with the equations for vortex dynamics.

7.6.6 Pressure field

We have thus far examined essentially only the kinematics of vortices. We have actually used dynamics in our incorporation of the Helmholtz equation and Kelvin's theorem, but their simple results really only justify the use of a simple kinematics. Dynamics asks what are the forces that give rise to the motion. Here, we will assume there is no body force, that the fluid is inviscid, in which case it must be pressure forces that give rise to the motion, and that the fluid is at rest at infinity. We have the proper conditions for which Bernoulli's equation can be used to give the pressure field. We consider two cases, a single stationary point vortex, and a group of N moving point vortices.

7.6.6.1 Single stationary vortex

If we take $p = p_\infty$ in the far field and $f_i = \mathbf{g} = \mathbf{0}$, this steady flow gives us

$$\frac{1}{2}\mathbf{v}^T \cdot \mathbf{v} + \frac{p}{\rho} = \frac{1}{2}\mathbf{v}_\infty^T \cdot \mathbf{v}_\infty + \frac{p_\infty}{\rho}, \tag{7.217}$$

$$\frac{1}{2}\left(\frac{\Gamma_o}{2\pi r}\right)^2 + \frac{p}{\rho} = 0 + \frac{p_\infty}{\rho}, \tag{7.218}$$

$$p(r) = p_\infty - \frac{\rho\Gamma_o^2}{8\pi^2}\frac{1}{r^2}. \tag{7.219}$$

The pressure goes to negative infinity at the origin. This is obviously unphysical. It can be corrected by including viscous effects, that turn out not to substantially alter our main conclusions.

7.6.6.2 Group of N vortices

For a collection of N vortices, the flow is certainly not steady, and we must in general retain the time-dependent velocity potential in Bernoulli's equation, Eq. 6.153), yielding

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi)^T \cdot \nabla\phi + \frac{p}{\rho} = f(t). \tag{7.220}$$

[1] Jean-Baptiste Biot, 1774-1862, Paris-born applied mathematician.

[2] Felix Savart, 1791-1841, French mathematician who worked on magnetic fields and acoustics.

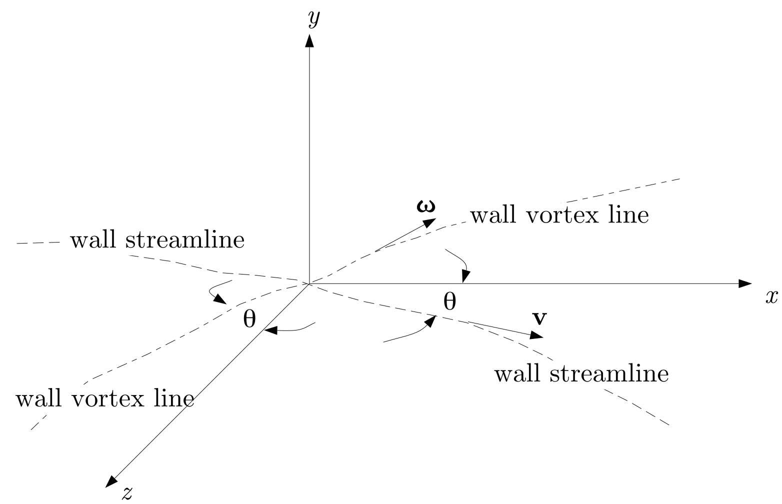

Figure 7.17: Wall streamlines and vortex lines at wall $y = 0$.

Now we require that as $r \to \infty$ that $p \to p_\infty$. We also know that as $r \to \infty$ that $\phi \to 0$, hence $\nabla\phi \to 0$ as well. Hence as $r \to \infty$, we have $p_\infty/\rho = f(t)$. So our final result is

$$p = p_\infty - \frac{1}{2}\rho(\nabla\phi)^T \cdot \nabla\phi - \frac{\partial\phi}{\partial t}. \tag{7.221}$$

So with a knowledge of the velocity field through ϕ, we can determine the pressure field that must have given rise to that velocity field.

7.7 Streamlines and vortex lines at walls

It seems odd that a streamline can be defined at a wall where the velocity is formally zero, but in the neighborhood of the wall, the fluid velocity is small but non-zero. We can extrapolate the position of streamlines near the wall to the wall to define a wall streamline. We shall also consider a so-called *vortex line*, a line everywhere parallel to the vorticity vector, at the wall.

We consider the geometry sketched in Fig. 7.17. Here the $x - z$ plane is locally attached to a wall at $y = 0$, and the y direction is normal to the wall. Wall streamlines and vortex lines are sketched in the figure.

Because the flow satisfies a no-slip condition, we have at the wall

$$u(x, y = 0, z) = 0, \qquad v(x, y = 0, z) = 0, \qquad w(x, y = 0, z) = 0. \tag{7.222}$$

Because of this, partial derivatives of all velocities with respect to either x or z will also be zero at $y = 0$:

$$\left.\frac{\partial u}{\partial x}\right|_{y=0} = \left.\frac{\partial u}{\partial z}\right|_{y=0} = \left.\frac{\partial v}{\partial x}\right|_{y=0} = \left.\frac{\partial v}{\partial z}\right|_{y=0} = \left.\frac{\partial w}{\partial x}\right|_{y=0} = \left.\frac{\partial w}{\partial z}\right|_{y=0} = 0. \tag{7.223}$$

Near the wall, the velocity is near zero, so the Mach number is small, and the flow is well modeled as incompressible. So here, the mass conservation equation implies that $\nabla^T \cdot \mathbf{v} = 0$, so applying this at the wall, we get

$$\underbrace{\left.\frac{\partial u}{\partial x}\right|_{y=0}}_{=0} + \left.\frac{\partial v}{\partial y}\right|_{y=0} + \underbrace{\left.\frac{\partial w}{\partial z}\right|_{y=0}}_{=0} = 0, \qquad \text{so} \tag{7.224}$$

$$\left.\frac{\partial v}{\partial y}\right|_{y=0} = 0. \tag{7.225}$$

Now let us examine the behavior of u, v, and w, as we leave the wall in the y direction. Consider a Taylor series of each:

$$u = \underbrace{u|_{y=0}}_{=0} + \left.\frac{\partial u}{\partial y}\right|_{y=0} y + \frac{1}{2}\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} y^2 + \ldots, \tag{7.226}$$

$$v = \underbrace{v|_{y=0}}_{=0} + \underbrace{\left.\frac{\partial v}{\partial y}\right|_{y=0}}_{=0} y + \frac{1}{2}\left.\frac{\partial^2 v}{\partial y^2}\right|_{y=0} y^2 + \ldots, \tag{7.227}$$

$$w = \underbrace{w|_{y=0}}_{=0} + \left.\frac{\partial w}{\partial y}\right|_{y=0} y + \frac{1}{2}\left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0} y^2 + \ldots. \tag{7.228}$$

So we get

$$u = \left.\frac{\partial u}{\partial y}\right|_{y=0} y + \ldots, \tag{7.229}$$

$$v = \frac{1}{2}\left.\frac{\partial^2 v}{\partial y^2}\right|_{y=0} y^2 + \ldots, \tag{7.230}$$

$$w = \left.\frac{\partial w}{\partial y}\right|_{y=0} y + \ldots. \tag{7.231}$$

Now for streamlines, we must have

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \tag{7.232}$$

For the streamline near the wall, consider just $dx/u = dz/w$, and also tag the streamline as dz_s, so that the slope of the wall streamline, that is the tangent of the angle θ between the wall streamline and the x axis is

$$\tan\theta = \left.\frac{dz_s}{dx}\right|_{y=0} = \lim_{y\to 0}\frac{w}{u} = \frac{\left.\frac{\partial w}{\partial y}\right|_{y=0}}{\left.\frac{\partial u}{\partial y}\right|_{y=0}}. \tag{7.233}$$

Now consider the vorticity vector evaluated at the wall:

$$\omega_x|_{y=0} = \left.\frac{\partial w}{\partial y}\right|_{y=0} - \underbrace{\left.\frac{\partial v}{\partial z}\right|_{y=0}}_{=0} = \left.\frac{\partial w}{\partial y}\right|_{y=0}, \tag{7.234}$$

$$\omega_y|_{y=0} = \underbrace{\left.\frac{\partial u}{\partial z}\right|_{y=0}}_{=0} - \underbrace{\left.\frac{\partial w}{\partial x}\right|_{y=0}}_{=0} = 0, \tag{7.235}$$

$$\omega_z|_{y=0} = \underbrace{\left.\frac{\partial v}{\partial x}\right|_{y=0}}_{=0} - \left.\frac{\partial u}{\partial y}\right|_{y=0} = -\left.\frac{\partial u}{\partial y}\right|_{y=0}. \tag{7.236}$$

So we see that on the wall at $y = 0$, the vorticity vector has no component in the y direction. Hence, it must be parallel to the wall itself. Further, we can then define the slope of the vortex line, dz_v/dx, at the wall in the same fashion as we define a streamline:

$$\left.\frac{dz_v}{dx}\right|_{y=0} = \frac{\omega_z}{\omega_x} = -\frac{\left.\frac{\partial u}{\partial y}\right|_{y=0}}{\left.\frac{\partial w}{\partial y}\right|_{y=0}} = -\frac{1}{\left.\frac{dz_s}{dx}\right|_{y=0}}. \tag{7.237}$$

Because the slope of the vortex line is the negative reciprocal of the slope of the streamline, we have that at a no-slip wall, streamlines are orthogonal to vortex lines. We also note that streamlines are orthogonal to vortex lines for flow with variation in the x and y directions only. For general three-dimensional flows away from walls, we do not expect the two lines to be orthogonal.

This motivates a local coordinate system attached to the wall with the x axis is aligned with the wall streamline and the z axis aligned with the wall vortex line. As before the y axis is normal to the wall. The coordinate system aligned with the wall streamlines and vortex lines is sketched in Fig. 7.18. In the figure we take the direction $\mathbf{n}$ to be normal to the wall.

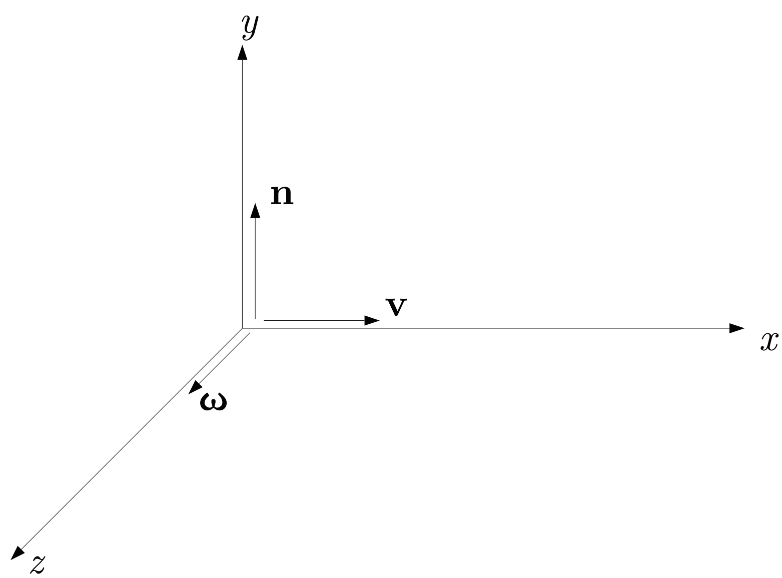

Figure 7.18: Coordinate system aligned with wall streamlines and vortex lines.

Chapter 8

One-dimensional compressible flow

see Shapiro (1953), Chapters 4-8,
see Kuethe and Chow, Chapter 9,
see Yih, Chapter 6,
see Liepmann and Roshko, Chapter 2,
see Whitaker, Chapter 10,
see Courant and Friedrichs,
see White (1986), Chapters 7-9.

This chapter will focus on one-dimensional flow of a compressible fluid. The following topics will be covered:

- thermodynamics of general compressible fluids,
- development of generalized one-dimensional flow equations,
- isentropic flow with area change,
- flow with normal shock waves, and
- the method of characteristics.

We will assume for this chapter:

- $v \equiv 0, w \equiv 0, \partial/\partial y \equiv 0, \partial/\partial z \equiv 0$; one-dimensional flow.

Friction and heat transfer will not be modeled rigorously. Instead, they will be modeled in a fashion that loosely captures the relevant physics and retains analytic tractability. Mathematically, we will not model friction and heat transfer as a classical diffusion processes; consequently, we will consider $\mu \equiv 0$ and $k \equiv 0$. However we will introduce simpler, less rigorous, new terms to model friction and heat transfer. They will will have a different mathematical character. As a consequence, our solutions will not represent rational limiting cases of the more fundamental Navier-Stokes equations. Direct comparison of results using our modeling approximations will never completely agree with equivalent (and expensive) predictions of compressible Navier-Stokes equations. Further, we will ignore the influences of an external body force, $f_i = 0$. Our model will best be seen as an adaptation of the Euler equations of Ch. 6.4. It will have the advantage of yielding rapid and non-intuitive insight into how actual fluids behave under the extreme conditions of flow near or above the speed of sound.

8.1 Thermodynamics of general compressible fluids

Here let us briefly consider compressible fluids with general equations of state. The topic is broad, but we will restrict attention to the speed of sound as well as how a typical thermodynamic analysis is affected by non-ideal equations of state.

8.1.1 Maxwell relation

Let us develop a necessary *Maxwell*[1] *relation*. The topic of Maxwell relations is non-trivial and draws upon a systematic knowledge of thermodynamic potentials. Additional details are given by Powers (2016). We will only develop here what is necessary for our purposes. Consider first the *Helmholtz free energy*, a, defined in terms of other thermodynamic variables as

$$a = e - Ts. \tag{8.1}$$

Then differentiating, we get

$$da = de - T\ ds - s\ dT. \tag{8.2}$$

Now use the Gibbs equation, Eq. (4.161), to eliminate de to get

$$\begin{aligned} da &= \underbrace{(-p\ d\hat{v} + T\ ds)}_{de} - T\ ds - s\ dT, \quad (8.3)\\ &= -p\ d\hat{v} - s\ dT. \quad (8.4) \end{aligned}$$

We can think of a as $a = a(\hat{v}, T)$, because intensive thermodynamic properties are functions of two independent state variables for simple compressible substances. Then calculus tells us

$$da = \left.\frac{\partial a}{\partial \hat{v}}\right|_T d\hat{v} + \left.\frac{\partial a}{\partial T}\right|_{\hat{v}} dT. \tag{8.5}$$

Comparing to Eq. (8.4), we see that we must have

$$-p = \left.\frac{\partial a}{\partial \hat{v}}\right|_T, \quad -s = \left.\frac{\partial a}{\partial T}\right|_{\hat{v}}. \tag{8.6}$$

[1] James Clerk Maxwell, 1831-1879, Scottish mathematical physicist.

Differentiating the first with respect to T and the second with respect to $\hat{v}$ gives

$$-\left.\frac{\partial p}{\partial T}\right|_{\hat{v}} = \frac{\partial^2 a}{\partial T \partial \hat{v}}, \qquad -\left.\frac{\partial s}{\partial \hat{v}}\right|_{T} = \frac{\partial^2 a}{\partial \hat{v} \partial T}. \tag{8.7}$$

Assuming a is continuous and sufficiently differentiable, the order of differentiation of the mixed second partials does not matter, thus giving the Maxwell relation

$$\left.\frac{\partial p}{\partial T}\right|_{\hat{v}} = \left.\frac{\partial s}{\partial \hat{v}}\right|_{T}. \tag{8.8}$$

This is useful because $\partial p/\partial T|_{\hat{v}}$ is available from the thermal equation of state, and it will be required in analysis of the next section where we find caloric equations of state that are consistent with a given thermal equation of state.

8.1.2 Internal energy from thermal equation of state

Let us first find the internal energy $e(T, \hat{v})$ for a general material whose thermal equation of state, $p = p(\hat{v}, T)$, is known. Starting then with e, we have

$$e = e(T, \hat{v}), \tag{8.9}$$

$$de = \left.\frac{\partial e}{\partial T}\right|_{\hat{v}} dT + \left.\frac{\partial e}{\partial \hat{v}}\right|_{T} d\hat{v}. \tag{8.10}$$

Specific heat at constant volume, c_v, for a general material is defined as

$$c_v = \left.\frac{\partial e}{\partial T}\right|_{\hat{v}}. \tag{8.11}$$

Using this, Eq. (8.10) becomes

$$de = c_v\ dT + \left.\frac{\partial e}{\partial \hat{v}}\right|_{T} d\hat{v}. \tag{8.12}$$

Now from the Gibbs equation, Eq. (4.161), we find

$$de = T\ ds - p\ d\hat{v}, \tag{8.13}$$

$$\frac{de}{d\hat{v}} = T\frac{ds}{d\hat{v}} - p, \tag{8.14}$$

$$\left.\frac{\partial e}{\partial \hat{v}}\right|_{T} = T\left.\frac{\partial s}{\partial \hat{v}}\right|_{T} - p. \tag{8.15}$$

Substitute from the Maxwell relation, Eq. (8.8), to get:

$$\left.\frac{\partial e}{\partial \hat{v}}\right|_{T} = T\left.\frac{\partial p}{\partial T}\right|_{\hat{v}} - p. \tag{8.16}$$

Thus, Eq. (8.12) can be rewritten as

$$de = c_v \, dT + \underbrace{\left(T \left. \frac{\partial p}{\partial T} \right|_{\hat{v}} - p \right)}_{\left. \frac{\partial e}{\partial \hat{v}} \right|_T} d\hat{v}. \tag{8.17}$$

Integrating, we then get the caloric equation of state:

$$\int_{e_o}^{e} d\hat{e} = \int_{T_o}^{T} c_v(\hat{T}) \, d\hat{T} + \int_{\hat{v}_o}^{\hat{v}} \left(T \left. \frac{\partial p}{\partial T} \right|_{\tilde{v}} - p \right) \, d\tilde{v}, \tag{8.18}$$

$$e(T, \hat{v}) = e_o + \int_{T_o}^{T} c_v(\hat{T}) \, d\hat{T} + \int_{\hat{v}_o}^{\hat{v}} \left(T \left. \frac{\partial p}{\partial T} \right|_{\tilde{v}} - p \right) d\tilde{v}. \tag{8.19}$$

This is the caloric equation of state that is thermodynamically consistent with the given thermal equation of state.

Example 8.1

Find a general expression for $e(T, \hat{v})$ if we have an ideal gas:

$$P(T, \hat{v}) = \frac{RT}{\hat{v}}. \tag{8.20}$$

Proceed as follows:

$$\left. \frac{\partial p}{\partial T} \right|_v = \frac{R}{\hat{v}}, \tag{8.21}$$

$$T \left. \frac{\partial p}{\partial T} \right|_{\hat{v}} - p = \frac{RT}{\hat{v}} - p, \tag{8.22}$$

$$= \frac{RT}{\hat{v}} - \frac{RT}{\hat{v}} = 0. \tag{8.23}$$

Thus, e is

$$e(T) = e_o + \int_{T_o}^{T} c_v(\hat{T}) \, d\hat{T}. \tag{8.24}$$

Iff c_v is a constant, then we have CPIG, and the caloric equation of state is

$$e(T) = e_o + c_v(T - T_o). \tag{8.25}$$

Example 8.2

Find a general expression for $e(T, \hat{v})$ for a van der Waals[2] gas:

$$p(T, \hat{v}) = \frac{RT}{\hat{v} - b} - \frac{a}{\hat{v}^2}. \tag{8.26}$$

[2]Johannes Diderik van der Waals, 1837-1923, Dutch thermodynamicist and 1910 Nobel laureate in physics for his work in developing his celebrated equation of state.

Proceed as before:

$$\left.\frac{\partial p}{\partial T}\right|_{\hat{v}} = \frac{R}{\hat{v}-b}, \tag{8.27}$$

$$T\left.\frac{\partial p}{\partial T}\right|_{\hat{v}} - p = \frac{RT}{\hat{v}-b} - p, \tag{8.28}$$

$$= \frac{RT}{\hat{v}-b} - \left(\frac{RT}{\hat{v}-b} - \frac{a}{\hat{v}^2}\right), \tag{8.29}$$

$$= \frac{a}{\hat{v}^2}. \tag{8.30}$$

Thus, the caloric equation of state for e is

$$e(T,\hat{v}) = e_o + \int_{T_o}^{T} c_v(\hat{T})\, d\hat{T} + \int_{\hat{v}_o}^{\hat{v}} \frac{a}{\tilde{v}^2}\, d\tilde{v}, \tag{8.31}$$

$$= e_o + \int_{T_o}^{T} c_v(\hat{T})\, d\hat{T} + a\left(\frac{1}{\hat{v}_o} - \frac{1}{\hat{v}}\right). \tag{8.32}$$

If c_v is constant, the caloric equation of state for the van der Waals gas reduces to

$$e(T,\hat{v}) = e_o + c_v(T - T_o) + a\left(\frac{1}{\hat{v}_o} - \frac{1}{\hat{v}}\right). \tag{8.33}$$

Example 8.3

A van der Waals gas with $R = 200$ J/kg/K, $a = 150$ Pa m^6/kg^2, $b = 0.001$ m^3/kg, $c_v = (350 + 0.2(T - 300 \text{ K}))$ J/kg/K begins at $T_1 = 300$ K, $p_1 = 10^5$ Pa. It is isothermally compressed to state 2 where $p_2 = 10^6$ Pa. It is then isochorically heated to state 3 where $T_3 = 1000$ K. Find w_{13}, q_{13}, and $s_3 - s_1$. Assume the surroundings are at 1000 K.

Recall

$$p = \frac{RT}{\hat{v}-b} - \frac{a}{\hat{v}^2}. \tag{8.34}$$

So at state 1

$$10^5 \text{ Pa} = \frac{\left(200 \frac{\text{J}}{\text{kg K}}\right)(300 \text{ K})}{\hat{v}_1 - 0.001 \frac{\text{m}^3}{\text{kg}}} - \frac{150 \text{ Pa m}^6/\text{kg}^2}{\hat{v}_1^2}. \tag{8.35}$$

or expanding, and ignoring the units

$$-0.15 + 150\hat{v} - 60100\hat{v}^2 + 100000\hat{v}^3 = 0. \tag{8.36}$$

This is a cubic equation, and it thus has three solutions:

$$\hat{v}_1 = 0.598 \ \frac{\text{m}^3}{\text{kg}}, \tag{8.37}$$

$$\hat{v}_1 = 0.00125 - 0.0097i \ \frac{\text{m}^3}{\text{kg}}, \qquad \text{not physical}, \tag{8.38}$$

$$\hat{v}_1 = 0.00125 + 0.0097i \ \frac{\text{m}^3}{\text{kg}}, \qquad \text{not physical}. \tag{8.39}$$

Now at state 2 we know p_2 and T_2 so we can determine $\hat{v}_2$

$$10^6 \text{ Pa} = \frac{\left(200 \ \frac{\text{J}}{\text{kg K}}\right)(300 \text{ K})}{\hat{v}_2 - 0.001 \ \frac{\text{m}^3}{\text{kg}}} - \frac{150 \text{ Pa m}^6/\text{kg}^2}{\hat{v}_2^2}. \tag{8.40}$$

The physical solution is $\hat{v}_2 = 0.0585 \text{ m}^3/\text{kg}$. Now at state 3 we know $\hat{v}_3 = \hat{v}_2$ and T_3. Determine p_3:

$$p_3 = \frac{\left(200 \ \frac{\text{J}}{\text{kg K}}\right)(1000 \text{ K})}{0.0585 \ \frac{\text{m}^3}{\text{kg}} - 0.001 \ \frac{\text{m}^3}{\text{kg}}} - \frac{150 \text{ Pa m}^6/\text{kg}^2}{\left(0.0585 \ \frac{\text{m}^3}{\text{kg}}\right)^2} = 3434430 \text{ Pa} = 3.43 \times 10^6 \text{ Pa}. \tag{8.41}$$

Now $w_{13} = w_{12} + w_{23} = \int_1^2 p \ d\hat{v} + \int_2^3 p \ d\hat{v} = \int_1^2 p \ d\hat{v}$ because $2-3$ is at constant volume. So

$$w_{13} = \int_{\hat{v}_1}^{\hat{v}_2} \left(\frac{RT}{\hat{v} - b} - \frac{a}{\hat{v}^2} \right) \ d\hat{v}, \tag{8.42}$$

$$= RT_1 \int_{\hat{v}_1}^{\hat{v}_2} \frac{d\hat{v}}{\hat{v} - b} - a \int_{\hat{v}_1}^{\hat{v}_2} \frac{d\hat{v}}{\hat{v}^2}, \tag{8.43}$$

$$= RT_1 \ln \left(\frac{\hat{v}_2 - b}{\hat{v}_1 - b} \right) + a \left(\frac{1}{\hat{v}_2} - \frac{1}{\hat{v}_1} \right), \tag{8.44}$$

$$= 200 \times 300 \ln \left(\frac{0.0585 - 0.001}{0.598 - 0.001} \right) + 150 \left(\frac{1}{0.0585} - \frac{1}{0.598} \right), \tag{8.45}$$

$$= -140408 \ \frac{\text{J}}{\text{kg}} + 2313 \ \frac{\text{J}}{\text{kg}}, \tag{8.46}$$

$$= -138095 \ \frac{\text{J}}{\text{kg}}, \tag{8.47}$$

$$= -138 \ \frac{\text{kJ}}{\text{kg}}. \tag{8.48}$$

The gas is compressed, so the work is negative. Because e is a state property:

$$e_3 - e_1 = \int_{T_1}^{T_3} c_v(T) \ dT + a \left(\frac{1}{v_1} - \frac{1}{v_3} \right). \tag{8.49}$$

Now

$$c_v = 350 + 0.2(T - 300) = 290 + \frac{1}{5}T, \tag{8.50}$$

so

$$e_3 - e_1 = \int_{T_1}^{T_3} \left(290 + \frac{1}{5}T \right) \ dT + a \left(\frac{1}{v_1} - \frac{1}{v_3} \right), \tag{8.51}$$

$$
\begin{aligned}
&= 290\,(T_3 - T_1) + \frac{1}{10}\left(T_3^2 - T_1^2\right) + a\left(\frac{1}{v_1} - \frac{1}{v_3}\right), && (8.52)\\
&= 290\,(1000 - 300) + \frac{1}{10}\left(1000^2 - 300^2\right) + 150\left(\frac{1}{0.598} - \frac{1}{0.0585}\right), && (8.53)\\
&= 203000 + 91000 - 2313, && (8.54)\\
&= 291687\ \frac{\text{J}}{\text{kg}}, && (8.55)\\
&= 292\ \frac{\text{kJ}}{\text{kg}}. && (8.56)
\end{aligned}
$$

Now from the first law, we have

$$
\begin{aligned}
e_3 - e_1 &= q_{13} - w_{13}, && (8.57)\\
q_{13} &= e_3 - e_1 + w_{13}, && (8.58)\\
&= 292 - 138, && (8.59)\\
&= 154\ \frac{\text{kJ}}{\text{kg}}. && (8.60)
\end{aligned}
$$

The heat transfer is positive as heat was added to the system.

Now find the entropy change. Manipulate the Gibbs equation, Eq. (4.162):

$$
\begin{aligned}
T\ ds &= de + p\ d\hat{v}, && (8.61)\\
ds &= \frac{1}{T}de + \frac{p}{T}\ d\hat{v}, && (8.62)\\
&= \frac{1}{T}\left(c_v(T)\ dT + \frac{a}{\hat{v}^2}\ d\hat{v}\right) + \frac{p}{T}\ d\hat{v}, && (8.63)\\
&= \frac{1}{T}\left(c_v(T)\ dT + \frac{a}{\hat{v}^2}\ d\hat{v}\right) + \frac{1}{T}\left(\frac{RT}{v-b} - \frac{a}{\hat{v}^2}\right)\ d\hat{v}, && (8.64)\\
&= \frac{c_v(T)}{T}\ dT + \frac{R}{\hat{v}-b}\ d\hat{v}, && (8.65)\\
s_3 - s_1 &= \int_{T_1}^{T_3} \frac{c_v(T)}{T}\ dT + R\ln\frac{\hat{v}_3 - b}{\hat{v}_1 - b}, && (8.66)\\
&= \int_{300}^{1000}\left(\frac{290}{T} + \frac{1}{5}\right)\ dT + R\ln\frac{\hat{v}_3 - b}{\hat{v}_1 - b}, && (8.67)\\
&= 290\ln\frac{1000}{300} + \frac{1}{5}(1000-300) + 200\ln\frac{0.0585 - 0.001}{0.598 - 0.001}, && (8.68)\\
&= 349 + 140 - 468, && (8.69)\\
&= 21\ \frac{\text{J}}{\text{kg K}} = 0.021\ \frac{\text{kJ}}{\text{kg K}}. && (8.70)
\end{aligned}
$$

Is the second law satisfied for each portion of the process? First look at $1 \to 2$:

$$
\begin{aligned}
e_2 - e_1 &= q_{12} - w_{12}, && (8.71)\\
q_{12} &= e_2 - e_1 + w_{12}, && (8.72)\\
&= \left(\int_{T_1}^{T_2} c_v(T)\ dT + a\left(\frac{1}{\hat{v}_1} - \frac{1}{\hat{v}_2}\right)\right) + \left(RT_1\ln\left(\frac{\hat{v}_2 - b}{\hat{v}_1 - b}\right) + a\left(\frac{1}{\hat{v}_2} - \frac{1}{\hat{v}_1}\right)\right). && (8.73)
\end{aligned}
$$

Because $T_1 = T_2$ and that fact that we can cancel the terms in a, we get

$$
q_{12} = RT_1\ln\left(\frac{\hat{v}_2 - b}{\hat{v}_1 - b}\right) = 200 \times 300\ln\left(\frac{0.0585 - 0.001}{0.598 - 0.001}\right) = -140408\ \frac{\text{J}}{\text{kg}}. \qquad (8.74)
$$

Because the process is isothermal, we find

$$s_2 - s_1 = R \ln\left(\frac{\hat{v}_2 - b}{\hat{v}_1 - b}\right), \tag{8.75}$$

$$= 200 \ln\left(\frac{0.0585 - 0.001}{0.598 - 0.001}\right), \tag{8.76}$$

$$= -468.0\ \frac{\text{J}}{\text{kg K}}. \tag{8.77}$$

Entropy *drops* because heat was transferred *out* of the system.

Check the second law. Note that in this portion of the process in which the heat is transferred out of the system, that the surroundings must have $T_{surr} \leq 300$ K. For this portion of the process let us take $T_{surr} = 300$ K.

$$s_2 - s_1 \geq \frac{q_{12}}{T}? \tag{8.78}$$

$$-468.0\ \frac{\text{J}}{\text{kg K}} \geq \frac{-140408\ \frac{\text{J}}{\text{kg}}}{300\ \text{K}}, \tag{8.79}$$

$$-468.0\ \frac{\text{J}}{\text{kg K}} \geq -468.0\ \frac{\text{J}}{\text{kg K}}, \qquad \text{ok.} \tag{8.80}$$

Next look at $2 \rightarrow 3$

$$q_{23} = e_3 - e_2 + w_{23}, \tag{8.81}$$

$$= \left(\int_{T_2}^{T_3} c_v(T)\ dT + a\left(\frac{1}{\hat{v}_2} - \frac{1}{\hat{v}_3}\right)\right) + \left(\int_{\hat{v}_2}^{\hat{v}_3} p\ d\hat{v}\right). \tag{8.82}$$

Because the process is isochoric, we have

$$q_{23} = \int_{T_2}^{T_3} c_v(T)\ dT, \tag{8.83}$$

$$= \int_{300}^{1000} \left(290 + \frac{T}{5}\right) dT, \tag{8.84}$$

$$= 294000\ \frac{\text{J}}{\text{K}}. \tag{8.85}$$

Now look at the entropy change for the isochoric process:

$$s_3 - s_2 = \int_{T_2}^{T_3} \frac{c_v(T)}{T}\ dT, \tag{8.86}$$

$$= \int_{T_2}^{T_3} \left(\frac{290}{T} + \frac{1}{5}\right)\ dT, \tag{8.87}$$

$$= 290 \ln \frac{1000}{300} + \frac{1}{5}(1000 - 300), \tag{8.88}$$

$$= 489\ \frac{\text{J}}{\text{kg K}}. \tag{8.89}$$

Entropy *rises* because heat was transferred *into* the system.

In order to transfer heat into the system we must have a different thermal reservoir. This one must have $T_{surr} \geq 1000$ K. Assume here that the heat transfer was from a reservoir held at 1000 K to assess the influence of the second law.

$$s_3 - s_2 \geq \frac{q_{23}}{T}? \tag{8.90}$$

$$489 \frac{\text{J}}{\text{kg K}} \geq \frac{294000 \frac{\text{J}}{\text{kg}}}{1000 \text{ K}}, \tag{8.91}$$

$$\geq 294 \frac{\text{J}}{\text{kg K}}, \qquad \text{ok.} \tag{8.92}$$

8.1.3 Sound speed

Let us find the sound speed $c(T, \rho)$ for a general material with known thermal equation of state $p(\rho, T)$. Later in Sec. 8.2.3, we will see how one can obtain the sound speed from a general caloric equation of state $e(p, \rho)$. At this point, c is best thought of as a thermodynamic property. Later in Sec. 8.4.6, we will see how it represents the speed of propagation of small acoustic disturbances. Let us define c as

$$c = \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_s}, \qquad c^2 = \left.\frac{\partial p}{\partial \rho}\right|_s. \tag{8.93}$$

Use the Gibbs relation, Eq. (4.161):

$$T\ ds = de + p\ d\hat{v}. \tag{8.94}$$

Now use Eq. (8.17) to eliminate de:

$$T\ ds = \underbrace{\left(c_v\ dT + \left(T \left.\frac{\partial p}{\partial T}\right|_{\hat{v}} - p\right)\ d\hat{v}\right)}_{de} + p\ d\hat{v}, \tag{8.95}$$

$$= c_v\ dT + T \left.\frac{\partial p}{\partial T}\right|_{\hat{v}}\ d\hat{v}, \tag{8.96}$$

$$= c_v\ dT - \frac{T}{\rho^2} \left.\frac{\partial p}{\partial T}\right|_{\rho}\ d\rho. \tag{8.97}$$

Because $p = p(T, \hat{v})$, we can also say $p = p(T, \rho)$, and then we get by calculus

$$dp = \left.\frac{\partial p}{\partial T}\right|_{\rho}\ dT + \left.\frac{\partial p}{\partial \rho}\right|_{T}\ d\rho, \tag{8.98}$$

$$dT = \frac{dp - \left.\frac{\partial p}{\partial \rho}\right|_T\ d\rho}{\left.\frac{\partial p}{\partial T}\right|_{\rho}}. \tag{8.99}$$

Thus substituting for dT in Eq. (8.97), we find

$$T\ ds = c_v \left(\frac{dp - \left.\frac{\partial p}{\partial \rho}\right|_T\ d\rho}{\left.\frac{\partial p}{\partial T}\right|_\rho} \right) - \frac{T}{\rho^2} \left.\frac{\partial p}{\partial T}\right|_\rho\ d\rho. \tag{8.100}$$

Grouping terms in dp and $d\rho$, we get

$$T\ ds = \left(\frac{c_v}{\left.\frac{\partial p}{\partial T}\right|_\rho} \right)\ dp - \left(c_v \frac{\left.\frac{\partial p}{\partial \rho}\right|_T}{\left.\frac{\partial p}{\partial T}\right|_\rho} + \frac{T}{\rho^2} \left.\frac{\partial p}{\partial T}\right|_\rho \right)\ d\rho. \tag{8.101}$$

For the isentropic sound speed, we must have $ds \equiv 0$; we thus obtain

$$c^2 = \left.\frac{\partial p}{\partial \rho}\right|_s = \frac{1}{c_v} \left.\frac{\partial p}{\partial T}\right|_\rho \left(c_v \frac{\left.\frac{\partial p}{\partial \rho}\right|_T}{\left.\frac{\partial p}{\partial T}\right|_\rho} + \frac{T}{\rho^2} \left.\frac{\partial p}{\partial T}\right|_\rho \right), \tag{8.102}$$

$$= \left.\frac{\partial p}{\partial \rho}\right|_T + \frac{T}{c_v \rho^2} \left(\left.\frac{\partial p}{\partial T}\right|_\rho \right)^2. \tag{8.103}$$

So the isentropic sound speed for a general thermal equation of state is

$$c(T,\rho) = \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_s} = \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_T + \frac{T}{c_v \rho^2} \left(\left.\frac{\partial p}{\partial T}\right|_\rho \right)^2}. \tag{8.104}$$

Without the benefit of modern understanding of thermodynamics, in 1687 Newton[3] concluded the speed of sound was $\sqrt{\partial p/\partial \rho|_T}$. So with benefit of the known Boyle's[4] Law, Newton could predict an isothermal sound speed of $\sqrt{RT}$. We might call that the isothermal sound speed as opposed to the isentropic sound speed. He then went to considerable effort to measure the sound speed, but could not reconcile the discrepancy of his measurements with his isothermal theory. Newton's approach was corrected by Laplace in 1816, who generated what amounts to our adiabatic prediction, long before notions of thermodynamics were settled. Laplace's notions rested on an uncertain theoretical foundation; he in fact adjusted his theory often, and it was not until thermodynamics was well established several decades later that our understanding of sound waves was clarified. The interested reader can consult Finn[5].

[3] I. Newton, 1934, *Principia*, Cajori's revised translation of Motte's 1729 translation, U. California Press, Berkeley.

[4] Robert Boyle, 1627-1691, Anglo-Irish natural philosopher and landlord of the author's County Waterford ancestors.

[5] B. S. Finn, 1964, "Laplace and the speed of sound," *Isis*, 55(1): 7-19.

Example 8.4

Find the sound speed for an ideal gas:

$$p(T,\rho) = \rho R T. \tag{8.105}$$

The necessary partials are

$$\left.\frac{\partial p}{\partial \rho}\right|_T = RT, \qquad \left.\frac{\partial p}{\partial T}\right|_\rho = \rho R. \tag{8.106}$$

so

$$c(T,\rho) = \sqrt{RT + \frac{T}{c_v \rho^2}(\rho R)^2}, \tag{8.107}$$

$$= \sqrt{RT + \frac{R^2 T}{c_v}}, \tag{8.108}$$

$$= \sqrt{RT\left(1 + \frac{R}{c_v}\right)}, \tag{8.109}$$

$$= \sqrt{RT\left(1 + \frac{c_P - c_v}{c_v}\right)}, \tag{8.110}$$

$$= \sqrt{RT\left(\frac{c_v + c_P - c_v}{c_v}\right)}, \tag{8.111}$$

$$= \sqrt{\gamma R T}. \tag{8.112}$$

Sound speed depends on temperature alone for the ideal gas.

Example 8.5

Find the sound speed of a so-called *virial gas*:

$$p(T,\rho) = \rho R T\,(1 + b\rho)\,. \tag{8.113}$$

The necessary partials are

$$\left.\frac{\partial p}{\partial \rho}\right|_T = RT + 2b\rho RT, \qquad \left.\frac{\partial p}{\partial T}\right|_\rho = \rho R\,(1 + b\rho)\,. \tag{8.114}$$

Thus,

$$c(T,\rho) = \sqrt{RT + 2b\rho RT + \frac{T}{c_v \rho^2}\left(\rho R\,(1+b\rho)\right)^2}, \tag{8.115}$$

$$= \sqrt{RT\left(1 + 2b\rho + \frac{R}{c_v}(1+b\rho)^2\right)}. \tag{8.116}$$

The sound speed of a virial gas depends on both temperature and density.

In Sec. 8.6.1, we shall need to consider $p = p(\rho, s)$, taking advantage of the fact that in thermodynamics, one can cast any intensive thermodynamic variable in terms of two other independent intensive thermodynamic variables.

Example 8.6

For a CPIG, find $p = p(\rho, s)$.

Start with the Gibbs equation, Eq. (4.161), $T\ ds = de + p\ d\hat{v}$. For a CPIG, we have $de = c_v\ dT$, $p = RT/\hat{v}$, so the Gibbs equation reduces to

$$ds = c_v \frac{dT}{T} + R\frac{d\hat{v}}{\hat{v}}, \tag{8.117}$$

$$= c_v \frac{dT}{T} + R\frac{-\frac{1}{\rho^2}\ d\rho}{\frac{1}{\rho}}, \tag{8.118}$$

$$= c_v \frac{dT}{T} - R\frac{d\rho}{\rho}. \tag{8.119}$$

Now because $p = \rho RT$, we also have

$$dp = \rho R\ dT + RT\ d\rho, \tag{8.120}$$

$$\frac{dp}{p} = \frac{dT}{T} + \frac{d\rho}{\rho}, \tag{8.121}$$

$$\frac{dT}{T} = \frac{dp}{p} - \frac{d\rho}{\rho}. \tag{8.122}$$

Substitute this into Eq. (8.119) to get

$$ds = c_v\left(\frac{dp}{p} - \frac{d\rho}{\rho}\right) - R\frac{d\rho}{\rho}, \tag{8.123}$$

$$= c_v\frac{dp}{p} - (c_v + R)\frac{d\rho}{\rho}, \tag{8.124}$$

$$= c_v\frac{dp}{p} - c_p\frac{d\rho}{\rho}, \tag{8.125}$$

$$\frac{ds}{c_v} = \frac{dp}{p} - \gamma\frac{d\rho}{\rho}, \tag{8.126}$$

$$\frac{s - s_o}{c_v} = \ln\frac{p}{p_o} - \gamma\ln\frac{\rho}{\rho_o}, \tag{8.127}$$

$$\ln\frac{p}{p_o} = \ln\left(\frac{\rho}{\rho_o}\right)^\gamma + \frac{s - s_o}{c_v}, \tag{8.128}$$

$$p(\rho, s) = p_o\left(\frac{\rho}{\rho_o}\right)^\gamma \exp\left(\frac{s - s_o}{c_v}\right). \tag{8.129}$$

8.2 Generalized one-dimensional equations

Now let us introduce the conservation principles of mass, momentum, and energy to our thermodynamics to better understand the dynamics of compressible flow. Here we will derive in a conventional way the one-dimensional equations of flow with area change. The development is guided by Shapiro (1953), Ch. 8. Although for the geometry we use, it will appear that we should be using at least two-dimensional equations, our results will be approximately correct when we interpret them as an average value at a given x location. Our results will be valid as long as the area changes slowly relative to how fast the flow can adjust to area changes.

We can be guided by our equations from Ch. 6.2. However, the *ad hoc* nature of friction and heat transfer commonly we will here employ makes a fresh derivation essential. The flow we wish to consider, flow with area change, heat transfer, and wall friction, is illustrated by the following sketch of a control volume, Fig. 8.1.

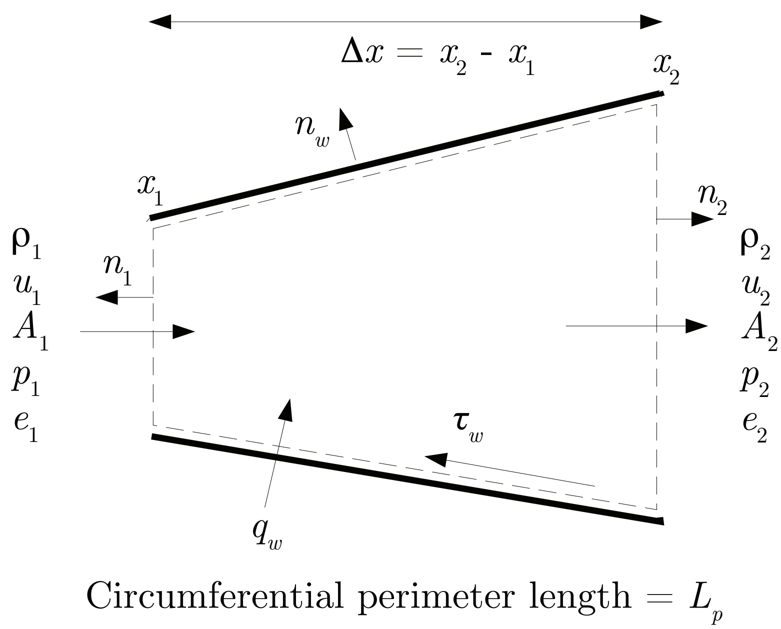

Figure 8.1: Control volume sketch for one-dimensional compressible flow with area change, heat transfer, and wall friction.

For this flow, we will adopt the following conventions

- surface 1 and 2 are open and allow fluxes of mass, momentum, and energy,
- surface w is a closed wall; no mass flux through the wall is allowed,
- external heat flux q_w (energy/area/time: W/m^2) *through* the wall is allowed: q_w is a known, *fixed* parameter,

- diffusive, longitudinal heat transfer is ignored, $q_x = 0$; thus, thermal conductivity $k = 0$,
- wall shear τ_w (force/area: N/m^2) is allowed: τ_w is a known, *fixed* parameter,
- diffusive viscous stress is not allowed, $\tau_{xx} = 0$; thus, viscosity $\mu = 0$, and
- cross-sectional area is a known, *fixed* function: $A(x)$.

8.2.1 Mass

Take the over-bar notation to indicate a volume-averaged quantity. The amount of mass in a control volume after a time increment Δt is equal to the original amount of mass plus that which came in minus that which left:

$$\bar{\rho}\bar{A}\Delta x\big|_{t+\Delta t} = \bar{\rho}\bar{A}\Delta x\big|_{t} + \rho_1 A_1 \left(u_1 \Delta t\right) - \rho_2 A_2 \left(u_2 \Delta t\right). \tag{8.130}$$

Rearrange and divide by $\Delta x \Delta t$:

$$\frac{\bar{\rho}\bar{A}\big|_{t+\Delta t} - \bar{\rho}\bar{A}\big|_{t}}{\Delta t} + \frac{\rho_2 A_2 u_2 - \rho_1 A_1 u_1}{\Delta x} = 0. \tag{8.131}$$

Taking the limit as $\Delta t \to 0, \Delta x \to 0$, we get

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A u) = 0. \tag{8.132}$$

If the flow is steady, then

$$\begin{aligned}
\frac{d}{dx}\left(\rho A u\right) &= 0, && (8.133)\\
Au\frac{d\rho}{dx} + \rho u \frac{dA}{dx} + \rho A \frac{du}{dx} &= 0, && (8.134)\\
\frac{1}{\rho}\frac{d\rho}{dx} + \frac{1}{A}\frac{dA}{dx} + \frac{1}{u}\frac{du}{dx} &= 0. && (8.135)
\end{aligned}$$

Now integrate Eq. (8.133) from x_1 to x_2 to get

$$\begin{aligned}
\int_{x_1}^{x_2} \frac{d}{dx}\left(\rho A u\right) dx &= \int_{x_1}^{x_2} 0 \; dx, && (8.136)\\
\int_1^2 d\left(\rho A u\right) &= 0, && (8.137)\\
\rho_2 u_2 A_2 - \rho_1 u_1 A_1 &= 0, && (8.138)\\
\rho_2 u_2 A_2 &= \rho_1 u_1 A_1 \equiv \dot{m}. && (8.139)
\end{aligned}$$

Here $\dot{m}$ is the *mass flux* with units kg/s. For steady flow, it is a constant.

8.2.2 Linear momentum

Newton's second law of motion says the time rate of change of linear momentum of a body equals the sum of the forces acting on the body. In the x direction this is roughly as follows, for a system:

$$\frac{d}{dt}(mu) = \sum F_x. \tag{8.140}$$

In discrete form this would be

$$\frac{mu|_{t+\Delta t} - mu|_t}{\Delta t} = \sum F_x, \tag{8.141}$$

$$mu|_{t+\Delta t} = mu|_t + \left(\sum F_x\right)\Delta t. \tag{8.142}$$

For a control volume containing fluid, we must also account for the momentum that enters and leaves the control volume. The amount of momentum in a control volume after a time increment Δt is equal to the original amount of momentum plus that which came in minus that which left plus that introduced by the forces acting on the control volume. Note that the

- pressure force at surface 1 *pushes* the fluid,
- pressure force at surface 2 *restrains* the fluid,
- force due to the reaction of the wall to the pressure force *pushes* the fluid if the area change is positive, and
- force due to the reaction of the wall to the shear force *restrains* the fluid.

We write the linear momentum principle as

$$\begin{aligned}\left(\bar{\rho}\bar{A}\Delta x\right)\bar{u}\big|_{t+\Delta t} = & \left(\bar{\rho}\bar{A}\Delta x\right)\bar{u}\big|_t \\ & + \left(\rho_1 A_1\left(u_1\Delta t\right)\right)u_1 \\ & - \left(\rho_2 A_2\left(u_2\Delta t\right)\right)u_2 \\ & + \left(p_1 A_1\right)\Delta t - \left(p_2 A_2\right)\Delta t \\ & + \left(\bar{p}\left(A_2 - A_1\right)\right)\Delta t \\ & - \left(\tau_w \bar{L}_p \Delta x\right)\Delta t.\end{aligned} \tag{8.143}$$

Rearrange and divide by $\Delta x \Delta t$ to get

$$\frac{\bar{\rho}\bar{A}\bar{u}\big|_{t+\Delta t} - \bar{\rho}\bar{A}\bar{u}\big|_t}{\Delta t} + \frac{\rho_2 A_2 u_2^2 - \rho_1 A_1 u_1^2}{\Delta x} = -\frac{p_2 A_2 - p_1 A_1}{\Delta x} + \bar{p}\frac{A_2 - A_1}{\Delta x} - \tau_w \bar{L}_p. \tag{8.144}$$

In the limit $\Delta x \to 0, \Delta t \to 0$ we get

$$\frac{\partial}{\partial t}(\rho A u) + \frac{\partial}{\partial x}\left(\rho A u^2\right) = -\frac{\partial}{\partial x}(pA) + p\frac{\partial A}{\partial x} - \tau_w L_p. \tag{8.145}$$

In steady state, we find

$$\frac{d}{dx}\left(\rho A u^2\right) = -\frac{d}{dx}(pA) + p\frac{dA}{dx} - \tau_w L_p, \tag{8.146}$$

$$\rho A u\frac{du}{dx} + u\underbrace{\frac{d}{dx}(\rho A u)}_{=0} = -p\frac{dA}{dx} - A\frac{dp}{dx} + p\frac{dA}{dx} - \tau_w L_p, \tag{8.147}$$

$$\rho u\frac{du}{dx} = -\frac{dp}{dx} - \tau_w\frac{L_p}{A}, \tag{8.148}$$

$$\rho u\ du + dp = -\tau_w\frac{L_p}{A}\ dx, \tag{8.149}$$

$$du + \frac{1}{\rho u}\ dp = -\tau_w\frac{L_p}{\dot{m}}\ dx, \tag{8.150}$$

$$\rho\ d\left(\frac{u^2}{2}\right) + dp = -\tau_w\frac{L_p}{A}\ dx. \tag{8.151}$$

If there is no wall shear, then Eq. (8.151) reduces to

$$dp = -\rho\ d\left(\frac{u^2}{2}\right), \qquad \text{no wall shear.} \tag{8.152}$$

An increase in velocity magnitude decreases the pressure. We can equivalently rewrite Eq. (8.149) for $\tau_w = 0$ as

$$\rho u\frac{du}{dx} + \frac{dp}{dx} = 0, \qquad \text{no wall shear.} \tag{8.153}$$

For flow with no area change, $dA/dx = 0$. For that limit, along with the limit of $\tau_w = 0$, Eq. (8.146) reduces to

$$\frac{d}{dx}\left(\rho A u^2\right) = -\frac{d}{dx}\left(pA\right), \tag{8.154}$$

$$A\frac{d}{dx}\left(\rho u^2 + p\right) = 0, \tag{8.155}$$

$$\frac{d}{dx}\left(\rho u^2 + p\right) = 0, \tag{8.156}$$

$$\rho u^2 + p = \rho_1 u_1^2 + p_1, \qquad \text{no wall shear, no area change.} \tag{8.157}$$

8.2.3 Energy

The first law of thermodynamics states that the change of total energy of a body equals the heat transferred to the body minus the work done by the body:

$$E_2 - E_1 = Q - W, \tag{8.158}$$

$$E_2 = E_1 + Q - W. \tag{8.159}$$

So for our control volume this becomes the following when we also account for the energy flux in and out of the control volume in addition to the work and heat transfer:

$$\begin{aligned}\left(\bar{\rho}\bar{A}\Delta x\right)\left(\bar{e}+\frac{\bar{u}^2}{2}\right)\bigg|_{t+\Delta t} &= \left(\bar{\rho}\bar{A}\Delta x\right)\left(\bar{e}+\frac{\bar{u}^2}{2}\right)\bigg|_{t} \\ &+\rho_1 A_1\left(u_1\Delta t\right)\left(e_1+\frac{u_1^2}{2}\right)-\rho_2 A_2\left(u_2\Delta t\right)\left(e_2+\frac{u_2^2}{2}\right) \\ &+q_w\left(\bar{L}_p\Delta x\right)\Delta t+\left(p_1A_1\right)\left(u_1\Delta t\right)-\left(p_2A_2\right)\left(u_2\Delta t\right). \quad (8.160)\end{aligned}$$

Note:

- the mean pressure times area difference does no work because it is acting on a stationary boundary, and
- the work done by the wall shear force is not included.[6]

Rearrange and divide by $\Delta t\Delta x$:

$$\frac{\bar{\rho}\bar{A}\left(\bar{e}+\frac{\bar{u}^2}{2}\right)\Big|_{t+\Delta t}-\bar{\rho}\bar{A}\left(\bar{e}+\frac{\bar{u}^2}{2}\right)\Big|_{t}}{\Delta t}+\frac{\rho_2A_2u_2\left(e_2+\frac{u_2^2}{2}+\frac{p_2}{\rho_2}\right)-\rho_1A_1u_1\left(e_1+\frac{u_1^2}{2}+\frac{p_1}{\rho_1}\right)}{\Delta x} = q_w\bar{L}_p. \quad (8.161)$$

In differential form as $\Delta x\to 0, \Delta t\to 0$

$$\frac{\partial}{\partial t}\left(\rho A\left(e+\frac{u^2}{2}\right)\right)+\frac{\partial}{\partial x}\left(\rho Au\left(e+\frac{u^2}{2}+\frac{p}{\rho}\right)\right)=q_wL_p. \quad (8.162)$$

In steady state:

$$\frac{d}{dx}\left(\rho Au\left(e+\frac{u^2}{2}+\frac{p}{\rho}\right)\right) = q_wL_p, \quad (8.163)$$

$$\rho Au\frac{d}{dx}\left(e+\frac{u^2}{2}+\frac{p}{\rho}\right)+\left(e+\frac{u^2}{2}+\frac{p}{\rho}\right)\underbrace{\frac{d}{dx}(\rho Au)}_{=0} = q_wL_p, \quad (8.164)$$

$$\rho u\frac{d}{dx}\left(e+\frac{u^2}{2}+\frac{p}{\rho}\right) = \frac{q_wL_p}{A}, \quad (8.165)$$

$$\rho u\left(\frac{de}{dx}+u\frac{du}{dx}+\frac{1}{\rho}\frac{dp}{dx}-\frac{p}{\rho^2}\frac{d\rho}{dx}\right) = \frac{q_wL_p}{A}. \quad (8.166)$$

[6]In neglecting work done by the wall shear force, I have taken an approach that is nearly universal, but fundamentally difficult to defend. At this stage of the development of these notes, I am not ready to enter into a grand battle with all established authors and probably confuse the student; consequently, results for flow with friction will be consistent with those of other sources. The argument typically used to justify this is that the real fluid satisfies no-slip at the boundary; thus, the wall shear actually does no work. However, one can easily argue that within the context of the one-dimensional model that has been posed that the shear force behaves as an external force that reduces the fluid's mechanical energy. Moreover, it is possible to show that neglect of this term results in the loss of frame invariance, a serious defect indeed. To model the work of the wall shear, one would include the term $\left(\tau_w\left(\bar{L}_p\Delta x\right)\right)\left(\bar{u}\Delta t\right)$ in the energy equation.

Now consider the product of velocity and momentum from Eq. (8.148) to get an equation for the mechanical energy:

$$\rho u^2 \frac{du}{dx} + u\frac{dp}{dx} = -\frac{\tau_w L_p u}{A}. \tag{8.167}$$

Subtract this, the mechanical energy, from Eq. (8.166) to get an equation for the thermal energy

$$\rho u \frac{de}{dx} - \frac{pu}{\rho}\frac{d\rho}{dx} = \frac{q_w L_p}{A} + \frac{\tau_w L_p u}{A}, \tag{8.168}$$

$$\frac{de}{dx} - \frac{p}{\rho^2}\frac{d\rho}{dx} = \frac{(q_w + \tau_w u) L_p}{\dot{m}}. \tag{8.169}$$

Because $e = e(p, \rho)$, we have

$$de = \left.\frac{\partial e}{\partial \rho}\right|_p d\rho + \left.\frac{\partial e}{\partial p}\right|_\rho dp, \tag{8.170}$$

$$\frac{de}{dx} = \left.\frac{\partial e}{\partial \rho}\right|_p \frac{d\rho}{dx} + \left.\frac{\partial e}{\partial p}\right|_\rho \frac{dp}{dx}. \tag{8.171}$$

so the steady energy equation becomes

$$\underbrace{\left.\frac{\partial e}{\partial \rho}\right|_p \frac{d\rho}{dx} + \left.\frac{\partial e}{\partial p}\right|_\rho \frac{dp}{dx}}_{\frac{de}{dx}} - \frac{p}{\rho^2}\frac{d\rho}{dx} = \frac{(q_w + \tau_w u) L_p}{\dot{m}}, \tag{8.172}$$

$$\frac{dp}{dx} - \underbrace{\left(\frac{\frac{p}{\rho^2} - \left.\frac{\partial e}{\partial \rho}\right|_p}{\left.\frac{\partial e}{\partial p}\right|_\rho} \right)}_{\equiv c^2} \frac{d\rho}{dx} = \frac{(q_w + \tau_w u) L_p}{\dot{m} \left.\frac{\partial e}{\partial p}\right|_\rho}. \tag{8.173}$$

Now let us consider the term in braces, which we label c^2, in the previous equation. It will be seen to be the square of the sound speed. We can put that term in a more common form by considering the Gibbs equation, Eq. (4.162):

$$T\ ds = de - \frac{p}{\rho^2}\ d\rho, \tag{8.174}$$

along with a general caloric equation of state $e = e(p, \rho)$, from which we get

$$de = \left.\frac{\partial e}{\partial p}\right|_\rho dp + \left.\frac{\partial e}{\partial \rho}\right|_p d\rho. \tag{8.175}$$

Substituting into the Gibbs equation, we get

$$T\ ds = \underbrace{\left.\frac{\partial e}{\partial p}\right|_{\rho} dp + \left.\frac{\partial e}{\partial \rho}\right|_{p} d\rho}_{de} - \frac{p}{\rho^2}\ d\rho. \tag{8.176}$$

Taking s to be constant and dividing by $d\rho$, we get

$$0 = \left.\frac{\partial e}{\partial p}\right|_{\rho} \left.\frac{\partial p}{\partial \rho}\right|_{s} + \left.\frac{\partial e}{\partial \rho}\right|_{p} - \frac{p}{\rho^2}. \tag{8.177}$$

Rearranging, we get

$$\left.\frac{\partial p}{\partial \rho}\right|_{s} = c^2 = \frac{\frac{p}{\rho^2} - \left.\frac{\partial e}{\partial \rho}\right|_{p}}{\left.\frac{\partial e}{\partial p}\right|_{\rho}}, \tag{8.178}$$

so

$$\frac{dp}{dx} - c^2\frac{d\rho}{dx} = \frac{(q_w + \tau_w u)\,L_p}{\dot{m}\,\left.\frac{\partial e}{\partial p}\right|_{\rho}}, \tag{8.179}$$

$$\frac{dp}{dx} - c^2\frac{d\rho}{dx} = \frac{(q_w + \tau_w u)\,L_p}{\rho u A\,\left.\frac{\partial e}{\partial p}\right|_{\rho}}. \tag{8.180}$$

Here c is the isentropic sound speed, a thermodynamic property of the material. We shall see later in Sec. 8.4.6 why it is appropriate to interpret this property as the propagation speed of small disturbances. At this point, it should simply be thought of as a state property.

Example 8.7

Find the speed of sound for a CPIG.

For a CPIG, we have

$$p = \rho RT, \qquad e = c_v T + e_o, \qquad c_p - c_v = R. \tag{8.181}$$

So

$$e = c_v\frac{p}{\rho R} + e_o, \tag{8.182}$$

$$= c_v\frac{p}{(c_p - c_v)\rho} + e_o, \tag{8.183}$$

$$= \frac{1}{\frac{c_p}{c_v} - 1}\frac{p}{\rho} + e_o, \tag{8.184}$$

$$= \frac{1}{\gamma - 1}\frac{p}{\rho} + e_o. \tag{8.185}$$

The relevant partial derivatives are

$$\left.\frac{\partial e}{\partial \rho}\right|_p = -\frac{1}{\gamma-1}\frac{p}{\rho^2}, \qquad \left.\frac{\partial e}{\partial p}\right|_\rho = \frac{1}{\gamma-1}\frac{1}{\rho}. \tag{8.186}$$

So by Eq. (8.178), we have

$$c^2 = \frac{\frac{p}{\rho^2} - \left(-\frac{1}{\gamma-1}\frac{p}{\rho^2}\right)}{\frac{1}{\gamma-1}\frac{1}{\rho}}, \tag{8.187}$$

$$= \frac{(\gamma-1)\frac{p}{\rho^2} + \frac{p}{\rho^2}}{\frac{1}{\rho}}, \tag{8.188}$$

$$= \gamma\frac{p}{\rho}, \tag{8.189}$$

$$= \gamma\frac{\rho RT}{\rho}, \tag{8.190}$$

$$= \gamma RT, \tag{8.191}$$

$$c = \sqrt{\gamma RT}. \tag{8.192}$$

Example 8.8

For $q_w = 0$, $\tau_w = 0$, find a relation between p and ρ for the steady flow of a CPIG.

Start with the energy equation, Eq. (8.180), in the limit as $q_w = 0$, $\tau_w = 0$:

$$\frac{dp}{dx} = c^2\frac{d\rho}{dx}. \tag{8.193}$$

Now from the previous example, we know for a CPIG that $c^2 = \gamma p/\rho$, so

$$\frac{dp}{dx} = \gamma\frac{p}{\rho}\frac{d\rho}{dx}, \tag{8.194}$$

$$\frac{1}{p}\frac{dp}{dx} = \gamma\frac{1}{\rho}\frac{d\rho}{dx}, \tag{8.195}$$

$$\frac{dp}{p} = \gamma\frac{d\rho}{\rho}, \tag{8.196}$$

$$\ln\frac{p}{p_o} = \gamma\ln\frac{\rho}{\rho_o}, \tag{8.197}$$

$$= \ln\left(\frac{\rho}{\rho_o}\right)^\gamma, \tag{8.198}$$

$$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^\gamma, \tag{8.199}$$

$$\frac{p}{\rho^\gamma} = \frac{p_o}{\rho_o^\gamma}. \tag{8.200}$$

This is equivalent to what we have earlier derived in Eq. (6.92). In terms of specific volume, we could say

$$p\hat{v}^{\gamma} = p_o\hat{v}_o^{\gamma}. \tag{8.201}$$

This is the equation for a so-called *polytropic process* in which the polytropic exponent is γ.

Consider now the special case of flow with no heat transfer $q_w \equiv 0$. We still allow area change and wall friction is allowed (see earlier footnote, p. 265):

$$\rho u \frac{d}{dx}\left(e + \frac{u^2}{2} + \frac{p}{\rho}\right) = 0, \tag{8.202}$$

$$e + \frac{u^2}{2} + \frac{p}{\rho} = e_1 + \frac{u_1^2}{2} + \frac{p_1}{\rho_1}, \tag{8.203}$$

$$h + \frac{u^2}{2} = h_1 + \frac{u_1^2}{2}. \tag{8.204}$$

8.2.4 Summary of equations

We can summarize the one-dimensional compressible flow equations in various forms here. In the equations below, we assume $A(x)$, τ_w, q_w, and L_p are all known.

8.2.4.1 Unsteady conservative form

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A u) = 0, \tag{8.205}$$

$$\frac{\partial}{\partial t}(\rho A u) + \frac{\partial}{\partial x}\left(\rho A u^2 + pA\right) = p\frac{\partial A}{\partial x} - \tau_w L_p, \tag{8.206}$$

$$\frac{\partial}{\partial t}\left(\rho A\left(e + \frac{u^2}{2}\right)\right) + \frac{\partial}{\partial x}\left(\rho A u\left(e + \frac{u^2}{2} + \frac{p}{\rho}\right)\right) = q_w L_p, \tag{8.207}$$

$$e = e(\rho, p), \tag{8.208}$$

$$p = p(\rho, T). \tag{8.209}$$

8.2.4.2 Unsteady non-conservative form

$$\frac{d\rho}{dt} = -\frac{\rho}{A}\frac{\partial}{\partial x}(Au), \tag{8.210}$$

$$\rho\frac{du}{dt} = -\frac{\partial p}{\partial x} - \frac{\tau_w L_p}{A}, \tag{8.211}$$

$$\rho\frac{de}{dt} - \frac{p}{\rho}\frac{d\rho}{dt} = \frac{(q_w + \tau_w u)L_p}{A}, \tag{8.212}$$

$$e = e(\rho, p), \tag{8.213}$$

$$p = p(\rho, T). \tag{8.214}$$

8.2.4.3 Steady conservative form

$$\frac{d}{dx}(\rho A u) = 0, \tag{8.215}$$

$$\frac{d}{dx}\left(\rho A u^2 + pA\right) = p\frac{dA}{dx} - \tau_w L_p, \tag{8.216}$$

$$\frac{d}{dx}\left(\rho A u\left(e + \frac{u^2}{2} + \frac{p}{\rho}\right)\right) = q_w L_p, \tag{8.217}$$

$$e = e(\rho, p), \tag{8.218}$$

$$p = p(\rho, T). \tag{8.219}$$

8.2.4.4 Steady non-conservative form

$$u\frac{d\rho}{dx} = -\frac{\rho}{A}\frac{d}{dx}(Au), \tag{8.220}$$

$$\rho u\frac{du}{dx} = -\frac{dp}{dx} + \frac{\tau_w L_p}{A}, \tag{8.221}$$

$$\rho u\frac{de}{dx} - \frac{pu}{\rho}\frac{d\rho}{dx} = \frac{(q_w + \tau_w u)L_p}{A}, \tag{8.222}$$

$$e = e(\rho, p), \tag{8.223}$$

$$p = p(\rho, T). \tag{8.224}$$

In whatever form we consider, we have five equations in five unknown dependent variables: ρ, u, p, e, and T. We can always use the thermal and caloric state equations to eliminate e and T to give rise to three equations in three unknowns.

Example 8.9

Let us consider the flow of air with heat addition.

Given: Air initially at $p_1 = 100$ kPa, $T_1 = 300$ K, $u_1 = 10$ m/s flows in a duct of length 100 m. The duct has a constant circular cross sectional area of $A = 0.02$ m^2 and is isobarically heated with a constant heat flux q_w along the entire surface of the duct. At the end of the duct the flow has $p_2 = 100$ kPa, $T_2 = 500$ K.

Find: the mass flow rate $\dot{m}$, the wall heat flux q_w and the entropy change $s_2 - s_1$; check for satisfaction of the second law.

Assume: CPIG, $R = 0.287$ kJ/(kg K), $c_p = 1.0035$ kJ/(kg K).

We begin by considering the geometry.

$$A = \pi r^2, \tag{8.225}$$

$$r = \sqrt{\frac{A}{\pi}}, \tag{8.226}$$

$$L_p = 2\pi r = 2\sqrt{\pi A} = 2\sqrt{\pi\,(0.02 \text{ m}^2)} = 0.501 \text{ m}. \tag{8.227}$$

Now get the mass flux.

$$p_1 = \rho_1 R T_1, \tag{8.228}$$

$$\rho_1 = \frac{p_1}{RT_1} = \frac{100 \text{ kPa}}{\left(0.287 \frac{\text{kJ}}{\text{kg K}}\right)(300 \text{ K})}, \tag{8.229}$$

$$= 1.161 \frac{\text{kg}}{\text{m}^3}. \tag{8.230}$$

So

$$\dot{m} = \rho_1 u_1 A_1 = \left(1.161 \frac{\text{kg}}{\text{m}^3}\right)\left(10 \frac{\text{m}}{\text{s}}\right)\left(0.02 \text{ m}^2\right) = 0.2322 \frac{\text{kg}}{\text{s}}. \tag{8.231}$$

Now get the flow variables at state 2:

$$\rho_2 = \frac{p_2}{RT_2}, = \frac{100 \text{ kPa}}{\left(0.287 \frac{\text{kJ}}{\text{kg K}}\right)(500 \text{ K})}, \tag{8.232}$$

$$= 0.6969 \frac{\text{kg}}{\text{m}^3}, \tag{8.233}$$

$$\rho_2 u_2 A_2 = \rho_1 u_1 A_1, \tag{8.234}$$

$$u_2 = \frac{\rho_1 u_1 A_1}{\rho_2 A_2} = \frac{\rho_1 u_1}{\rho_2}, \tag{8.235}$$

$$= \frac{\left(1.161 \frac{\text{kg}}{\text{m}^3}\right)\left(10 \frac{\text{m}}{\text{s}}\right)}{0.6969 \frac{\text{kg}}{\text{m}^3}} = 16.67 \frac{\text{m}}{\text{s}}. \tag{8.236}$$

Now consider the energy equation:

$$\rho u \frac{d}{dx}\left(e + \frac{u^2}{2} + \frac{p}{\rho}\right) = \frac{q_w L_p}{A}, \tag{8.237}$$

$$\frac{d}{dx}\left(h + \frac{u^2}{2}\right) = \frac{q_w L_p}{\dot{m}}, \tag{8.238}$$

$$\int_0^L \frac{d}{dx}\left(h + \frac{u^2}{2}\right) dx = \int_0^L \frac{q_w L_p}{\dot{m}} \, dx, \tag{8.239}$$

$$h_2 + \frac{u_2^2}{2} - h_1 - \frac{u_1^2}{2} = \frac{q_w L L_p}{\dot{m}}, \tag{8.240}$$

$$c_p (T_2 - T_1) + \frac{u_2^2}{2} - \frac{u_1^2}{2} = \frac{q_w L L_p}{\dot{m}}. \tag{8.241}$$

Solving for q_w, we get

$$q_w = \left(\frac{\dot{m}}{L L_p}\right)\left(c_p (T_2 - T_1) + \frac{u_2^2}{2} - \frac{u_1^2}{2}\right), \tag{8.242}$$

$$= \left(\frac{0.2322 \frac{\text{kg}}{\text{s}}}{(100 \text{ m})(0.501 \text{ m})}\right)\left(\left(1003.5 \frac{\text{J}}{\text{kg K}}\right)(500 \text{ K} - 300 \text{ K}) + \frac{\left(16.67 \frac{\text{m}}{\text{s}}\right)^2}{2} - \frac{\left(10 \frac{\text{m}}{\text{s}}\right)^2}{2}\right), \tag{8.243}$$

$$= 0.004635 \frac{\text{kg}}{\text{m}^2 \text{ s}}\left(200700 \frac{\text{J}}{\text{kg}} - 88.9 \frac{\text{m}^2}{\text{s}^2}\right), \tag{8.244}$$

$$= 0.004635 \frac{\text{kg}}{\text{m}^2 \text{ s}}\left(200700 \frac{\text{J}}{\text{kg}} - 88.9 \frac{\text{J}}{\text{kg}}\right), \tag{8.245}$$

$$= 930 \frac{\text{W}}{\text{m}^2}. \tag{8.246}$$

The heat flux is positive, that indicates a transfer of thermal energy *into* the air.

Now find the entropy change.

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right), \tag{8.247}$$

$$s_2 - s_1 = \left(1003.5 \ \frac{\text{J}}{\text{kg K}}\right) \ln\left(\frac{500 \text{ K}}{300 \text{ K}}\right) - \left(287 \ \frac{\text{J}}{\text{kg K}}\right) \ln\left(\frac{100 \text{ kPa}}{100 \text{ kPa}}\right), \tag{8.248}$$

$$s_2 - s_1 = 512.6 - 0 = 512.6 \ \frac{\text{J}}{\text{kg K}}. \tag{8.249}$$

Is the second law satisfied? Assume the heat transfer takes place from a reservoir held at 500 K. The reservoir would have to be *at least* at 500 K in order to bring the fluid to its final state of 500 K. It could be greater than 500 K and still satisfy the second law.

$$S_2 - S_1 \geq \frac{Q_{12}}{T}, \tag{8.250}$$

$$\dot{S}_2 - \dot{S}_1 \geq \frac{\dot{Q}_{12}}{T}, \tag{8.251}$$

$$\dot{m}(s_2 - s_1) \geq \frac{\dot{Q}_{12}}{T}, \tag{8.252}$$

$$\geq \frac{q_w A_{tot}}{T}, \tag{8.253}$$

$$\geq \frac{q_w L L_p}{T}, \tag{8.254}$$

$$s_2 - s_1 \geq \frac{q_w L L_p}{\dot{m} T}, \tag{8.255}$$

$$512.6 \ \frac{\text{J}}{\text{kg K}} \geq \frac{\left(930 \ \frac{\text{J}}{\text{s m}^2}\right)(100 \text{ m})(0.501 \text{ m})}{\left(0.2322 \ \frac{\text{kg}}{\text{s}}\right)(500 \text{ K})}, \tag{8.256}$$

$$512.6 \ \frac{\text{J}}{\text{kg K}} \geq 401.3 \ \frac{\text{J}}{\text{kg K}}. \tag{8.257}$$

8.2.5 Influence coefficients

Now, let us uncouple the steady one-dimensional equations. First let us summarize again, in a slightly different manner than before, by rearranging the mass, Eq. (8.134), momentum, Eq. (8.148), and energy, Eq. (8.180), equations:

$$u\frac{d\rho}{dx} + \rho\frac{du}{dx} = -\frac{\rho u}{A}\frac{dA}{dx}, \tag{8.258}$$

$$\rho u\frac{du}{dx} + \frac{dp}{dx} = -\frac{\tau_w L_p}{A}, \tag{8.259}$$

$$\frac{dp}{dx} - c^2\frac{d\rho}{dx} = \frac{(q_w + \tau_w u) L_p}{\rho u A \left.\frac{\partial e}{\partial p}\right|_\rho}. \tag{8.260}$$

In matrix form, these are recast as

$$\begin{pmatrix} u & \rho & 0 \\ 0 & \rho u & 1 \\ -c^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{d\rho}{dx} \\ \frac{du}{dx} \\ \frac{dp}{dx} \end{pmatrix} = \begin{pmatrix} -\frac{\rho u}{A}\frac{dA}{dx} \\ -\frac{\tau_w L_p}{A} \\ \frac{(q_w+\tau_w u)L_p}{\rho u A \left.\frac{\partial e}{\partial p}\right|_\rho} \end{pmatrix}. \tag{8.261}$$

Use Cramer's rule to solve for the derivatives. First calculate the determinant of the coefficient matrix:

$$u\left((\rho u)(1)-(1)(0)\right)-\rho\left((0)(1)-(-c^2)(1)\right)=\rho\left(u^2-c^2\right). \tag{8.262}$$

Implementing Cramer's rule, we find after detailed calculation:

$$\frac{d\rho}{dx} = \frac{\rho u\left(-\frac{\rho u}{A}\frac{dA}{dx}\right)-\rho\left(-\frac{\tau_w L_p}{A}\right)+\rho\left(\frac{(q_w+\tau_w u)L_p}{\rho u A\left.\frac{\partial e}{\partial p}\right|_\rho}\right)}{\rho\left(u^2-c^2\right)}, \tag{8.263}$$

$$\frac{du}{dx} = \frac{-c^2\left(-\frac{\rho u}{A}\frac{dA}{dx}\right)+u\left(-\frac{\tau_w L_p}{A}\right)-u\left(\frac{(q_w+\tau_w u)L_p}{\rho u A\left.\frac{\partial e}{\partial p}\right|_\rho}\right)}{\rho\left(u^2-c^2\right)}, \tag{8.264}$$

$$\frac{dp}{dx} = \frac{\rho u c^2\left(-\frac{\rho u}{A}\frac{dA}{dx}\right)-\rho c^2\left(-\frac{\tau_w L_p}{A}\right)+\rho u^2\left(\frac{(q_w+\tau_w u)L_p}{\rho u A\left.\frac{\partial e}{\partial p}\right|_\rho}\right)}{\rho\left(u^2-c^2\right)}. \tag{8.265}$$

Simplify to find

$$\frac{d\rho}{dx} = \frac{1}{A}\frac{-\rho u^2\frac{dA}{dx}+\tau_w L_p+\frac{(q_w+\tau_w u)L_p}{\rho u\left.\frac{\partial e}{\partial p}\right|_\rho}}{u^2-c^2}, \tag{8.266}$$

$$\frac{du}{dx} = \frac{1}{A}\frac{c^2\rho u\frac{dA}{dx}-u\tau_w L_p-\frac{(q_w+\tau_w u)L_p}{\rho\left.\frac{\partial e}{\partial p}\right|_\rho}}{\rho\left(u^2-c^2\right)}, \tag{8.267}$$

$$\frac{dp}{dx} = \frac{1}{A}\frac{-c^2\rho u^2\frac{dA}{dx}+c^2\tau_w L_p+\frac{(q_w+\tau_w u)L_p u}{\rho\left.\frac{\partial e}{\partial p}\right|_\rho}}{u^2-c^2}. \tag{8.268}$$

We have

- a system of coupled non-linear ordinary differential equations,
- in standard form for dynamic system analysis: $d\mathbf{u}/dx = \mathbf{f}(\mathbf{u})$,
- valid for *general* equations of state, and
- *singular* when the fluid particle velocity is sonic $u = c$.

8.3 Flow with area change

This section will consider flow with area change with an emphasis on isentropic flow. Some problems will involve non-isentropic flow but a detailed discussion of such flows will be delayed.

8.3.1 Isentropic Mach number relations

Take the special case of

- $\tau_w = 0$,
- $q_w = 0$,
- CPIG.

Then we can recast Eqs. (8.215-8.217) as

$$\frac{d}{dx}(\rho u A) = 0, \tag{8.269}$$

$$\rho u \frac{du}{dx} + \frac{dp}{dx} = 0, \tag{8.270}$$

$$\frac{d}{dx}\left(e + \frac{u^2}{2} + \frac{p}{\rho}\right) = 0. \tag{8.271}$$

Integrate the energy equation using Eq. (4.136), $h = e + p/\rho$, to get

$$h + \frac{u^2}{2} = h_o + \frac{u_o^2}{2}. \tag{8.272}$$

If we define the "o" condition to be a condition of rest, then $u_o \equiv 0$. This is a stagnation condition, as introduced on p. 102 and p. 206. In an unfortunate choice of nomenclature, properties evaluated for the fluid in motion are named *static* properties. So

$$h + \frac{u^2}{2} = h_o, \tag{8.273}$$

$$(h - h_o) + \frac{u^2}{2} = 0. \tag{8.274}$$

Because we have a CPIG,

$$c_p (T - T_o) + \frac{u^2}{2} = 0, \tag{8.275}$$

$$T - T_o + \frac{u^2}{2c_p} = 0, \tag{8.276}$$

$$1 - \frac{T_o}{T} + \frac{u^2}{2c_p T} = 0. \tag{8.277}$$

Now note that

$$c_p = c_p \frac{c_p - c_v}{c_p - c_v} = \frac{c_p}{c_v} \frac{c_p - c_v}{\frac{c_p}{c_v} - 1} = \frac{\gamma R}{\gamma - 1}, \tag{8.278}$$

so

$$1 - \frac{T_o}{T} + \frac{\gamma - 1}{2} \frac{u^2}{\gamma R T} = 0, \tag{8.279}$$

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} \frac{u^2}{\gamma R T}. \tag{8.280}$$

Recall the sound speed and Mach number for a CPIG:

$$c^2 = \gamma R T, \qquad \text{if} \qquad p = \rho R T, \qquad e = c_v T + \hat{e}, \tag{8.281}$$

$$M^2 \equiv \left(\frac{u}{c}\right)^2. \tag{8.282}$$

Thus, the ratio of stagnation temperature T_o to static temperature T is

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} M^2, \tag{8.283}$$

$$\frac{T}{T_o} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-1}. \tag{8.284}$$

Now if the flow is isentropic and for a CPIG, it can be inferred from Eq. (8.199) that ratios of static to stagnation values are

$$\frac{T}{T_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma - 1} = \left(\frac{p}{p_o}\right)^{\frac{\gamma - 1}{\gamma}}. \tag{8.285}$$

Thus,

$$\frac{\rho}{\rho_o} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{1}{\gamma - 1}}, \tag{8.286}$$

$$\frac{p}{p_o} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{\gamma}{\gamma - 1}}. \tag{8.287}$$

For air $\gamma = 7/5$, so

$$\frac{T}{T_o} = \left(1 + \frac{1}{5} M^2\right)^{-1}, \tag{8.288}$$

$$\frac{\rho}{\rho_o} = \left(1 + \frac{1}{5} M^2\right)^{-\frac{5}{2}}, \tag{8.289}$$

$$\frac{p}{p_o} = \left(1 + \frac{1}{5} M^2\right)^{-\frac{7}{2}}. \tag{8.290}$$

Figures 8.2, 8.3, and 8.4 show the variation of T, ρ and p with M^2 for isentropic flow. Other thermodynamic properties can be determined from these, e.g. the sound speed:

$$\frac{c}{c_o} = \sqrt{\frac{\gamma RT}{\gamma RT_o}} = \sqrt{\frac{T}{T_o}} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-1/2}. \tag{8.291}$$

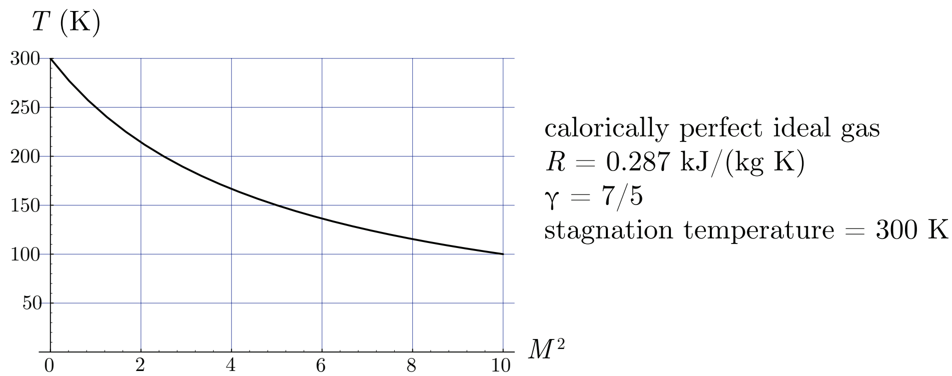

Figure 8.2: Static temperature versus Mach number squared.

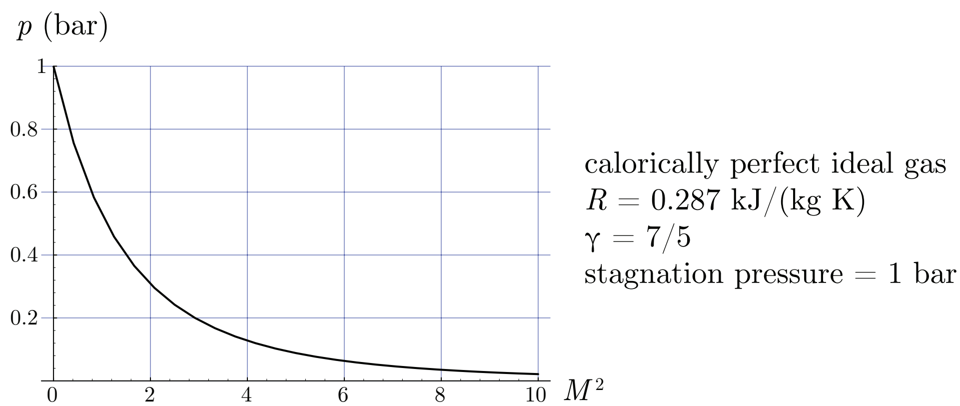

Figure 8.3: Static pressure versus Mach number squared.

Example 8.10

Show that how the isentropic relation for the ratio of stagnation to static pressures reduces to the incompressible Bernoulli's equation in the limit of low Mach number, and show how it deviates from this as the Mach number rises.

Begin with the reciprocal of Eq. (8.287):

$$\frac{p_o}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}}. \tag{8.292}$$

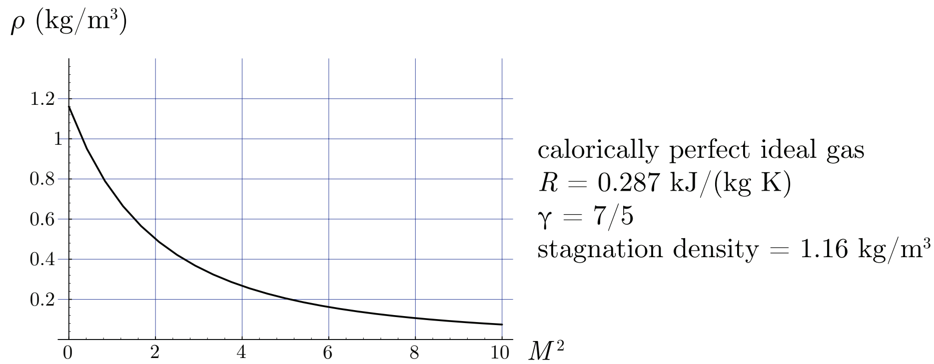

Figure 8.4: Static density versus Mach number squared.

Taylor series of this equation about $M = 0$ yields

$$\frac{p_o}{p} = 1 + \frac{\gamma}{2}M^2 + \frac{\gamma}{8}M^4 + \ldots, \quad (8.293)$$

$$= 1 + \frac{\gamma}{2}M^2\left(1 + \frac{1}{4}M^2 + \ldots\right), \quad (8.294)$$

$$= 1 + \frac{\gamma}{2}\frac{\rho v^2}{\gamma p}\left(1 + \frac{1}{4}M^2 + \ldots\right), \quad (8.295)$$

$$= 1 + \frac{\rho v^2}{2p}\left(1 + \frac{1}{4}M^2 + \ldots\right), \quad (8.296)$$

$$p_o = p + \frac{\rho v^2}{2}\left(1 + \frac{1}{4}M^2 + \ldots\right). \quad (8.297)$$

For $M = 0$, we recover the incompressible Bernoulli equation, and the correction at small finite Mach number to the incompressible Bernoulli equation estimate for stagnation pressure is evident.

Example 8.11

Given: An airplane is flying into still air at $u = 200$ m/s. The ambient air is at 288 K and 101.3 kPa.
Find: Temperature, pressure, and density at nose of airplane.
Assume: Steady isentropic flow of a CPIG.

Analysis: In the steady wave frame, the ambient conditions are *static* while the nose conditions are *stagnation*.

$$M = \frac{u}{c} = \frac{u}{\sqrt{\gamma R T}} = \frac{200\ \frac{\text{m}}{\text{s}}}{\sqrt{\frac{7}{5}\left(287\ \frac{\text{J}}{\text{kg K}}\right) 288\ \text{K}}} = 0.588. \quad (8.298)$$

so

$$T_o = T\left(1 + \frac{1}{5}M^2\right) = (288\ \text{K})\left(1 + \frac{1}{5}0.588^2\right) = 307.9\ \text{K}, \quad (8.299)$$

$$\rho_o = \rho\left(1+\frac{1}{5}M^2\right)^{\frac{5}{2}} = \frac{101.3 \text{ kPa}}{\left(0.287 \frac{\text{kJ}}{\text{kg K}}\right)(288 \text{ K})\left(1+\frac{1}{5}0.588^2\right)^{\frac{5}{2}}} = 1.45 \frac{\text{kg}}{\text{m}^3}, \tag{8.300}$$

$$p_o = p\left(1+\frac{1}{5}M^2\right)^{\frac{7}{2}} = (101.3 \text{ kPa})\left(1+\frac{1}{5}0.588^2\right)^{\frac{7}{2}} = 128 \text{ kPa}. \tag{8.301}$$

The temperature, pressure, and density all rise in the isentropic process. In this wave frame, the kinetic energy of the flow is being converted isentropically to thermal energy.

8.3.2 Sonic properties

Let "*" denote a property at the sonic state $M^2 \equiv 1$. Then we get

$$\frac{T_*}{T_o} = \left(1+\frac{\gamma-1}{2}1^2\right)^{-1} = \frac{2}{\gamma+1}, \tag{8.302}$$

$$\frac{\rho_*}{\rho_o} = \left(1+\frac{\gamma-1}{2}1^2\right)^{-\frac{1}{\gamma-1}} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}, \tag{8.303}$$

$$\frac{p_*}{p_o} = \left(1+\frac{\gamma-1}{2}1^2\right)^{-\frac{\gamma}{\gamma-1}} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}, \tag{8.304}$$

$$\frac{c_*}{c_o} = \left(1+\frac{\gamma-1}{2}1^2\right)^{-1/2} = \sqrt{\frac{2}{\gamma+1}}, \tag{8.305}$$

$$u_* = c_* = \sqrt{\gamma R T_*} = \sqrt{\frac{2\gamma}{\gamma+1}RT_o}. \tag{8.306}$$

If the fluid is air, we have $\gamma = 7/5$ and

$$\frac{T_*}{T_o} = 0.8333, \tag{8.307}$$

$$\frac{\rho_*}{\rho_o} = 0.6339, \tag{8.308}$$

$$\frac{p_*}{p_o} = 0.5283, \tag{8.309}$$

$$\frac{c_*}{c_o} = 0.9129. \tag{8.310}$$

8.3.3 Effect of area change

To understand the effect of area change, the influence of the mass equation must be considered. So far we have really only looked at energy. In the isentropic limit the mass, momentum, and energy equations for a CPIG, Eqs. (8.135, 8.153, 8.196), reduce to

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0, \tag{8.311}$$

$$\rho u \; du + dp = 0, \tag{8.312}$$

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho}. \tag{8.313}$$

Substitute energy, then mass into momentum:

$$\rho u \; du + \underbrace{\gamma \frac{p}{\rho} \; d\rho}_{dp} = 0, \tag{8.314}$$

$$\rho u \; du + \gamma \frac{p}{\rho} \underbrace{\left(-\frac{\rho}{u} \; du - \frac{\rho}{A} \; dA \right)}_{d\rho} = 0, \tag{8.315}$$

$$du + \gamma \frac{p}{\rho} \left(-\frac{1}{u^2} \; du - \frac{1}{uA} \; dA \right) = 0, \tag{8.316}$$

$$du \left(1 - \frac{\gamma p/\rho}{u^2} \right) = \gamma \frac{p}{\rho} \frac{dA}{uA}, \tag{8.317}$$

$$\frac{du}{u} \left(1 - \frac{\gamma p/\rho}{u^2} \right) = \frac{\gamma p/\rho}{u^2} \frac{dA}{A}, \tag{8.318}$$

$$\frac{du}{u} \left(1 - \frac{1}{M^2} \right) = \frac{1}{M^2} \frac{dA}{A}, \tag{8.319}$$

$$\frac{du}{u} \left(M^2 - 1 \right) = \frac{dA}{A}, \tag{8.320}$$

$$\frac{du}{u} = \frac{1}{M^2 - 1} \frac{dA}{A}. \tag{8.321}$$

Figure 8.5 shows the performance of a fluid in a variable area duct. We note

- there is a singularity when $M^2 = 1$,
- if $M^2 = 1$, we need $dA = 0$,
- area minimum necessary to transition from subsonic to supersonic flow,
- it can be shown an area maximum is not relevant.

Consider A at a sonic state. From the mass equation:

$$\rho u A = \rho_* u_* A_*, \tag{8.322}$$

$$\rho u A = \rho_* c_* A_*, \tag{8.323}$$

$$\frac{A}{A_*} = \frac{\rho_*}{\rho} c_* \frac{1}{u} = \frac{\rho_*}{\rho} \sqrt{\gamma R T_*} \frac{1}{u} = \frac{\rho_*}{\rho} \frac{\sqrt{\gamma R T_*}}{\sqrt{\gamma R T}} \frac{\sqrt{\gamma R T}}{u}, \tag{8.324}$$

$$\frac{A}{A_*} = \frac{\rho_*}{\rho} \sqrt{\frac{T_*}{T}} \frac{1}{M} = \frac{\rho_*}{\rho_o} \frac{\rho_o}{\rho} \sqrt{\frac{T_* T_o}{T_o T}} \frac{1}{M}. \tag{8.325}$$

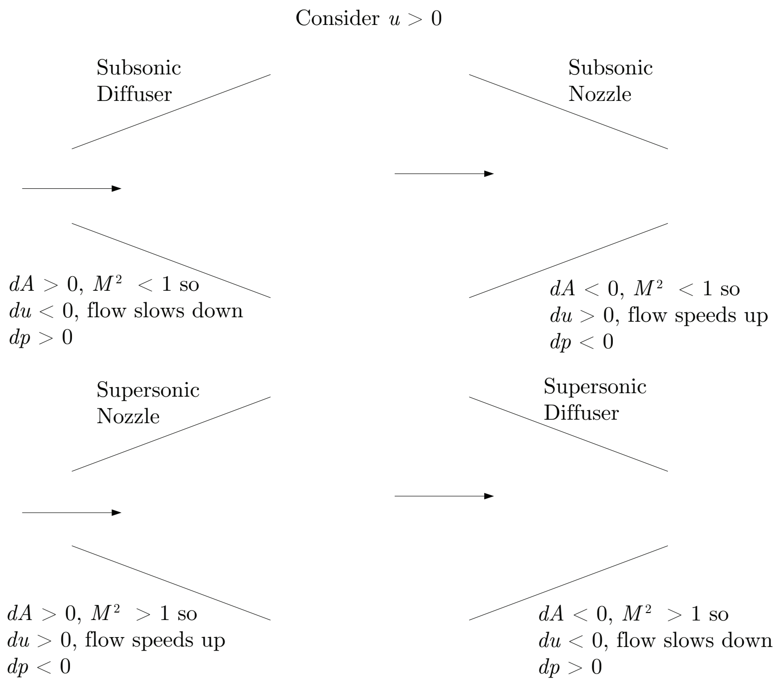

Figure 8.5: Behavior of fluid in sub- and supersonic nozzles and diffusers.

Substitute from earlier-developed relations and get

$$\frac{A}{A_*} = \frac{1}{M}\left(\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}. \tag{8.326}$$

Fig. 8.6 shows the performance of a fluid in a variable area duct.

Note that

- A/A_* has a minimum value of 1 at $M = 1$,
- For each $A/A_* > 1$, there exist *two* values of M, and
- $A/A_* \to \infty$ as $M \to 0$ or $M \to \infty$.

8.3.4 Choking

Consider mass flow rate variation with pressure difference. We have then

- small pressure difference gives small velocity and small mass flow,
- as pressure difference grows, velocity and mass flow rate grow,
- velocity is limited to sonic at a particular duct location,

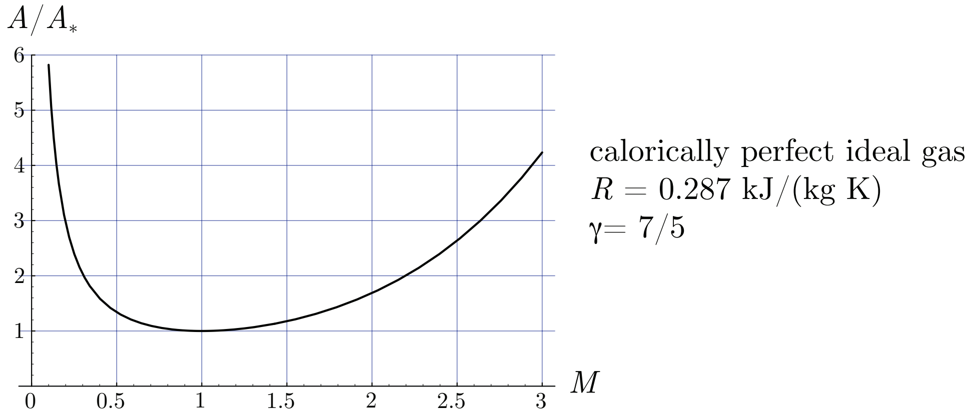

Figure 8.6: Area versus Mach number for a CPIG.

- this provides fundamental restriction on mass flow rate,
- it can be proven rigorously that sonic condition gives maximum mass flow rate.

$$\dot{m}_{max} = \rho_* u_* A_*, \tag{8.327}$$

$$\textit{if}\text{ ideal gas:} \quad = \rho_o \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \left(\sqrt{\frac{2\gamma}{\gamma+1}RT_o}\right) A_*, \tag{8.328}$$

$$= \rho_o \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \left(\frac{2}{\gamma+1}\right)^{1/2} \sqrt{\gamma R T_o} A_*, \tag{8.329}$$

$$= \rho_o \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}} \sqrt{\gamma R T_o} A_*. \tag{8.330}$$

A flow that has a maximum mass flow rate is known as *choked* flow. Flows will choke at area minima in a duct.

Example 8.12

Consider an isentropic area change problem with choking.[7]

Given: Air with stagnation conditions $p_o = 200$ kPa, $T_o = 500$ K flows through a throat to an exit Mach number of 2.5. The desired mass flow is 3.0 kg/s.

Find: a) throat area, b) exit pressure, c) exit temperature, d) exit velocity, and e) exit area.

Assume: CPIG, isentropic flow, $\gamma = 7/5$.

First find the stagnation density via the ideal gas law:

$$\rho_o = \frac{p_o}{RT_o} = \frac{200 \text{ kPa}}{\left(0.287 \frac{\text{kJ}}{\text{kg K}}\right)(500 \text{ K})} = 1.394 \frac{\text{kg}}{\text{m}^3}. \tag{8.331}$$

[7]adopted from White (1986), p. 529, Ex. 9.5.

Because it necessarily flows through a sonic throat:

$$\dot{m}_{max} = \rho_o \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}} \sqrt{\gamma R T_o} A_*, \quad (8.332)$$

$$A_* = \frac{\dot{m}_{max}}{\rho_o \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}} \sqrt{\gamma R T_o}}, \quad (8.333)$$

$$A_* = \frac{3\ \frac{\text{kg}}{\text{s}}}{\left(1.394\ \frac{\text{kg}}{\text{m}^3}\right)(0.5787)\sqrt{1.4\left(287\ \frac{\text{J}}{\text{kg K}}\right)(500\ \text{K})}} = 0.008297\ \text{m}^2. \quad (8.334)$$

Because we know M_e, use isentropic relations to find other exit conditions.

$$p_e = p_o\left(1+\frac{\gamma-1}{2}M_e^2\right)^{-\frac{\gamma}{\gamma-1}} = (200\ \text{kPa})\left(1+\frac{1}{5}2.5^2\right)^{-3.5} = 11.71\ \text{kPa}, \quad (8.335)$$

$$T_e = T_o\left(1+\frac{\gamma-1}{2}M_e^2\right)^{-1} = (500\ \text{K})\left(1+\frac{1}{5}2.5^2\right)^{-1} = 222.2\ \text{K}. \quad (8.336)$$

Note

$$\rho_e = \frac{p_e}{RT_e} = \frac{11.71\ \text{kPa}}{\left(0.287\ \frac{\text{kJ}}{\text{kg K}}\right)(222.2\ \text{K})} = 0.1834\ \frac{\text{kg}}{\text{m}^3}. \quad (8.337)$$

Now the exit velocity is simply

$$u_e = M_e c_e = M_e\sqrt{\gamma R T_e} = 2.5\sqrt{1.4\left(287\ \frac{\text{J}}{\text{kg K}}\right)(222.2\ \text{K})} = 747.0\ \frac{\text{m}}{\text{s}}. \quad (8.338)$$

Now determine the exit area.

$$A = \frac{A_*}{M_e}\left(\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_e^2\right)\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}, \quad (8.339)$$

$$= \frac{0.008297\ \text{m}^2}{2.5}\left(\frac{5}{6}\left(1+\frac{1}{5}2.5^2\right)\right)^3 = 0.0219\ \text{m}^2. \quad (8.340)$$

8.4 Normal shock waves

This section will develop relations for normal shock waves in fluids with general equations of state. It will be specialized to a CPIG to illustrate common features of the waves. Assume for this section we have

- one-dimensional flow,
- steady flow,

- no area change,
- viscous effects and wall friction do not have time to influence flow, and
- heat conduction and wall heat transfer do not have time to influence flow.

We will consider the problem in the context of the piston problem as sketched in Fig. 8.7. The physical problem is as follows:

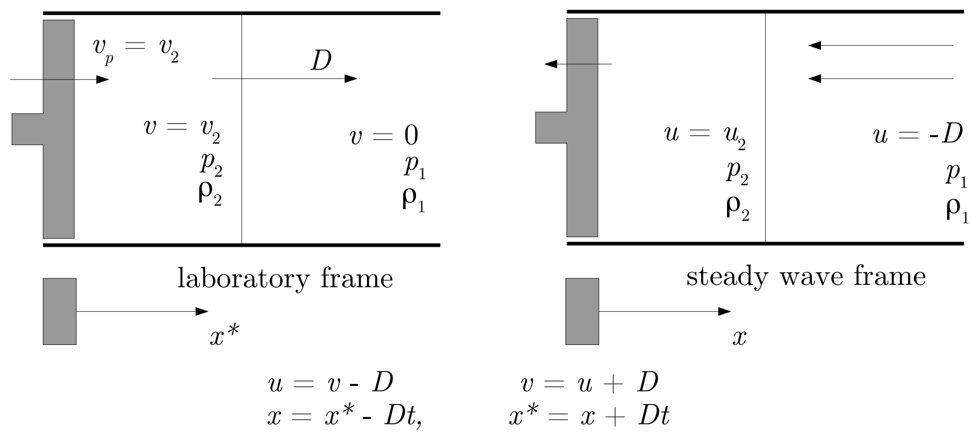

Figure 8.7: Normal shock sketch.

- Drive a piston with known velocity v_p into a fluid at rest ($v_1 = 0$) with known properties, p_1, ρ_1 in the x^* laboratory frame,
- Determine the disturbance speed D,
- Determine the disturbance properties v_2, p_2, ρ_2,
- in this frame of reference we have an *unsteady* problem.

Transformed Problem:

- use a Galilean transformation $x = x^* - Dt$, $u = v - D$ to transform to the frame in which the wave is at rest, therefore rending the problem *steady* in this frame,
- solve as though D is known to get downstream "2" conditions: $u_2(D), p_2(D), ...$,
- invert to solve for D as function of u_2, the transformed piston velocity: $D(u_2)$,
- back transform to get all variables as function of v_2, the laboratory piston velocity: $D(v_2), p_2(v_2), \rho_2(v_2), ...$

8.4.1 Rankine-Hugoniot equations

Under these assumptions, we can recast the conservation principles and equation of state in the steady frame, Eqs.(8.215-8.219), as follows:

$$\frac{d}{dx}(\rho u) = 0, \tag{8.341}$$

$$\frac{d}{dx}\left(\rho u^2 + p\right) = 0, \tag{8.342}$$

$$\frac{d}{dx}\left(\rho u\left(h + \frac{u^2}{2}\right)\right) = 0, \tag{8.343}$$

$$h = h(p, \rho). \tag{8.344}$$

Upstream conditions are $\rho = \rho_1$, $p = p_1$, $u = -D$. With knowledge of the equation of state, we get $h = h_1$. In what is a natural, but in fact naïve, step we can integrate the equations from upstream to state "2" to give the correct *Rankine-Hugoniot jump equations.*[8][9]

$$\rho_2 u_2 = -\rho_1 D, \tag{8.345}$$

$$\rho_2 u_2^2 + p_2 = \rho_1 D^2 + p_1, \tag{8.346}$$

$$h_2 + \frac{u_2^2}{2} = h_1 + \frac{D^2}{2}, \tag{8.347}$$

$$h_2 = h(p_2, \rho_2). \tag{8.348}$$

This analysis is straightforward and yields the correct result. In actuality, however, the analysis should be more nuanced. We are going to solve these algebraic equations to arrive at *discontinuous* shock jumps. Thus, we should be concerned about the validity of of differential equations in the vicinity of a discontinuity.

As described by LeVeque (1992), the proper way to arrive at the shock jump equations is to use a more primitive form of the conservation laws, expressed in terms of integrals of conserved quantities balanced by fluxes of those quantities, e.g. Eq. (4.4). If $\mathbf{q}$ is a set of conserved variables, and $\mathbf{f}(\mathbf{q})$ is the flux of $\mathbf{q}$ (e.g. for mass conservation, ρ is a conserved variable and ρu is the flux), then the primitive form of the conservation law can be written as

$$\frac{d}{dt}\int_{x_1}^{x_2} \mathbf{q}(x,t)\ dx = \mathbf{f}(\mathbf{q}(x_1,t)) - \mathbf{f}(\mathbf{q}(x_2,t)). \tag{8.349}$$

Here we have considered flow into and out of a one-dimensional box for $x \in [x_1, x_2]$. For the Euler equations, we have

$$\mathbf{q} = \begin{pmatrix} \rho \\ \rho u \\ \rho\left(e + \frac{1}{2}u^2\right) \end{pmatrix}, \qquad \mathbf{f}(\mathbf{q}) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u\left(e + \frac{1}{2}u^2 + \frac{p}{\rho}\right) \end{pmatrix}. \tag{8.350}$$

[8]William John Macquorn Rankine, 1820-1872, Scottish engineer and mechanician, pioneer of thermodynamics and steam engine theory, taught at University of Glasgow, studied fatigue in railway engine axles.

[9]Pierre Henri Hugoniot, 1851-1887, French engineer.

If we assume there is a discontinuity in the region $x \in [x_1, x_2]$ propagating at speed D, we can break up the integral into the form

$$\frac{d}{dt}\int_{x_1}^{x_1+Dt^-} \mathbf{q}(x,t)\;dx + \frac{d}{dt}\int_{x_1+Dt^+}^{x_2} \mathbf{q}(x,t)\;dx = \mathbf{f}(\mathbf{q}(x_1,t)) - \mathbf{f}(\mathbf{q}(x_2,t)). \tag{8.351}$$

Here $x_1 + Dt^-$ lies just before the discontinuity and $x_1 + Dt^+$ lies just past the discontinuity. Using Leibniz's rule, Eq. (2.274), we get

$$\mathbf{q}(x_1+Dt^-,t)D + 0 + \int_{x_1}^{x_1+Dt^-} \frac{\partial \mathbf{q}}{\partial t}\;dx + 0 - \mathbf{q}(x_1+Dt^+,t)D + \int_{x_1+Dt^+}^{x_2} \frac{\partial \mathbf{q}}{\partial t}\;dx \tag{8.352}$$
$$= \mathbf{f}(\mathbf{q}(x_1,t)) - \mathbf{f}(\mathbf{q}(x_2,t)).$$

Now if we assume that on either side of the discontinuity the volume of integration is sufficiently small so that the time and space variation of $\mathbf{q}$ is negligibly small, we get

$$\mathbf{q}(x_1)D - \mathbf{q}(x_2)D = \mathbf{f}(\mathbf{q}(x_1)) - \mathbf{f}(\mathbf{q}(x_2)), \tag{8.353}$$
$$D\left(\mathbf{q}(x_1) - \mathbf{q}(x_2)\right) = \mathbf{f}(\mathbf{q}(x_1)) - \mathbf{f}(\mathbf{q}(x_2)). \tag{8.354}$$

Defining next the notation for a jump as

$$[\![\mathbf{q}(x)]\!] \equiv \mathbf{q}(x_2) - \mathbf{q}(x_1), \tag{8.355}$$

the jump conditions are rewritten as

$$D\,[\![\mathbf{q}(x)]\!] = [\![\mathbf{f}(\mathbf{q}(x))]\!]\,. \tag{8.356}$$

If $D = 0$, as is the case when we transform to the frame where the wave is at rest, we simply recover

$$\mathbf{0} = \mathbf{f}(\mathbf{q}(x_1)) - \mathbf{f}(\mathbf{q}(x_2)), \tag{8.357}$$
$$\mathbf{f}(\mathbf{q}(x_1)) = \mathbf{f}(\mathbf{q}(x_2)), \tag{8.358}$$
$$[\![\mathbf{f}(\mathbf{q}(x))]\!] = \mathbf{0}. \tag{8.359}$$

That is the fluxes on either side of the discontinuity are equal. This is precisely what we obtained by our naïve analysis. We also get a more general result for $D \neq 0$, that is the well-known

$$D = \frac{\mathbf{f}(\mathbf{q}(x_2)) - \mathbf{f}(\mathbf{q}(x_1))}{\mathbf{q}(x_2) - \mathbf{q}(x_1)} = \frac{[\![\mathbf{f}(\mathbf{q}(x))]\!]}{[\![\mathbf{q}(x)]\!]}. \tag{8.360}$$

The general Rankine-Hugoniot equation then for the one-dimensional Euler equations across a non-stationary jump is given by

$$D\begin{pmatrix} \rho_2 - \rho_1 \\ \rho_2 u_2 - \rho_1 u_1 \\ \rho_2\left(e_2 + \frac{1}{2}u_2^2\right) - \rho_1\left(e_1 + \frac{1}{2}u_1^2\right) \end{pmatrix} = \begin{pmatrix} \rho_2 u_2 - \rho_1 u_1 \\ \rho_2 u_2^2 + p_2 - \rho_1 u_1^2 - p_1 \\ \rho_2 u_2\left(e_2 + \frac{1}{2}u_2^2 + \frac{p_2}{\rho_2}\right) - \rho_1 u_1\left(e_1 + \frac{1}{2}u_1^2 + \frac{p_1}{\rho_1}\right) \end{pmatrix}. \tag{8.361}$$

8.4.2 Rayleigh line

If we operate on the momentum equation as follows

$$p_2 = p_1 + \rho_1 D^2 - \rho_2 u_2^2, \tag{8.362}$$

$$p_2 = p_1 + \frac{\rho_1^2 D^2}{\rho_1} - \frac{\rho_2^2 u_2^2}{\rho_2}. \tag{8.363}$$

Because mass gives us $\rho_2^2 u_2^2 = \rho_1^2 D^2$ we get an equation for the *Rayleigh Line*,[10] a line in $(p, 1/\rho)$ space:

$$p_2 = p_1 + \rho_1^2 D^2 \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right). \tag{8.364}$$

Note that the Rayleigh line

- passes through ambient state,
- has *negative* slope,
- has a slope with magnitude proportional to square of the wave speed, and
- is independent of state and energy equations.

8.4.3 Hugoniot curve

Let us now work on the energy equation, using both mass and momentum to eliminate velocity. First eliminate u_2 via the mass equation:

$$h_2 + \frac{u_2^2}{2} = h_1 + \frac{D^2}{2}, \tag{8.365}$$

$$h_2 + \frac{1}{2}\left(\frac{\rho_1 D}{\rho_2} \right)^2 = h_1 + \frac{D^2}{2}, \tag{8.366}$$

$$h_2 - h_1 + \frac{D^2}{2}\left(\left(\frac{\rho_1}{\rho_2} \right)^2 - 1 \right) = 0, \tag{8.367}$$

$$h_2 - h_1 + \frac{D^2}{2}\left(\frac{\rho_1^2 - \rho_2^2}{\rho_2^2} \right) = 0, \tag{8.368}$$

$$h_2 - h_1 + \frac{D^2}{2}\left(\frac{(\rho_1 - \rho_2)(\rho_1 + \rho_2)}{\rho_2^2} \right) = 0. \tag{8.369}$$

[10]John William Strutt (Lord Rayleigh), 1842-1919, aristocratic-born English mathematician and physicist, studied at Cambridge, influenced by Stokes, toured the United States rather than the traditional continent of Europe, described correctly why the sky is blue, appointed Cavendish professor experimental physics at Cambridge, won the Nobel prize for the discovery of Argon, described traveling waves and solitons.

Now use the Rayleigh line, Eq. (8.364), to eliminate D^2:

$$D^2 = (p_2 - p_1)\left(\frac{1}{\rho_1^2}\right)\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^{-1}, \quad (8.370)$$

$$D^2 = (p_2 - p_1)\left(\frac{1}{\rho_1^2}\right)\left(\frac{\rho_2 - \rho_1}{\rho_1\rho_2}\right)^{-1}, \quad (8.371)$$

$$D^2 = (p_2 - p_1)\left(\frac{1}{\rho_1^2}\right)\left(\frac{\rho_1\rho_2}{\rho_2 - \rho_1}\right). \quad (8.372)$$

So the energy equation becomes

$$h_2 - h_1 + \frac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_1^2}\right)\left(\frac{\rho_1\rho_2}{\rho_2 - \rho_1}\right)\left(\frac{(\rho_1 - \rho_2)(\rho_1 + \rho_2)}{\rho_2^2}\right) = 0, \quad (8.373)$$

$$h_2 - h_1 - \frac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_1}\right)\left(\frac{\rho_1 + \rho_2}{\rho_2}\right) = 0, \quad (8.374)$$

$$h_2 - h_1 - \frac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right) = 0. \quad (8.375)$$

Regrouping to see what induces enthalpy changes, we get

$$h_2 - h_1 = (p_2 - p_1)\left(\frac{1}{2}\right)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right), \quad (8.376)$$

$$\underbrace{h_2 - h_1}_{\Delta h} = \underbrace{\left(\frac{\hat{v}_2 + \hat{v}_1}{2}\right)}_{\hat{v}_{mean}} \underbrace{(p_2 - p_1)}_{\Delta p}, \quad (8.377)$$

$$\Delta h = \hat{v}_{mean}\Delta p. \quad (8.378)$$

This equation is the *Hugoniot* equation. It

- holds that enthalpy change equals the product of the mean volume, and the pressure difference,[11],
- is independent of wave speed D and velocity u_2, and
- is independent of the equation of state.

8.4.4 Solution procedure for general equations of state

The shocked state can be determined by the following procedure:

[11]Note the similarity here between a common result for reversible thermodynamics. Using the definition of enthalpy, $h = e + p\hat{v}$ in the Gibbs equation gives $T\ ds = dh - \hat{v}\ dp$. For an isentropic change, we get $dh = \hat{v}\ dp$.

- specify the equation of state $h(p, \rho)$,
- substitute the equation of state into the Hugoniot, Eq. (8.376), to get a second relation between p_2 and ρ_2,
- use the Rayleigh line, Eq. (8.364), to eliminate p_2 in the Hugoniot so that the Hugoniot is a single equation in ρ_2,
- solve for ρ_2 as functions of "1" and D,
- back substitute to solve for p_2, u_2, h_2, T_2 as functions of "1" and D,
- invert to find D as function of "1" state and u_2,
- back transform to laboratory frame to get D as function of "1" state and piston velocity $v_2 = v_p$.

8.4.5 Calorically perfect ideal gas solutions

Let us follow this procedure for the special case of a CPIG.

$$h = c_p(T - T_o) + \hat{h}, \tag{8.379}$$
$$p = \rho R T. \tag{8.380}$$

Thus,

$$h = c_p\left(\frac{p}{R\rho} - \frac{p_o}{R\rho_o}\right) + \hat{h}, \tag{8.381}$$
$$h = \frac{c_p}{R}\left(\frac{p}{\rho} - \frac{p_o}{\rho_o}\right) + \hat{h}, \tag{8.382}$$
$$h = \frac{c_p}{c_p - c_v}\left(\frac{p}{\rho} - \frac{p_o}{\rho_o}\right) + \hat{h}, \tag{8.383}$$
$$h = \frac{\gamma}{\gamma - 1}\left(\frac{p}{\rho} - \frac{p_o}{\rho_o}\right) + \hat{h}. \tag{8.384}$$

Evaluate at states 1 and 2 and substitute into the Hugoniot equation, Eq. (8.376):

$$\left(\frac{\gamma}{\gamma-1}\left(\frac{p_2}{\rho_2} - \frac{p_o}{\rho_o}\right) + \hat{h}\right) - \left(\frac{\gamma}{\gamma-1}\left(\frac{p_1}{\rho_1} - \frac{p_o}{\rho_o}\right) + \hat{h}\right) = (p_2 - p_1)\left(\frac{1}{2}\right)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right). \tag{8.385}$$

Rearranging, we find

$$\frac{\gamma}{\gamma-1}\left(\frac{p_2}{\rho_2}-\frac{p_1}{\rho_1}\right)-(p_2-p_1)\left(\frac{1}{2}\right)\left(\frac{1}{\rho_2}+\frac{1}{\rho_1}\right) = 0, \tag{8.386}$$

$$p_2\left(\frac{\gamma}{\gamma-1}\frac{1}{\rho_2}-\frac{1}{2\rho_2}-\frac{1}{2\rho_1}\right)-p_1\left(\frac{\gamma}{\gamma-1}\frac{1}{\rho_1}-\frac{1}{2\rho_2}-\frac{1}{2\rho_1}\right) = 0, \tag{8.387}$$

$$p_2\left(\frac{\gamma+1}{2(\gamma-1)}\frac{1}{\rho_2}-\frac{1}{2\rho_1}\right)-p_1\left(\frac{\gamma+1}{2(\gamma-1)}\frac{1}{\rho_1}-\frac{1}{2\rho_2}\right) = 0, \tag{8.388}$$

$$p_2\left(\frac{\gamma+1}{\gamma-1}\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)-p_1\left(\frac{\gamma+1}{\gamma-1}\frac{1}{\rho_1}-\frac{1}{\rho_2}\right) = 0. \tag{8.389}$$

The final form is

$$p_2 = p_1\frac{\frac{\gamma+1}{\gamma-1}\frac{1}{\rho_1}-\frac{1}{\rho_2}}{\frac{\gamma+1}{\gamma-1}\frac{1}{\rho_2}-\frac{1}{\rho_1}}. \tag{8.390}$$

We see the Hugoniot for a CPIG

- is a *hyperbola* in $(p, 1/\rho)$ space,
- has as $1/\rho_2 \to (\gamma-1)/(\gamma+1)(1/\rho_1)$ causes $p_2 \to \infty$, note for $\gamma = 7/5$, we get $\rho_2 \to 6\rho_1$ for infinite pressure, and
- has as $1/\rho_2 \to \infty, p_2 \to -p_1(\gamma-1)/(\gamma+1)$; note negative pressure, not physical here.

The Rayleigh line and Hugoniot curve are sketched in Fig. 8.8. Note:

- intersections of the two curves are solutions to the equations,
- the ambient state "1" is one solution,
- the other solution "2" is known as the shock solution,
- the shock solution has higher pressure and higher density,
- higher wave speed implies higher pressure and higher density,
- a minimum wave speed exists, it
  - occurs when the Rayleigh line is tangent to the Hugoniot curve,
  - occurs for infinitesimally small pressure changes,
  - corresponds to a sonic wave speed, and
  - has disturbances that are *acoustic*.
- if pressure increases, it can be shown that entropy increases, and

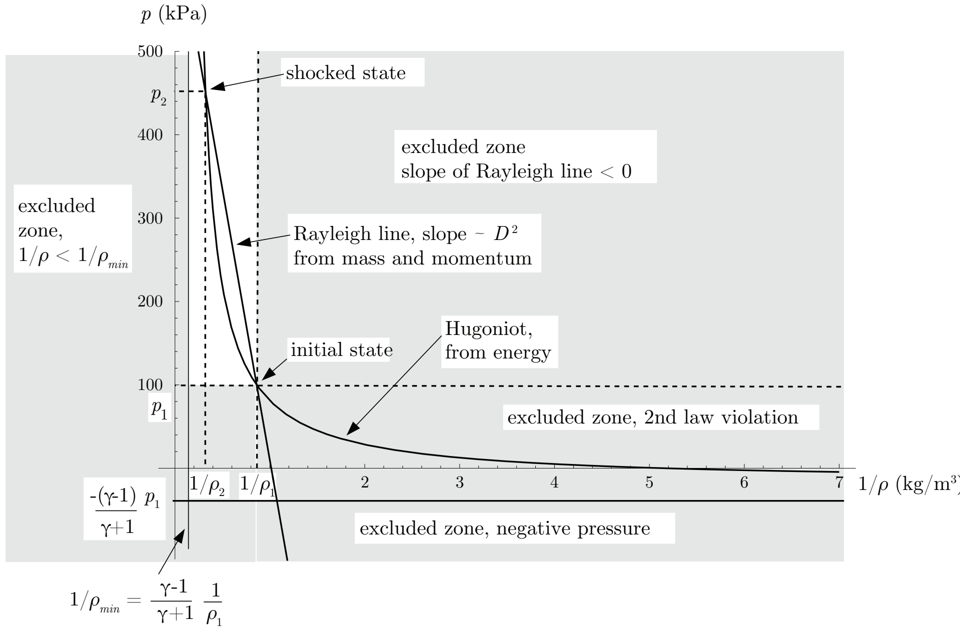

Figure 8.8: Rayleigh line and Hugoniot curve for a typical shocked gas.

- if pressure decreases (for wave speeds that are less than sonic), entropy decreases; this is non-physical.

Substitute the Rayleigh line into the Hugoniot equation to get a single equation for ρ_2:

$$p_1 + \rho_1^2 D^2 \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) = p_1 \frac{\frac{\gamma+1}{\gamma-1}\frac{1}{\rho_1} - \frac{1}{\rho_2}}{\frac{\gamma+1}{\gamma-1}\frac{1}{\rho_2} - \frac{1}{\rho_1}}. \tag{8.391}$$

This equation is quadratic in $1/\rho_2$ and factorizable. Use computer algebra to solve and get two solutions, one ambient $1/\rho_2 = 1/\rho_1$ and one shocked solution:

$$\frac{1}{\rho_2} = \frac{1}{\rho_1}\frac{\gamma-1}{\gamma+1}\left(1 + \frac{2\gamma}{(\gamma-1)\,D^2}\frac{p_1}{\rho_1}\right). \tag{8.392}$$

The shocked density ρ_2 is plotted against wave speed D for CPIG air in Fig. 8.9a. Note

- the density solution allows allows all wave speeds $0 < D < \infty$,
- the range, however, is $D \in [c_1, \infty)$,
- the Rayleigh line and Hugoniot show $D \geq c_1$,
- the solution for $D = D(v_p)$, *(to be shown)*, requires $D \geq c_1$,
- *strong shock limit*: $D^2 \to \infty, \rho_2 \to (\gamma + 1)/(\gamma - 1)$,

- *acoustic limit*: $D^2 \to \gamma p_1/\rho_1, \rho_2 \to \rho_1$, and
- *non-physical limit*: $D^2 \to 0, \rho_2 \to 0$.

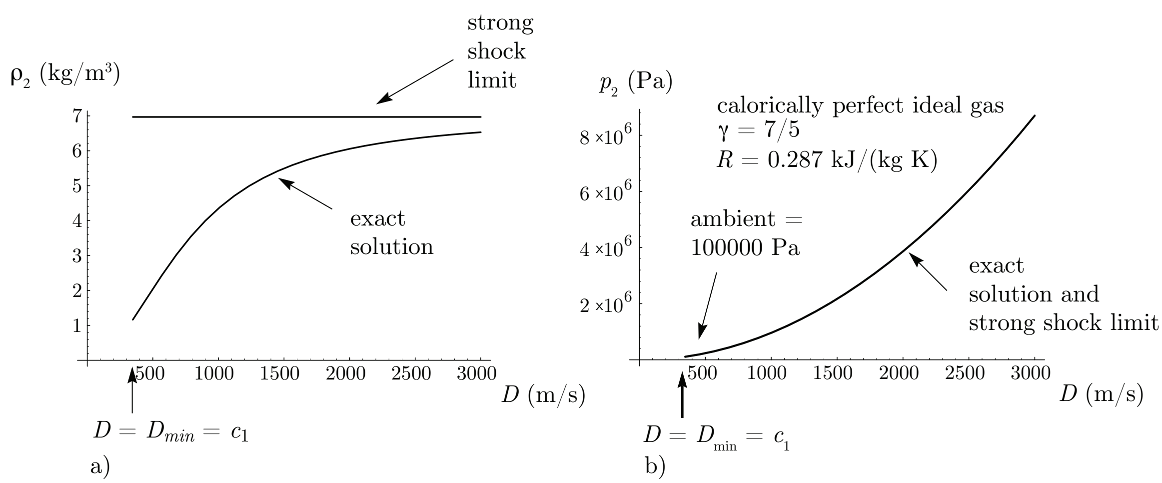

Figure 8.9: Shock a) density and b) pressure versus shock wave speed for a CPIG.

Back substitute into Rayleigh line and mass conservation to solve for the shocked pressure and the fluid velocity in the shocked wave frame:

$$p_2 = \frac{2}{\gamma+1}\rho_1 D^2 - \frac{\gamma-1}{\gamma+1}p_1, \tag{8.393}$$

$$u_2 = -D\frac{\gamma-1}{\gamma+1}\left(1+\frac{2\gamma}{(\gamma-1)\,D^2}\frac{p_1}{\rho_1}\right). \tag{8.394}$$

The shocked pressure p_2 is plotted against wave speed D for CPIG air in Fig. 8.9b including both the exact solution and the solution in the strong shock limit. For these parameters, the results are indistinguishable. The shocked wave frame fluid particle velocity u_2 is plotted against wave speed D for CPIG air in Fig. 8.10a. The shocked wave frame fluid particle Mach number, $M_2^2 = \rho_2 u_2^2/(\gamma p_2)$, is plotted against wave speed D for CPIG air in Fig. 8.10b. In the steady frame, the Mach number of the

- undisturbed flow is (and must be) > 1: *supersonic*, and
- shocked flow is (and must be) < 1: *subsonic*.

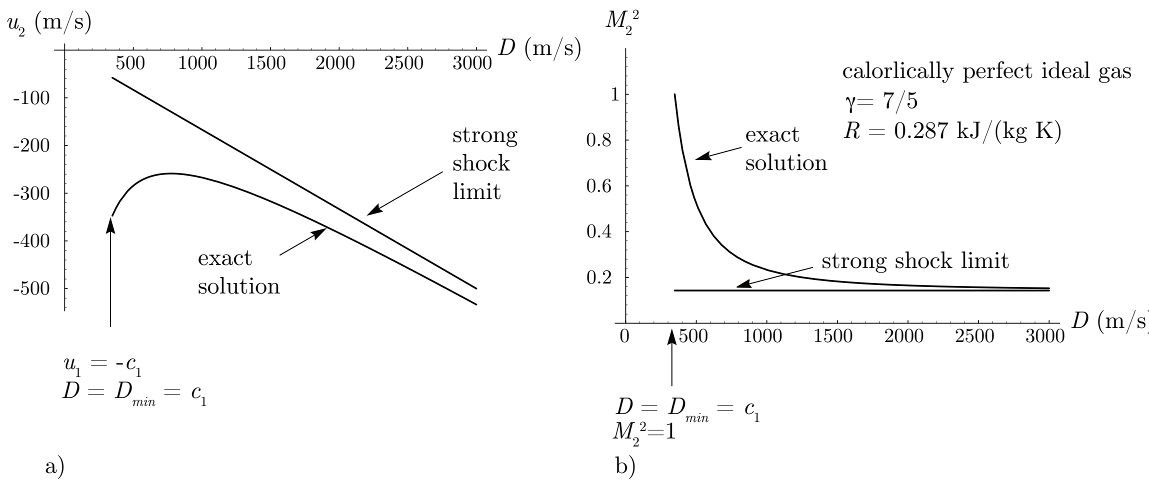

Figure 8.10: a) Shock wave frame fluid particle velocity and b) Mach number squared of shocked fluid particle versus shock wave speed for CPIG.

Transform back to the laboratory frame $u = v - D$:

$$v_2 - D = -D\frac{\gamma-1}{\gamma+1}\left(1+\frac{2\gamma}{(\gamma-1)D^2}\frac{p_1}{\rho_1}\right), \tag{8.395}$$

$$v_2 = D - D\frac{\gamma-1}{\gamma+1}\left(1+\frac{2\gamma}{(\gamma-1)D^2}\frac{p_1}{\rho_1}\right). \tag{8.396}$$

Manipulate this equation and solve the resulting quadratic equation for D and get

$$D = \frac{\gamma+1}{4}v_2 \pm \sqrt{\frac{\gamma p_1}{\rho_1} + v_2^2\left(\frac{\gamma+1}{4}\right)^2}. \tag{8.397}$$

Now if $v_2 > 0$, we expect $D > 0$ so take positive root, also set the velocity equal to the piston velocity $v_2 = v_p$.

$$D = \frac{\gamma+1}{4}v_p + \sqrt{\frac{\gamma p_1}{\rho_1} + v_p^2\left(\frac{\gamma+1}{4}\right)^2}. \tag{8.398}$$

Note:

- *acoustic limit*: as $v_p \to 0$, $D \to c_1$; the shock speed approaches the sound speed, and
- *strong shock limit*: as $v_p \to \infty, D \to v_p(\gamma+1)/2$.

The shock speed D is plotted against piston velocity v_p for CPIG air in Fig. 8.11a. Both the exact solution and strong shock limit are shown. If we define the Mach number of the shock

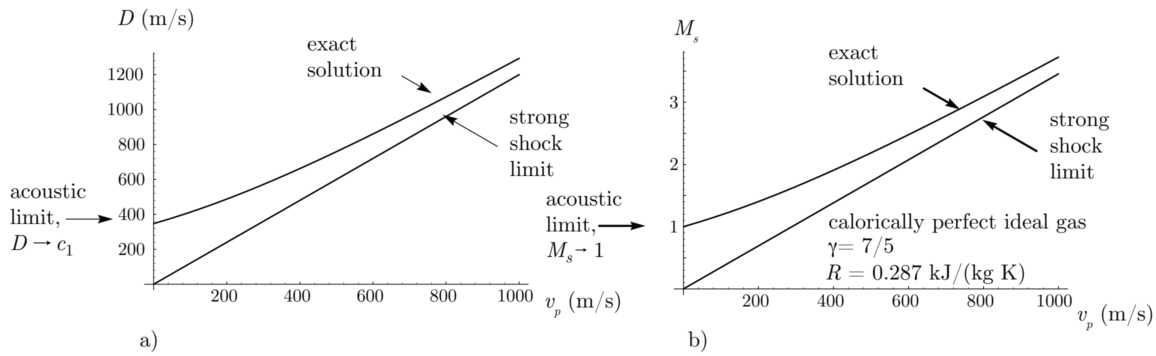

Figure 8.11: a) Shock speed and b) shock Mach number versus piston velocity for CPIG.

as

$$M_s \equiv \frac{D}{c_1}, \tag{8.399}$$

we get

$$M_s = \frac{\gamma+1}{4}\frac{v_p}{\sqrt{\gamma R T_1}} + \sqrt{1 + \frac{v_p^2}{\gamma R T_1}\left(\frac{\gamma+1}{4}\right)^2}. \tag{8.400}$$

The shock Mach number M_s is plotted against piston velocity v_p for CPIG air in Fig. 8.11b. Both the exact solution and strong shock limit are shown.

Let us find the entropy change induced by a shock for a CPIG. We first need an expression for the entropy change. Begin with the Gibbs equation, Eq. (4.161):

$$T\ ds = de + p\ d\hat{v}, \tag{8.401}$$

$$ds = \frac{de}{T} + \frac{p}{T}\ d\hat{v}. \tag{8.402}$$

Now invoke the CPIG assumption to get

$$ds = c_v \frac{dT}{T} + R\frac{d\hat{v}.}{\hat{v}}. \tag{8.403}$$

Now for the ideal gas with $p\hat{v} = RT$, we get

$$p\ d\hat{v} + \hat{v}\ dp = R\ dT. \tag{8.404}$$

Divide the left side by $p\hat{v}$ and the right side by the equivalent RT to get

$$\frac{d\hat{v}}{\hat{v}} + \frac{dp}{p} = \frac{dT}{T}. \tag{8.405}$$

Use this to eliminate temperature in Eq. (8.403) to get

$$\begin{aligned}
ds &= c_v\left(\frac{d\hat{v}}{\hat{v}}+\frac{dp}{p}\right)+R\frac{d\hat{v}.}{\hat{v}}, && (8.406)\\
&= (c_v+R)\frac{d\hat{v}}{\hat{v}}+c_v\frac{dp}{p}, && (8.407)\\
&= (c_v+(c_p-c_v))\frac{d\hat{v}}{\hat{v}}+c_v\frac{dp}{p}, && (8.408)\\
&= c_p\frac{d\hat{v}}{\hat{v}}+c_v\frac{dp}{p}, && (8.409)\\
ds &= c_v\left(\gamma\frac{d\hat{v}}{\hat{v}}+\frac{dp}{p}\right), && (8.410)\\
s_2-s_1 &= c_v\left(\gamma\ln\frac{\hat{v}_2}{\hat{v}_1}+\ln\frac{p_2}{p_1}\right), && (8.411)\\
s_2-s_1 &= c_v\left(\ln\left(\frac{\hat{v}_2}{\hat{v}_1}\right)^{\gamma}+\ln\frac{p_2}{p_1}\right), && (8.412)\\
&= c_v\left(\ln\left(\frac{\rho_1}{\rho_2}\right)^{\gamma}+\ln\frac{p_2}{p_1}\right), && (8.413)\\
&= c_v\ln\left(\left(\frac{\rho_1}{\rho_2}\right)^{\gamma}\frac{p_2}{p_1}\right). && (8.414)
\end{aligned}$$

Then we use Eqs. (8.392, 8.393, and 8.399) to eliminate the pressure and density ratios in favor of M_s, and follow this with algebraic reduction to arrive at

$$\frac{s_2-s_1}{c_v}=\ln\left(\frac{\left(\frac{(\gamma-1)M_s^2+2}{(\gamma+1)M_s^2}\right)^{\gamma}\left(\gamma\left(2M_s^2-1\right)+1\right)}{\gamma+1}\right). \qquad (8.415)$$

For $\gamma = 7/5$, we plot $(s_2 - s_1)/c_v$ as a function of M_s in Fig. 8.12. Clearly for $M_s = 1$, we have $(s_2 - s_1)/c_v = 0$, so the sonic wave is isentropic. And clearly for $M_s > 1$, the entropy rises, thus satisfying the second law for what is an adiabatic irreversible compression. For $M_s < 1$, the entropy is predicted to fall for an adiabatic expansion. This is not observed in nature and violates the second law of thermodynamics. So we must have $M_s \geq 1$ for a propagating discontinuity. The equation for entropy jump is complicated. We can better understand it by performing a Taylor series expansion in the neighborhood of $M_s = 1$. Doing so yields

$$\frac{s_2-s_1}{c_v}=\frac{16\gamma\left(\gamma-1\right)}{3(\gamma+1)^2}(M_s-1)^3+\mathcal{O}\left((M_s-1)^4\right). \qquad (8.416)$$

Clearly for general $\gamma > 1$, the entropy rises for $M_s > 1$ and falls for $M_s < 1$, and the local behavior is cubic in the deviation of M_s from unity.

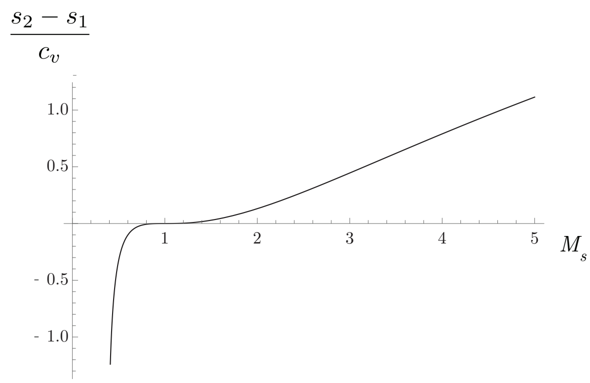

Figure 8.12: Scaled entropy jump through a discontinuity as a function of M_s for a CPIG with $\gamma = 7/5$.

8.4.6 Acoustic limit

Consider that state 2 is a small perturbation of state 1 so that

$$\rho_2 = \rho_1 + \Delta\rho, \tag{8.417}$$
$$u_2 = u_1 + \Delta u_1, \tag{8.418}$$
$$p_2 = p_1 + \Delta p. \tag{8.419}$$

Substituting into the normal shock equations, we get

$$(\rho_1 + \Delta\rho)(u_1 + \Delta u) = \rho_1 u_1, \tag{8.420}$$
$$(\rho_1 + \Delta\rho)(u_1 + \Delta u)^2 + (p_1 + \Delta p) = \rho_1 {u_1}^2 + p_1, \tag{8.421}$$
$$\frac{\gamma}{\gamma - 1}\frac{p_1 + \Delta p}{\rho_1 + \Delta\rho} + \frac{1}{2}(u_1 + \Delta u)^2 = \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{1}{2}{u_1}^2. \tag{8.422}$$

Expanding, we get

$$\rho_1 u_1 + u_1(\Delta\rho) + \rho_1(\Delta u) + (\Delta\rho)(\Delta u) = \rho_1 u_1, \tag{8.423}$$
$$\left(\rho_1 {u_1}^2 + 2\rho_1 u_1(\Delta u) + {u_1}^2(\Delta\rho) + \rho_1(\Delta u)^2 + 2u_1(\Delta u)(\Delta\rho) + (\Delta\rho)(\Delta u)^2\right) + (p_1 + \Delta p) = \rho_1 {u_1}^2 + p_1, \tag{8.424}$$
$$\frac{\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1} + \frac{1}{\rho_1}\Delta p - \frac{p_1}{\rho_1^2}\Delta\rho + \ldots\right) + \frac{1}{2}\left({u_1}^2 + 2u_1(\Delta u) + (\Delta u)^2\right) \tag{8.425}$$
$$= \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{1}{2}{u_1}^2. \tag{8.426}$$

Subtracting the base state and eliminating products of small quantities yields

$$u_1(\Delta\rho) + \rho_1(\Delta u) = 0, \tag{8.427}$$
$$2\rho_1 u_1(\Delta u) + {u_1}^2(\Delta\rho) + \Delta p = 0, \tag{8.428}$$
$$\frac{\gamma}{\gamma - 1}\left(\frac{1}{\rho_1}\Delta p - \frac{p_1}{\rho_1^2}\Delta\rho\right) + u_1(\Delta u) = 0. \tag{8.429}$$

In matrix form this is

$$\begin{pmatrix} u_1 & \rho_1 & 0 \\ {u_1}^2 & 2\rho_1 u_1 & 1 \\ -\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1^2} & u_1 & \frac{\gamma}{\gamma-1}\frac{1}{\rho_1} \end{pmatrix} \begin{pmatrix} \Delta\rho \\ \Delta u \\ \Delta p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{8.430}$$

As the right hand side is zero, the determinant must be zero and there must be a linear dependency of the solution. First check the determinant:

$$u_1\left(\frac{2\gamma}{\gamma-1}u_1 - u_1\right) - \rho_1\left(\frac{\gamma}{\gamma-1}\frac{{u_1}^2}{\rho_1} + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1^2}\right) = 0, \tag{8.431}$$

$$\frac{{u_1}^2}{\gamma-1}\left(2\gamma-(\gamma-1)\right) - \frac{1}{\gamma-1}\left(\gamma\ {u_1}^2 + \gamma\ \frac{p_1}{\rho_1}\right) = 0, \tag{8.432}$$

$${u_1}^2(\gamma+1) - \left(\gamma {u_1}^2 + \gamma\ \frac{p_1}{\rho_1}\right) = 0, \tag{8.433}$$

$${u_1}^2 = \gamma\frac{p_1}{\rho_1} = c_1^2. \tag{8.434}$$

So the velocity is necessarily sonic for a small disturbance.

Take Δu to be known and solve a resulting 2×2 system:

$$\begin{pmatrix} u_1 & 0 \\ -\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1^2} & \frac{\gamma}{\gamma-1}\frac{1}{\rho_1} \end{pmatrix} \begin{pmatrix} \Delta\rho \\ \Delta p \end{pmatrix} = \begin{pmatrix} -\rho_1\Delta u \\ -u_1\Delta u \end{pmatrix}. \tag{8.435}$$

Solving yields

$$\Delta\rho = -\frac{\rho_1\Delta u}{\sqrt{\gamma\frac{p_1}{\rho_1}}} = -\rho_1\frac{\Delta u}{c_1}, \tag{8.436}$$

$$\Delta p = -\rho_1\sqrt{\gamma\frac{p_1}{\rho_1}}\ \Delta u = -\rho_1 c_1\Delta u. \tag{8.437}$$

8.5 Flow with area change and normal shocks

This section will consider flow from a reservoir with the fluid at stagnation conditions to a constant pressure environment. The pressure of the environment is commonly known as the *back pressure*: p_b.

Generic problem: Given $A(x)$, stagnation conditions and p_b, find the pressure, temperature, density at all points in the duct and the mass flow rate.

8.5.1 Converging nozzle

A converging nozzle operating at several different values of p_b is sketched in Fig. 8.13. The flow through the duct can be solved using the following procedure:

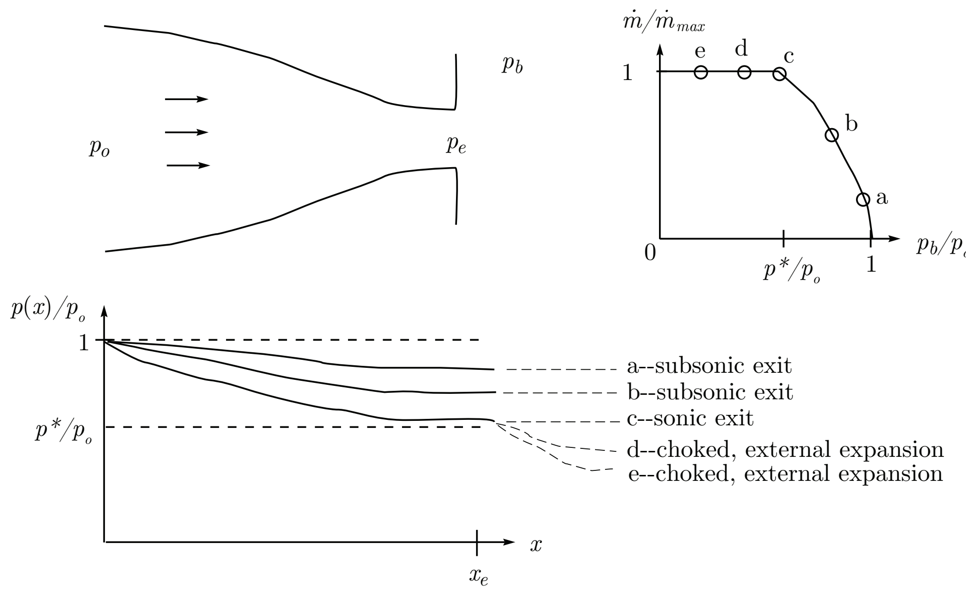

Figure 8.13: Converging nozzle sketch.

- check if $p_b \geq p_*$,
- if so, set $p_e = p_b$,
- determine M_e from isentropic flow relations,
- determine A_* from A/A_* relation,
- at any point in the flow where A is known, compute A/A_* and then invert A/A_* relation to find local M.

Note:

- These flows are subsonic throughout and correspond to points a and b in Fig. 8.13.
- If $p_b = p_*$ then the flow is sonic at the exit and just choked. This corresponds to point c in Fig. 8.13.
- If $p_b < p_*$, then the flow chokes, is sonic at the exit, and continues to expand outside of the nozzle. This corresponds to points d and e in Fig. 8.13.

8.5.2 Converging-diverging nozzle

A converging-diverging nozzle operating at several different values of p_b is sketched in Fig. 8.14.

The flow through the duct can be solved using the a similar following procedure

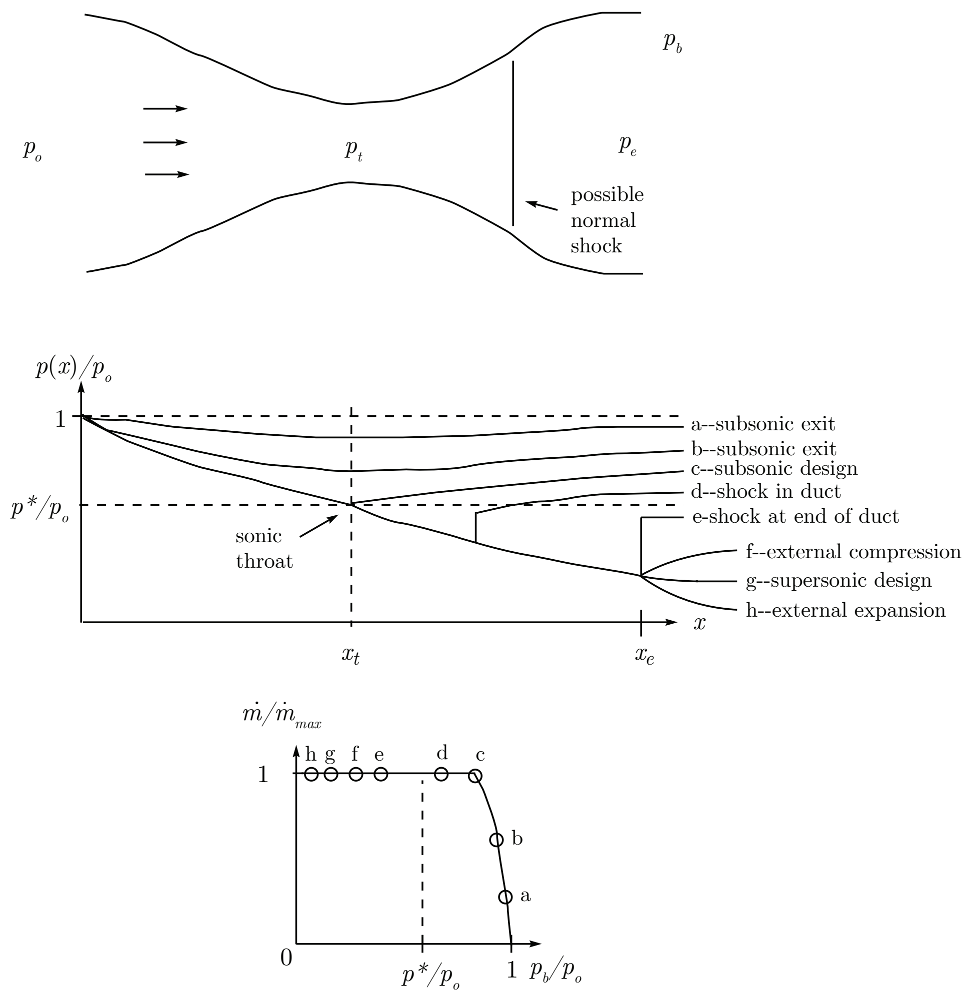

Figure 8.14: Converging-diverging nozzle sketch.

- set $A_t = A_*$,
- with this assumption, calculate A_e/A_*,
- determine M_{esub}, M_{esup}, both supersonic and subsonic, from A/A_* relation,
- determine p_{esub}, p_{esup}, from M_{esub}, M_{esup}; these are the supersonic and subsonic design pressures,
- if $p_b > p_{esub}$, the flow is subsonic throughout and the throat is not sonic. Use same procedure as for converging duct: Determine M_e by setting $p_e = p_b$ and using isentropic relations,
- if $p_{esub} > p_b > p_{esup}$, the procedure is complicated.
  - estimate the pressure with a normal shock at the end of the duct, p_{esh}.

  - If $p_b \geq p_{esh}$, there is a normal shock inside the duct,
  - If $p_b < p_{esh}$, the duct flow is shockless, and there may be compression outside the duct.

- if $p_{esup} = p_b$, the flow is at supersonic design conditions and the flow is shockless, and
- if $p_b < p_{esup}$, the flow in the duct is isentropic and there is expansion outside the duct.

8.6 Method of characteristics

Here we discuss how to use the so-called *method of characteristics* to model a one-dimensional unsteady, inviscid, non-heat conducting fluid. The emphasis will be on rarefaction waves. This analysis is a good deal more rigorous than much of traditional one-dimensional gas dynamics, and draws upon some of the more difficult mathematical methods we will encounter.

In assuming no diffusive transport, we have eliminated all mechanisms for entropy generation; consequently, we will be able to model the process as isentropic. We note that even without diffusion, shocks can generate entropy. However, the expansion waves are inherently continuous, and do remain isentropic. We will consider a general equation of state, and later specialize to a CPIG. The problem is inherently non-linear and is modeled by partial differential equations of the type that is known as *hyperbolic*. Such problems, in contrast to say Laplace's equation, that requires boundary conditions, require initial data only, and no boundary data.

8.6.1 Inviscid one-dimensional equations

The equations to be considered are shown here in non-conservative form. These are one-dimensional limits of the system developed in Ch. 6.4:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0, \tag{8.438}$$

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0, \tag{8.439}$$

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = 0, \tag{8.440}$$

$$p = p(\rho, s). \tag{8.441}$$

Here we have written the energy equation in terms of entropy. The development of this was shown in Ch. 4.4.5.6. We have also utilized the general result from thermodynamics that any intensive property can be written as a function of two other independent thermodynamic properties. Here we have chosen to write pressure as a function of density and entropy, as we did in Eq. (8.129) for a special case. Thus, we have four equations for the four unknowns, ρ, u, p, s.

Now we note that

$$dp = \left.\frac{\partial p}{\partial \rho}\right|_s d\rho + \left.\frac{\partial p}{\partial s}\right|_\rho ds, \qquad \text{so,} \tag{8.442}$$

$$\left.\frac{\partial p}{\partial x}\right|_t = \left.\frac{\partial p}{\partial \rho}\right|_s \left.\frac{\partial \rho}{\partial x}\right|_t + \left.\frac{\partial p}{\partial s}\right|_\rho \left.\frac{\partial s}{\partial x}\right|_t. \tag{8.443}$$

Now, let us define thermodynamic properties c^2 and ζ as follows

$$c^2 \equiv \left.\frac{\partial p}{\partial \rho}\right|_s, \qquad \zeta \equiv \left.\frac{\partial p}{\partial s}\right|_\rho. \tag{8.444}$$

We will see that ζ will be unimportant, and will be able to ascribe to c the physical significance of the speed of propagation of small disturbances, the so-called sound speed, that we have already encountered in acoustics. If we know the equation of state, then we can think of c^2 and ζ as known thermodynamic functions of ρ and s. Our definitions give us

$$\frac{\partial p}{\partial x} = c^2 \frac{\partial \rho}{\partial x} + \zeta \frac{\partial s}{\partial x}. \tag{8.445}$$

Substituting into our governing equations, we see that pressure can be eliminated to give three equations in three unknowns:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0, \tag{8.446}$$

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \underbrace{c^2\frac{\partial \rho}{\partial x} + \zeta\frac{\partial s}{\partial x}}_{\frac{\partial p}{\partial x}} = 0, \tag{8.447}$$

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = 0. \tag{8.448}$$

Now we can say that if $s = s(x,t)$,

$$ds = \frac{\partial s}{\partial t}\, dt + \frac{\partial s}{\partial x}\, dx, \tag{8.449}$$

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} + \frac{dx}{dt}\frac{\partial s}{\partial x}, \tag{8.450}$$

$$= \frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x}. \tag{8.451}$$

Thus, on curves where $dx/dt = u$ (that by definition are particle pathlines), we have from substituting Eq. (8.451) into the energy equation (8.448)

$$\frac{ds}{dt} = 0. \tag{8.452}$$

Thus we have converted the partial differential equation for energy conservation into an ordinary differential equation. This can be integrated to give us

$$s = C, \qquad \text{on a particle pathline,} \quad \frac{dx}{dt} = u. \tag{8.453}$$

This scenario is sketched on the so-called $x - t$ diagram of Fig. 8.15.

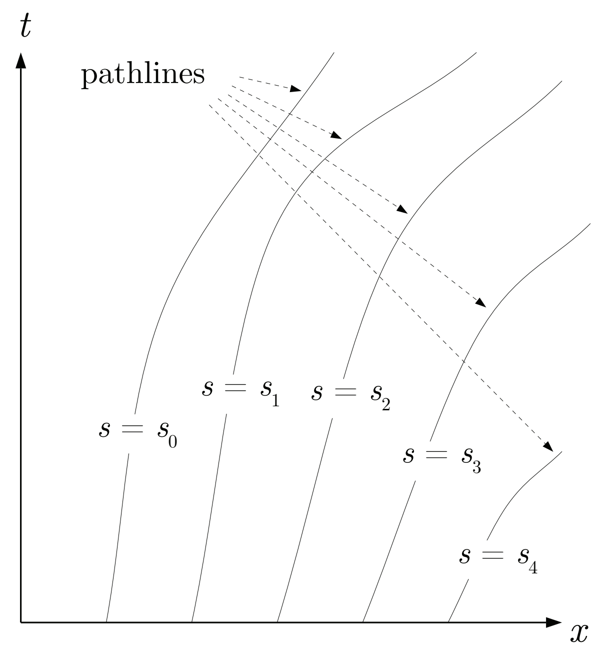

Figure 8.15: $x - t$ diagram showing maintenance of entropy s along particle pathlines $dx/dt = u$ for isentropic flow.

This result is satisfying, but not complete, as we do not in general know where the pathlines are. Let us try to apply this technique to the system in general. Consider our equations in matrix form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial \rho}{\partial t} \\ \frac{\partial u}{\partial t} \\ \frac{\partial s}{\partial t} \end{pmatrix} + \begin{pmatrix} u & \rho & 0 \\ c^2 & \rho u & \zeta \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} \frac{\partial \rho}{\partial x} \\ \frac{\partial u}{\partial x} \\ \frac{\partial s}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{8.454}$$

These equations are of the form

$$A_{ij} \frac{\partial u_j}{\partial t} + B_{ij} \frac{\partial u_j}{\partial x} = f_i. \tag{8.455}$$

As described by Whitham,[12] there is a general technique to analyze such equations. First pre-multiply both sides of the equation by a yet to be determined vector of variables ℓ_i:

$$\ell_i A_{ij} \frac{\partial u_j}{\partial t} + \ell_i B_{ij} \frac{\partial u_j}{\partial x} = \ell_i f_i. \tag{8.456}$$

[12]Gerald Beresford Whitham, 1927-2014, applied mathematician and developer of theory for non-linear wave propagation.

Now, this method will work if we can choose ℓ_i to render this product to be of the form similar to $\partial/\partial t + u(\partial/\partial x)$. Let us take

$$\ell_i A_{ij}\frac{\partial u_j}{\partial t} + \ell_i B_{ij}\frac{\partial u_j}{\partial x} = m_j\left(\frac{\partial u_j}{\partial t} + \lambda\frac{\partial u_j}{\partial x}\right), \tag{8.457}$$

$$= m_j\frac{du_j}{dt} \quad \text{on} \quad \frac{dx}{dt} = \lambda. \tag{8.458}$$

So comparing terms, we see that

$$\ell_i A_{ij} = m_j, \qquad \ell_i B_{ij} = \lambda m_j, \tag{8.459}$$

$$\lambda \ell_i A_{ij} = \lambda m_j, \tag{8.460}$$

so, we get by eliminating m_j that

$$\ell_i\left(\lambda A_{ij} - B_{ij}\right) = 0. \tag{8.461}$$

This is a left eigenvalue problem. We set the determinant of $\lambda A_{ij} - B_{ij}$ to zero for a non-trivial solution and find

$$\begin{vmatrix} \lambda - u & -\rho & 0 \\ -c^2 & \rho(\lambda-u) & -\zeta \\ 0 & 0 & \lambda - u \end{vmatrix} = 0. \tag{8.462}$$

Evaluating, we get

$$(\lambda-u)\left(\rho(\lambda-u)^2\right) + \rho(\lambda-u)(-c^2) = 0, \tag{8.463}$$

$$\rho(\lambda-u)\left((\lambda-u)^2 - c^2\right) = 0. \tag{8.464}$$

Solving we get

$$\lambda = u, \qquad \lambda = u \pm c. \tag{8.465}$$

Now the left eigenvectors ℓ_i give us the actual equations. First for $\lambda = u$, we get

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \end{pmatrix}\begin{pmatrix} u-u & -\rho & 0 \\ -c^2 & \rho(u-u) & -\zeta \\ 0 & 0 & u-u \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \tag{8.466}$$

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \end{pmatrix}\begin{pmatrix} 0 & -\rho & 0 \\ -c^2 & 0 & -\zeta \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}. \tag{8.467}$$

Two of the equations require that $\ell_1 = 0$ and $\ell_2 = 0$. There is no restriction on ℓ_3. We will select a normalized solution so that

$$\ell_i = (0, 0, 1). \tag{8.468}$$

Thus, $\ell_i A_{ij}(\partial u_j/\partial t) + \ell_i B_{ij}(\partial u_j/\partial x) = \ell_i f_i$ gives

$$\begin{aligned}
\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial \rho}{\partial t} \\ \frac{\partial u}{\partial t} \\ \frac{\partial s}{\partial t} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u & \rho & 0 \\ c^2 & \rho u & \zeta \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} \frac{\partial \rho}{\partial x} \\ \frac{\partial u}{\partial x} \\ \frac{\partial s}{\partial x} \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\
\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial \rho}{\partial t} \\ \frac{\partial u}{\partial t} \\ \frac{\partial s}{\partial t} \end{pmatrix} + \begin{pmatrix} 0 & 0 & u \end{pmatrix} \begin{pmatrix} \frac{\partial \rho}{\partial x} \\ \frac{\partial u}{\partial x} \\ \frac{\partial s}{\partial x} \end{pmatrix} &= 0, \\
\frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} &= 0. \qquad (8.469)
\end{aligned}$$

So as before with $s = s(x,t)$, we have $ds = (\partial s/\partial t)dt + (\partial s/\partial x)dx$, and $ds/dt = \partial s/\partial t + (dx/dt)(\partial s/\partial x)$. Now *if* we require dx/dt to be a particle pathline, $dx/dt = u$, then our energy equation, Eq. (8.469), gives us

$$\frac{ds}{dt} = 0, \qquad \text{on} \qquad \frac{dx}{dt} = u. \qquad (8.470)$$

The special case in which the pathlines are straight in $x-t$ space, corresponding to a uniform velocity field of $u(x,t) = u_o$, is sketched in the $x-t$ diagram of Fig. 8.16.

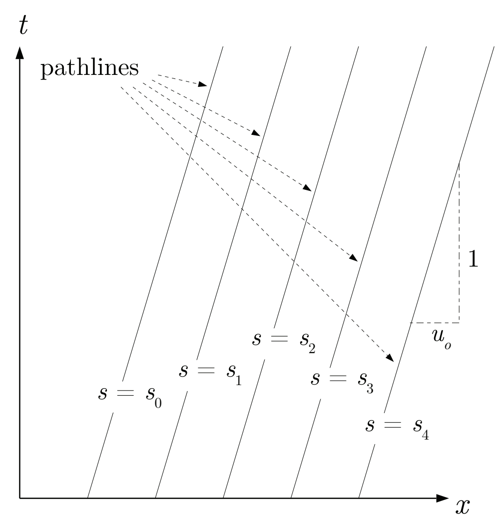

Figure 8.16: $x - t$ diagram showing maintenance of entropy s along particle pathlines $dx/dt = u_o$ for isentropic flow.

Now let us look at the remaining eigenvalues, $\lambda = u \pm c$.

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \end{pmatrix} \begin{pmatrix} u \pm c - u & -\rho & 0 \\ -c^2 & \rho(u \pm c - u) & -\zeta \\ 0 & 0 & u \pm c - u \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \qquad (8.471)$$

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \end{pmatrix} \begin{pmatrix} \pm c & -\rho & 0 \\ -c^2 & \pm \rho c & -\zeta \\ 0 & 0 & \pm c \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}. \qquad (8.472)$$

As one of the components of the left eigenvector should be arbitrary, we will take $\ell_1 = 1$; we arrive at the following equations then

$$\pm c - c^2\ell_2 = 0, \Longrightarrow \ell_2 = \pm\frac{1}{c}, \tag{8.473}$$

$$-\rho \pm \rho c\ell_2 = 0, \Longrightarrow \ell_2 = \pm\frac{1}{c}, \tag{8.474}$$

$$-\zeta\ell_2 \pm c\ell_3 = 0, \Longrightarrow \ell_3 = \frac{\zeta}{c^2}. \tag{8.475}$$

Thus, $\ell_i A_{ij}(\partial u_j/\partial t) + \ell_i B_{ij}(\partial u_j/\partial x) = \ell_i f_i$ gives

$$\begin{pmatrix} 1 & \pm\frac{1}{c} & \frac{\zeta}{c^2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial\rho}{\partial t} \\ \frac{\partial u}{\partial t} \\ \frac{\partial s}{\partial t} \end{pmatrix} + \begin{pmatrix} 1 & \pm\frac{1}{c} & \frac{\zeta}{c^2} \end{pmatrix} \begin{pmatrix} u & \rho & 0 \\ c^2 & \rho u & \zeta \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} \frac{\partial\rho}{\partial x} \\ \frac{\partial u}{\partial x} \\ \frac{\partial s}{\partial x} \end{pmatrix} = \begin{pmatrix} 1 & \pm\frac{1}{c} & \frac{\zeta}{c^2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{8.476}$$

$$\begin{pmatrix} 1 & \pm\frac{\rho}{c} & \frac{\zeta}{c^2} \end{pmatrix} \begin{pmatrix} \frac{\partial\rho}{\partial t} \\ \frac{\partial u}{\partial t} \\ \frac{\partial s}{\partial t} \end{pmatrix} + \begin{pmatrix} u \pm c & \rho \pm \rho\frac{u}{c} & \pm\frac{\zeta}{c} + \frac{\zeta u}{c^2} \end{pmatrix} \begin{pmatrix} \frac{\partial\rho}{\partial x} \\ \frac{\partial u}{\partial x} \\ \frac{\partial s}{\partial x} \end{pmatrix} = 0, \tag{8.477}$$

$$\frac{\partial\rho}{\partial t} + (u \pm c)\frac{\partial\rho}{\partial x} \pm \frac{\rho}{c}\frac{\partial u}{\partial t} + \rho\left(1 \pm \frac{u}{c}\right)\frac{\partial u}{\partial x} + \frac{\zeta}{c^2}\frac{\partial s}{\partial t} + \left(\frac{\zeta u}{c^2} \pm \frac{\zeta}{c}\right)\frac{\partial s}{\partial x} = 0, \tag{8.478}$$

$$\left(\frac{\partial\rho}{\partial t} + (u \pm c)\frac{\partial\rho}{\partial x}\right) \pm \frac{\rho}{c}\left(\frac{\partial u}{\partial t} + (u \pm c)\frac{\partial u}{\partial x}\right) + \frac{\zeta}{c^2}\left(\frac{\partial s}{\partial t} + (u \pm c)\frac{\partial s}{\partial x}\right) = 0, \tag{8.479}$$

$$c^2\underbrace{\left(\frac{\partial\rho}{\partial t} + (u \pm c)\frac{\partial\rho}{\partial x}\right)}_{d\rho/dt} \pm \rho c\underbrace{\left(\frac{\partial u}{\partial t} + (u \pm c)\frac{\partial u}{\partial x}\right)}_{du/dt} + \zeta\underbrace{\left(\frac{\partial s}{\partial t} + (u \pm c)\frac{\partial s}{\partial x}\right)}_{ds/dt} = 0. \tag{8.480}$$

Now on lines where $dx/dt = u \pm c$, we get a transformation of the partial differential equations to ordinary differential equations:

$$c^2\frac{d\rho}{dt} \pm \rho c\frac{du}{dt} + \zeta\frac{ds}{dt} = 0, \qquad \text{on} \qquad \frac{dx}{dt} = u \pm c. \tag{8.481}$$

A sketch of the *characteristics*, the lines on which the differential equations are obtained, is given in the $x - t$ diagram of Fig. 8.17.

8.6.2 Homeoentropic flow of a calorically perfect ideal gas

The equations developed so far are valid for a general equation of state. Here let us now consider the flow of a CPIG, so $p = \rho RT$ and $e = c_v T + \hat{e}$. Consequently, we have the standard relation for the square of the sound speed of a CPIG:

$$c^2 = \gamma\frac{p}{\rho}. \tag{8.482}$$

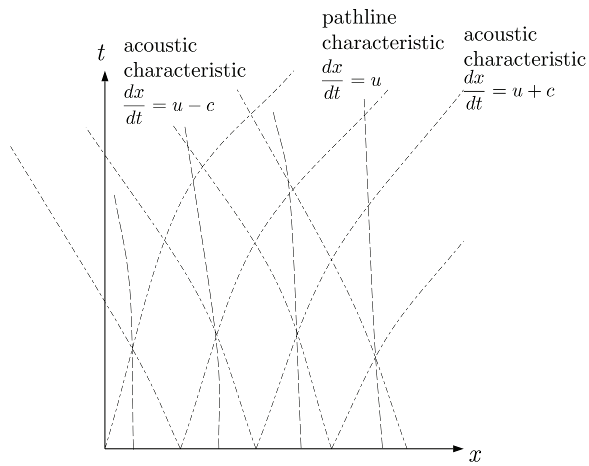

Figure 8.17: $x - t$ diagram showing characteristics for pathlines $dx/dt = u$ and acoustic waves $dx/dt = u \pm c$.

Further let us take the flow to be homeoentropic (as introduced in Ch. 6.4.3), that is to say, not only does the entropy remain constant on pathlines, which is isentropic, but it has the same value on each streamline. That is the entropy field is a constant. Consequently, we have the standard relation for a CPIG:

$$\frac{p}{\rho^\gamma} = A, \tag{8.483}$$

where A is a constant. Because of homeoentropy, we no longer need consider the energy equation, and the linear combination of mass and linear momentum equations, Eq. (8.481), reduces to

$$c^2 \frac{d\rho}{dt} \pm \rho c \frac{du}{dt} = 0, \qquad \text{on} \qquad \frac{dx}{dt} = u \pm c. \tag{8.484}$$

Rearranging, we get

$$\frac{du}{dt} = \mp \frac{d\rho}{dt} \frac{c}{\rho}, \qquad \text{on} \qquad \frac{dx}{dt} = u \pm c. \tag{8.485}$$

Now $c^2 = \gamma p/\rho = \gamma A \rho^{\gamma - 1}$; thus, $c = \sqrt{\gamma A} \rho^{\frac{\gamma-1}{2}}$, so

$$\frac{du}{dt} = \mp \sqrt{\gamma A} \rho^{\frac{\gamma-1}{2}} \rho^{-1} \frac{d\rho}{dt}, \tag{8.486}$$

$$= \mp \sqrt{\gamma A} \frac{2}{\gamma - 1} \frac{d}{dt} \left(\rho^{\frac{\gamma-1}{2}} \right). \tag{8.487}$$

Regrouping, we find

$$\frac{d}{dt} \left(u \pm \sqrt{\gamma A} \frac{2}{\gamma - 1} \rho^{\frac{\gamma-1}{2}} \right) = 0, \tag{8.488}$$

$$\frac{d}{dt} \left(u \pm \frac{2}{\gamma - 1} c \right) = 0. \tag{8.489}$$

Following notation used by Courant[13] and Friedrichs,[14] (1976) we then integrate each of these equations, both of which are homogeneous, along characteristics to obtain algebraic relations

$$u + \frac{2}{\gamma - 1} c = 2r, \quad \text{on} \quad \frac{dx}{dt} = u + c, \quad C^+ \text{ characteristic}, \tag{8.490}$$

$$u - \frac{2}{\gamma - 1} c = -2s, \quad \text{on} \quad \frac{dx}{dt} = u - c, \quad C^- \text{ characteristic}. \tag{8.491}$$

Courant and Friedrichs' s has no relation to entropy; it is just a new variable introduced for convenience. A sketch of the characteristics is given in the $x-t$ diagram of Fig. 8.18. Now r and s can take on different values, depending on which characteristic we are on. On

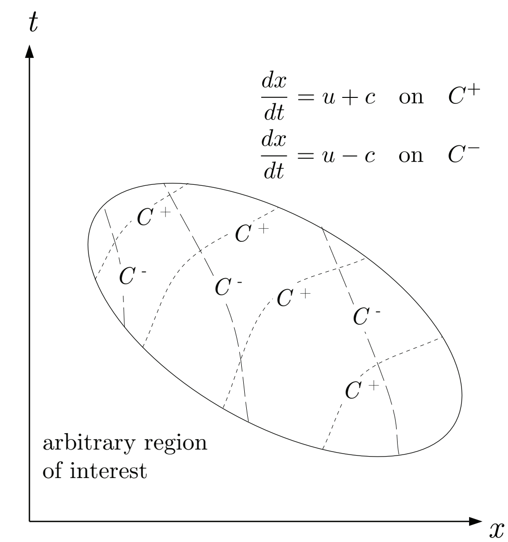

Figure 8.18: $x-t$ diagram showing C^+ and C^- characteristics $dx/dt = u \pm c$.

a given characteristic, they remain constant. Let us define additional parameters α and β to identify which characteristic we are on. So we have

$$u + \frac{2}{\gamma - 1} c = 2r(\beta), \quad \text{on} \quad \frac{dx}{dt} = u + c, \quad C^+ \text{ characteristic}, \tag{8.492}$$

$$u - \frac{2}{\gamma - 1} c = -2s(\alpha), \quad \text{on} \quad \frac{dx}{dt} = u - c, \quad C^- \text{ characteristic}. \tag{8.493}$$

[13]Richard Courant, 1888-1972, Prussian-born German mathematician, received Ph.D. under David Hilbert at Göttingen, compiled Hilbert's course notes into classic two-volume text of applied mathematics, drafted into German army in World War I, where half of his unit was killed in action, developed telegraph system that used the earth as a conductor for use in the trenches of the Western front, expelled from Göttingen by the Nazis in 1933, fled Germany, and founded the Courant Institute of Mathematical Sciences at New York University, author of classic mathematical text on supersonic fluid mechanics.

[14]Kurt Otto Friedrichs, 1901-1982, German-born mathematician who emigrated to the United States in 1937, student of Richard Courant's at Göttingen, taught at Aachen, Braunschweig, and New York University, worked on partial differential equations of mathematical physics and fluid mechanics.

These quantities are known as *Riemann invariants*.[15]

8.6.3 Simple waves

Simple waves are defined to exist when either $r(\beta)$ or $s(\alpha)$ are constant everywhere in $x-t$ space and not just on characteristics. For example, say $s(\alpha) = s_o$. Then the Riemann invariant

$$u - \frac{2}{\gamma-1}c = -2s_o, \qquad \text{everywhere}, \tag{8.494}$$

is actually invariant over all of $x-t$ space. Now the other Riemann invariant,

$$u + \frac{2}{\gamma-1}c = 2r(\beta), \qquad \text{on } C^+, \tag{8.495}$$

takes on many values depending on β. However, it is easily shown that for the simple wave that the characteristics have a constant slope in the $x-t$ plane as sketched in the $x-t$ diagram of Fig. 8.19.

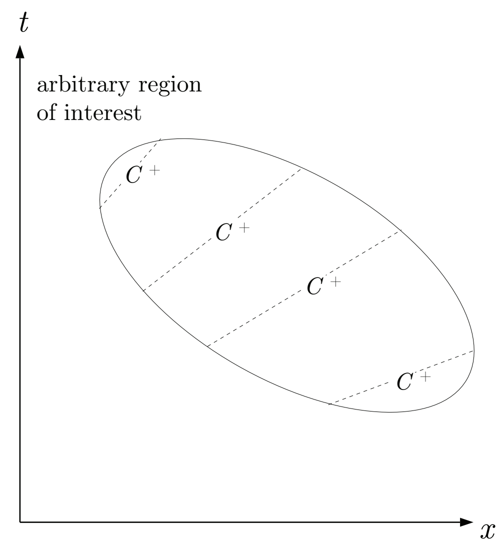

Figure 8.19: $x-t$ diagram showing C^+ for a simple wave.

Now consider a rarefaction with a *prescribed* piston motion $u = u_p(t)$. A sketch is given in the $x-t$ diagram of Fig. 8.20. For this configuration, the Riemann invariant $u - 2c/(\gamma-1) = -2s_o$ is valid everywhere. Let us evaluate s_o in terms of more fundamental variables. For the piston problem we are considering, when $t = 0$, we have $u = 0$, $c = c_o$, so

$$u - \frac{2}{\gamma-1}c = -\frac{2}{\gamma-1}c_o, \qquad \text{everywhere}. \tag{8.496}$$

[15]Georg Friedrich Bernhard Riemann, 1826-1866, German mathematician and geometer whose work in non-Euclidean geometry was critical to Einstein's theory of general relativity, produced the first major study of shock waves.

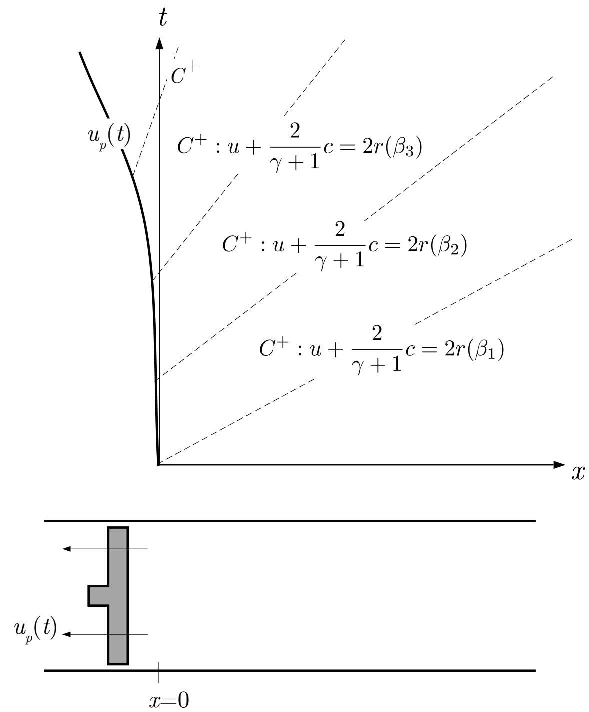

Figure 8.20: $x - t$ diagram showing C^+ characteristics for isentropic rarefaction problem, along with piston cylinder arrangement.

Thus, $s_o = c_o/(\gamma - 1)$.

Consider now a special characteristic $\hat{C}^+$ at $t = \hat{t}$. See Fig. 8.21. At this time the piston moves with velocity $\hat{u}_p$, and the fluid velocity at the piston face is

$$u_{face}(\hat{t}) = \hat{u}_p. \tag{8.497}$$

We get $c_{face}(\hat{t})$ from Eq. (8.496), which must be valid everywhere, including the face of the piston:

$$\underbrace{u_{face}}_{\hat{u}_p} - \frac{2}{\gamma - 1} c_{face} = -\frac{2}{\gamma - 1} c_o, \tag{8.498}$$

$$c_{face}(t = \hat{t}) = c_o + \frac{\gamma - 1}{2} \hat{u}_p. \tag{8.499}$$

Also from Eq. (8.496), we have

$$c = c_o + \frac{\gamma - 1}{2} u, \qquad \text{everywhere}, \tag{8.500}$$

that is valid everywhere.

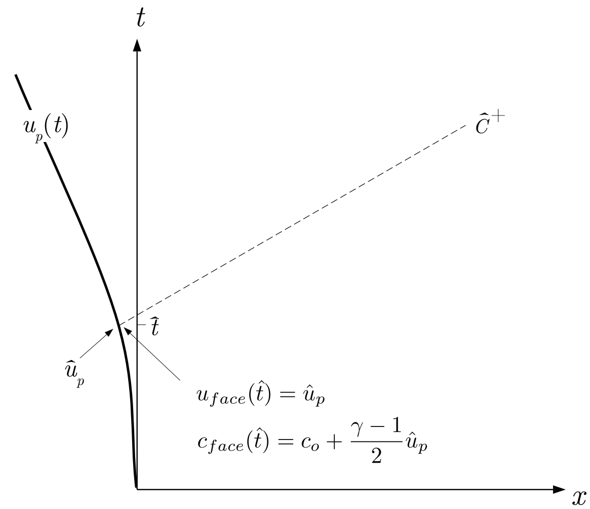

Figure 8.21: $x - t$ diagram showing $\hat{C}^+$ for our rarefaction problem.

Now on $\hat{C}^+$, we have from Eq. (8.495) that

$$u + \frac{2}{\gamma-1}c = \left(u_{face} + \frac{2}{\gamma-1}c_{face}\right)_{t=\hat{t}}, \tag{8.501}$$

$$u + \frac{2}{\gamma-1}\underbrace{\left(c_o + \frac{\gamma-1}{2}u\right)}_{c} = \hat{u}_p + \frac{2}{\gamma-1}\underbrace{\left(c_o + \frac{\gamma-1}{2}\hat{u}_p\right)}_{c_{face}}, \tag{8.502}$$

$$2u + \frac{2}{\gamma-1}c_o = 2\hat{u}_p + \frac{2}{\gamma-1}c_o, \tag{8.503}$$

$$u = \hat{u}_p \quad \text{on} \quad \hat{C}^+. \tag{8.504}$$

So on $\hat{C}^+$, we have from Eq. (8.500) that

$$c = c_o + \frac{\gamma-1}{2}\hat{u}_p. \tag{8.505}$$

So for $\hat{C}^+$, we get

$$\frac{dx}{dt} = u + c, \tag{8.506}$$

$$= \hat{u}_p + \underbrace{c_o + \frac{\gamma-1}{2}\hat{u}_p}_{c}, \tag{8.507}$$

$$= \frac{\gamma+1}{2}\hat{u}_p + c_o. \tag{8.508}$$

for a particular characteristic, this slope is a constant, as was earlier suggested.

Now for our *prescribed* motion, $\hat{u}_p$ decreases with time and becomes more negative; hence the slope of our $\hat{C}^+$ characteristic decreases, and the characteristics *diverge* in $x-t$ space. The slope of the leading characteristic is c_o, the ambient sound speed. The characteristic we consider, $\hat{C}^+$ is sketched in the $x-t$ diagram of Fig. 8.21.

We can use our Riemann invariant along with isentropic relations to obtain other flow variables. From Eq. (8.496), we get

$$\frac{c}{c_o} = 1 + \frac{\gamma-1}{2}\frac{u}{c_o}. \tag{8.509}$$

Because the flow is homeoentropic, we have $c/c_o = (\rho/\rho_o)^{\frac{\gamma-1}{2}}$ and $p/p_o = (\rho/\rho_o)^{\gamma}$, so

$$\frac{p}{p_o} = \left(1 + \frac{\gamma-1}{2}\frac{u}{c_o}\right)^{\frac{2\gamma}{\gamma-1}}, \tag{8.510}$$

$$\frac{\rho}{\rho_o} = \left(1 + \frac{\gamma-1}{2}\frac{u}{c_o}\right)^{\frac{2}{\gamma-1}}. \tag{8.511}$$

8.6.4 Centered rarefaction

If the piston is *suddenly* accelerated to a constant velocity, then a family of characteristics clusters at the origin on the $x-t$ diagram and fans out in a *centered rarefaction*. This can also be studied using the similarity transformation $\xi = x/t$ that reduces the partial differential equations to ordinary differential equations. Relevant sketches comparing centered to non-centered rarefactions are shown in in the $x-t$ diagram of Fig. 8.22.

Example 8.13

Analyze a centered rarefaction fan propagating into CPIG air for a piston suddenly accelerated from rest to $u_p = -100$ m/s. Take the ambient air to be at $p_o = 10^5$ Pa, $T_o = 300$ K.

The ideal gas law gives $\rho_o = p_o/RT_o = (10^5 \text{ Pa})/((287 \text{ J/kg/K})(300 \text{ K})) = 1.16 \text{ kg/m}^3$. Now

$$c_o = \sqrt{\gamma R T_o} = \sqrt{\frac{7}{5}\left(287\ \frac{\text{J}}{\text{kg K}}\right)(300\text{ K})} = 347\ \frac{\text{m}}{\text{s}}. \tag{8.512}$$

On the final characteristic of the fan, C_f^+: $u = u_p = -100$ m/s. So

$$c = c_o + \frac{\gamma-1}{2}u_p = \left(347\ \frac{\text{m}}{\text{s}}\right) + \frac{7/5-1}{2}\left(-100\ \frac{\text{m}}{\text{s}}\right) = 327\ \frac{\text{m}}{\text{s}}. \tag{8.513}$$

Now the final pressure is

$$\frac{p_f}{p_o} = \left(1 + \frac{\gamma-1}{2}\frac{u_f}{c_o}\right)^{\frac{2\gamma}{\gamma-1}} = \left(1 + \frac{7/5-1}{2}\frac{\left(-100\ \frac{\text{m}}{\text{s}}\right)}{347\ \frac{\text{m}}{\text{s}}}\right)^{\frac{2(7/5)}{7/5-1}} = 0.660. \tag{8.514}$$

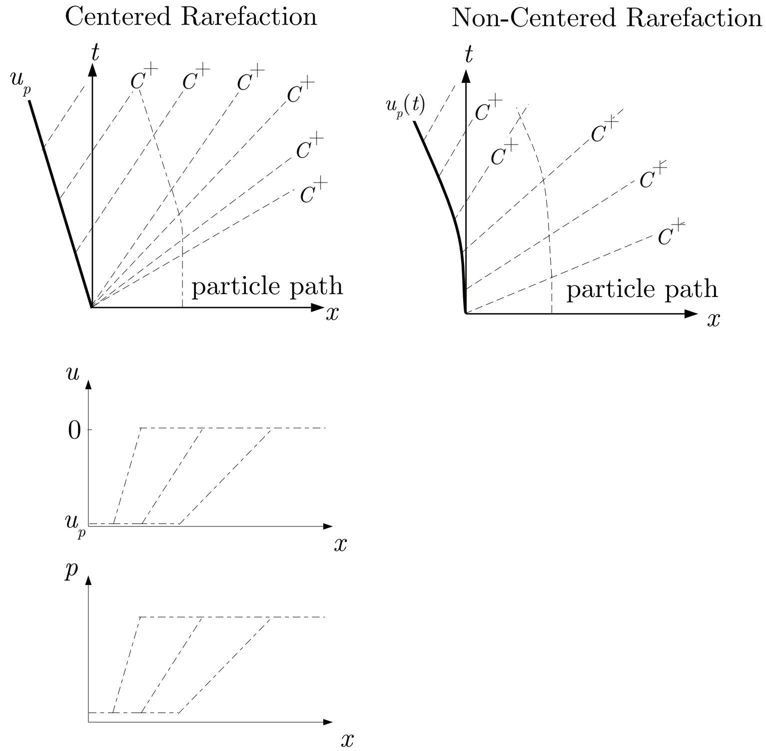

Figure 8.22: $x-t$ diagram centered and non-centered rarefactions, along with pressure and velocity profiles for centered fans.

Hence $p_f = 6.6 \times 10^4$ Pa. Because the flow is homeoentropic, we get

$$\rho_f = \rho_o \left(\frac{p_f}{p_o}\right)^{\frac{1}{\gamma}} = \left(1.16 \ \frac{\text{kg}}{\text{m}^3}\right)(0.660)^{5/7} = 0.863 \ \frac{\text{kg}}{\text{m}^3}. \tag{8.515}$$

And the final temperature is

$$T_f = \frac{p_f}{\rho_f R} = \frac{66.0 \times 10^3 \ \text{Pa}}{\left(0.863 \ \frac{\text{kg}}{\text{m}^3}\right)\left(287 \ \frac{\text{J}}{\text{kg K}}\right)} = 266.4 \ \text{K}. \tag{8.516}$$

From linear acoustic theory, Sec. 8.4.6, we deduce that

$$\Delta\rho \sim -\rho_o \frac{\Delta u}{c_o}, \qquad \Delta p \sim -\rho_o c_o \Delta u, \qquad \Delta T \sim -(\gamma-1) T_o \frac{\Delta u}{c_o}. \tag{8.517}$$

We compare the results of this problem with the estimates of linear acoustic theory. and see

$$\Delta\rho_{exact} = -0.298 \ \frac{\text{kg}}{\text{m}^3}, \qquad \Delta\rho_{linear} = -0.335 \ \frac{\text{kg}}{\text{m}^3}, \tag{8.518}$$

$$\Delta p_{exact} = -34.0 \times 10^3 \ \text{Pa}, \qquad \Delta p_{linear} = -40.3 \times 10^3 \ \text{Pa}, \tag{8.519}$$

$$\Delta T_{exact} = -33.6 \ \text{K}, \qquad \Delta T_{linear} = -34.6 \ \text{K}. \tag{8.520}$$

8.6.5 Simple compression

We sketch a simple compression generated by a piston gradually accelerating from rest in the $x - t$ diagram of Fig. 8.23. Here the leading wave is at the ambient sonic speed. It

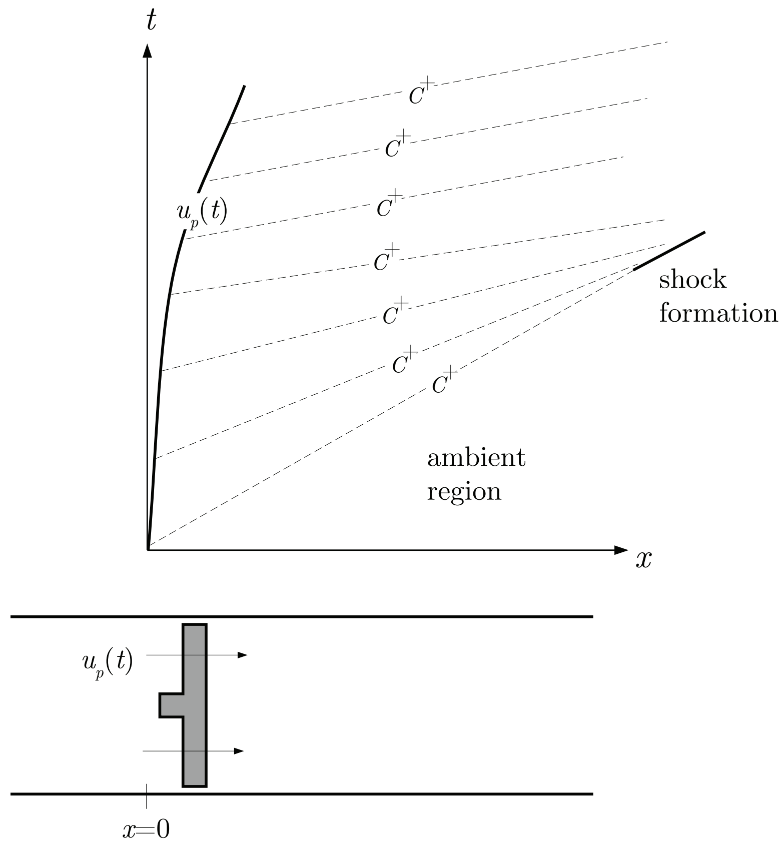

Figure 8.23: $x - t$ diagram for simple compression.

slightly raises the temperature downstream. So the local sound speed increases slightly. This happens for each subsequent compression. So each compression that follows travels slightly faster. At a downstream point, these compression waves coalesce to form a shock wave. After the piston reaches a steady velocity, it sends a series of compression waves into the flow, all propagating at the same speed, and all interacting with the lead shock wave. This is the mechanism by which energy from the piston is transmitted to the shock front to support its propagation.

8.6.6 Two interacting expansions

We sketch two interacting expansion waves from two suddenly accelerating pistons in the $x-t$ diagram of Fig. 8.24. Both pistons generate centered rarefactions, one right-propagating, the other left-propagating. In contrast to the compression wave, each small rarefaction slightly

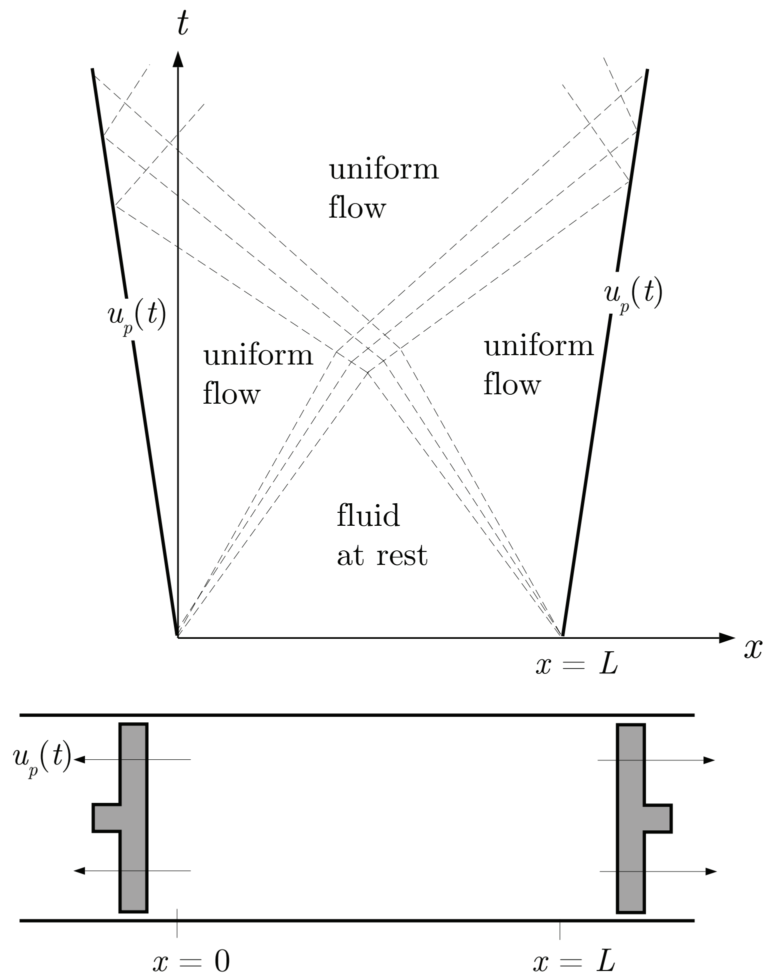

Figure 8.24: $x - t$ diagram for two interacting expansion waves.

cools the flow, thus lowering its sound speed. So later rarefactions travel at lower speeds, and there is no analog to the coalescence of compressions to form a shock. The rarefaction waves collide and interact in a complicated fashion. The modulated waves then are transmitted and reflect from the piston walls. This process would continue and generate a long train of ever-more complicated wave patterns; however, no shocks would be formed.

8.6.7 Wall interactions

We sketch an expansion wall interaction in the $x - t$ diagram of Fig. 8.25. Here a suddenly accelerated left-traveling piston generates a right-propagating centered rarefaction wave. This wave reflects from a wall at $x = L$. Though not sketched, the reflected wave will catch and reflect from the piston. Once again, because the volume is expanding, no shocks would be formed.

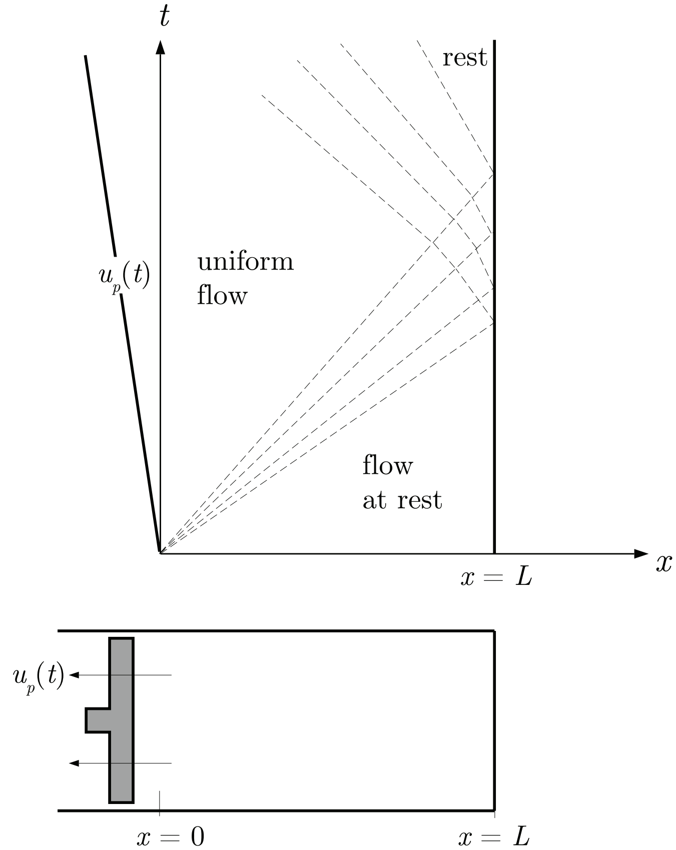

Figure 8.25: $x - t$ diagram for expansion wall interaction.

8.6.8 Shock tube

We sketch the behavior of the fluid in a shock tube in the diagrams of Fig. 8.26. In a shock tube, we have a gas initially segregated into two chambers, separated by a thin diaphragm, in the initial state. The initial temperature in both chambers is identical, and the gas is at rest in both chambers. The pressure on one side is different from that on the other, and this induces a density difference between one side and the other. At $t = 0^+$, the diaphragm is removed, and the pressure difference induces a fluid acceleration. As sketched in Fig. 8.26, where the initial state has higher pressure on the left, a shock wave propagates to the right, and a rarefaction propagates to the left. These two waves are nonlinear extensions of acoustic disturbances, and propagate at speeds close to the sound speed. There is also a contact discontinuity segregating the two regions of different density. This discontinuity also propagates, but at a speed near the local particle velocity, as it is associated with a so-called *entropy wave*. Across the contact discontinuity, it can be shown that pressure and velocity must be continuous, while density and temperature can suffer a jump. The shock and rarefaction both reflect from the walls and interact with each other as well as the contact discontinuity in a complicated manner.

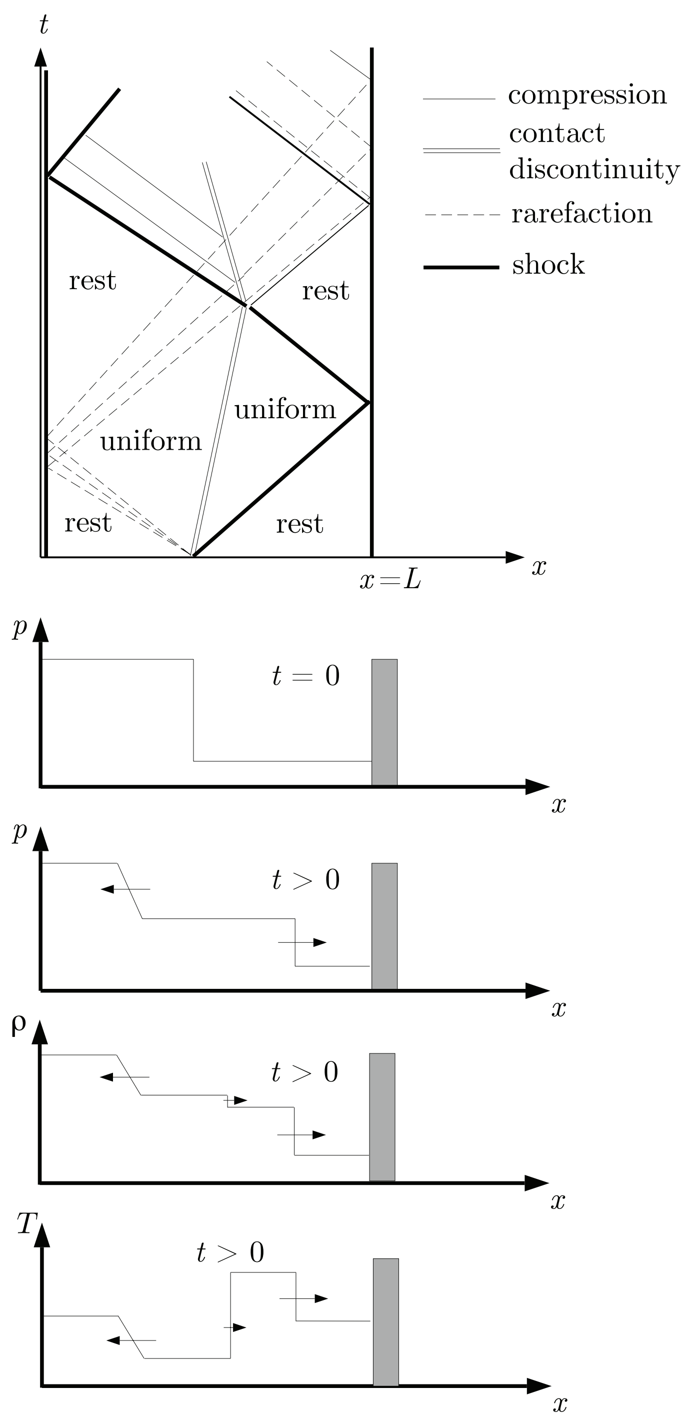

Figure 8.26: $x - t$ and p, ρ, T versus x behavior for a shock tube.

8.6.9 Inviscid Bateman-Burgers' equation solution

To this point, we have described a common and traditional approach to the method of characteristics. Using common notation, we have written what began as partial differential equations (PDEs) in the form of ordinary differential equations (ODEs), and it is often said that the method of characteristics is a way to *transform* PDEs into ODEs. However, the equations that result are certainly not in a standard form for ODEs; they are burdened with unusual side conditions.

It is in fact more sound to state that the method of characteristics transforms the PDEs in (x, t) space to another set of PDEs in a new space (ξ, τ) in which the integration is much easier. Consider for example a model equation that is hyperbolic, the inviscid Bateman[16]-Burgers'[17] equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0. \tag{8.521}$$

This is more commonly known as the inviscid Burgers' equation, but Bateman's work has priority.[18] The inviscid Burgers' equation does not apply directly to a fluid; however, it contains one of the key mathematical features of fluid models: advective nonlinearity that has profound effect on the dynamics.

Now consider a general transformation $(x, t) \to (\xi, \tau)$.

$$x = x(\xi, \tau), \tag{8.522}$$
$$t = t(\xi, \tau). \tag{8.523}$$

We assume the transformation to be unique and invertible. The chain rule gives

$$\begin{pmatrix} dx \\ dt \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \tau} \\ \frac{\partial t}{\partial \xi} & \frac{\partial t}{\partial \tau} \end{pmatrix}}_{\mathsf{J}} \begin{pmatrix} d\xi \\ d\tau \end{pmatrix}. \tag{8.524}$$

The Jacobian matrix of the transformation is

$$\mathsf{J} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \tau} \\ \frac{\partial t}{\partial \xi} & \frac{\partial t}{\partial \tau} \end{pmatrix}. \tag{8.525}$$

And we have for the Jacobian determinant J:

$$J = \det \mathsf{J} = \frac{\partial x}{\partial \xi}\frac{\partial t}{\partial \tau} - \frac{\partial x}{\partial \tau}\frac{\partial t}{\partial \xi}. \tag{8.526}$$

[16]Harry Bateman, 1882-1946, well regarded English mathematician who spent many years at Caltech about whom von Kármán said "He seemed to know everything but did nothing important. I liked him."

[17]Johannes Martinus Burgers, 1895-1981, Dutch physicist.

[18]The viscous version of the model equation, $\partial u/\partial t + u\partial u/\partial x = \nu \partial^2 u/\partial x^2$, is widely known as Burgers' equation and is often cited as originating from J. M. Burgers, 1948, A mathematical model illustrating the theory of turbulence, *Advances in Applied Mathematics*, 1: 171-199. However, the viscous version was given earlier by H. Bateman, 1915, Some recent researches in the motion of fluids, *Monthly Weather Review*, 43(4): 163-170.

Now from Eq. (2.284), we can deduce

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial t} \end{pmatrix} = (\mathsf{J}^T)^{-1} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \tau} \end{pmatrix} = \frac{1}{J} \begin{pmatrix} \frac{\partial t}{\partial \tau} & -\frac{\partial t}{\partial \xi} \\ -\frac{\partial x}{\partial \tau} & \frac{\partial x}{\partial \xi} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \tau} \end{pmatrix} = \frac{1}{J} \begin{pmatrix} \frac{\partial t}{\partial \tau}\frac{\partial}{\partial \xi} - \frac{\partial t}{\partial \xi}\frac{\partial}{\partial \tau} \\ -\frac{\partial x}{\partial \tau}\frac{\partial}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial}{\partial \tau} \end{pmatrix}. \tag{8.527}$$

With these transformation rules, Eq (8.521) is rewritten as

$$\underbrace{\frac{1}{J}\left(-\frac{\partial x}{\partial \tau}\frac{\partial u}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial u}{\partial \tau}\right)}_{\partial u/\partial t} + u \underbrace{\frac{1}{J}\left(\frac{\partial t}{\partial \tau}\frac{\partial u}{\partial \xi} - \frac{\partial t}{\partial \xi}\frac{\partial u}{\partial \tau}\right)}_{\partial u/\partial x} = 0. \tag{8.528}$$

Now by assumption, $J \neq 0$, so we can multiply by J to get

$$-\frac{\partial x}{\partial \tau}\frac{\partial u}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial u}{\partial \tau} + u\frac{\partial t}{\partial \tau}\frac{\partial u}{\partial \xi} - u\frac{\partial t}{\partial \xi}\frac{\partial u}{\partial \tau} = 0. \tag{8.529}$$

Let us now restrict our transformation to satisfy the following requirements:

$$\frac{\partial x}{\partial \tau} = u\frac{\partial t}{\partial \tau}, \tag{8.530}$$

$$t(\xi, \tau) = \tau. \tag{8.531}$$

The first says that if we insist that ξ is held fixed, that the ratio of the change in x to the change in t will be u; this is equivalent to the more standard statement that on a characteristic line we have $dx/dt = u$. The second is a convenience simply equating τ to t. Applying the second restriction to the first, we can also say

$$\frac{\partial x}{\partial \tau} = u. \tag{8.532}$$

With these restrictions, our inviscid Burgers' equation becomes

$$-\underbrace{\frac{\partial x}{\partial \tau}}_{u}\frac{\partial u}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial u}{\partial \tau} + u\underbrace{\frac{\partial t}{\partial \tau}}_{1}\frac{\partial u}{\partial \xi} - u\underbrace{\frac{\partial t}{\partial \xi}}_{0}\frac{\partial u}{\partial \tau} = 0, \tag{8.533}$$

$$-u\frac{\partial u}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial u}{\partial \tau} + u\frac{\partial u}{\partial \xi} = 0, \tag{8.534}$$

$$\frac{\partial x}{\partial \xi}\frac{\partial u}{\partial \tau} = 0. \tag{8.535}$$

Let us further require that $\partial x/\partial \xi \neq 0$. Then we have

$$\frac{\partial u}{\partial \tau} = 0, \tag{8.536}$$

$$u = f(\xi). \tag{8.537}$$

Here f is an arbitrary function. Substitute this into Eq. (8.530) to get

$$\frac{\partial x}{\partial \tau} = f(\xi)\frac{\partial t}{\partial \tau}. \tag{8.538}$$

We can integrate Eq. (8.538) to get

$$x = f(\xi)t + g(\xi). \tag{8.539}$$

Here $g(\xi)$ is an arbitrary function. Note the coordinate transformation can be chosen for our convenience. To this end, remove t in favor of τ and set $g(\xi) = \xi$ so that x maps to ξ when $t = \tau = 0$ giving

$$x(\xi, \tau) = f(\xi)\tau + \xi. \tag{8.540}$$

We can then state the solution to the inviscid Burgers' equation, Eq. (8.521), parametrically as

$$u(\xi, \tau) = f(\xi), \tag{8.541}$$
$$x(\xi, \tau) = f(\xi)\tau + \xi, \tag{8.542}$$
$$t(\xi, \tau) = \tau. \tag{8.543}$$

For this transformation, we have from Eq. (8.525) that

$$\mathsf{J} = \begin{pmatrix} 1 + \frac{df}{d\xi}\tau & f(\xi) \\ 0 & 1 \end{pmatrix}. \tag{8.544}$$

Thus

$$J = \det \mathsf{J} = 1 + \frac{df}{d\xi}\tau. \tag{8.545}$$

We have a singularity in the coordinate transformation whenever $J = 0$, implying a difficulty when

$$\tau = -\frac{1}{\frac{df}{d\xi}}. \tag{8.546}$$

Example 8.14

Solve the inviscid Burgers' equation, $\partial u/\partial t + u\partial u/\partial x = 0$, Eq. (8.521), if

$$u(x, 0) = 1 + \sin \pi x, \qquad x \in [0, 1] \tag{8.547}$$

Let us not be concerned with that portion of u which at $t = 0$ has $x < 0$ or $x > 1$. The analysis is easily modified to address this.

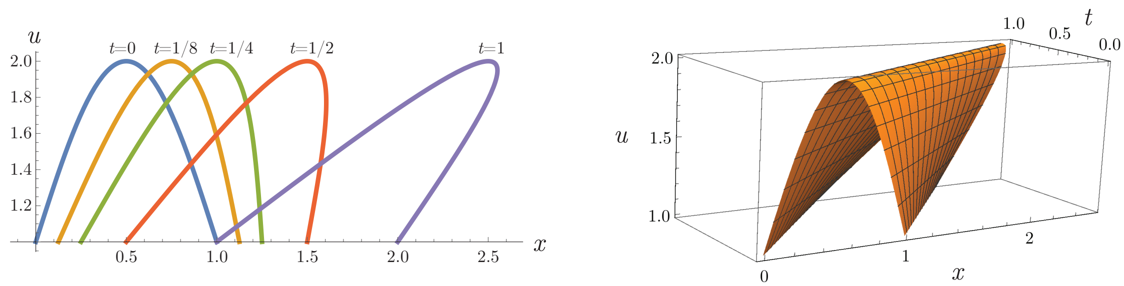

Figure 8.27: Solution to $\partial u/\partial t + u\partial u/\partial x$ with $u(x, 0) = 1 + \sin \pi x$.

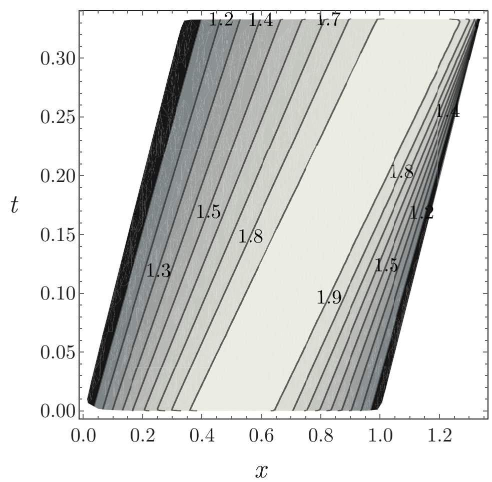

Figure 8.28: Early time solution to $\partial u/\partial t + u\partial u/\partial x$ with $u(x, 0) = 1 + \sin \pi x$ in the form of a contour plot in $x - t$ space giving contours of constant u.

We know the solution is given in general by Eqs. (8.541-8.543). At $t = 0$, we have $\tau = 0$, and thus $x = \xi$. And we have

$$f(\xi) = 1 + \sin \pi\xi. \tag{8.548}$$

Thus we can say by inspection that the solution is

$$u(\xi, \tau) = 1 + \sin \pi\xi, \tag{8.549}$$
$$x(\xi, \tau) = (1 + \sin \pi\xi)\, \tau + \xi, \tag{8.550}$$
$$t(\xi, \tau) = \tau. \tag{8.551}$$

Results for $u(x, t)$ are plotted in Fig. 8.27. Another way to view the results is in the $x - t$ diagram that gives contours of u in Fig. 8.28 These contours are generated before the transformation has become singular. The curves of constant u are the characteristics. Clearly those on the right of the maximum of u are coalescing, while those to the left are diverging. The coalescence corresponds to shock formation, and the divergence corresponds to a rarefaction.

One notes the following:

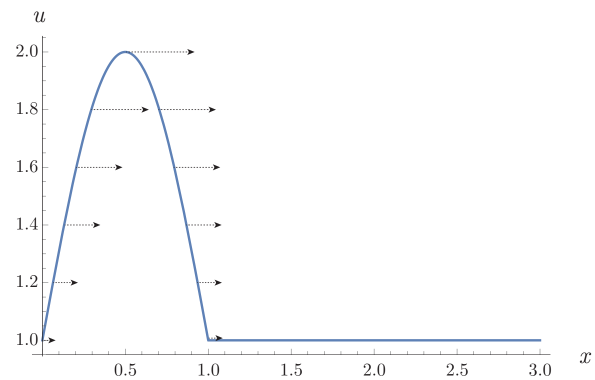

Figure 8.29: Sketch of response of u which satisfies the inviscid Burgers' equation $\partial u/\partial t + u\partial u/\partial x$ with $u(x, 0) = 1 + \sin \pi x$.

- The signal propagates to the right; this is a consequence of $u > 0$ in the domain we consider.
- Portions of the signal with higher u propagate faster.
- The signal distorts as t increases.
- The wave appears to "break" at $t = t_s$, where $1/4 \lesssim t_s \lesssim 1/2$. For $t > t_s$, it is possible to find multiple values of u at a given x and t. If u were a physical variable, we would not expect to see such multivaluedness in nature.

It appears to be challenging to write an explicit formula for $u(x, t)$. However, for small ξ, one can write a useful approximation. Taylor series expansion of Eq. (8.550) for small ξ yields

$$x(\xi, \tau) \sim (1 + \pi\tau)\xi + \tau + \ldots. \tag{8.552}$$

We invert this and use $\tau = t$ to find

$$\xi(x, t) \sim \frac{x - t}{1 + \pi t} + \ldots. \tag{8.553}$$

Then, because $u = f(\xi)$, we get

$$u(x, t) \sim 1 + \sin \frac{\pi(x - t)}{1 + \pi t} + \ldots. \tag{8.554}$$

This itself has a series expansion for small x and t of

$$u(x, t) \sim 1 + \pi(x - t) + \ldots. \tag{8.555}$$

The sketch of Fig. 8.29 shows how one can envision the portion of the initial sine wave with $x > 1/2$ steepening, while that portion with $x < 1/2$ flattens. We place arrows whose magnitude is proportional to the local value of u on the plot itself.

For our value of $f(\xi)$, we have from from Eq. (8.545) that

$$J = 1 + \pi\tau \cos \pi\xi. \tag{8.556}$$

Clearly, there exist values of (ξ, τ) for which $J = 0$. At such points, we can expect difficulties in our solution. In Fig. 8.30, we plot a portion of the locus of points for which $J = 0$ in the (ξ, τ) plane.

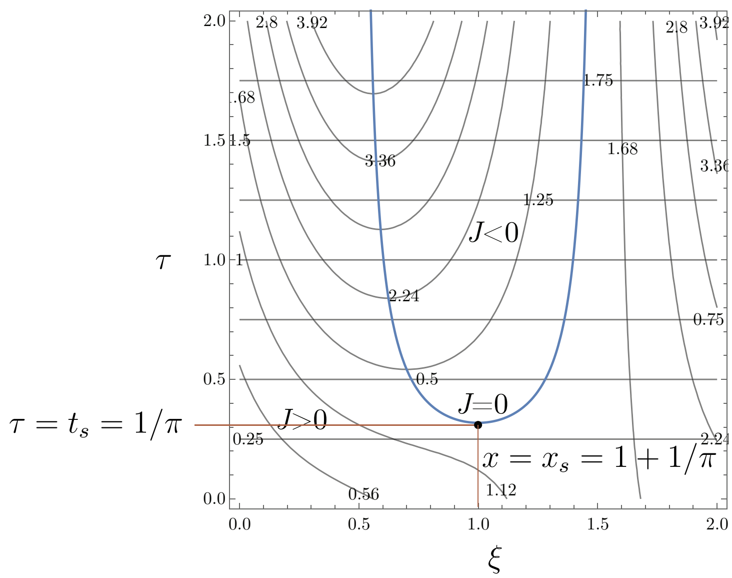

Figure 8.30: Curves where $J = 0$ and of constant x and t in the (ξ, τ) plane for our coordinate transformation.

We also see portions of this plane where the transformation is orientation-preserving, for which $J > 0$, and orientation-reversing, for which $J < 0$. Also shown in Fig. 8.30 are contours of constant x and t. Clearly when $J = 0$, the contours of constant x are parallel to those of constant t, and there are not enough linearly independent vectors to form a basis.

From Eq. (8.546), we can expect a singular coordinate transformation when

$$\tau = -\frac{1}{\frac{df}{d\xi}} = -\frac{1}{\pi \cos \pi\xi}. \tag{8.557}$$

We then substitute this into Eqs. (8.550, 8.551) to get a parametric curve for when the transformation is singular, $x_s(\xi), t_s(\xi)$:

$$x_s(\xi) = -\frac{1 + \sin \pi\xi}{\pi \cos \pi\xi} + \xi, \tag{8.558}$$

$$t_s(\xi) = -\frac{1}{\pi \cos \pi\xi}. \tag{8.559}$$

A portion of this curve for where the transformation is singular is shown in Fig. 8.31. Figure 8.31a plots $x_s(\xi)$ from Eq. (8.558). Figure 8.31b plots $t_s(\xi)$ from Eq. (8.559). We see a parametric plot of the same quantities in Fig. 8.31c. At early time the system is free of singularities. It is easily shown that both $x_s(\xi)$ and $t_s(\xi)$ have a local minimum at $\xi = 1$, at which point, we have

$$x_s(1) = 1 + \frac{1}{\pi}, \tag{8.560}$$

$$t_s(1) = \frac{1}{\pi}. \tag{8.561}$$

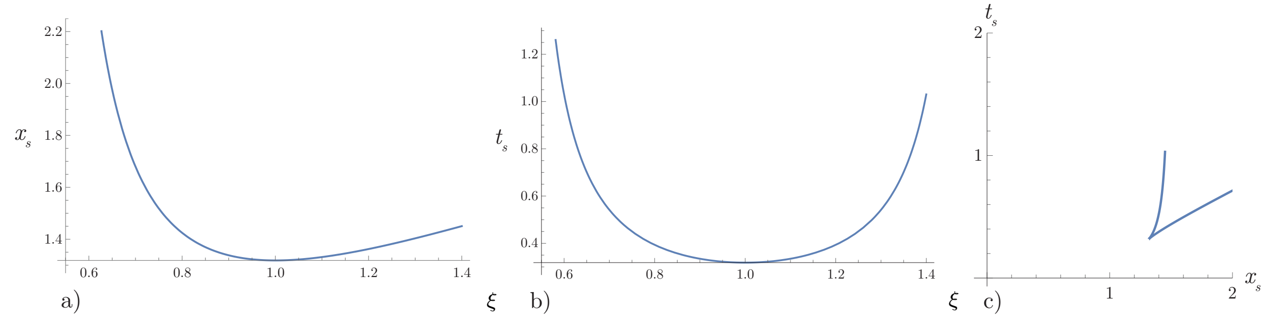

Figure 8.31: Plots indicating where the coordinate transformation of Eqs. (8.550,8.551) is singular: a) $x_s(\xi)$ from Eq. (8.558), b) $t_s(\xi)$ from Eq. (8.559), c) representation of the curve of singularity in (x,t) space.

Examining Fig. 8.27, this appears to be the point at which the solution becomes multivalued. Examining Fig. 8.30, this is the point on the curve $J = 0$ that is a local minimum. So while x_s and t_s are well-behaved as functions of ξ for the domain considered, when the curves are projected into the (x,t) plane, there is a cusp at $(x,t) = (x_s(1), t_s(1)) = (1 + 1/\pi, 1/\pi)$.

Let us examine with Taylor series the behavior of $\partial u/\partial x$ in the neighborhood of the singularity. Our expectation is that the slope approaches infinity as the singularity is approached. From Eq. (8.528), we see that

$$\frac{\partial u}{\partial x} = \frac{1}{J}\left(\frac{\partial t}{\partial \tau}\frac{\partial u}{\partial \xi} - \frac{\partial t}{\partial \xi}\frac{\partial u}{\partial \tau}\right). \tag{8.562}$$

For our transformation, we have $\partial t/\partial\tau = 1$, $\partial t/\partial \xi = 0$, so

$$\frac{\partial u}{\partial x} = \frac{1}{J}\frac{\partial u}{\partial \xi}. \tag{8.563}$$

Now use our solution for u, Eq. (8.541), and for J, Eq. (8.545), to say

$$\frac{\partial u}{\partial x} = \frac{1}{1 + \frac{df}{d\xi}\tau}\frac{df}{d\xi}. \tag{8.564}$$

With $f(\xi)$ for our example from Eq. (8.548), we get

$$\frac{\partial u}{\partial x} = \frac{\pi\cos(\pi\xi)}{1 + \pi\tau\cos(\pi\xi)}. \tag{8.565}$$

We use computer algebra to perform a Taylor series expansion of $\partial u/\partial \xi$ about $\xi = 1$, $\tau = 1/\pi$ to find the behavior near the singularity to be

$$\frac{\partial u}{\partial x} = \frac{1}{\tau - \frac{1}{\pi}} + \left(\frac{\pi}{2\left(\tau - \frac{1}{\pi}\right)^2}\right)(\xi - 1)^2 + \ldots \tag{8.566}$$

By inspection we thus see that

$$\lim_{\tau \to \frac{1}{\pi}} \frac{\partial u}{\partial x} = -\infty. \tag{8.567}$$

It is necessary for τ to be increasing towards $1/\pi$ for the slope to be negative. This is the physically relevant approach as τ begins at zero and is increasing.

This procedure can be extended to the Euler equations, though it is more complicated. For the Euler equations, Courant and Friedrichs (1976) give some special solutions for rarefactions.

8.6.10 Viscous Bateman-Burgers' equation solution

Our predictions of $u(x,t)$ change dramatically when diffusion is introduced. Consider the viscous Bateman-Burgers' equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}. \tag{8.568}$$

When we discretized the partial differential equations and simulate via standard numerical methods the same problem whose diffusion-free solution is plotted in Fig. 8.27 for which $u(x,0) = 1 + \sin \pi x$, we obtain the results plotted in Fig. 8.32 for four different values of $\nu = 1/1000$, $1/100$, $1/10$, and 1. While an exact solution to the viscous Burgers' equation is available, in practice, it is complicated. It is often easier to obtain results by numerical discretization, and that is what we did here. The scheme used was sufficiently resolved to capture the thin zones present when ν was small. For the case where $\nu = 1/100$, we plot the $x - t$ diagram, where the shading is proportional to the local value of u, in Fig. 8.33.

We note:

- We restricted our study to positive values of ν, which can be shown to be necessary for a stable solution as $t \to \infty$.
- If $\nu = 0$, our viscous Burgers' equation reduces to the inviscid Burgers' equation.
- As $\nu \to 0$, solutions to the viscous Burgers' equation seem to relax to a solution with an infinitely thin discontinuity; they do not relax to those solutions displayed in Fig. 8.27.
- For all values of ν, the solution $u(x,t)$ at a given time has a single value of u for a single value of x, in contrast to multi-valued solutions exhibited by the diffusion-free analog.
- As $\nu \to 0$, the peaks retain a larger magnitude. Thus one can conclude that enhancing ν smears peaks.
- At early time the solutions to the viscous Burgers' equation resemble those of the inviscid Burgers' equation.

Let us try to understand this behavior. Fundamentally, it will be seen that in many cases, nonlinearity, manifested in $u\partial u/\partial x$ can serve to steepen a waveform. If that steepening is unchecked by diffusion, either a formal discontinuity is admitted, or multi-valued solutions.

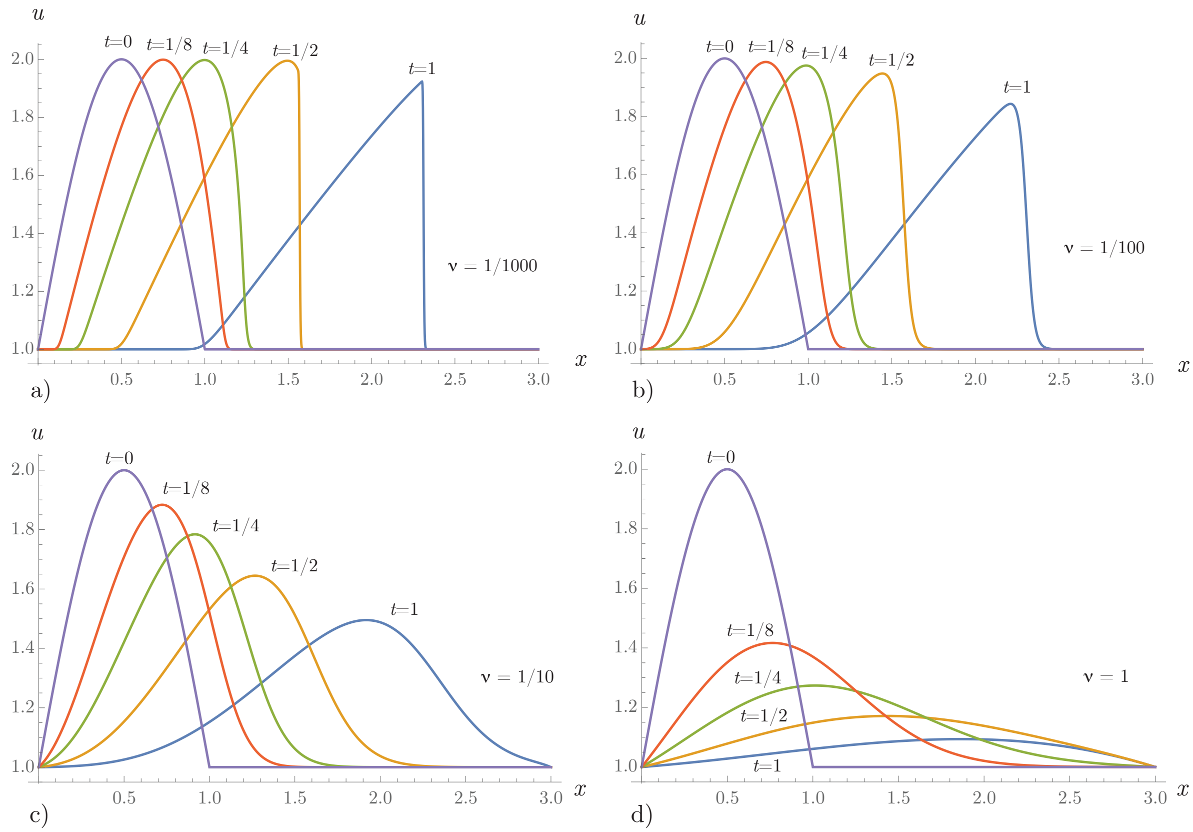

Figure 8.32: Numerical solution to the viscous Burgers' equation $\partial u/\partial t + u\partial u/\partial x = \nu\partial^2 u/\partial x^2$ with $u(x,0) = 1 + \sin \pi x$ and various values of ν: a) 1/1000, b) 1/100, c) 1/10, d) 1.

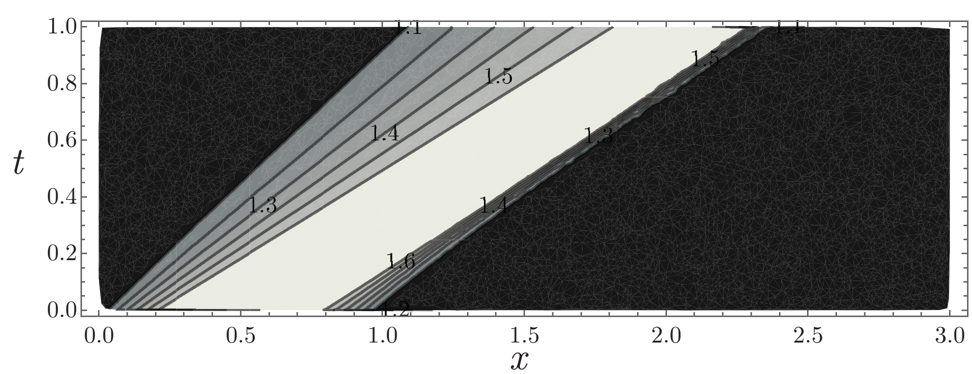

Figure 8.33: $x - t$ diagram with contours of u for solution to the viscous Burgers' equation $\partial u/\partial t + u\partial u/\partial x = \nu\partial^2 u/\partial x^2$ with $u(x,0) = 1 + \sin \pi x$, $\nu = 1/100$.

Now diffusion acts most strongly when gradients are steep, that is when $\partial u/\partial x$ has large magnitude. As a wave steepens due to nonlinear effects, diffusion, which many have been initially unimportant, can reassert its importance and serve to suppress the growth due to the nonlinearity.

Chapter 9

Potential flow

see Panton, Chapter 18,
see Yih, Chapter 4,
see Lamb, Chapter 4,
see Kuethe and Chow, Chapter 4.

This chapter will consider *potential flow*. Such flows can be characterized by a scalar *potential field*. Knowledge of this scalar field is sufficient to deduce all flow variables. As is typical for such fields, gradients of the potential induce flow. A good deal of highly developed and beautiful mathematical theory was generated for potential flows in the nineteenth century. Additionally, these solutions can be applied in highly disparate fields, as the equations governing potential flow of a fluid are identical in form to those governing some forms of energy and mass diffusion, as well as electro-magnetics.

Despite its beauty, in some ways it is impractical for many engineering applications, though not all. As the theory necessarily ignores all vorticity generating mechanisms, it must ignore viscous effects. Consequently, the theory is incapable of predicting drag forces on solid bodies. Consequently, those who needed to know the drag resorted in the nineteenth century to more empirically based methods.

In the early twentieth century, Prandtl took steps to reconcile the practical viscous world of engineering with the more mathematical world of potential flow with his viscous boundary layer theory. He showed that indeed potential flow solutions could be of value away from no-slip walls, and provided a recipe to fix the solutions in the neighborhood of the wall. In so doing, he opened a new field of applied mathematics known as matched asymptotic analysis.

So why study potential flows? The following arguments offer some justification.

- low speed aerodynamics are often well described by potential flow theory,
- portions of many real flow fields are captured by potential theory, and those that are not can often be remedied by application of a viscous boundary layer theory,
- study of potential flow solutions can give great insight into fluid behavior and aid in the honing of a more precise intuition,

- potential flow solutions are useful as test cases for verification of numerical methods, and
- there is pedantic and historical value in understanding potential flow.

9.1 Stream functions and velocity potentials

We first consider *stream functions* and *velocity potentials*. We have seen velocity potentials before in Ch. 7.6 in study of ideal vortices. In this chapter, we will adopt the same assumption of irrotationality, and further require that the flow be two-dimensional. Recall if a flow velocity is confined to the $x-y$ plane, then the vorticity vector is confined to the z direction and takes the form first shown in Eq. (3.203):

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix}. \tag{9.1}$$

So if the flow is two-dimensional and irrotational, we have

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \tag{9.2}$$

Moreover, because of irrotationality, we can express the velocity vector $\mathbf{v}$ as the gradient of a potential ϕ, the velocity potential, as first shown in Eq. (6.148):

$$\mathbf{v} = \nabla\phi. \tag{9.3}$$

With this definition, fluid flows from regions of low velocity potential to regions of high velocity potential. The scalar velocity components are

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}. \tag{9.4}$$

We see by substitution into the irrotationality condition, Eq. (9.2), that this is true identically:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = 0. \tag{9.5}$$

Now for two-dimensional incompressible flows, we have by specializing Eq. (6.31) that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{9.6}$$

Substituting for u and v in favor of ϕ, we get Laplace's equation for ϕ, a special case of that seen earlier in Eq. (7.187):

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = 0, \tag{9.7}$$

$$\nabla^2\phi = 0. \tag{9.8}$$

Now if the flow is incompressible, we can also define the stream function ψ as follows:

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{9.9}$$

Direct substitution into the incompressible mass conservation equation, Eq. (9.6), shows that this yields an identity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0. \tag{9.10}$$

Now, in an equation that will be critically important soon, we can set our definitions of u and v in terms of ϕ and ψ, Eqs. (9.4, 9.9), equal to each other, as they must be:

$$\underbrace{\frac{\partial \phi}{\partial x}}_{u} = \underbrace{\frac{\partial \psi}{\partial y}}_{u}, \tag{9.11}$$

$$\underbrace{\frac{\partial \phi}{\partial y}}_{v} = \underbrace{-\frac{\partial \psi}{\partial x}}_{v}. \tag{9.12}$$

Now if we differentiate the first equation with respect to y, and the second with respect to x we see

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \psi}{\partial y^2}, \tag{9.13}$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{\partial^2 \psi}{\partial x^2}. \tag{9.14}$$

Now subtract the second from the first to get Laplace's equation for ψ:

$$0 = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2}, \tag{9.15}$$

$$\nabla^2 \psi = 0. \tag{9.16}$$

Let us now examine lines of constant ϕ (equipotential lines) and lines of constant ψ (that we will see are streamlines). So take $\phi = C_1$, $\psi = C_2$. Because $\phi = \phi(x, y)$, we take the total derivative on a curve of constant ϕ and get

$$d\phi = \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy = 0, \tag{9.17}$$

$$= u\, dx + v\, dy = 0, \tag{9.18}$$

$$\left.\frac{dy}{dx}\right|_{\phi = C_1} = -\frac{u}{v}. \tag{9.19}$$

Now for $\psi = \psi(x, y)$ we similarly get on curves of constant ψ that

$$
\begin{aligned}
d\psi &= \frac{\partial \psi}{\partial x}\, dx + \frac{\partial \psi}{\partial y}\, dy = 0, && (9.20)\\
&= -v\, dx + u\, dy = 0, && (9.21)\\
\left.\frac{dy}{dx}\right|_{\psi=C_2} &= \frac{v}{u}. && (9.22)
\end{aligned}
$$

We note two features:

- $\left.\frac{dy}{dx}\right|_{\phi=C_1} = -\frac{1}{\left.\frac{dy}{dx}\right|_{\psi=C_2}}$; hence, lines of constant ϕ are orthogonal to lines of constant ψ, and
- on $\psi = C_2$, we see that $dx/u = dy/v$; hence, by Eq. (3.49), lines of $\psi = C_2$ must be streamlines.

As an aside, we note that the definition of the stream function $u = \partial\psi/\partial y, v = -\partial\psi/\partial x$, can be rewritten as

$$
\frac{\partial \psi}{\partial y} = \frac{dx}{dt}, \qquad \frac{\partial \psi}{\partial x} = -\frac{dy}{dt}. \tag{9.23}
$$

This is a common form from classical dynamics in which we can interpret ψ as the *Hamiltonian*[1] of the system. We shall not pursue this path, but note that a significant literature exists for Hamiltonian systems; see Powers and Sen, (2015), Ch. 9.6.6, for an introduction or Goldstein (1950), Ch. 7, for a detailed discussion.

Now the study of ϕ and ψ is essentially kinematics. The only incursion of dynamics is that we must have irrotational flow. Recalling the Helmholtz equation, Eq. (7.155), we realize that we can only have potential flow when the vorticity generating mechanisms (three-dimensional effects, non-conservative body forces, baroclinic effects, and viscous effects) are suppressed. In that case, the dynamics, that is the driving force for the fluid motion, can be understood in the context of the unsteady Bernoulli equation, Eq. (6.153, taken for incompressible flow and negligible body force, in which limit, Eq. (6.139) reduces to $\Upsilon = p/\rho$:

$$
\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla^T \phi \cdot \nabla \phi + \frac{p}{\rho} = f(t). \tag{9.24}
$$

We do not have to require steady flow to have a potential flow field. It is also easy to correct for the presence of a conservative body force.

Now solutions to the two key equations of potential flow $\nabla^2\phi = 0, \nabla^2\psi = 0$, are most efficiently studied using methods involving complex variables. We will delay discussing solutions until we have reviewed the necessary mathematics.

[1]William Rowan Hamilton, 1805-1865, Dublin-based Anglo-Irish mathematician. Discovered the quaternion group, an extension of complex numbers, while walking along a canal and engraved it in the Broom Bridge. He invented the notion of the dot and cross products, coined the terms "scalar" and "tensor," and was the first to use "vector" in the modern sense. Introduced the ∇ operator in 1837, albeit in a rotated form, $\lhd$, based on the harp. Some believe quaternions were satirized in Lewis Carroll's *Alice in Wonderland* and that the Mad Hatter's tea party was to argue for a return to Euclidean geometry, rather than the topsy-turvy universe described by quaternion methods.

9.2 Mathematics of complex variables

Here we briefly introduce relevant elements of complex variable theory. Recall that the imaginary number i is defined such that

$$i^2 = -1, \qquad i = \sqrt{-1}. \tag{9.25}$$

9.2.1 Euler's formula

We can get the useful *Euler's formula* by considering the following Taylor expansions of common functions about $t = 0$:

$$\begin{aligned} e^t &= 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5 \ldots, & (9.26)\\ \sin t &= 0 + t + 0\frac{1}{2!}t^2 - \frac{1}{3!}t^3 + 0\frac{1}{4!}t^4 + \frac{1}{5!}t^5 \ldots, & (9.27)\\ \cos t &= 1 + 0t - \frac{1}{2!}t^2 + 0\frac{1}{3!}t^3 + \frac{1}{4!}t^4 + 0\frac{1}{5!}t^5 \ldots & (9.28)\end{aligned}$$

With these expansions, now consider the following combinations: $(\cos t + i \sin t)_{t=\theta}$ and $e^t|_{t=i\theta}$:

$$\begin{aligned} \cos\theta + i\sin\theta &= 1 + i\theta - \frac{1}{2!}\theta^2 - i\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + i\frac{1}{5!}\theta^5 + \ldots, & (9.29)\\ e^{i\theta} &= 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \ldots, & (9.30)\\ &= 1 + i\theta - \frac{1}{2!}\theta^2 - i\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + i\frac{1}{5!}\theta^5 + \ldots & (9.31)\end{aligned}$$

As the two series are identical, we have Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{9.32}$$

9.2.2 Polar and Cartesian representations

We take $x \in \mathbb{R}^1$, $y \in \mathbb{R}^1$ and define the complex number z to be

$$z = x + iy. \tag{9.33}$$

We say that $z \in \mathbb{C}^1$. We define the operator $\Re$ as selecting the real part of a complex number and $\Im$ as selecting the imaginary part of a complex number. For Eq. (9.33), we see

$$\Re(z) = x, \qquad \Im(z) = y. \tag{9.34}$$

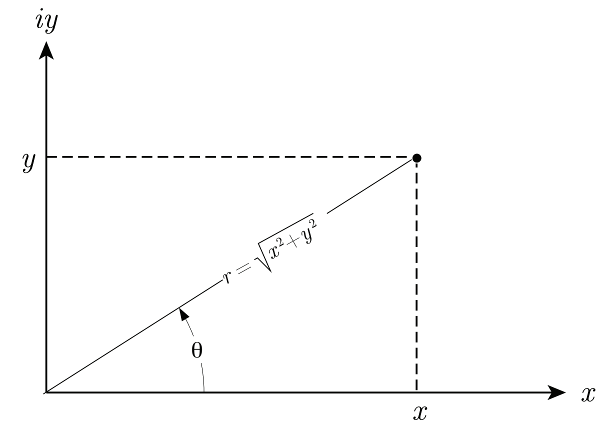

Figure 9.1: Polar and Cartesian representation of a complex number z.

Both operators $\Re$ and $\Im$ take $\mathbb{C}^1 \rightarrow \mathbb{R}^1$. We can multiply and divide Eq. (9.33) by $\sqrt{x^2+y^2}$ to obtain

$$z = \underbrace{\sqrt{x^2+y^2}}_{r}\left(\underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{\cos\theta} + i\underbrace{\frac{y}{\sqrt{x^2+y^2}}}_{\sin\theta}\right). \tag{9.35}$$

Noting the similarities between this and the transformation between Cartesian and polar coordinates suggests we adopt

$$r = \sqrt{x^2+y^2}, \qquad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}, \qquad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}. \tag{9.36}$$

Thus, we have

$$z = r\left(\cos\theta + i\sin\theta\right), \tag{9.37}$$
$$= re^{i\theta}. \tag{9.38}$$

We often say that a complex number can be characterized by its magnitude $|z|$ and its argument, θ; we say then

$$r = |z|, \tag{9.39}$$
$$\theta = \arg z. \tag{9.40}$$

Here, $r \in \mathbb{R}^1$ and $\theta \in \mathbb{R}^1$. Note that $|e^{i\theta}| = 1$. If $x > 0$, the function $\arg z$ is identical to $\arctan(y/x)$ and is suggested by the polar and Cartesian representation of z as shown in Fig. 9.1. However, we recognize that the ordinary arctan (also known as $\tan^{-1}$) function maps onto the range $[-\pi/2, \pi/2]$, while we would like arg to map onto $[-\pi, \pi]$. For example, to capture the entire unit circle if $r = 1$, we need $\theta \in [-\pi, \pi]$. This can be achieved if we

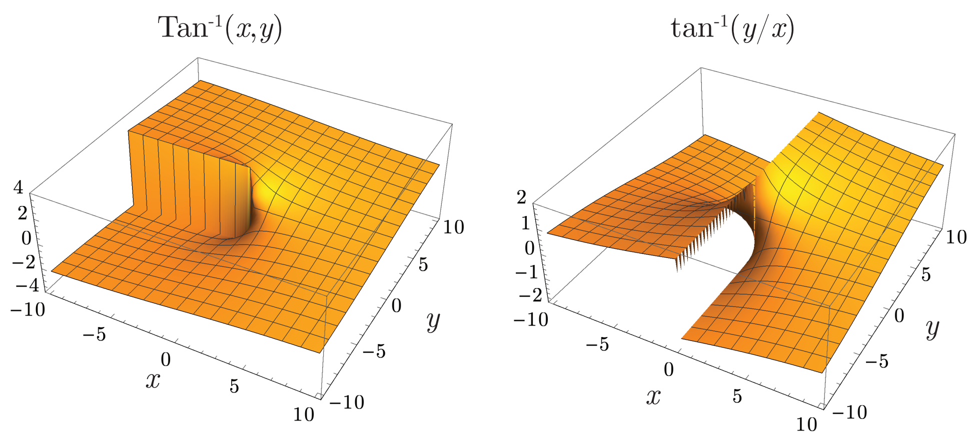

Figure 9.2: Comparison of $\mathrm{Tan}^{-1}(x,y)$ and $\tan^{-1}(y/x)$.

define arg, also known as Tan^{-1} as follows:

$$\arg z = \arg(x+iy) = \mathrm{Tan}^{-1}(x,y) = 2\arctan\left(\frac{y}{x+\sqrt{x^2+y^2}}\right). \tag{9.41}$$

Iff $x > 0$, this reduces to the more typical

$$\arg z = \arg(x+iy) = \mathrm{Tan}^{-1}(x,y) = \arctan\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{y}{x}\right), \qquad x > 0. \tag{9.42}$$

The preferred and more general form is Eq. (9.41). We give simple function evaluations involving arctan and Tan^{-1} for selected values of x and y in Table 9.1. Use of Tan^{-1}

Table 9.1: Comparison of the action of arg, Tan^{-1}, and arctan.

x	y	$\arg(x+iy)$	$\mathrm{Tan}^{-1}(x,y)$	$\arctan(y/x)$
1	1	$\pi/4$	$\pi/4$	$\pi/4$
-1	1	$3\pi/4$	$3\pi/4$	$-\pi/4$
-1	-1	$-3\pi/4$	$-3\pi/4$	$\pi/4$
1	-1	$-\pi/4$	$-\pi/4$	$-\pi/4$

effectively captures the correct quadrant of the complex plane corresponding to different positive and negative values of x and y. The function is sometimes known as Arctan or atan2. A comparison of $\mathrm{Tan}^{-1}(x,y)$ and $\tan^{-1}(y/x)$ is given in Fig. 9.2.

Now we can define the *complex conjugate* $\overline{z}$ as

$$\overline{z} = x - iy, \tag{9.43}$$

$$= \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} - i\frac{y}{\sqrt{x^2+y^2}}\right), \quad (9.44)$$

$$= r\left(\cos\theta - i\sin\theta\right), \quad (9.45)$$

$$= r\left(\cos(-\theta) + i\sin(-\theta)\right), \quad (9.46)$$

$$= re^{-i\theta}. \quad (9.47)$$

Note now that

$$z\overline{z} = (x+iy)(x-iy) = x^2+y^2 = |z|^2, \quad (9.48)$$

$$= re^{i\theta}re^{-i\theta}, \quad (9.49)$$

$$= r^2, \quad (9.50)$$

$$= |z|^2. \quad (9.51)$$

We also have

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}, \quad (9.52)$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}. \quad (9.53)$$

Example 9.1

Use the polar representation of z to find all roots to the algebraic equation

$$z^4 = 1. \quad (9.54)$$

We know that $z = re^{i\theta}$. We also note that the constant 1 can be represented as

$$1 = e^{2n\pi i}, \qquad n = 0, 1, 2, ... \quad (9.55)$$

This will be useful in finding all roots to our equation. With this representation, Eq. (9.54) becomes

$$r^4e^{4i\theta} = e^{2n\pi i}, \qquad n = 0, 1, 2, ... \quad (9.56)$$

We have a solution when

$$r = 1, \qquad \theta = \frac{n\pi}{2}, \qquad n = 0, 1, 2, ... \quad (9.57)$$

There are unique solutions for $n = 0, 1, 2, 3$. For larger n, the solutions repeat. So we have four solutions

$$z = e^{0i}, \quad z = e^{i\pi/2}, \quad z = e^{i\pi}, \quad z = e^{3i\pi/2}. \quad (9.58)$$

In Cartesian form, the four solutions are

$$z = \pm 1, \quad z = \pm i. \quad (9.59)$$

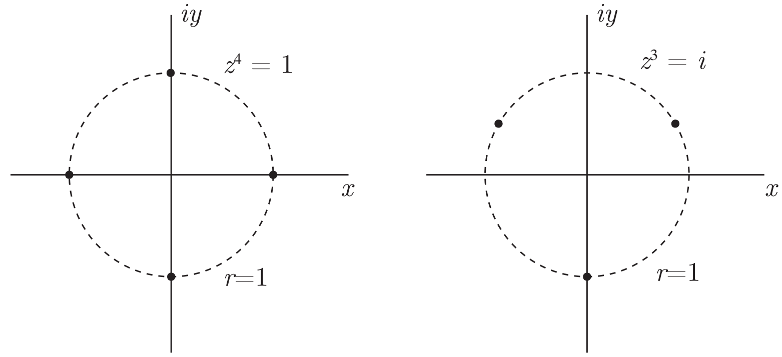

Figure 9.3: Sketch of solutions to $z^4 = 1$ and $z^3 = i$ in the complex plane.

Example 9.2

Find all roots to

$$z^3 = i. \tag{9.60}$$

We proceed in a similar fashion as for the previous example. We know that

$$i = e^{i(\pi/2+2n\pi)}, \quad n = 0, 1, 2, \ldots \tag{9.61}$$

Substituting this into Eq. (9.60), we get

$$r^3 e^{3i\theta} = e^{i(\pi/2+2n\pi)}, \quad n = 0, 1, 2, \ldots \tag{9.62}$$

Solving, we get

$$r = 1, \qquad \theta = \frac{\pi}{6} + \frac{2n\pi}{3}. \tag{9.63}$$

There are only three unique values of θ, those being $\theta = \pi/6$, $\theta = 5\pi/6$, $\theta = 3\pi/2$. So the three roots are

$$z = e^{i\pi/6}, \qquad z = e^{5i\pi/6}, \qquad z = e^{3i\pi/2}. \tag{9.64}$$

In Cartesian form these roots are

$$z = \frac{\sqrt{3} + i}{2}, \qquad z = \frac{-\sqrt{3} + i}{2}, \qquad z = -i. \tag{9.65}$$

Sketches of the solutions to this and the previous example are shown in Fig. 9.3. For both examples, the roots are uniformly distributed about the unit circle, with four roots for the quartic equation and three for the cubic.

9.2.3 Cauchy-Riemann equations

Now it is possible to define complex functions of complex variables $W(z)$. For example, take a complex function to be defined as

$$\begin{aligned} W(z) &= z^2 + z, && (9.66)\\ &= (x+iy)^2 + (x+iy), && (9.67)\\ &= x^2 + 2xyi - y^2 + x + iy, && (9.68)\\ &= \left(x^2 + x - y^2\right) + i\,(2xy + y)\,. && (9.69) \end{aligned}$$

In general, we can say

$$W(z) = \phi(x, y) + i\psi(x, y). \quad (9.70)$$

Here ϕ and ψ are *real* functions of *real* variables.

Now $W(z)$ is defined as *analytic* at z_o if dW/dz exists at z_o and is independent of the direction in which it was calculated. That is, using the definition of the derivative

$$\left.\frac{dW}{dz}\right|_{z_o} = \frac{W(z_o + \Delta z) - W(z_o)}{\Delta z}. \quad (9.71)$$

Now there are many paths that we can choose to evaluate the derivative. Let us consider two distinct paths, $y = C_1$ and $x = C_2$. We will get a result that can be shown to be valid for arbitrary paths. For $y = C_1$, we have $\Delta z = \Delta x$, so

$$\left.\frac{dW}{dz}\right|_{z_o} = \frac{W(x_o + iy_o + \Delta x) - W(x_o + iy_o)}{\Delta x} = \left.\frac{\partial W}{\partial x}\right|_y. \quad (9.72)$$

For $x = C_2$, we have $\Delta z = i\Delta y$, so

$$\left.\frac{dW}{dz}\right|_{z_o} = \frac{W(x_o + iy_o + i\Delta y) - W(x_o + iy_o)}{i\Delta y} = \frac{1}{i}\left.\frac{\partial W}{\partial y}\right|_x = -i\left.\frac{\partial W}{\partial y}\right|_x. \quad (9.73)$$

Now for an analytic function, we need

$$\left.\frac{\partial W}{\partial x}\right|_y = -i\left.\frac{\partial W}{\partial y}\right|_x, \quad (9.74)$$

or, expanding, we need

$$\begin{aligned} \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} &= -i\left(\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y}\right), && (9.75)\\ &= \frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y}. && (9.76) \end{aligned}$$

For equality, and thus path independence of the derivative, we require

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \quad (9.77)$$

These are the well known *Cauchy*[2]*-Riemann* equations for analytic functions of complex variables. *They are identical to our kinematic equations, Eqs. (9.11, 9.12), for incompressible irrotational fluid mechanics.* Consequently, *any analytic complex function is guaranteed to be a physical solution.* There are an infinite number of functions from which to choose!

We define the *complex velocity potential* as

$$W(z) = \phi(x, y) + i\psi(x, y), \tag{9.78}$$

and taking a derivative of this potential, we have

$$\frac{dW}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}, \tag{9.79}$$

$$= u - iv. \tag{9.80}$$

Because the direction of the derivative does not matter, we can equivalently say

$$\frac{dW}{dz} = -i\left(\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y}\right) = \left(\frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y}\right) = u - iv. \tag{9.81}$$

We can associate the velocity magnitude with the magnitude of dW/dz:

$$\left|\frac{dW}{dz}\right|^2 = \overline{\frac{dW}{dz}}\frac{dW}{dz}, \tag{9.82}$$

$$= (u + iv)(u - iv), \tag{9.83}$$

$$= u^2 + v^2, \tag{9.84}$$

$$\left|\frac{dW}{dz}\right| = \sqrt{u^2 + v^2}. \tag{9.85}$$

Now most common functions are easily shown to be analytic. For example for the function of Eq. (9.66)

$$W(z) = z^2 + z, \tag{9.86}$$

that we have seen can be expressed as

$$W(z) = (x^2 + x - y^2) + i(2xy + y), \tag{9.87}$$

we have

$$\phi(x, y) = x^2 + x - y^2, \qquad \psi(x, y) = 2xy + y, \tag{9.88}$$

$$\frac{\partial \phi}{\partial x} = 2x + 1, \qquad \frac{\partial \psi}{\partial x} = 2y, \tag{9.89}$$

$$\frac{\partial \phi}{\partial y} = -2y, \qquad \frac{\partial \psi}{\partial y} = 2x + 1. \tag{9.90}$$

The fields of ψ and ϕ are plotted in Fig. 9.4. The Cauchy-Riemann equations are satisfied

[2]Augustin-Louis Cauchy, 1789-1857, French mathematician and military engineer, worked in complex analysis, optics, and theory of elasticity.

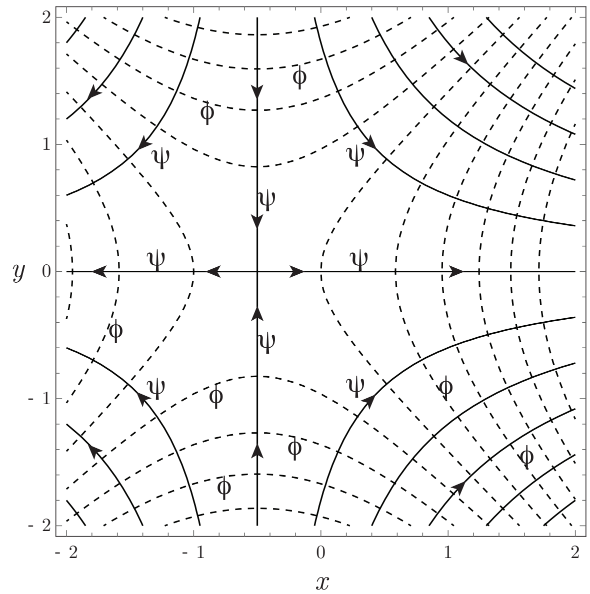

Figure 9.4: Fields of $\psi(x, y)$, $\phi(x, y)$ corresponding to the complex potential $W(z) = z^2 + 1$.

because $\partial\phi/\partial x = \partial\psi/\partial y$ and $\partial\phi/\partial y = -\partial\psi/\partial x$. Moreover,

$$\nabla^2\phi = \frac{\partial}{\partial x}(2x+1) + \frac{\partial}{\partial y}(-2y) = 2 - 2 = 0, \tag{9.91}$$

$$\nabla^2\psi = \frac{\partial}{\partial x}(2y) + \frac{\partial}{\partial y}(2x+1) = 0 + 0 = 0, \tag{9.92}$$

$$\mathbf{v} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2x+1 \\ -2y \end{pmatrix}. \tag{9.93}$$

So the derivative is independent of direction, and we can say

$$\frac{dW}{dz} = \left.\frac{\partial W}{\partial x}\right|_y = (2x+1) + i(2y) = 2(x+iy) + 1 = 2z + 1. \tag{9.94}$$

We could get this result by ordinary rules of derivatives for real functions.

For an example of a non-analytic function consider $W(z) = \overline{z}$. Thus,

$$W(z) = x - iy. \tag{9.95}$$

So $\phi = x$ and $\psi = -y$, $\partial\phi/\partial x = 1$, $\partial\phi/\partial y = 0$, and $\partial\psi/\partial x = 0$, $\partial\psi/\partial y = -1$. Because $\partial\phi/\partial x \neq \partial\psi/\partial y$, the Cauchy-Riemann equations are not satisfied, and the derivative depends on direction.

9.3 Elementary complex potentials

Let us examine some simple analytic functions and see the fluid mechanics to which they correspond.

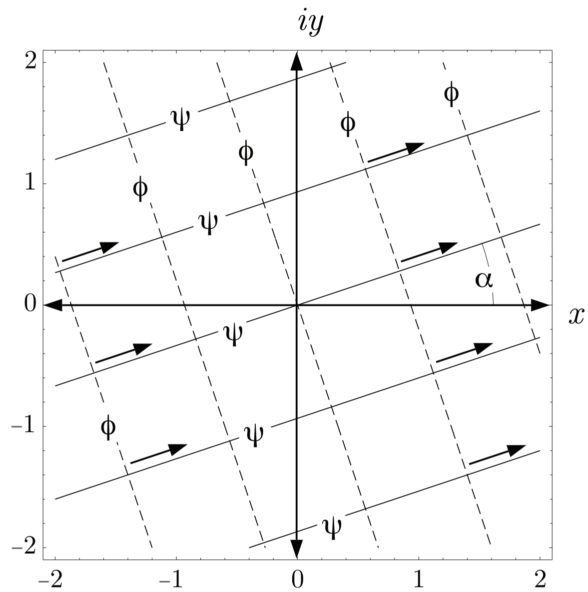

Figure 9.5: Streamlines and equipotential lines for uniform flow.

9.3.1 Uniform flow

Take

$$W(z) = Bz, \qquad \text{with} \qquad B \in \mathbb{C}^1. \tag{9.96}$$

Then

$$\frac{dW}{dz} = B = u - iv. \tag{9.97}$$

Because B is complex, we can say

$$B = U_o e^{-i\alpha} = U_o \cos\alpha - iU_o \sin\alpha. \tag{9.98}$$

Thus we get

$$u = U_o \cos\alpha, \qquad v = U_o \sin\alpha. \tag{9.99}$$

This represents a spatially uniform flow with streamlines inclined at angle α to the x axis. The flow is sketched in Fig. 9.5.

9.3.2 Sources and sinks

Take

$$W(z) = B \ln z, \qquad \text{with} \qquad B \in \mathbb{R}^1. \tag{9.100}$$

With $z = re^{i\theta}$ we have $\ln z = \ln r + i\theta$. So

$$W(z) = B \ln r + iB\theta. \tag{9.101}$$

Consequently, we have for the velocity potential and stream function

$$\phi = B \ln r, \qquad \psi = B\theta. \tag{9.102}$$

Now $\mathbf{v} = \nabla\phi$, so

$$v_r = \frac{\partial \phi}{\partial r} = \frac{B}{r}, \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0. \tag{9.103}$$

So the velocity is all radial, and becomes infinite at $r = 0$. We can show that the volume flow rate per unit depth is bounded, and is in fact a constant. For this two-dimensional flow, we really want to consider the volumetric flow rate per unit depth. We shall call this Q and recognize it has units of m^2/s.[3] Then with dA as a differential area per unit depth, we have $dA = r\ d\theta$, and the volume flow rate per unit depth Q through a surface is

$$Q = \int_A \mathbf{v}^T \cdot \mathbf{n}\ dA = \int_0^{2\pi} v_r r\ d\theta = \int_0^{2\pi} \frac{B}{r} r\ d\theta = 2\pi B. \tag{9.104}$$

The volume flow rate is a constant. If $B > 0$, we have a source. If $B < 0$, we have a sink. The potential for a source/sink is often written as

$$W(z) = \frac{Q}{2\pi} \ln z. \tag{9.105}$$

For a source located at a point z_o that is not at the origin, we can say

$$W(z) = \frac{Q}{2\pi} \ln(z - z_o). \tag{9.106}$$

The flow is sketched in Fig. 9.6.

9.3.3 Point vortices

For an ideal point vortex, identical to what we studied in Ch. 7.6.4, we have

$$W(z) = iB \ln z, \qquad \text{with} \qquad B \in \mathbb{R}^1. \tag{9.107}$$

So

$$W(z) = iB\,(\ln r + i\theta) = -B\theta + iB \ln r. \tag{9.108}$$

Consequently,

$$\phi = -B\theta, \qquad \psi = B \ln r. \tag{9.109}$$

We get the velocity field from

$$v_r = \frac{\partial \phi}{\partial r} = 0, \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{B}{r}. \tag{9.110}$$

[3] It is also common to interpret Q as a volume flow rate with units of m^3/s. Our notation would have to be adjusted if we took this interpretation.

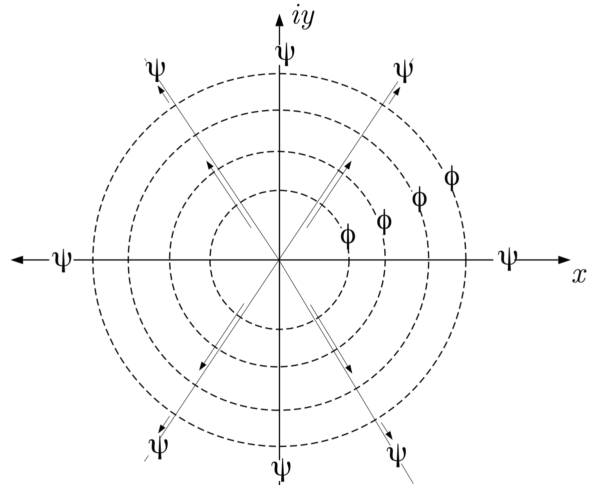

Figure 9.6: Streamlines and equipotential lines for source flow.

So we see that the streamlines are circles about the origin, and there is no radial component of velocity. Consider the circulation of this flow, Eq. (7.2):

$$\Gamma = \oint_C \mathbf{v}^T \cdot d\mathbf{r} = \int_0^{2\pi} -\frac{B}{r} r \; d\theta = -2\pi B. \tag{9.111}$$

So we often write the complex potential in terms of the ideal vortex strength Γ_o:

$$W(z) = -\frac{i\Gamma_o}{2\pi} \ln z. \tag{9.112}$$

For an ideal vortex not at $z = z_o$, we say

$$W(z) = -\frac{i\Gamma_o}{2\pi} \ln(z - z_o). \tag{9.113}$$

The point vortex flow is sketched in Fig. 9.7.

9.3.4 Superposition of sources

Because the equation for velocity potential is linear, we can use the method of superposition to create new solutions as summations of elementary solutions. Say we want to model the effect of a wall on a source as sketched in Fig. 9.8. At the wall we want $u(0, y) = 0$. That is

$$\Re\left\{\frac{dW}{dz}\right\} = \Re\left\{u - iv\right\} = 0, \qquad \text{on} \qquad z = iy. \tag{9.114}$$

Now let us place a source at $z = a$ and superpose a source at $z = -a$, where a is a real scalar; $a \in \mathbb{R}^1$. So we have for the complex potential

$$W(z) = \underbrace{\frac{Q}{2\pi}\ln(z-a)}_{\text{original}} + \underbrace{\frac{Q}{2\pi}\ln(z+a)}_{\text{image}}, \tag{9.115}$$

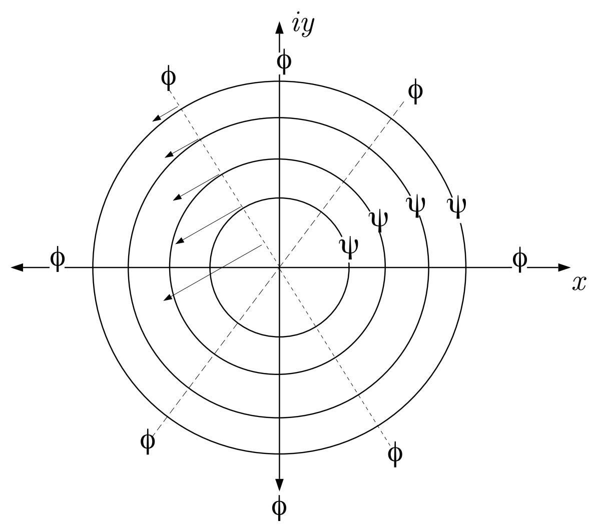

Figure 9.7: Streamlines, equipotential, and velocity vectors lines for a point vortex.

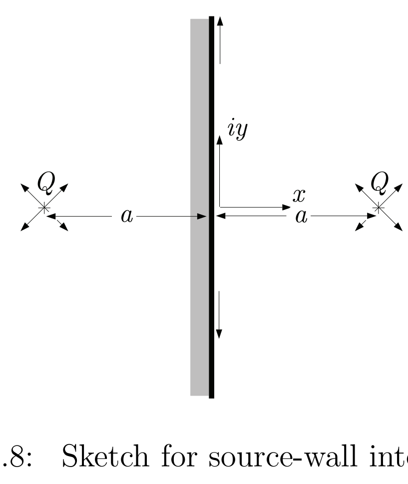

Figure 9.8: Sketch for source-wall interaction.

$$= \frac{Q}{2\pi}\left(\ln(z-a)+\ln(z+a)\right), \quad (9.116)$$

$$= \frac{Q}{2\pi}\ln\left((z-a)(z+a)\right), \quad (9.117)$$

$$= \frac{Q}{2\pi}\ln(z^2-a^2), \quad (9.118)$$

$$\frac{dW}{dz} = \frac{Q}{2\pi}\frac{2z}{z^2-a^2}. \quad (9.119)$$

Now on $z = iy$, that is the location of the wall, we have

$$\frac{dW}{dz} = \frac{Q}{2\pi}\left(\frac{2iy}{-y^2-a^2}\right) = u - iv. \quad (9.120)$$

The term is purely imaginary; hence, the real part is zero, and we have $u = 0$ on the wall, as desired.

On the wall we do have a non-zero y component of velocity. Hence the wall is not a no-slip wall. On the wall we have then

$$v = \frac{Q}{\pi}\frac{y}{y^2+a^2}. \quad (9.121)$$

We find the location on the wall of the maximum v velocity by setting the derivative with respect to y to be zero,

$$\frac{\partial v}{\partial y} = \frac{Q}{\pi}\frac{(y^2+a^2)-y(2y)}{(y^2+a^2)^2} = 0. \quad (9.122)$$

Solving, we find critical points at $y = \pm a$. It can be shown that v is a local maximum at $y = a$ and a local minimum at $y = -a$. So on the wall we have

$$\frac{1}{2}(u^2+v^2) = \frac{1}{2}\frac{Q^2}{\pi^2}\frac{y^2}{(y^2+a^2)^2}. \quad (9.123)$$

We can use Bernoulli's equation to find the pressure field, assuming steady flow and that $p \to p_o$ as $r \to \infty$. So Bernoulli's equation in this limit

$$\frac{1}{2}\nabla^T\phi\cdot\nabla\phi + \frac{p}{\rho} = \frac{p_o}{\rho}, \quad (9.124)$$

reduces to

$$p = p_o - \frac{1}{2}\rho\frac{Q^2}{\pi^2}\frac{y^2}{\left(y^2+a^2\right)^2}. \quad (9.125)$$

The pressure is p_o at $y = 0$ and is p_o as $y \to \infty$. By integrating the pressure over the wall surface, one would find the net force on the wall induced by the source.

9.3.5 Flow in corners

Flow in or around a corner can be modeled by the complex potential

$$\begin{aligned} W(z) &= Bz^n, \qquad \text{with } B \in \mathbb{R}^1, & (9.126)\\ &= B\left(re^{i\theta}\right)^n, & (9.127)\\ &= Br^n e^{in\theta}, & (9.128)\\ &= Br^n(\cos(n\theta) + i\sin(n\theta)). & (9.129) \end{aligned}$$

So we have

$$\phi = Br^n \cos n\theta, \qquad \psi = Br^n \sin n\theta. \quad (9.130)$$

Now recall that lines on which ψ is constant are streamlines. Examining the stream function, we obviously have streamlines when $\psi = 0$ that occurs whenever $\theta = 0$ or $\theta = \pi/n$.

For example if $n = 2$, we model a stream striking a flat wall; kinematics of this have been previously described in Ch. 3.11.8. For this flow, we have

$$\begin{aligned} W(z) &= Bz^2, & (9.131)\\ &= B(x + iy)^2, & (9.132)\\ &= B((x^2 - y^2) + i(2xy)), & (9.133)\\ \phi &= B(x^2 - y^2), \qquad \psi = B(2xy). & (9.134) \end{aligned}$$

So the streamlines are hyperbolas. For the velocity field, we take

$$\frac{dW}{dz} = 2Bz = 2B(x + iy) = u - iv, \quad (9.135)$$

$$u = 2Bx, \qquad v = -2By. \quad (9.136)$$

This flow actually represents flow in a corner formed by a right angle or flow striking a flat plate, or the impingement of two streams. For $n = 2$, streamlines are sketched in in Fig. 9.9.

Example 9.3

Explore kinematics and dynamics of the flow field induced by the complex potential $W(z) = z^2$.

We have

$$\begin{aligned} W(z) &= z^2, & (9.137)\\ &= (x + iy)^2, & (9.138)\\ &= (x^2 - y^2) + 2xyi. & (9.139) \end{aligned}$$

Thus we have

$$\phi(x, y) = x^2 - y^2, \qquad \psi(x, y) = 2xy. \quad (9.140)$$

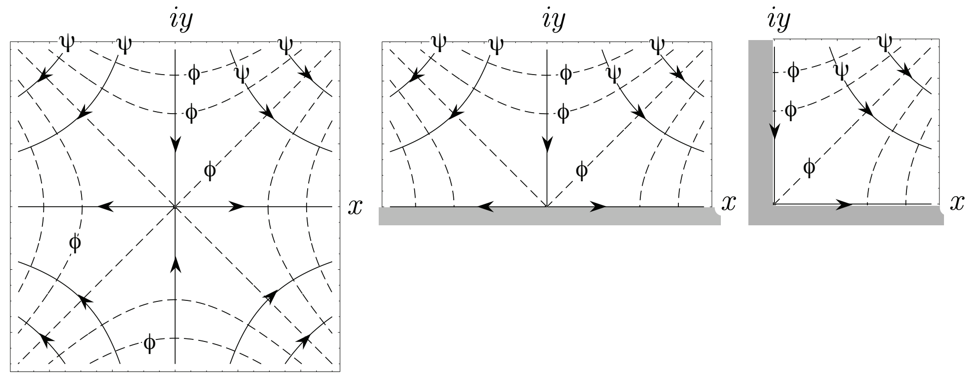

Figure 9.9: Sketch for impingement flow, stagnation flow, and flow in a corner, $n = 2$.

We see that the velocity field is given by

$$u = \frac{\partial \phi}{\partial x} = 2x, \qquad v = \frac{\partial \phi}{\partial y} = -2y. \tag{9.141}$$

We can also get it from the stream function

$$u = \frac{\partial \psi}{\partial y} = 2x, \qquad v = -\frac{\partial \psi}{\partial x} = -2y. \tag{9.142}$$

We also can get the velocity field directly via

$$\frac{dW}{dz} = 2z = u - iv. \tag{9.143}$$

Thus

$$2(x + iy) = u - iv. \tag{9.144}$$

Comparing, we see that

$$u = 2x, \qquad v = -2y. \tag{9.145}$$

So the velocity vector is

$$\mathbf{v} = \begin{pmatrix} 2x \\ -2y \end{pmatrix}. \tag{9.146}$$

The velocity vector is zero at the origin, which is a stagnation point. We also see the Laplacian equations are satisfied because

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2 - 2 \quad = \quad 0, \tag{9.147}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 + 0 \quad = \quad 0. \tag{9.148}$$

Incompressibility is satisfied as

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2 - 2 = 0. \tag{9.149}$$

Irrotationality is satisfied because

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 - 0 = 0. \tag{9.150}$$

Let us examine the deformation tensor, which is the symmetric part of the velocity gradient tensor:

$$\mathsf{D} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}. \tag{9.151}$$

The deformation tensor is already in diagonal form, so there is no necessity to rotate the axes to identify the principal axes. The principal axes are the eigenvectors associated with the tensor. We can normalize them and take them to be the unit vectors **i** and **j**. There is positive extensional strain aligned with **i**. This extensional strain is exactly counterbalanced by negative extensional strain aligned with **j**. So the volume is preserved of the deforming fluid particle, as required by incompressibility. For the fluid element aligned with the coordinate axes, there is no shear deformation.

The local acceleration vector of a fluid particle is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \begin{pmatrix} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix}. \tag{9.152}$$

The acceleration vector is zero at the origin, and points outward from the origin when away from the origin. A fluid particle on a stagnation streamline has no curvature, and its acceleration vector is parallel to its velocity vector. Streamlines that are not stagnation streamlines have non-zero curvature. The acceleration of such a fluid particle has a centripetal component that points towards the local instantaneous center of curvature.

Dynamics tells us that acceleration vectors must be induced by net force. In our problem, in which we neglect viscous and body forces, the only net force in play is that induced by the gradient of pressure. Let us find the pressure field associated with this flow field. Bernoulli's equation gives us

$$p_o = p + \frac{1}{2}\rho(u^2 + v^2), \tag{9.153}$$

$$= p + \frac{1}{2}\rho(4x^2 + 4y^2), \tag{9.154}$$

$$p - p_o = -2\rho(x^2 + y^2). \tag{9.155}$$

By inspection isobars are circles. The peak pressure is at the origin; thus the pressure decreases with the square of the distance from the origin. Fluid accelerates from regions of high pressure to regions of low pressure, and the acceleration vector points in the opposite direction of the pressure gradient vector:

$$\nabla p = -4\rho\left(x\mathbf{i} + y\mathbf{j}\right), \tag{9.156}$$

$$= -\rho\mathbf{a}. \tag{9.157}$$

Thus we recover our linear momenta equation

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla p. \tag{9.158}$$

A sketch of streamlines, equipotential lines, isobars, velocity vectors, and acceleration vectors is given in Fig. 9.10

Let us focus on one particular streamline, that for which $\psi = 2$. The equation of this streamline is

$$y = \frac{1}{x}. \tag{9.159}$$

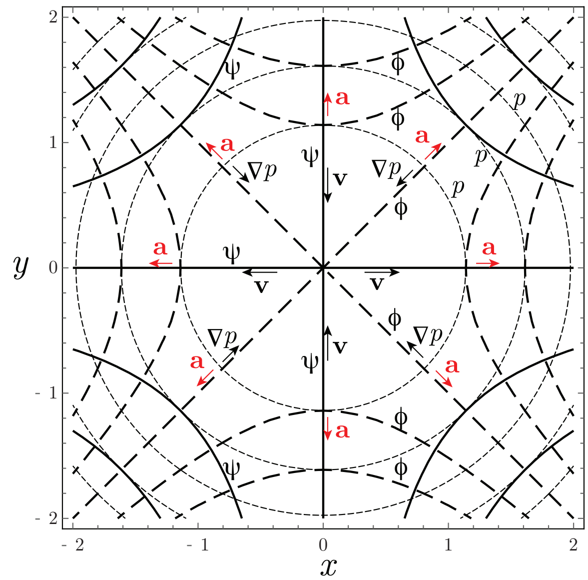

Figure 9.10: Sketch for $W(z) = z^2$ of streamlines, equipotential lines, isobars, velocity vectors, acceleration vectors, pressure gradient vectors.

We can use standard notions from calculus to define the curvature, κ. The general formula for curvature is

$$\kappa = \frac{\frac{d^2y}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}. \tag{9.160}$$

For this particular $\psi = 2$ streamline, we find

$$\kappa = \frac{2}{x^3\left(1+\frac{1}{x^4}\right)^{3/2}}. \tag{9.161}$$

Taylor series reveals that

$$\lim_{x\to 0} \kappa = 0, \qquad \lim_{x\to\infty} \kappa = 0. \tag{9.162}$$

This makes sense as the streamline $\psi = 1$ in the first quadrant approaches the x axis for large x and the y axis for small positive x. These axes have no curvature. For $x \in [0, +\infty)$, it is easy to show with calculus that κ has a maximum value given by

$$\kappa_{max} = \kappa(x = 1) = \frac{\sqrt{2}}{2}. \tag{9.163}$$

For this streamline, when $x = 1$, we have $y = 1/x = 1/1 = 1$. So the point of maximum streamline curvature is at an angle of $\pi/4$ from the x axis.

The velocity magnitude is given by

$$|\mathbf{v}| = \sqrt{4x^2 + 4y^2}. \tag{9.164}$$

For the $y = 1/x$ streamline, this gives

$$|\mathbf{v}| = \sqrt{4x^2 + \frac{4}{x^2}}. \tag{9.165}$$

Calculus reveals this has a local minimum of $2\sqrt{2}$ at the point $(x, y) = (1, 1)$.

From this analysis, it is easy to see the following are true.

- On the x axis, with $y = 0$ and for $x \to \infty$, $\mathbf{v}$ and $\mathbf{a}$ are parallel and point in positive x direction. The fluid acceleration is parallel to the pressure gradient vector, and both are parallel to the streamwise direction.
- On the y axis, with $x = 0$ and for $y \to \infty$, $\mathbf{v}$ and $\mathbf{a}$ are parallel and point opposite directions. The fluid acceleration is parallel to the pressure gradient vector, and both are parallel to the streamwise direction.
- For points on the curve $\theta = \pi/4$, $\mathbf{v}$ is orthogonal to $\mathbf{a}$. The fluid acceleration is parallel to the pressure gradient vector, and both are in the stream-normal direction. Here the acceleration is all centripetal and due to streamline curvature. And at such points the velocity magnitude has a local minimum.
- For intermediate points, $\mathbf{v}$ is neither parallel nor orthogonal to $\mathbf{a}$. The acceleration is parallel to the pressure gradient vector, and there are non-zero components of acceleration in the streamwise and stream-normal directions.

We can get the streamlines by direct integration. Because we know the velocity field, $u = 2x$, $v = -2y$, we then have the system of differential equations for the streamlines, pathlines, and streaklines for this steady flow. Let us find the streamline that passes through $(x, y)^T = (1, 1)^T$. We have the kinematics cast as a dynamical system as discussed in Ch. 3.13.

$$\frac{dx}{dt} = 2x, \qquad x(0) = 1, \tag{9.166}$$

$$\frac{dy}{dt} = -2y, \qquad y(0) = 1. \tag{9.167}$$

Integrating, we get

$$x(t) = e^{2t}, \tag{9.168}$$

$$y(t) = e^{-2t}. \tag{9.169}$$

This is a parametric solution for the streamline, streakline, and pathline that passes through $(1, 1)^T$ at $t = 0$. Obviously $1/x = e^{-2t} = y$, so this streamline is given, as expected, by

$$y = \frac{1}{x}. \tag{9.170}$$

The streamline $y = 1/x$ along with the local velocity and acceleration vectors and a few pressure contours are plotted in Fig. 9.10. The unit tangent to the streamline is

$$\boldsymbol{\alpha}_t = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{2x\mathbf{i} - 2y\mathbf{j}}{\sqrt{4x^2 + 4y^2}} \frac{2x\mathbf{i} - \frac{2}{x}\mathbf{j}}{\sqrt{4x^2 + \frac{4}{x^2}}} = \frac{x^2}{\sqrt{1 + x^4}}\mathbf{i} - \frac{1}{\sqrt{1 + x^4}}\mathbf{j} = \begin{pmatrix} \frac{x^2}{\sqrt{1+x^4}} \\ -\frac{1}{\sqrt{1+x^4}} \end{pmatrix} \tag{9.171}$$

We can specialize Eq. (3.148) to find the stretching rate $\mathcal{D}_t$ in the streamwise direction along the streamline $y = 1/x$. This is

$$\mathcal{D}_t = \boldsymbol{\alpha}_t^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_t = \begin{pmatrix} \frac{x^2}{\sqrt{1+x^4}} & -\frac{1}{\sqrt{1+x^4}} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} \frac{x^2}{\sqrt{1+x^4}} \\ -\frac{1}{\sqrt{1+x^4}} \end{pmatrix} = 2\frac{x^4 - 1}{x^4 + 1}. \tag{9.172}$$

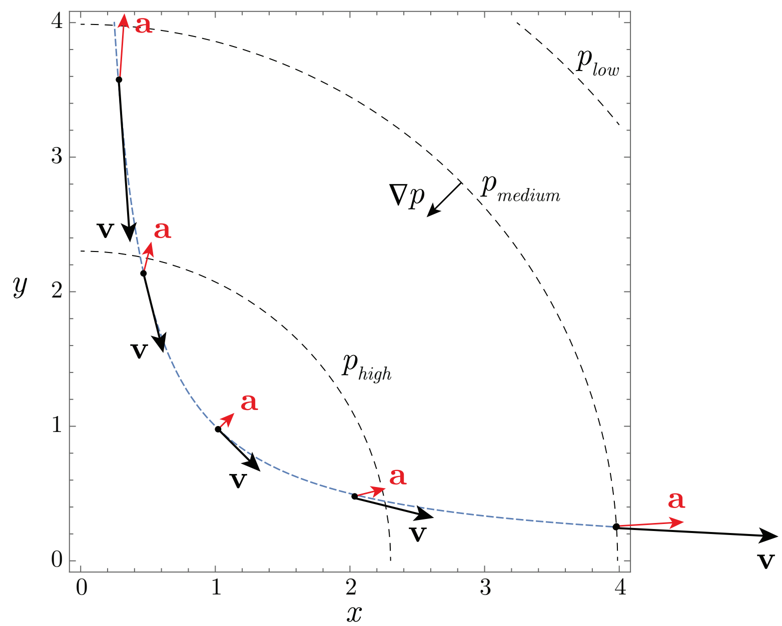

Figure 9.11: Kinematics and dynamics on the streamline $y = 1/x$ for the flow field defined by the complex potential $W(z) = z^2$.

There is no streamwise stretching at $x = 1$. As $x \to \infty$, $\mathcal{D}_t \to 2$, and as $x \to 0$, $\mathcal{D}_t \to -2$.

We see by inspection that the unit vector normal to the streamline $\boldsymbol{\alpha}_n$ must be

$$\boldsymbol{\alpha}_n = \begin{pmatrix} \frac{1}{\sqrt{1+x^4}} \\ \frac{x^2}{\sqrt{1+x^4}} \end{pmatrix}. \tag{9.173}$$

This vector obviously has $\boldsymbol{\alpha}_n^T \cdot \boldsymbol{\alpha}_t = 0$, and $\boldsymbol{\alpha}_n^T \cdot \boldsymbol{\alpha}_n = 1$. Moreover $\boldsymbol{\alpha}_n$ points toward the center of curvature of the streamline. We can also find the stretching rate $\mathcal{D}_n$ in the stream-normal direction along the streamline $y = 1/x$. This is

$$\mathcal{D}_n = \boldsymbol{\alpha}_n^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_n = \begin{pmatrix} \frac{1}{\sqrt{1+x^4}} & \frac{x^2}{\sqrt{1+x^4}} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+x^4}} \\ \frac{x^2}{\sqrt{1+x^4}} \end{pmatrix} = 2\frac{1-x^4}{x^4+1}. \tag{9.174}$$

There is no stream-normal stretching at $x = 1$. As $x \to \infty$, $\mathcal{D}_n \to -2$, and as $x \to 0$, $\mathcal{D}_n \to 2$. Moreover, because the flow is incompressible and thus volume-preserving, streamwise stretching is balanced by stream-normal stretching to that overall one has

$$\mathcal{D}_t + \mathcal{D}_n = 2\frac{x^4-1}{x^4+1} + 2\frac{1-x^4}{x^4+1} = 0. \tag{9.175}$$

Now let us perform an analysis for extreme values for $\mathcal{D}$ similar to that performed in generating Fig. 3.4. For a general direction $\boldsymbol{\alpha}$, we have

$$\mathcal{D} = \boldsymbol{\alpha}^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 2\alpha_1^2 - 2\alpha_2^2. \tag{9.176}$$

As before, we could plot contours for which $\mathcal{D}$ is constant in the (α_1, α_2) plane and get an infinite family of curves. In contrast to the ellipses of Fig. 3.4, here we have hyperbolas as contours. Once more, we

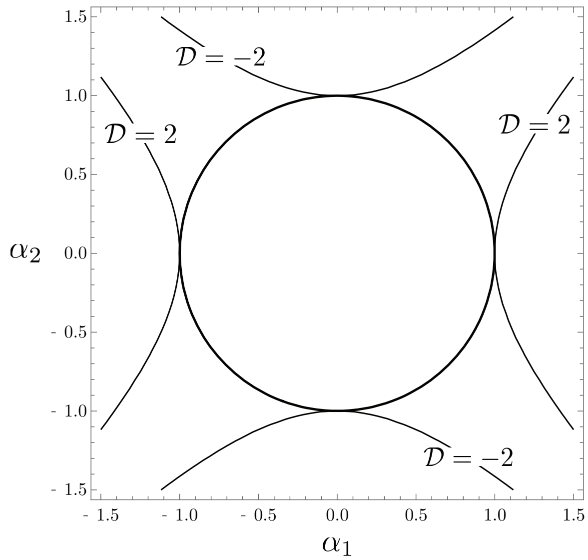

Figure 9.12: Two special contours of $\mathcal{D}$ along with the unit circle illustrating the extreme values of $\mathcal{D}$.

also have the constraint of $\alpha_1^2 + \alpha_2^2 = 1$. And once more, the eigenvalues of D are the special contours of $\mathcal{D}$, suggesting we examine the two special contours

$$2 = 2\alpha_1^2 - 2\alpha_2^2, \tag{9.177}$$
$$-2 = 2\alpha_1^2 - 2\alpha_2^2. \tag{9.178}$$

These two curves, along with the unit circle $\alpha_1^2 + \alpha_2^2 = 1$ are plotted in Fig. 9.12 An infinite family of contours of $\mathcal{D}$ exist. Many of them will also intersect the unit circle, and so are candidate solutions. However, the special contours we selected are extreme values. For intersection with the unit circle, we require

$$\mathcal{D} \in [\lambda_{min}, \lambda_{max}], \tag{9.179}$$
$$\in [-2, 2]. \tag{9.180}$$

Because D is already diagonal, the eigenvectors are aligned with the unrotated coordinate axes. It is straightforward to show for $\mathcal{D} = -2$, that $\boldsymbol{\alpha} = (0, 1)^T$ and for $\mathcal{D} = 2$, that $\boldsymbol{\alpha} = (1, 0)^T$.

9.3.6 Doublets

We can form what is known as a doublet flow by considering the superposition of a source and sink and let the two approach each other. Consider a source and sink of equal and opposite strength straddling the y axis, each separated from the origin by a distance ϵ as sketched in Fig. 9.13. The complex velocity potential is

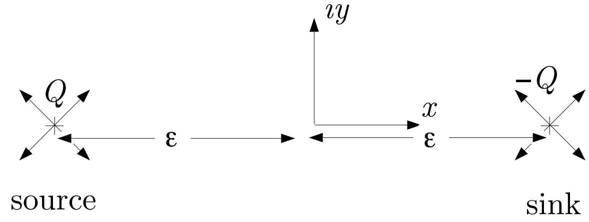

Figure 9.13: Source sink pair.

$$W(z) = \frac{Q}{2\pi}\ln(z+\epsilon) - \frac{Q}{2\pi}\ln(z-\epsilon), \tag{9.181}$$

$$= \frac{Q}{2\pi}\ln\left(\frac{z+\epsilon}{z-\epsilon}\right). \tag{9.182}$$

It can be shown by synthetic division that as $\epsilon \to 0$, that

$$\frac{z+\epsilon}{z-\epsilon} = 1 + \epsilon\frac{2}{z} + \epsilon^2\frac{2}{z^2} + \dots. \tag{9.183}$$

So the potential approaches

$$W(z) \sim \frac{Q}{2\pi}\ln\left(1 + \epsilon\frac{2}{z} + \epsilon^2\frac{2}{z^2} + \dots\right). \tag{9.184}$$

Now because $\ln(1+x) \to x$ as $x \to 0$, we get for small ϵ that

$$W(z) \sim \frac{Q}{2\pi}\epsilon\frac{2}{z} \sim \frac{Q\epsilon}{\pi z}. \tag{9.185}$$

Now if we require that

$$\lim_{\epsilon\to 0}\frac{Q\epsilon}{\pi} \to \mu, \tag{9.186}$$

we have

$$W(z) = \frac{\mu}{z} = \frac{\mu}{x+iy}\frac{x-iy}{x-iy} = \frac{\mu(x-iy)}{x^2+y^2}. \tag{9.187}$$

So

$$\phi(x,y) = \mu\frac{x}{x^2+y^2}, \qquad \psi(x,y) = -\mu\frac{y}{x^2+y^2}. \tag{9.188}$$

In polar coordinates, we then say

$$\phi = \mu\frac{\cos\theta}{r}, \qquad \psi = -\mu\frac{\sin\theta}{r}. \tag{9.189}$$

Streamlines and equipotential lines for a doublet are plotted in Fig. 9.14.

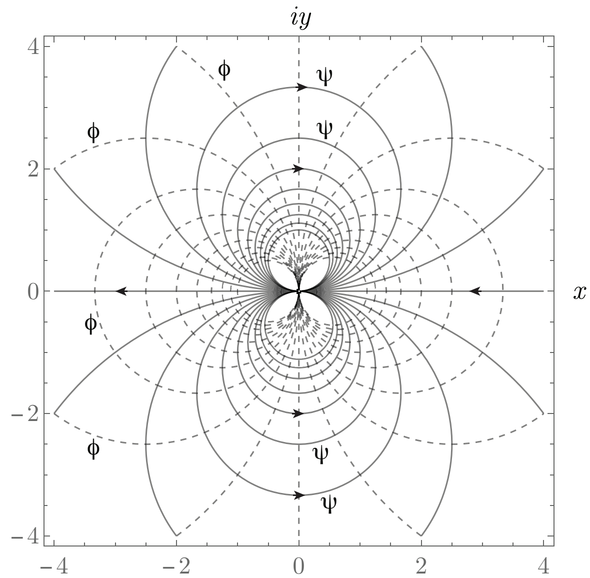

Figure 9.14: Streamlines and equipotential lines for a doublet. Notice because the sink is infinitesimally to the right of the source, there exists a directionality. This can be considered a type of *dipole moment*; in this case, the direction of the dipole is $-\mathbf{i}$.

9.3.7 Rankine half body

Now consider the superposition of a uniform stream and a source, that we define to be a Rankine half body:

$$\begin{aligned} W(z) &= Uz + \frac{Q}{2\pi}\ln z, \qquad \text{with} \qquad U, Q \in \mathbb{R}^1, && (9.190) \\ &= Ure^{i\theta} + \frac{Q}{2\pi}(\ln r + i\theta), && (9.191) \\ &= Ur(\cos\theta + i\sin\theta) + \frac{Q}{2\pi}(\ln r + i\theta), && (9.192) \\ &= \left(Ur\cos\theta + \frac{Q}{2\pi}\ln r\right) + i\left(Ur\sin\theta + \frac{Q}{2\pi}\theta\right). && (9.193) \end{aligned}$$

So

$$\phi = Ur\cos\theta + \frac{Q}{2\pi}\ln r, \qquad \psi = Ur\sin\theta + \frac{Q}{2\pi}\theta. \tag{9.194}$$

Streamlines for a Rankine half body are plotted in Fig. 9.15. Now for the Rankine half body, it is clear that there is a stagnation point somewhere on the x axis, along $\theta = \pi$. With the velocity given by

$$\frac{dW}{dz} = U + \frac{Q}{2\pi z} = u - iv, \tag{9.195}$$

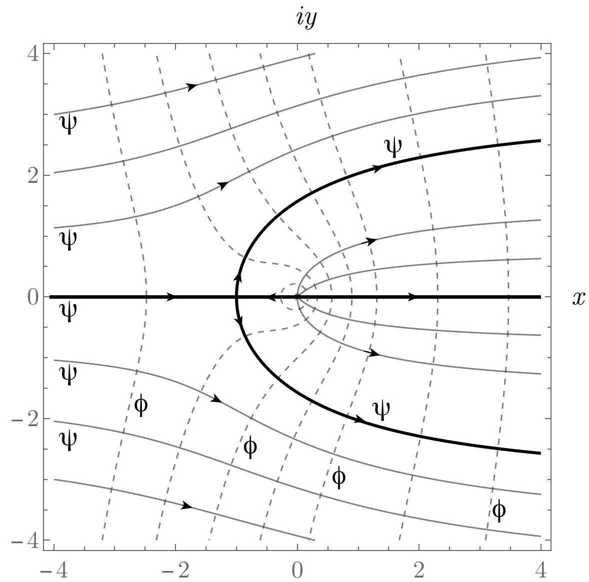

Figure 9.15: Streamlines for a Rankine half body.

we get

$$U + \frac{Q}{2\pi}\frac{1}{r}e^{-i\theta} = u - iv, \tag{9.196}$$

$$U + \frac{Q}{2\pi}\frac{1}{r}(\cos\theta - i\sin\theta) = u - iv, \tag{9.197}$$

$$u = U + \frac{Q}{2\pi r}\cos\theta, \qquad v = \frac{Q}{2\pi r}\sin\theta. \tag{9.198}$$

When $\theta = \pi$, we get $u = 0$ when;

$$0 = U + \frac{Q}{2\pi r}(-1), \tag{9.199}$$

$$r = \frac{Q}{2\pi U}. \tag{9.200}$$

9.3.8 Flow over a cylinder

We can model flow past a cylinder without circulation by superposing a uniform flow with a doublet. Defining $a^2 = \mu/U$, we write

$$W(z) = Uz + \frac{\mu}{z} = U\left(z + \frac{a^2}{z}\right), \tag{9.201}$$

$$= U\left(re^{i\theta} + \frac{a^2}{re^{i\theta}}\right), \tag{9.202}$$

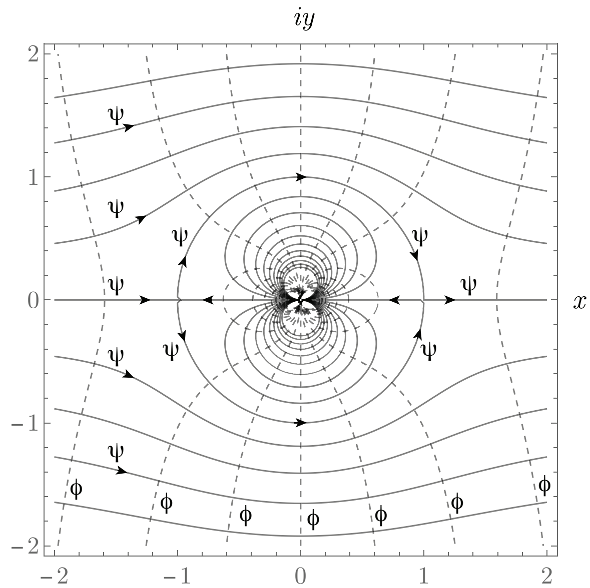

Figure 9.16: Streamlines and equipotential lines for flow over a cylinder without circulation.

$$= U\left(r(\cos\theta + i\sin\theta) + \frac{a^2}{r}(\cos\theta - i\sin\theta)\right), \tag{9.203}$$

$$= U\left(\left(r\cos\theta + \frac{a^2}{r}\cos\theta\right) + i\left(r\sin\theta - \frac{a^2}{r}\sin\theta\right)\right), \tag{9.204}$$

$$= Ur\left(\cos\theta\left(1 + \frac{a^2}{r^2}\right) + i\sin\theta\left(1 - \frac{a^2}{r^2}\right)\right). \tag{9.205}$$

So

$$\phi = Ur\cos\theta\left(1 + \frac{a^2}{r^2}\right), \qquad \psi = Ur\sin\theta\left(1 - \frac{a^2}{r^2}\right). \tag{9.206}$$

Now on $r = a$, we have $\psi = 0$. Because the stream function is constant here, the curve $r = a$, a circle, must be a streamline through which no mass can pass. A sketch of the streamlines and equipotential lines is plotted in Fig. 9.16.

For the velocities, we have

$$v_r = \frac{\partial\phi}{\partial r} = U\cos\theta\left(1 + \frac{a^2}{r^2}\right) + Ur\cos\theta\left(-2\frac{a^2}{r^3}\right), \tag{9.207}$$

$$= U\cos\theta\left(1 - \frac{a^2}{r^2}\right), \tag{9.208}$$

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta\left(1 + \frac{a^2}{r^2}\right). \tag{9.209}$$

So on $r = a$, we have $v_r = 0$, and $v_\theta = -2U\sin\theta$. Thus on the surface, we have

$$\nabla^T\phi \cdot \nabla\phi = 4U^2\sin^2\theta. \tag{9.210}$$

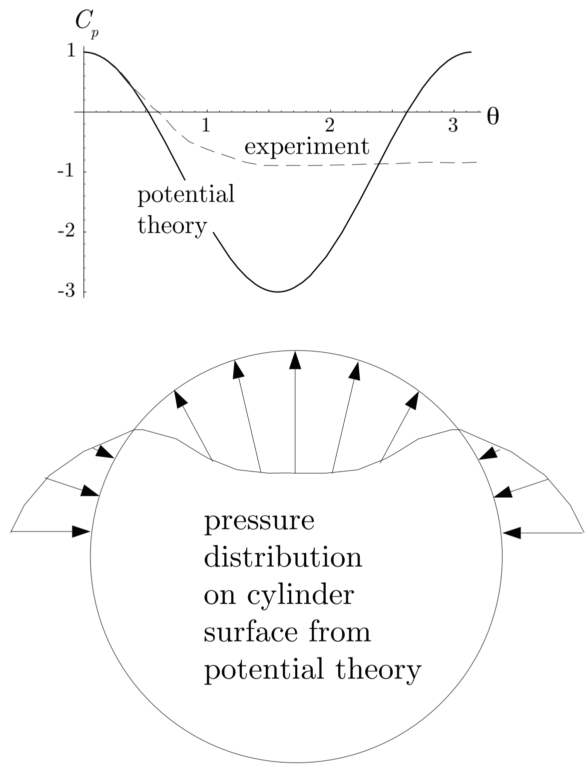

Figure 9.17: Pressure distribution for ideal flow over a cylinder without circulation.

Bernoulli's equation, Eq. (6.166), for a steady flow with $p \to p_\infty$ as $r \to \infty$ then gives

$$\frac{p}{\rho} + \frac{1}{2}\nabla^T\phi \cdot \nabla\phi = \frac{p_\infty}{\rho} + \frac{U^2}{2}, \tag{9.211}$$

$$p = p_\infty + \frac{1}{2}\rho U^2(1 - 4\sin^2\theta). \tag{9.212}$$

The pressure coefficient C_p, defined below, then is

$$C_p \equiv \frac{p - p_\infty}{\frac{1}{2}\rho U^2} = 1 - 4\sin^2\theta. \tag{9.213}$$

A sketch of the pressure distribution, both predicted and experimentally observed, is plotted in Fig. 9.17. We note that the potential theory predicts the pressure well on the front surface of the cylinder, but not so well on the back surface. This is because in most real fluids, a phenomenon known as flow separation manifests itself in regions of negative pressure gradients. Correct modeling of separation events requires a re-introduction of viscous stresses. A potential theory cannot predict separation.

Example 9.4

For a cylinder of radius c at rest in an accelerating potential flow field with a far field velocity of $U = a + bt$, find the pressure on the stagnation point of the cylinder.

The velocity potential and velocities for this flow are

$$\phi(r,\theta,t) = (a+bt)r\cos\theta\left(1+\frac{c^2}{r^2}\right), \tag{9.214}$$

$$v_r = \frac{\partial\phi}{\partial r} = (a+bt)\cos\theta\left(1-\frac{c^2}{r^2}\right), \tag{9.215}$$

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -(a+bt)\sin\theta\left(1+\frac{c^2}{r^2}\right), \tag{9.216}$$

$$\frac{1}{2}\nabla^T\phi\cdot\nabla\phi = \frac{1}{2}(a+bt)^2\left(\cos^2\theta\left(1-\frac{c^2}{r^2}\right)^2+\sin^2\theta\left(1+\frac{c^2}{r^2}\right)^2\right), \tag{9.217}$$

$$= \frac{1}{2}(a+bt)^2\left(1+\frac{c^4}{r^4}+\frac{2c^2}{r^2}\left(\sin^2\theta-\cos^2\theta\right)\right). \tag{9.218}$$

Also, because the flow is unsteady, we will need $\partial\phi/\partial t$:

$$\frac{\partial\phi}{\partial t} = br\cos\theta\left(1+\frac{c^2}{r^2}\right). \tag{9.219}$$

Now we note in the limit as $r\to\infty$ that

$$\frac{\partial\phi}{\partial t}\to br\cos\theta, \qquad \frac{1}{2}\nabla^T\phi\cdot\nabla\phi\to\frac{1}{2}(a+bt)^2. \tag{9.220}$$

We also note that on the surface of the cylinder

$$v_r(r=c,\theta,t)=0. \tag{9.221}$$

The unsteady Bernoulli equation, the incompressible, zero-body force version of Eq. (6.153), gives us

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\nabla^T\phi\cdot\nabla\phi+\frac{p}{\rho}=f(t). \tag{9.222}$$

We use the far field behavior to evaluate $f(t)$:

$$br\cos\theta+\frac{1}{2}(a+bt)^2+\frac{p}{\rho}=f(t). \tag{9.223}$$

Now if we make the non-intuitive choice of $f(t)=\frac{1}{2}(a+bt)^2+p_o/\rho$, we get

$$br\cos\theta+\frac{1}{2}(a+bt)^2+\frac{p}{\rho}=\frac{1}{2}(a+bt)^2+\frac{p_o}{\rho}. \tag{9.224}$$

So

$$p=p_o-\rho br\cos\theta=p_o-\rho bx. \tag{9.225}$$

Because the flow at infinity is accelerating, there must be a far-field pressure gradient to induce this acceleration. Consider the x momentum equation in the far field

$$\rho\frac{du}{dt} = -\frac{\partial p}{\partial x}, \tag{9.226}$$

$$\rho(b) = -(-\rho b). \tag{9.227}$$

So for the pressure field, we have

$$\underbrace{br\cos\theta\left(1+\frac{c^2}{r^2}\right)}_{\partial\phi/\partial t}+\underbrace{\frac{1}{2}(a+bt)^2\left(1+\frac{c^4}{r^4}+\frac{2c^2}{r^2}(\sin^2\theta-\cos^2\theta)\right)}_{\nabla^T\phi\cdot\nabla\phi/2}+\frac{p}{\rho}=\underbrace{\frac{1}{2}(a+bt)^2+\frac{p_o}{\rho}}_{f(t)}, \tag{9.228}$$

that can be solved for p to yield

$$p(r,\theta,t)=p_o-\rho br\cos\theta\left(1+\frac{c^2}{r^2}\right)-\frac{1}{2}\rho(a+bt)^2\left(\frac{c^4}{r^4}+\frac{2c^2}{r^2}(\sin^2\theta-\cos^2\theta)\right). \tag{9.229}$$

For the stagnation point, we evaluate as

$$p(c,\pi,t) = p_o-\rho bc(-1)(1+1)-\frac{1}{2}\rho(a+bt)^2(1+2(1)(0-1)), \tag{9.230}$$

$$= p_o+\frac{1}{2}\rho(a+bt)^2+2\rho bc, \tag{9.231}$$

$$= p_o+\frac{1}{2}\rho U^2+2\rho bc. \tag{9.232}$$

The first two terms would be predicted by a naïve extension of the steady Bernoulli's equation, Eq. (6.166). The final term however is not intuitive and is a purely unsteady effect.

9.4 Forces induced by potential flow

There are more basic ways to describe the force on bodies using complex variables directly. We shall give those methods, but first a discussion of the motivating complex variable theory is necessary.

9.4.1 Contour integrals

Consider the closed contour integral of a complex function in the complex plane. For such integrals, we have a useful theory that we will not prove, but will demonstrate here. Consider contour integrals enclosing the origin with a circle in the complex plane for four functions. The contour in each is

$$C: z=\hat{R}e^{i\theta}, \qquad \theta\in[0,2\pi). \tag{9.233}$$

For such a contour,

$$dz=i\hat{R}e^{i\theta}\,d\theta. \tag{9.234}$$

9.4.1.1 Simple pole

We describe a simple pole with the complex potential

$$W(z) = \frac{a}{z}. \tag{9.235}$$

and the contour integral is

$$\oint_C W(z)\ dz = \oint_C \frac{a}{z}\ dz = \int_{\theta=0}^{\theta=2\pi} \frac{a}{\hat{R}e^{i\theta}} i\hat{R}e^{i\theta}\ d\theta, \tag{9.236}$$

$$= ai \int_0^{2\pi} d\theta = 2\pi i a. \tag{9.237}$$

9.4.1.2 Constant potential

We describe a constant with the complex potential

$$W(z) = b. \tag{9.238}$$

and the contour integral is

$$\oint_C W(z)\ dz = \oint_C b\ dz = \int_{\theta=0}^{\theta=2\pi} bi\hat{R}e^{i\theta}\ d\theta, \tag{9.239}$$

$$= \left.\frac{bi\hat{R}}{i} e^{i\theta}\right|_0^{2\pi} = 0, \tag{9.240}$$

because $e^{0i} = e^{2\pi i} = 1$.

9.4.1.3 Uniform flow

We describe a constant with the complex potential

$$W(z) = cz. \tag{9.241}$$

and the contour integral is

$$\oint_C W(z)\ dz = \oint_C cz\ dz = \int_{\theta=0}^{\theta=2\pi} c\hat{R}e^{i\theta} i\hat{R}e^{i\theta}\ d\theta, \tag{9.242}$$

$$= ic\hat{R}^2 \int_0^{2\pi} e^{2i\theta}\ d\theta = \left.\frac{ic\hat{R}^2}{2i} e^{2i\theta}\right|_0^{2\pi} = 0. \tag{9.243}$$

because $e^{0i} = e^{4\pi i} = 1$.

9.4.1.4 Quadrupole

A quadrupole potential is described by

$$W(z) = \frac{k}{z^2}. \tag{9.244}$$

Taking the contour integral, we find

$$\oint_C \frac{k}{z^2}\,dz = k\int_0^{2\pi} \frac{i\hat{R}e^{i\theta}}{\hat{R}^2 e^{2i\theta}}\,d\theta, \tag{9.245}$$

$$= \frac{ki}{\hat{R}}\int_0^{2\pi} e^{-i\theta}\,d\theta = \frac{ki}{\hat{R}}\frac{1}{-i}e^{-i\theta}\Big|_0^{2\pi} = 0. \tag{9.246}$$

So the only non-zero contour integral is for functions of the form $W(z) = a/z$. We find all polynomial powers of z have a zero contour integral about the origin for arbitrary contours except this special one.

9.4.2 Laurent series

Now it can be shown that any function can be expanded, much as for a Taylor series, as a *Laurent series*:[4]

$$W(z) = \ldots + C_{-2}(z-z_o)^{-2} + C_{-1}(z-z_o)^{-1} + C_0(z-z_o)^0 + C_1(z-z_o)^1 + C_2(z-z_o)^2 + \ldots. \tag{9.247}$$

In compact summation notation, we can say

$$W(z) = \sum_{n=-\infty}^{n=\infty} C_n (z-z_o)^n. \tag{9.248}$$

Taking the contour integral of both sides we get

$$\oint_C W(z)\,dz = \oint_C \sum_{n=-\infty}^{n=\infty} C_n (z-z_o)^n\,dz, \tag{9.249}$$

$$= \sum_{n=-\infty}^{n=\infty} C_n \oint_C (z-z_o)^n\,dz. \tag{9.250}$$

From our just completed analysis, this has value $2\pi i$ only when $n=-1$, so

$$\oint_C W(z)\,dz = C_{-1}2\pi i. \tag{9.251}$$

[4]Pierre Alphonse Laurent, 1813-1854, Parisian engineer who worked on port expansion in Le Harve, submitted his work on Laurent series for a Grand Prize in 1842, with the recommendation of Cauchy, but was rejected because of a late submission.

Here C_{-1} is known as the *residue* of the Laurent series. In general we have the *Cauchy integral theorem* that holds that if $W(z)$ is analytic within and on a closed curve C except for a finite number of singular points, then

$$\oint_C W(z)\; dz = 2\pi i \sum \text{residues}. \tag{9.252}$$

Let us get a simple formula for C_n. We first exchange m for n in Eq. (9.248) and say

$$W(z) = \sum_{m=-\infty}^{m=\infty} C_m (z-z_o)^m. \tag{9.253}$$

Then we operate as follows:

$$\frac{W(z)}{(z-z_o)^{n+1}} = \sum_{m=-\infty}^{m=\infty} C_m (z-z_o)^{m-n-1}, \tag{9.254}$$

$$\oint_C \frac{W(z)}{(z-z_o)^{n+1}}\; dz = \oint_C \sum_{m=-\infty}^{m=\infty} C_m (z-z_o)^{m-n-1}\; dz, \tag{9.255}$$

$$= \sum_{m=-\infty}^{m=\infty} C_m \oint_C (z-z_o)^{m-n-1}\; dz. \tag{9.256}$$

Here C is any closed contour that has z_o in its interior. The contour integral on the right side only has a non-zero value when $n = m$. Let us then insist that $n = m$, giving

$$\oint_C \frac{W(z)}{(z-z_o)^{n+1}}\; dz = C_n \underbrace{\oint_C (z-z_o)^{-1}\; dz}_{=2\pi i}. \tag{9.257}$$

We know from earlier analysis that the contour integral enclosing a simple pole such as found on the right side has a value of $2\pi i$. Solving, we find then that

$$C_n = \frac{1}{2\pi i} \oint_C \frac{W(z)}{(z-z_o)^{n+1}}\; dz. \tag{9.258}$$

If the closed contour C encloses no poles, then

$$\oint_C W(z)\; dz = 0. \tag{9.259}$$

9.4.3 Pressure distribution for steady flow

For steady, irrotational, incompressible flow with no body force present, we have the Bernoulli equation. We recast Eq. (6.166) as

$$\frac{p}{\rho} + \frac{1}{2}\nabla^T\phi \cdot \nabla\phi = \frac{p_\infty}{\rho} + \frac{1}{2}U_\infty^2. \tag{9.260}$$

We can write this in terms of the complex potential in a simple fashion. First, recall that

$$\nabla^T \phi \cdot \nabla \phi = u^2 + v^2. \tag{9.261}$$

We also have $dW/dz = u - iv$, so $\overline{dW/dz} = u + iv$. Consequently,

$$\frac{dW}{dz}\overline{\frac{dW}{dz}} = u^2 + v^2 = \nabla^T \phi \cdot \nabla \phi. \tag{9.262}$$

So we get the pressure field from Bernoulli's equation to be

$$p = p_\infty + \frac{1}{2}\rho \left(U_\infty^2 - \frac{dW}{dz}\overline{\frac{dW}{dz}} \right). \tag{9.263}$$

The pressure coefficient C_p is

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2} = 1 - \frac{1}{U_\infty^2}\frac{dW}{dz}\overline{\frac{dW}{dz}}. \tag{9.264}$$

9.4.4 Blasius force theorem

For steady flows, we can find the net contribution of a pressure force on an arbitrary shaped solid body with the Blasius[5] force theorem. Consider the geometry sketched in Fig. 9.18. The surface of the arbitrarily shaped body is described by S_b, and C is a closed contour containing S_b. First consider the linear momenta equation for steady flow, no body forces, and no viscous forces,

$$\begin{aligned}
\rho \left(\mathbf{v}^T \cdot \nabla \right) \mathbf{v} &= -\nabla p, \qquad \text{add mass to get conservative form,} && (9.265)\\
\left(\nabla^T \cdot (\rho \mathbf{v}\mathbf{v}^T) \right)^T &= -\nabla p, \qquad \text{integrate over } V, && (9.266)\\
\int_V \left(\nabla^T \cdot (\rho \mathbf{v}\mathbf{v}^T) \right)^T dV &= -\int_V \nabla p \; dV, \qquad \text{use Gauss,} && (9.267)\\
\int_S \rho \mathbf{v}(\mathbf{v}^T \cdot \mathbf{n}) \; dS &= -\int_S p\mathbf{n} \; dS. && (9.268)
\end{aligned}$$

Now the surface integral here is really a line integral with unit depth b, $dS = b\; ds$. Moreover the surface enclosing the *fluid* has an inner contour S_b and an outer contour C. Now on C, that we prescribe, we will know $x(s)$ and $y(s)$, where s is arc length. So on C we also get the unit tangent $\boldsymbol{\alpha}$ and unit outward normal $\mathbf{n}$:

$$\boldsymbol{\alpha} = \begin{pmatrix} \frac{dx}{ds} \\ \frac{dy}{ds} \end{pmatrix}, \qquad \mathbf{n} = \begin{pmatrix} \frac{dy}{ds} \\ -\frac{dx}{ds} \end{pmatrix}, \qquad \text{on } C. \tag{9.269}$$

[5]Paul Richard Heinrich Blasius, 1883-1970, student of Ludwig Prandtl and long time teacher at the technical college of Hamburg whose 1907 Ph.D. thesis gave mathematical description of similarity solution to the boundary layer problem.

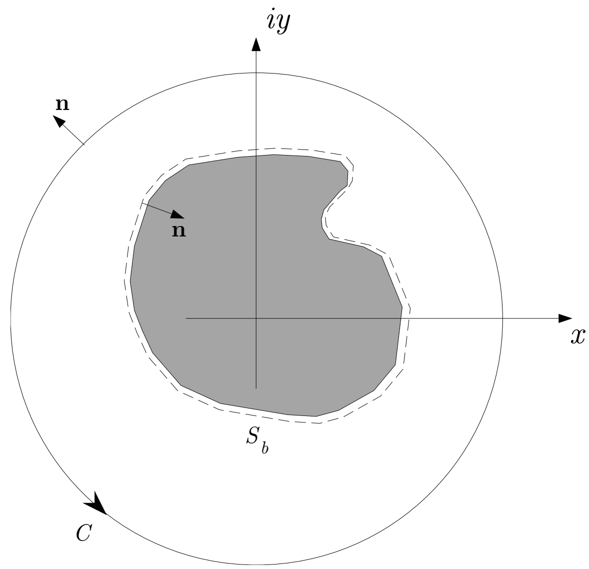

Figure 9.18: Potential flow about arbitrarily shaped two-dimensional body with fluid control volume indicated.

A loose analysis that can be verified more rigorously shows $\boldsymbol{\alpha}$ and $\mathbf{n}$ are unit vectors. Standard geometry tells us $ds^2 = dx^2 + dy^2$. So $\boldsymbol{\alpha} = (dx/\sqrt{dx^2+dy^2}, dy/\sqrt{dx^2+dy^2})^T$. By inspection $||\boldsymbol{\alpha}|| = 1$. A similar result holds for $\mathbf{n}$.

On S_b we have, because it is a solid surface

$$\mathbf{v}^T \cdot \mathbf{n} = 0, \qquad \text{on } S_b. \tag{9.270}$$

Now let the force on the body due to fluid pressure be $\mathbf{F}$:

$$\int_{S_b} p\mathbf{n}\, dS = \mathbf{F}. \tag{9.271}$$

Now return to our linear momentum equation

$$\int_S \rho\mathbf{v}\mathbf{v}^T \cdot \mathbf{n}\, dS = -\int_S p\mathbf{n}\, dS. \tag{9.272}$$

Break this up to get

$$\oint_{S_b} \rho\mathbf{v}\underbrace{\mathbf{v}^T \cdot \mathbf{n}}_{=0}\, dS + \oint_C \rho\mathbf{v}\mathbf{v}^T \cdot \mathbf{n}\, dS = -\underbrace{\oint_{S_b} p\mathbf{n}\, dS}_{=\mathbf{F}} - \oint_C p\mathbf{n}\, dS, \tag{9.273}$$

$$\oint_C \rho\mathbf{v}\mathbf{v}^T \cdot \mathbf{n}\, dS = -\mathbf{F} - \oint_C p\mathbf{n}\, dS. \tag{9.274}$$

We can break this into x and y components:

$$\oint_C \rho u \underbrace{\left(u\frac{dy}{ds} - v\frac{dx}{ds} \right)}_{\mathbf{v}^T\cdot\mathbf{n}} b\ ds = -F_x - \oint_C p \underbrace{\frac{dy}{ds}}_{n_x} b\ ds, \quad (9.275)$$

$$\oint_C \rho v \underbrace{\left(u\frac{dy}{ds} - v\frac{dx}{ds} \right)}_{\mathbf{v}^T\cdot\mathbf{n}} b\ ds = -F_y - \oint_C p \underbrace{\left(-\frac{dx}{ds}\right)}_{n_y} b\ ds. \quad (9.276)$$

Solving for F_x and F_y per unit depth, we get

$$\frac{F_x}{b} = \oint_C -p\ dy - \rho u^2\ dy + \rho uv\ dx, \quad (9.277)$$

$$\frac{F_y}{b} = \oint_C p\ dx + \rho v^2\ dx - \rho uv\ dy. \quad (9.278)$$

Now Bernoulli gives us $p = p_o - (1/2)\rho(u^2+v^2)$, where p_o is some constant. So the x force per unit depth becomes

$$\frac{F_x}{b} = \oint_C -p_o\ dy + \frac{1}{2}\rho(u^2+v^2)\ dy - \rho u^2\ dy + \rho uv\ dx. \quad (9.279)$$

Because the integral over a closed contour of a constant p_o is zero, we get

$$\frac{F_x}{b} = \oint_C \frac{1}{2}\rho(-u^2+v^2)\ dy + \rho uv\ dx, \quad (9.280)$$

$$= \frac{1}{2}\rho \oint_C (-u^2+v^2)\ dy + 2uv\ dx. \quad (9.281)$$

Similarly for the y direction, we get

$$\frac{F_y}{b} = \oint_C p_o\ dx - \frac{1}{2}\rho(u^2+v^2)\ dx + \rho v^2\ dx - \rho uv\ dy, \quad (9.282)$$

$$= \frac{1}{2}\rho \oint_C (-u^2+v^2)\ dx - 2uv\ dy. \quad (9.283)$$

Now consider the linear combination $(F_x - iF_y)/b$:

$$\frac{F_x - iF_y}{b} = \frac{1}{2}\rho \oint_C (-u^2+v^2)\ dy + 2uv\ dx - (-u^2+v^2)i\ dx + 2uvi\ dy, \quad (9.284)$$

$$= \frac{1}{2}\rho \oint_C (i(u^2-v^2) + 2uv)\ dx + ((-u^2+v^2) + 2uvi)\ dy, \quad (9.285)$$

$$= \frac{1}{2}\rho \oint_C (i(u^2-v^2) + 2uv)\ dx + (i(u^2-v^2) + 2uv)i\ dy, \quad (9.286)$$

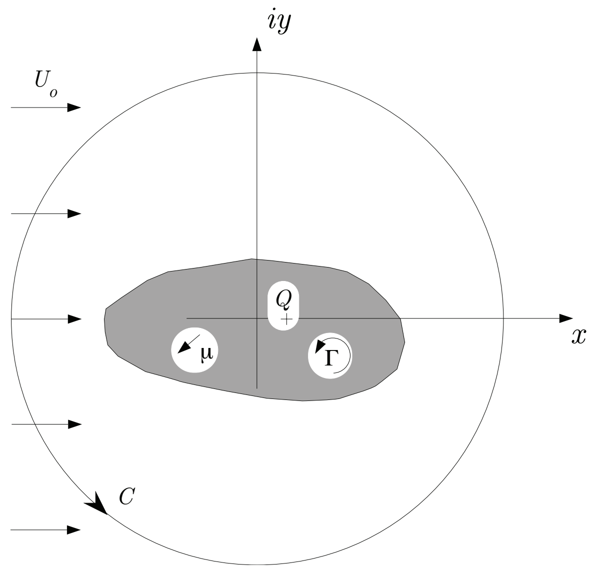

Figure 9.19: Potential flow about arbitrarily shaped two-dimensional body with distribution of sources, sinks, vortices, and dipoles.

$$= \frac{1}{2}\rho \oint_C (i(u^2 - v^2) + 2uv)(dx + i\ dy), \tag{9.287}$$

$$= \frac{1}{2}\rho \oint_C i(u - iv)^2(dx + i\ dy), \tag{9.288}$$

$$= \frac{1}{2}\rho i \oint_C \left(\frac{dW}{dz}\right)^2 dz. \tag{9.289}$$

So if we have the complex potential, we can easily get the force on a body.

9.4.5 Kutta-Zhukovsky lift theorem

Consider the geometry sketched in Fig. 9.19. Here we consider a flow with a freestream constant velocity of U_o. We take an arbitrary body shape to enclose a distribution of canceling source sink pairs, doublets, point vortices, quadruples, and any other non-mass adding potential flow term. This combination gives rise to some surface that is a streamline.

Now far from the body surface a contour sees all of these features as effectively concentrated at the origin. Then, the potential can be written as

$$W(z) \sim \underbrace{Uz}_{\text{uniform flow}} + \underbrace{\frac{Q}{2\pi}\ln z - \frac{Q}{2\pi}\ln z}_{\text{canceling source sink pair}} + \underbrace{\frac{i\Gamma}{2\pi}\ln z}_{\text{clockwise! vortex}} + \underbrace{\frac{\mu}{z}}_{\text{doublet}} + \ldots \tag{9.290}$$

The sign convention for Γ has been violated here, by tradition. Now let us take D to be the so-called drag force per unit depth and L to be the so-called lift force per unit depth, so in terms of F_x and F_y, we have

$$\frac{F_x}{b} = D, \qquad \frac{F_y}{b} = L. \tag{9.291}$$

Now by the Blasius force theorem, we have

$$D - iL = \frac{1}{2}\rho i \oint_C \left(\frac{dW}{dz}\right)^2 dz, \tag{9.292}$$

$$= \frac{1}{2}\rho i \oint_C \left(U + \frac{i\Gamma}{2\pi z} - \frac{\mu}{z^2} + \dots\right)^2 dz, \tag{9.293}$$

$$= \frac{1}{2}\rho i \oint_C \left(U^2 + \frac{i\Gamma U}{\pi z} - \frac{1}{z^2}\left(\frac{\Gamma^2}{4\pi^2} + 2U\mu\right) + \dots\right) dz. \tag{9.294}$$

Now the Cauchy integral theorem, Eq. (9.252), gives us the contour integral to be $2\pi i \sum$ residues. Here the residue is $i\Gamma U/\pi$. So we get

$$D - iL = \frac{1}{2}\rho i \left(2\pi i \left(\frac{i\Gamma U}{\pi}\right)\right), \tag{9.295}$$

$$= -i\rho\Gamma U. \tag{9.296}$$

We see that

$$D = 0, \tag{9.297}$$

$$L = \rho U \Gamma. \tag{9.298}$$

This is a remarkably simple and elegant result! Note that

- Γ is associated with *clockwise* circulation here. This is something of a tradition in aerodynamics.
- Because for airfoils $\Gamma \sim U$, we get the lift force $L \sim \rho U^2$,
- For steady inviscid flow, there is no drag. Consideration of either unsteady or viscous effects would lead to a non-zero x component of force.

Example 9.5

Consider the flow over a cylinder of radius a with clockwise circulation Γ.

To do so, we can superpose a point vortex onto the potential for flow over a cylinder in the following fashion:

$$W(z) = U\left(z + \frac{a^2}{z}\right) + \frac{i\Gamma}{2\pi}\ln\left(\frac{z}{a}\right). \tag{9.299}$$

Breaking this up as before into real and complex parts, we get

$$W(z) = \left(Ur\cos\theta \left(1 + \frac{a^2}{r^2}\right)\right) + i\left(Ur\sin\theta\left(1 - \frac{a^2}{r^2}\right)\right) + \frac{i\Gamma}{2\pi}\left(\ln\left(\frac{r}{a}\right) + i\theta\right). \quad (9.300)$$

So, we find

$$\psi = \Im(W(z)) = Ur\sin\theta\left(1 - \frac{a^2}{r^2}\right) + \frac{\Gamma}{2\pi}\ln\left(\frac{r}{a}\right). \quad (9.301)$$

On $r = a$, we find that $\psi = 0$, so the addition of the circulation in the way we have proposed maintains the cylinder surface to be a streamline. It is important to note that this is valid for *arbitrary* Γ. That is the potential flow solution for flow over a cylinder is *non-unique*. In aerodynamics, this is used to advantage to add just enough circulation to enforce the so-called *Kutta condition*.[6] The Kutta condition is an experimentally observed fact that for a steady flow, the trailing edge of an airfoil is a stagnation point.

The Kutta-Zhukovsky[7] lift theorem tells us whenever we add circulation, that a lift force $L = \rho U\Gamma$ is induced. This is consistent with the phenomena observed in baseball that the "fastball" rises. The fastball leaves the pitcher's hand traveling towards the batter and rotating towards the pitcher. The induced aerodynamic force is opposite to the force of gravity.

Let us get the lift force the hard way and verify the Kutta-Zhukovsky theorem. We can easily get the velocity field from the velocity potential:

$$\phi = \Re(W(z)) = Ur\cos\theta\left(1 + \frac{a^2}{r^2}\right) - \frac{\Gamma\theta}{2\pi}. \quad (9.302)$$

Thus, we differentiate ϕ appropriately to find v_r and v_θ:

$$v_r = \frac{\partial\phi}{\partial r} = Ur\cos\theta\left(-\frac{2a^2}{r^3}\right) + U\cos\theta\left(1 + \frac{a^2}{r^2}\right), \quad (9.303)$$

$$v_r|_{r=a} = U\cos\theta\left(-\frac{2a^3}{a^3} + 1 + \frac{a^2}{a^2}\right), \quad (9.304)$$

$$= 0, \quad (9.305)$$

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{1}{r}\left(-Ur\sin\theta\left(1 + \frac{a^2}{r^2}\right) - \frac{\Gamma}{2\pi}\right), \quad (9.306)$$

$$v_\theta|_{r=a} = -U\sin\theta\left(1 + \frac{a^2}{a^2}\right) - \frac{\Gamma}{2\pi a}, \quad (9.307)$$

$$= -2U\sin\theta - \frac{\Gamma}{2\pi a}. \quad (9.308)$$

We get the pressure on the cylinder surface from Bernoulli's equation:

$$p = p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\nabla^T\phi\cdot\nabla\phi, \quad (9.309)$$

$$= p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left(-2U\sin\theta - \frac{\Gamma}{2\pi a}\right)^2. \quad (9.310)$$

Now for a small element of the cylinder at $r = a$, the surface area is $dA = br\ d\theta = ba\ d\theta$. This is sketched in Fig. 9.20. We also note that the x and y forces depend on the orientation of the element,

[6]Martin Wilhelm Kutta, 1867-1944, Silesian-born German mechanician, studied at Breslau, taught mainly at Stuttgart, co-developer of Runge-Kutta method for integrating ordinary differential equations.

[7]Nikolai Egorovich Zhukovsky, 1847-1921, Russian applied mathematician and mechanician, father of Russian aviation, purchased glider from Lilienthal, developed lift theorem independently of Kutta, organized Central Aerohydrodynamic Institute in 1918.

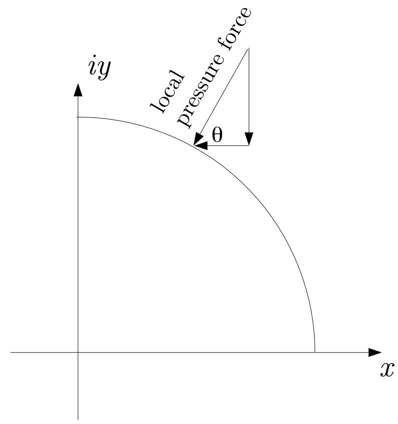

Figure 9.20: Pressure force on a differential area element of cylindrical surface.

given by θ. Elementary trigonometry shows that the elemental x and y forces per depth are

$$\frac{dF_x}{b} = -p(\cos\theta)a\ d\theta, \tag{9.311}$$
$$\frac{dF_y}{b} = -p(\sin\theta)a\ d\theta. \tag{9.312}$$

So integrating over the entire cylinder, we obtain,

$$\frac{F_x}{b} = \int_0^{2\pi} -\underbrace{\left(p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left(-2U\sin\theta - \frac{\Gamma}{2\pi a}\right)^2\right)}_{p}(\cos\theta)a\ d\theta, \tag{9.313}$$
$$\frac{F_y}{b} = \int_0^{2\pi} -\underbrace{\left(p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left(-2U\sin\theta - \frac{\Gamma}{2\pi a}\right)^2\right)}_{p}(\sin\theta)a\ d\theta. \tag{9.314}$$

Integration via computer algebra gives

$$\frac{F_x}{b} = 0, \tag{9.315}$$
$$\frac{F_y}{b} = \rho U\Gamma. \tag{9.316}$$

This is identical to the result we expect from the Kutta-Zhukovsky lift theorem, Eq. (9.298).

Chapter 10

Viscous incompressible laminar flow

see Panton, Chapters 7 and 11,
see Yih, Chapter 7,
see Segel, Chapter 3,
see White (2006), Chapters 3 and 4.

Here we consider a few standard problems in viscous incompressible laminar flow. For this entire chapter, we will make the following assumptions:

- the flow is incompressible,
- body forces are negligible, and
- the fluid properties, c, μ and k, are constants.

10.1 Fully developed, one-dimensional solutions

The first type of solution we will consider is known as a one-dimensional *fully developed* solution. These are commonly considered in first courses in fluid mechanics and heat transfer. The flows here are essentially one-dimensional, but not absolutely, as they were in the chapter on one-dimensional compressible flow. In this section, we will further enforce that

- the flow is time-independent, $\partial_o = 0$,
- the velocity and temperature gradients in the x and z directions are zero, $\partial \mathbf{v}/\partial x = \mathbf{0}$, $\partial \mathbf{v}/\partial z = \mathbf{0}$, $\partial T/\partial x = 0$, $\partial T/\partial z = 0$.

We will see that these assumptions give rise to flows with a non-zero x velocity u that varies in the y direction, and that other velocities v, and w, will be zero.

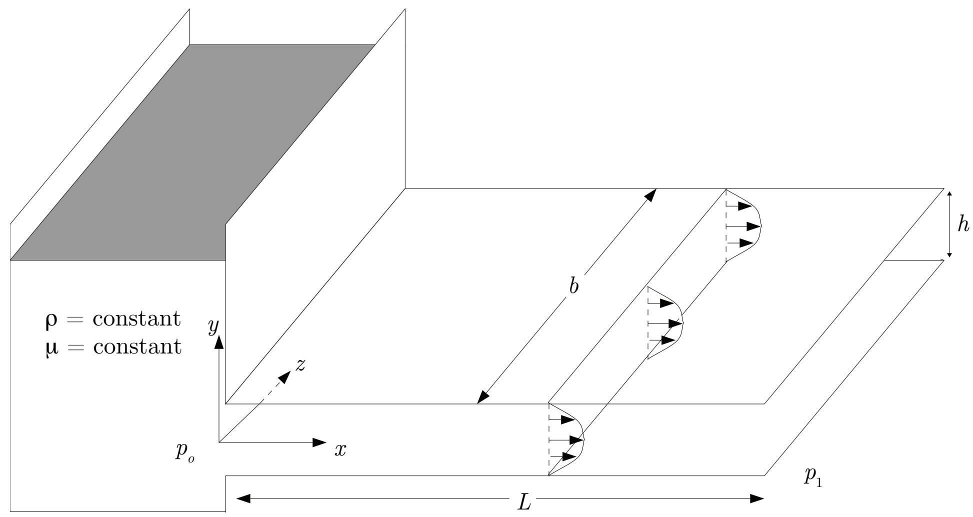

Figure 10.1: Pressure gradient-driven flow in a slot.

10.1.1 Pressure gradient-driven flow in a slot

Consider the flow sketched in Fig. 10.1. Here we have a large reservoir of fluid with a long narrow slot located around $y = 0$. We take the length of the slot in the z direction, b, to be long relative to the slot width in the y direction h. Attached to the slot are two parallel plates, separated by distance in the y direction h. The length of the plates in the x direction is L. We take $L >> h$. Because of gravity forces, that we neglect in the slot, the pressure at the entrance of the slot p_o is higher than atmospheric. At the end of the slot, the fluid expels to the atmosphere that is at p_1. Hence, there is a pressure gradient in the x direction, that drives the flow in the slot. We will see that the flow is resisted by viscous stresses. An analogous flow in a circular duct is defined as a *Hagen*[1]*-Poiseuille*[2] *flow*.

Near $x = 0$, the flow accelerates in what is known as the *entrance length*. If L is sufficiently long, we observe that sufficiently downstream of $x = 0$, the fluid particles no longer accelerate. It is at this point where the viscous shear forces exactly balance the pressure forces to give rise to the fully developed velocity field.

For this flow, let us make the additional assumptions that

- there is no imposed pressure gradient in the z direction, and
- the walls are held at a constant temperature, T_o.

Incorporating some of these assumptions, we recast the incompressible constant property Navier-Stokes equations of Ch. 6.3.4 as

$$\partial_i v_i = 0, \tag{10.1}$$

[1]Gotthilf Ludwig Hagen, 1797-1884, German engineer who measured velocity of water in small diameter tubes.

[2]Jean Louis Poiseuille, 1799-1869, French physician who repeated experiments of Hagen for simulated blood flow.

$$\rho \partial_o v_i + \rho v_j \partial_j v_i = -\partial_i p + \mu \partial_j \partial_j v_i, \tag{10.2}$$
$$\rho c \partial_o T + \rho c v_j \partial_j T = k \partial_i \partial_i T + 2\mu \partial_{(i} v_{j)} \partial_{(i} v_{j)}. \tag{10.3}$$

Here we have five equations in five unknowns, v_i, p, and T.

As for all incompressible flows with constant properties, we can get the velocity field by only considering the mass and momenta equations; velocity is only coupled one way to the energy equation. The mass equation, recalling that gradients in x and z are zero, gives us

$$\underbrace{\frac{\partial}{\partial x}}_{=0} u + \frac{\partial}{\partial y} v + \underbrace{\frac{\partial}{\partial z}}_{=0} w = 0. \tag{10.4}$$

So the mass equation gives us

$$\frac{\partial v}{\partial y} = 0. \tag{10.5}$$

Now, from our assumptions of steady and fully developed flow, we know that v cannot be a function of x, z, or t. So the partial becomes a total derivative, and mass conservation holds that $dv/dy = 0$. Integrating, we find that $v(y) = C$. The constant C must be zero, because we must satisfy a no-penetration boundary condition at either wall that $v(y = h/2) = v(y = -h/2) = 0$. Hence, mass conservation, coupled with the no-penetration boundary condition gives us

$$v = 0. \tag{10.6}$$

Now consider the x momentum equation:

$$\rho \underbrace{\frac{\partial}{\partial t}}_{=0} u + \rho u \underbrace{\frac{\partial}{\partial x}}_{=0} u + \rho \underbrace{v}_{=0} \frac{\partial}{\partial y} u + \rho w \underbrace{\frac{\partial}{\partial z}}_{=0} u = -\frac{\partial p}{\partial x} + \mu \left(\underbrace{\frac{\partial^2}{\partial x^2}}_{=0} u + \frac{\partial^2}{\partial y^2} u + \underbrace{\frac{\partial^2}{\partial z^2}}_{=0} u \right), \tag{10.7}$$
$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}. \tag{10.8}$$

For this fully developed flow the acceleration, that is the material derivative of velocity, is formally zero, and the equation gives rise to a balance of pressure and viscous surface forces.

For the y momentum equation, we get

$$\rho \underbrace{\frac{\partial}{\partial t}}_{=0} \underbrace{v}_{=0} + \rho u \underbrace{\frac{\partial}{\partial x}}_{=0} \underbrace{v}_{=0} + \rho \underbrace{v}_{=0} \frac{\partial}{\partial y} \underbrace{v}_{=0} + \rho w \underbrace{\frac{\partial}{\partial z}}_{=0} \underbrace{v}_{=0} = -\frac{\partial p}{\partial y} \tag{10.9}$$
$$+\mu \left(\underbrace{\frac{\partial^2}{\partial x^2}}_{=0} v + \frac{\partial^2}{\partial y^2} \underbrace{v}_{=0} + \underbrace{\frac{\partial^2}{\partial z^2}}_{=0} v \right),$$
$$0 = \frac{\partial p}{\partial y}. \tag{10.10}$$

Hence, $p = p(x, z)$, but because we have assumed there is no pressure gradient in the z direction, we have at most that

$$p = p(x). \tag{10.11}$$

For the z momentum equation we get:

$$\rho \underbrace{\frac{\partial}{\partial t}}_{=0} w + \rho u \underbrace{\frac{\partial}{\partial x}}_{=0} w + \rho \underbrace{v}_{=0} \frac{\partial}{\partial y} w + \rho w \underbrace{\frac{\partial}{\partial z}}_{=0} w = -\underbrace{\frac{\partial p}{\partial z}}_{=0} + \mu \left(\underbrace{\frac{\partial^2}{\partial x^2}}_{=0} w + \frac{\partial^2}{\partial y^2} w + \underbrace{\frac{\partial^2}{\partial z^2}}_{=0} w \right), \tag{10.12}$$

$$0 = \frac{\partial^2 w}{\partial y^2}. \tag{10.13}$$

Solution of this partial differential equation gives us

$$w = f(x, z) y + g(x, z). \tag{10.14}$$

Now to satisfy the no-slip condition, we must have $w = 0$ at $y = \pm h/2$. This leads us to two linear equations for f and g:

$$\begin{pmatrix} \frac{h}{2} & 1 \\ -\frac{h}{2} & 1 \end{pmatrix} \begin{pmatrix} f(x, z) \\ g(x, z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10.15}$$

Because the determinant of the coefficient matrix, $h/2 + h/2 = h$, is non-zero, the only solution is the trivial solution $f(x, z) = g(x, z) = 0$. Hence,

$$w = 0. \tag{10.16}$$

Next consider how the energy equation reduces:

$$\rho c \underbrace{\frac{\partial}{\partial t}}_{=0} T + \rho c \left(u \underbrace{\frac{\partial}{\partial x}}_{=0} T + \underbrace{v}_{=0} \frac{\partial}{\partial y} T + \underbrace{w}_{=0} \underbrace{\frac{\partial}{\partial z}}_{=0} T \right) = k \left(\underbrace{\frac{\partial^2}{\partial x^2}}_{=0} T + \frac{\partial^2}{\partial y^2} T + \underbrace{\frac{\partial^2}{\partial z^2}}_{=0} T \right) + 2\mu \partial_{(i} v_{j)} \partial_{(i} v_{j)}, \tag{10.17}$$

$$0 = k \frac{\partial^2 T}{\partial y^2} + 2\mu \partial_{(i} v_{j)} \partial_{(i} v_{j)}. \tag{10.18}$$

There is no tendency for a particle's temperature to increase. There is a balance between thermal energy generated by viscous dissipation and that conducted away by energy diffusion. Thus the energy path is 1) viscous work is done to generate thermal energy, 2) thermal energy diffuses throughout the channel and out the boundary. Now consider the viscous dissipation term for this flow.

$$\partial_i v_j = \begin{pmatrix} \underbrace{\partial_1}_{=0} v_1 & \underbrace{\partial_1}_{=0}\underbrace{v_2}_{=0} & \underbrace{\partial_1}_{=0}\underbrace{v_3}_{=0} \\ \partial_2 v_1 & \partial_2 \underbrace{v_2}_{=0} & \partial_2 \underbrace{v_3}_{=0} \\ \underbrace{\partial_3}_{=0} v_1 & \underbrace{\partial_3}_{=0}\underbrace{v_2}_{=0} & \underbrace{\partial_3}_{=0}\underbrace{v_3}_{=0} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \partial_2 v_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (10.19)$$

$$\partial_{(i} v_{j)} = \begin{pmatrix} 0 & \frac{1}{2}\left(\partial_2 v_1 + \underbrace{\partial_1 v_2}_{=0}\right) & 0 \\ \frac{1}{2}\left(\partial_2 v_1 + \underbrace{\partial_1 v_2}_{=0}\right) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\frac{\partial u}{\partial y} & 0 \\ \frac{1}{2}\frac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (10.20)$$

Further,

$$\partial_{(i} v_{j)} \partial_{(i} v_{j)} = \left(\frac{1}{2}\frac{\partial u}{\partial y}\right)^2 + \left(\frac{1}{2}\frac{\partial u}{\partial y}\right)^2 = \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2. \quad (10.21)$$

So the energy equation becomes finally

$$0 = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2. \quad (10.22)$$

At this point we have the x momentum and energy equations as the only two that seem to have any substance.

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}, \quad (10.23)$$

$$0 = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2. \quad (10.24)$$

This looks like two equations in three unknowns. One peculiarity of incompressible equations is that there is always some side condition, that ultimately hinges on the mass equation, that really gives a third equation. Without going into details, it involves for general flows solving a Poisson[3] equation for pressure that is of the form $\nabla^2 p = f(u, v)$. Sec. 10.2 will give a few of these details. The Poisson equation involves second derivatives of pressure. Here we can obtain a simple form of this general equation by taking the partial derivative with respect to x of the x momentum equation:

$$0 = -\frac{\partial^2 p}{\partial x^2} + \mu\frac{\partial}{\partial x}\frac{\partial^2 u}{\partial y^2}, \quad (10.25)$$

[3]Siméon Denis Poisson, 1781-1840, French mathematician taught by Laplace, Lagrange, and Legendre, studied partial differential equations, potential theory, elasticity, and electrodynamics.

$$0 = -\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2}{\partial y^2} \underbrace{\frac{\partial u}{\partial x}}_{=0}. \tag{10.26}$$

The viscous term here is zero because of our assumption of fully developed flow. Moreover, because $p = p(x)$ only, we then get

$$\frac{d^2 p}{dx^2} = 0, \qquad p(0) = p_o, \; p(L) = p_1, \tag{10.27}$$

that has a solution showing the pressure field must be linear in x:

$$p(x) = p_o - \frac{p_o - p_1}{L} x, \tag{10.28}$$

$$\frac{dp}{dx} = -\frac{p_o - p_1}{L}. \tag{10.29}$$

Now, because u is at most a function of y, we can convert partial derivatives to ordinary derivatives, and write the x momentum equation and energy equation as two ordinary differential equations in two unknowns with appropriate boundary conditions at the wall $y = \pm h/2$:

$$\frac{d^2 u}{dy^2} = -\frac{p_o - p_1}{\mu L}, \qquad u\left(\frac{h}{2}\right) = 0, \quad u\left(-\frac{h}{2}\right) = 0, \tag{10.30}$$

$$\frac{d^2 T}{dy^2} = -\frac{\mu}{k}\left(\frac{du}{dy}\right)^2, \qquad T\left(\frac{h}{2}\right) = T_o, \quad T\left(-\frac{h}{2}\right) = T_o. \tag{10.31}$$

We could solve these equations directly, but instead let us first cast them in dimensionless form. This will give our results some universality and efficiency. Moreover, it will reveal more fundamental groups of terms that govern the fluid behavior. Let us select scales such that dimensionless variables, denoted by a * subscript, are as follows

$$y_* = \frac{y}{h}, \qquad T_* = \frac{T - T_o}{T_o}, \qquad u_* = \frac{u}{u_c}. \tag{10.32}$$

We have yet to determine the characteristic velocity u_c. The dimensionless temperature has been chosen to render it zero at the boundaries. With these choices, the x momentum equation becomes

$$\frac{u_c}{h^2}\frac{d^2 u_*}{dy_*^2} = -\frac{p_o - p_1}{\mu L}, \tag{10.33}$$

$$\frac{d^2 u_*}{dy_*^2} = -\frac{(p_o - p_1) h^2}{\mu L u_c}, \tag{10.34}$$

$$u_c u_*(x_* h = h/2) = u_c u_*(x_* h = -h/2) = 0, \tag{10.35}$$

$$u_*(x_* = 1/2) = u_*(x_* = -1/2) = 0. \tag{10.36}$$

Let us now choose the characteristic velocity to render the x momentum equation to have a simple form:

$$u_c \equiv \frac{(p_o - p_1)h^2}{\mu L}. \tag{10.37}$$

Now scale the energy equation:

$$\frac{T_o}{h^2}\frac{d^2T_*}{dy_*^2} = -\frac{\mu u_c^2}{kh^2}\left(\frac{du_*}{dy_*}\right)^2, \tag{10.38}$$

$$\frac{d^2T_*}{dy_*^2} = -\frac{\mu u_c^2}{kT_o}\left(\frac{du_*}{dy_*}\right)^2, \tag{10.39}$$

$$= -\frac{\mu c}{k}\frac{u_c^2}{cT_o}\left(\frac{du_*}{dy_*}\right)^2, \tag{10.40}$$

$$= -Pr Ec\left(\frac{du_*}{dy_*}\right)^2, \tag{10.41}$$

$$T_*\left(-\frac{1}{2}\right) = T_*\left(\frac{1}{2}\right) = 0. \tag{10.42}$$

Here we have grouped terms so that the Prandtl number, Eq. (6.123), $Pr = \mu c/k$, explicitly appears. Further, we have defined the Eckert[4] number Ec as

$$Ec = \frac{u_c^2}{cT_o} = \frac{\left(\frac{(p_o-p_1)h^2}{\mu L}\right)^2}{cT_o}. \tag{10.43}$$

In summary our dimensionless differential equations and boundary conditions are

$$\frac{d^2u_*}{dy_*^2} = -1, \qquad u\left(\pm\frac{1}{2}\right) = 0, \tag{10.44}$$

$$\frac{d^2T_*}{dy_*^2} = -Pr Ec\left(\frac{du_*}{dy_*}\right)^2, \qquad T_*\left(\pm\frac{1}{2}\right) = 0. \tag{10.45}$$

These boundary conditions are homogeneous; hence, they do not contribute to a non-trivial solution. The pressure gradient is an inhomogeneous forcing term in the momentum equation, and the viscous dissipation is a forcing term in the energy equation.

The solution for the velocity field that satisfies the differential equation and boundary conditions is quadratic in y_* and is

$$u_* = \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 - y_*^2\right). \tag{10.46}$$

[4]Ernst R. G. Eckert, 1904-2004, scholar of convective heat transfer.

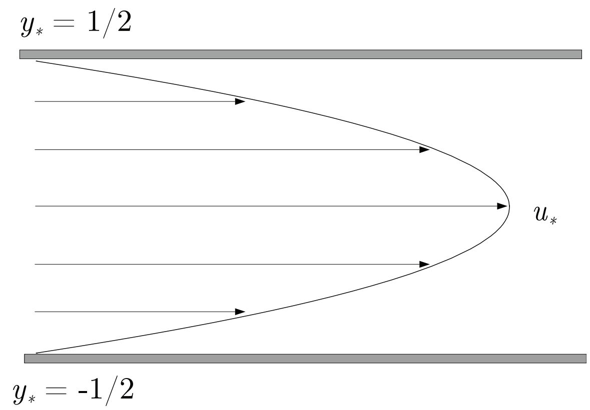

Figure 10.2: Velocity profile for pressure gradient-driven flow in a slot.

The maximum velocity occurs at $y_* = 0$ and has value

$$u_{*max} = \frac{1}{8}. \tag{10.47}$$

The mean velocity is found through integrating the velocity field to arrive at

$$\begin{aligned} u_{*mean} &= \int_{-1/2}^{1/2} u_*(y_*)\ dy_*, & (10.48)\\ &= \int_{-1/2}^{1/2} \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 - y_*^2\right) dy_*, & (10.49)\\ &= \frac{1}{2}\left(\frac{1}{4}y_* - \frac{1}{3}y_*^3\right)\Bigg|_{-1/2}^{1/2}, & (10.50)\\ &= \frac{1}{12}. & (10.51) \end{aligned}$$

We could have scaled the velocity field in such a fashion that either the maximum or the mean velocity was unity. The scaling we chose gave rise to a non-unity value of both. In dimensional terms we could say

$$\frac{u}{\frac{(p_o - p_1)h^2}{\mu L}} = \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 - \left(\frac{y}{h}\right)^2\right). \tag{10.52}$$

The velocity profile is sketched in Fig. 10.2. This flow is rotational. For the two-dimensional flow, the only component of vorticity is in the z_* direction, and we have

$$\begin{aligned} \omega_{z_*} &= \underbrace{\frac{\partial v_*}{\partial x_*}}_{=0} - \frac{\partial u_*}{\partial y_*}, & (10.53)\\ &= y_*. & (10.54) \end{aligned}$$

The vorticity magnitude is maximum at the solid walls at $y_* = \pm 1/2$, and it is zero at the centerline, $y_* = 0$. The deformation tensor is

$$\mathsf{D} = \begin{pmatrix} \frac{\partial u_*}{\partial x_*} & \frac{1}{2}\left(\frac{\partial u_*}{\partial y_*} + \frac{\partial v_*}{\partial x_*}\right) \\ \frac{1}{2}\left(\frac{\partial u_*}{\partial y_*} + \frac{\partial v_*}{\partial x_*}\right) & \frac{\partial v_*}{\partial y_*} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{y_*}{2} \\ -\frac{y_*}{2} & 0 \end{pmatrix}. \tag{10.55}$$

It is easy to show the eigenvalues of D are given by $\lambda = \pm y_*/2$ and the eigenvectors are at angles of $\pi/4$ and $3\pi/4$ to the horizontal. So on these axes exists the rate of extreme extensional straining.

Now let us get the temperature field.

$$\begin{aligned}
\frac{d^2 T_*}{dy_*^2} &= -Pr Ec \left(\frac{d}{dy_*}\left(\frac{1}{2}\left(\left(\frac{1}{2}\right)^2 - y_*^2\right)\right)\right)^2, && (10.56)\\
&= -Pr Ec \, (-y_*)^2, && (10.57)\\
&= -Pr Ec \, y_*^2, && (10.58)\\
\frac{dT_*}{dy_*} &= -\frac{1}{3} Pr Ec \, y_*^3 + C_1, && (10.59)\\
T_* &= -\frac{1}{12} Pr Ec \, y_*^4 + C_1 y_* + C_2, && (10.60)\\
0 &= -\frac{1}{12} Pr Ec \, \frac{1}{16} + C_1 \frac{1}{2} + C_2, \qquad y_* = \frac{1}{2}, && (10.61)\\
0 &= -\frac{1}{12} Pr Ec \, \frac{1}{16} - C_1 \frac{1}{2} + C_2, \qquad y_* = -\frac{1}{2}, && (10.62)\\
C_1 &= 0, \qquad C_2 = \frac{Pr Ec}{192}. && (10.63)
\end{aligned}$$

Regrouping, we find that

$$T_* = \frac{Pr Ec}{12}\left(\left(\frac{1}{2}\right)^4 - y_*^4\right). \tag{10.64}$$

In terms of dimensional quantities, we can say

$$\frac{T - T_o}{T_o} = \frac{(p_o - p_1)^2 h^4}{12 \mu L^2 k T_o}\left(\left(\frac{1}{2}\right)^4 - \left(\frac{y}{h}\right)^4\right). \tag{10.65}$$

The temperature profile is sketched in Fig. 10.3.

From knowledge of the velocity and temperature field, we can calculate other quantities of interest. Let us calculate the field of shear stress and heat flux, and then evaluate both at the wall. First for the shear stress, recall that in dimensional form we have

$$\begin{aligned}
\tau_{ij} &= 2\mu \partial_{(i} v_{j)} + \lambda \underbrace{\partial_k v_k}_{=0} \delta_{ij}, && (10.66)\\
&= 2\mu \partial_{(i} v_{j)}. && (10.67)
\end{aligned}$$

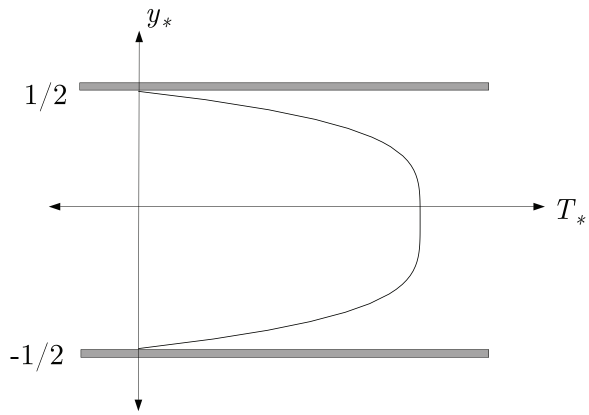

Figure 10.3: Temperature profile for pressure gradient-driven flow in a slot.

We have already seen the only non-zero components of the symmetric part of the velocity gradient tensor are the 12 and 21 components. Thus the 21 stress component is

$$\tau_{21} = 2\mu\partial_{(2}v_{1)} = 2\mu\left(\frac{\partial_2 v_1 + \underbrace{\partial_1 v_2}_{=0}}{2}\right), \quad (10.68)$$

$$= \mu\partial_2 v_1. \quad (10.69)$$

In (x, y) space, we then say here that

$$\tau_{yx} = \mu\frac{du}{dy}. \quad (10.70)$$

This is a stress on the y (tangential) face that points in the x direction; hence, it is certainly a shearing stress. In dimensionless terms, we can define a characteristic shear stress τ_c, so that the scale shear is $\tau_* = \tau_{yx}/\tau_c$. Thus, our equation for shear becomes

$$\tau_c\tau_* = \frac{\mu u_c}{h}\frac{du_*}{dy_*}. \quad (10.71)$$

Now take

$$\tau_c \equiv \frac{\mu u_c}{h} = \frac{\mu(p_o - p_1)h^2}{h\mu L} = (p_o - p_1)\left(\frac{h}{L}\right). \quad (10.72)$$

With this definition, we get

$$\tau_* = \frac{du_*}{dy_*}. \quad (10.73)$$

Evaluating for the velocity profile of the pressure gradient-driven flow, we find

$$\tau_* = -y_*. \quad (10.74)$$

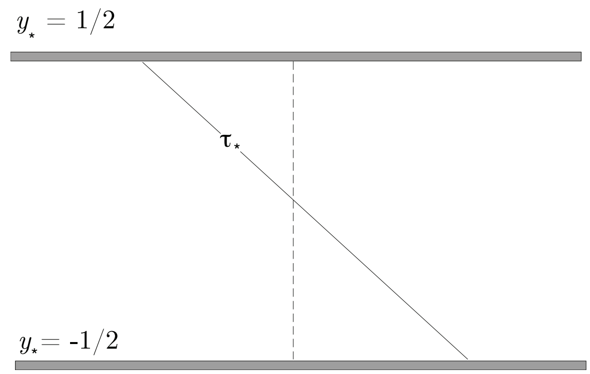

Figure 10.4: Shear stress profile for pressure gradient-driven flow in a slot.

The stress is zero at the centerline $y_* = 0$ and has maximum magnitude of 1/2 at either wall, $y_* = \pm 1/2$. In dimensional terms, the wall shear stress τ_w is

$$\tau_w = -\frac{1}{2}(p_o - p_1)\left(\frac{h}{L}\right). \tag{10.75}$$

The wall shear stress is governed by the pressure difference and not the viscosity. However, the viscosity plays a determining role in selecting the maximum fluid velocity. The shear profile is sketched in Fig. 10.4.

Next, let us calculate the heat flux vector. Recall that, for this flow, with no x or z variation of T, we have the only non-zero component of the heat flux vector as

$$q_y = -k\frac{dT}{dy}. \tag{10.76}$$

Now define scale the heat flux by a characteristic heat flux q_c, to be determined, to obtain a dimensionless heat flux:

$$q_* = \frac{q_y}{q_c}. \tag{10.77}$$

So,

$$q_c q_* = -\frac{kT_o}{h}\frac{dT_*}{dy_*}, \tag{10.78}$$

$$q_* = -\frac{kT_o}{hq_c}\frac{dT_*}{dt_*}. \tag{10.79}$$

Let $q_c \equiv kT_o/h$, so

$$q_* = -\frac{dT_*}{dy_*}, \tag{10.80}$$

$$q_* = \frac{1}{3}PrEc\, y_*^3. \tag{10.81}$$

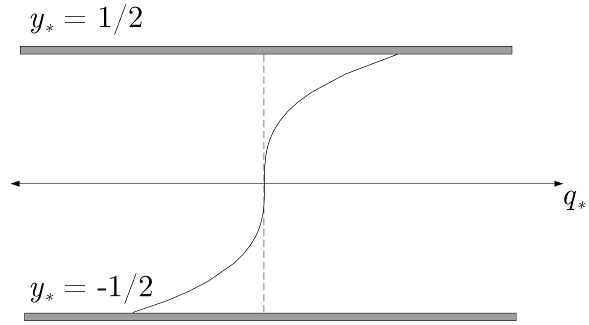

Figure 10.5: Heat flux profile for pressure gradient-driven flow in a slot.

For our flow, we have a cubic variation of the heat flux vector magnitude. There is no heat flux at the centerline, that corresponds to this being a region of no shear. The magnitude of the heat flux is maximum at the wall, the region of maximum shear. At the upper wall, we have

$$q_*|_{y_*=1/2} = \frac{1}{24} Pr Ec. \tag{10.82}$$

The heat flux profile is sketched in Fig. 10.5. In dimensional terms we have

$$\frac{q_w}{\frac{kT_o}{h}} = \frac{1}{24}\frac{(p_o - p_1)^2 h^4}{\mu L^2 k T_o}, \tag{10.83}$$

$$q_w = \frac{1}{24}\frac{(p_o - p_1)^2 h^3}{\mu L^2}. \tag{10.84}$$

10.1.2 Couette flow with pressure gradient

We next consider Couette flow with a pressure gradient. Couette flow implies that there is a moving plate at one boundary and a fixed plate at the other. It is a common experimental configuration, and used often to actually determine a fluid's viscosity. Here we will take the same assumptions as for pressure gradient-driven flow in a slot, expect for the boundary condition at the upper surface, that we will require to have a constant velocity U. We will also shift the coordinates so that $y = 0$ matches the lower plate surface and $y = h$ matches the upper plate surface. The configuration for this flow is shown in Fig. 10.6.

Our equations governing this flow are

$$\frac{d^2u}{dy^2} = -\frac{p_o - p_1}{\mu L}, \qquad u(0) = 0, \quad u(h) = U, \tag{10.85}$$

$$\frac{d^2T}{dy^2} = -\frac{\mu}{k}\left(\frac{du}{dy}\right)^2, \qquad T(0) = T_o, \quad T(h) = T_o. \tag{10.86}$$

Once again in momentum, there is no acceleration, and viscous stresses balance shear stresses. In energy, there is no energy increase, and generation of thermal energy due to viscous work

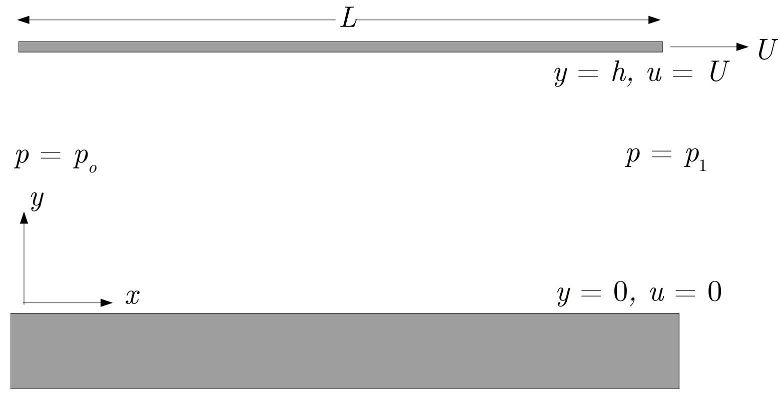

Figure 10.6: Configuration for Couette flow with pressure gradient.

is balanced by diffusion of the thermal energy, ultimately out of the system through the boundaries. Here there are inhomogeneities in both the forcing terms and the boundary conditions. In terms of work, both the pressure gradient and the pulling of the plate induce work.

Once again let us scale the equations. This time, we have a natural velocity scale, U, the upper plate velocity. So take

$$y_* = \frac{y}{h}, \qquad T_* = \frac{T - T_o}{T_o}, \qquad u_* = \frac{u}{U}. \tag{10.87}$$

The momentum equation becomes

$$\frac{U}{h^2}\frac{d^2u_*}{dy_*^2} = -\frac{p_o - p_1}{\mu L}, \tag{10.88}$$

$$\frac{d^2u_*}{dy_*^2} = -\frac{(p_o - p_1)h^2}{\mu U L}. \tag{10.89}$$

With dimensionless pressure gradient

$$\mathcal{P} \equiv \frac{(p_o - p_1)h^2}{\mu U L}, \tag{10.90}$$

we get

$$\frac{d^2u_*}{dy_*^2} = -\mathcal{P}, \tag{10.91}$$

$$u_*(0) = 0, \quad u_*(1) = 1. \tag{10.92}$$

This has solution

$$u_* = -\frac{1}{2}\mathcal{P}y_*^2 + C_1 y_* + C_2. \tag{10.93}$$

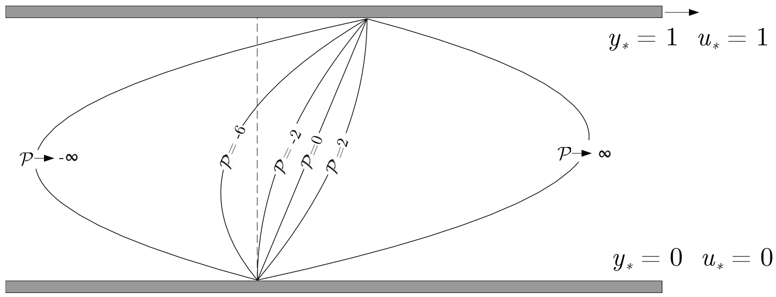

Figure 10.7: Velocity profiles for various values of $\mathcal{P}$ for Couette flow with pressure gradient.

Applying the boundary conditions, we get

$$0 = -\frac{1}{2}\mathcal{P}(0)^2 + C_1(0) + C_2, \tag{10.94}$$

$$= C_2, \tag{10.95}$$

$$1 = -\frac{1}{2}\mathcal{P}(1)^2 + C_1(1), \tag{10.96}$$

$$C_1 = 1 + \frac{1}{2}\mathcal{P}, \tag{10.97}$$

$$u_* = -\frac{1}{2}\mathcal{P}y_*^2 + \left(1 + \frac{1}{2}\mathcal{P}\right) y_*, \tag{10.98}$$

$$= \underbrace{\frac{1}{2}\mathcal{P}y_*(1 - y_*)}_{\text{pressure effect}} + \underbrace{y_*}_{\text{Couette effect}}. \tag{10.99}$$

We see that the pressure gradient generates a velocity profile that is quadratic in y_*. This is distinguished from the Couette effect, that is the effect of the upper plate's motion, that gives a linear profile. Because our governing equation here is linear, it is appropriate to think of these as superposed solutions. Velocity profiles for various values of $\mathcal{P}$ are shown in Fig. 10.7.

Let us now calculate the shear stress profile. With $\tau = \mu(du/dy)$, and taking $\tau_* = \tau/\tau_c$, we get

$$\tau_c\tau_* = \frac{\mu U}{h}\frac{du_*}{dy_*}, \tag{10.100}$$

$$\tau_* = \frac{\mu U}{h\tau_c}\frac{du_*}{dy_*}, \tag{10.101}$$

$$\text{taking} \quad \tau_c \equiv \frac{\mu U}{h}, \tag{10.102}$$

$$\tau_* = \frac{du_*}{dy_*}, \qquad \text{so here,} \tag{10.103}$$

$$\tau_* = -\mathcal{P}y_* + \frac{1}{2}\mathcal{P} + 1, \qquad \text{and} \tag{10.104}$$

$$\tau_*|_{y_*=0} = \frac{1}{2}\mathcal{P} + 1, \tag{10.105}$$
$$\tau_*|_{y_*=1} = -\frac{1}{2}\mathcal{P} + 1. \tag{10.106}$$

The wall shear has a pressure gradient effect and a Couette effect as well. In fact we can select a pressure gradient to balance the Couette effect at one or the other wall, but not both.

We can also calculate the dimensionless volume flow rate Q_*, that for incompressible flow, is directly proportional to the mass flux. Ignoring how the scaling would be done, we arrive at

$$Q_* = \int_0^1 u_* \, dy_*, \tag{10.107}$$
$$= \int_0^1 \left(-\frac{1}{2}\mathcal{P}y_*^2 + \left(1 + \frac{1}{2}\mathcal{P}\right) y_* \right) dy_*, \tag{10.108}$$
$$= \left(-\frac{1}{6}\mathcal{P}y_*^3 \right)\Big|_0^1 + \left(1 + \frac{1}{2}\mathcal{P}\right) \frac{y_*^2}{2}\Big|_0^1, \tag{10.109}$$
$$= -\frac{\mathcal{P}}{6} + \left(1 + \frac{1}{2}\mathcal{P}\right)\frac{1}{2}, \tag{10.110}$$
$$= \frac{\mathcal{P}}{12} + \frac{1}{2}. \tag{10.111}$$

Again there is a pressure gradient contribution and a Couette contribution, and we could select $\mathcal{P}$ to give no net volume flow rate. We summarize some of the special cases as follows

- $\mathcal{P} \to -\infty$: $u_* = (1/2)\mathcal{P}y_*(1 - y_*)$; $\tau_* = \mathcal{P}\,(1/2 - y_*)$, $Q_* = \mathcal{P}/12$. Here the fluid flows in the opposite direction as driven by the plate because of the large pressure gradient.
- $\mathcal{P} = -6$. Here we get no net mass flow and $u_* = 3y_*^2 - 2y_*$, $\tau_* = 2y_*$, $Q_* = 0$.
- $\mathcal{P} = -2$. Here we get no shear at the bottom wall and $u_* = y_*^2$, $\tau_* = 2y_*$, $Q_* = 1/3$.
- $\mathcal{P} = 0$. Here we have no pressure gradient and $u_* = y_*$, $\tau_* = 1$, $Q_* = 1/2$.
- $\mathcal{P} = 2$. Here we get no shear at the top wall and $u_* = -y_*^2 + 2y_*$, $\tau * = -2y_* + 2$, $Q_* = 2/3$.
- $\mathcal{P} \to \infty$: $u_* = (1/2)\mathcal{P}y_*(1 - y_*)$; $\tau_* = \mathcal{P}\,(1/2 - y_*)$, $Q_* = \mathcal{P}/12$. Here the fluid flows in the same direction as driven by the plate.

We now consider the heat transfer problem. Scaling, we get

$$\frac{T_o}{h^2}\frac{d^2T_*}{dy_*^2} = -\frac{\mu U^2}{kh^2}\left(\frac{du_*}{dy_*}\right)^2, \qquad T_*(0) = T_*(1) = 0, \tag{10.112}$$

$$\frac{d^2T_*}{dy_*^2} = -\frac{\mu U^2}{kT_o}\left(\frac{du_*}{dy_*}\right)^2, \tag{10.113}$$

$$= -\frac{\mu c}{k}\frac{U^2}{cT_o}\left(\frac{du_*}{dy_*}\right)^2, \tag{10.114}$$

$$= -Pr Ec\left(\frac{du_*}{dy_*}\right)^2, \tag{10.115}$$

$$= -Pr Ec\ \tau_*^2, \tag{10.116}$$

$$= -Pr Ec\left(-\mathcal{P}y_* + \frac{1}{2}\mathcal{P} + 1\right)^2, \tag{10.117}$$

$$= -Pr Ec\left(\mathcal{P}^2 y_*^2 - 2\mathcal{P}\left(\frac{1}{2}\mathcal{P}+1\right)y_* + \left(1+\frac{1}{2}\mathcal{P}\right)^2\right), \tag{10.118}$$

$$\frac{dT_*}{dy_*} = -Pr Ec\left(\frac{\mathcal{P}^2}{3}y_*^3 - \mathcal{P}\left(\frac{1}{2}\mathcal{P}+1\right)y_*^2 + \left(1+\frac{1}{2}\mathcal{P}\right)^2 y_*\right) + C_1, \tag{10.119}$$

$$T_* = -Pr Ec\left(\frac{\mathcal{P}^2}{12}y_*^4 - \frac{\mathcal{P}}{3}\left(\frac{1}{2}\mathcal{P}+1\right)y_*^3 + \frac{1}{2}\left(1+\frac{1}{2}\mathcal{P}\right)^2 y_*^2\right) + C_1 y_* + C_2, \tag{10.120}$$

$$T_*(0) = 0 = C_2, \tag{10.121}$$

$$T_*(1) = 0 = -Pr Ec\left(\frac{\mathcal{P}^2}{12} - \frac{\mathcal{P}}{3}\left(\frac{1}{2}\mathcal{P}+1\right) + \frac{1}{2}\left(1+\frac{1}{2}\mathcal{P}\right)^2\right) + C_1, \tag{10.122}$$

$$C_1 = Pr Ec\left(\frac{1}{2} + \frac{\mathcal{P}}{6} + \frac{\mathcal{P}^2}{24}\right), \tag{10.123}$$

$$T_* = -Pr Ec\left(\frac{\mathcal{P}^2}{12}y_*^4 - \frac{\mathcal{P}}{3}\left(\frac{1}{2}\mathcal{P}+1\right)y_*^3 + \frac{1}{2}\left(1+\frac{1}{2}\mathcal{P}\right)^2 y_*^2\right) + Pr Ec\left(\frac{1}{2} + \frac{\mathcal{P}}{6} + \frac{\mathcal{P}^2}{24}\right)y_*. \tag{10.124}$$

Factoring, we can write the temperature profile as

$$T_* = \frac{Pr Ec}{24}y_*(1-y_*)(12 + 4\mathcal{P} + \mathcal{P}^2 - 8\mathcal{P}y_* - 2\mathcal{P}^2 y_* + 2\mathcal{P}^2 y_*^2). \tag{10.125}$$

For the wall heat transfer, recall $q_y = -k(dT/dy)$. Scaling, we get

$$q_c q_* = -\frac{kT_o}{h}\frac{dT_*}{dy_*}, \tag{10.126}$$

$$q_* = -\frac{kT_o}{hq_c}\frac{dT_*}{dy_*}. \tag{10.127}$$

Now choose q_c such that

$$q_c \equiv \frac{kT_o}{h}, \tag{10.128}$$

$$q_* = -\frac{dT_*}{dy_*}. \tag{10.129}$$

So

$$q_* = PrEc\left(\frac{\mathcal{P}^2}{3}y_*^3 - \mathcal{P}\left(\frac{1}{2}\mathcal{P}+1\right)y_*^2 + \left(1+\frac{1}{2}\mathcal{P}\right)^2 y_* - \frac{1}{2} - \frac{\mathcal{P}}{6} - \frac{\mathcal{P}^2}{24}\right). \tag{10.130}$$

At the bottom wall $y_* = 0$, we get for the heat transfer vector

$$q_*|_{y_*=0} = -PrEc\left(\frac{1}{2} + \frac{\mathcal{P}}{6} + \frac{\mathcal{P}^2}{24}\right). \tag{10.131}$$

10.2 Poisson equation for pressure

In the numerical solution of incompressible Navier-Stokes equations in multiple dimensions, one is often required to solve a Poisson equation for the pressure field. Let us see how such an equation arises. We will not solve this equation as there are complicated issues associated with the boundary conditions.

Let us first take the divergence of the incompressible, constant property linear momenta equation, Eq. (10.2):

$$\partial_i\left(\rho\partial_o v_i + \rho v_j\partial_j v_i\right) = \partial_i\left(-\partial_i p + \mu\partial_j\partial_j v_i\right), \tag{10.132}$$

$$\rho\partial_o \underbrace{\partial_i v_i}_{=0} + \rho\partial_i\left(v_j\partial_j v_i\right) = -\partial_i\partial_i p + \mu\partial_j\partial_j \underbrace{\partial_i v_i}_{=0}, \tag{10.133}$$

Because mass is conserved, we have $\partial_i v_i = 0$, by Eq. (10.1), so

$$\rho\partial_i\left(v_j\partial_j v_i\right) = -\partial_i\partial_i p, \tag{10.134}$$

$$\rho(v_j\partial_j \underbrace{\partial_i v_i}_{=0} + (\partial_i v_j)(\partial_j v_i)) = -\partial_i\partial_i p, \tag{10.135}$$

$$\rho(\partial_i v_j)(\partial_j v_i) = -\partial_i\partial_i p, \tag{10.136}$$

$$\partial_i\partial_i p = -\rho(\partial_i v_j)(\partial_j v_i). \tag{10.137}$$

In Gibbs notation, we have

$$\nabla^2 p = -\rho(\nabla\mathbf{v})^T : \nabla\mathbf{v}. \tag{10.138}$$

In two-dimensional flow, this reduces to

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\rho\left(\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial y}\right)^2\right). \tag{10.139}$$

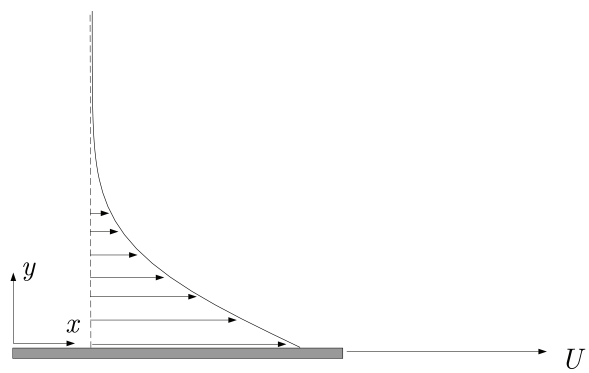

Figure 10.8: Schematic for Stokes' first problem of a suddenly accelerated plate diffusing linear momentum into a fluid at rest.

10.3 Similarity solutions

In this section, we will consider problems that can be addressed by what is known as a similarity transformation. The problems themselves will be fundamental ones that have variation in either time and one spatial coordinate, or with two spatial coordinates. This is in contrast with solutions of the previous section that varied only with one spatial coordinate.

Because two coordinates are involved, we must resort to solving partial differential equations. The similarity transformation actually reveals a hidden symmetry of the partial differential equations by defining a new independent variable, that is a grouping of the original independent variables, under which the partial differential equations transform into ordinary differential equations. We then solve the resulting ordinary differential equations by standard techniques.

10.3.1 Stokes' first problem

The first problem we will consider that uses a similarity transformation is known as Stokes' first problem, as Stokes addressed it in his original work that developed the Navier-Stokes equations in the mid-nineteenth century.[5] The problem is described as follows, and is sketched in Fig. 10.8. Consider a flat plate of infinite extent lying at rest for $t < 0$ on the $y = 0$ plane in $x - y - z$ space. In the volume described by $y > 0$ exists a fluid of semi-infinite extent that is at rest at time $t < 0$. At $t = 0$, the flat plate is suddenly accelerated to a constant velocity of U, entirely in the x direction. Because the no-slip condition is satisfied for the viscous flow, this induces the fluid at the plate surface to acquire an instantaneous velocity of $u(0) = U$. Because of diffusion of linear x momentum via tangential viscous shear forces, the fluid in the region above the plate begins to acquire a positive velocity in the x direction as well. We will use the Navier-Stokes equations to quantify this behavior. Let

[5]Stokes, G. G., 1851, "On the effect of the internal friction of fluids on the motion of pendulums," *Transactions of the Cambridge Philosophical Society*, 9(2): 8-106.

us make identical assumptions as we did in the previous section, except that 1) we will not neglect time derivatives, as they are an obviously important feature of the flow, and 2) we will assume all pressure gradients are zero; hence the fluid has a constant pressure.

Under these assumptions, the x momentum equation,

$$\rho\frac{\partial}{\partial t}u + \rho u \underbrace{\frac{\partial}{\partial x}}_{=0} u + \rho \underbrace{v}_{=0} \frac{\partial}{\partial y}u + \rho w \underbrace{\frac{\partial}{\partial z}}_{=0} u = -\underbrace{\frac{\partial p}{\partial x}}_{=0} + \mu\left(\underbrace{\frac{\partial^2}{\partial x^2}}_{=0} u + \frac{\partial^2}{\partial y^2}u + \underbrace{\frac{\partial^2}{\partial z^2}}_{=0} u\right), \tag{10.140}$$

is the only relevant component of linear momenta, and reduces to

$$\underbrace{\rho\frac{\partial u}{\partial t}}_{\text{(mass)(acceleration)}} = \underbrace{\mu\frac{\partial^2 u}{\partial y^2}}_{\text{shear force}}. \tag{10.141}$$

The energy equation reduces as follows

$$\rho c\frac{\partial}{\partial t}T + \rho c\left(u \underbrace{\frac{\partial}{\partial x}}_{=0} T + \underbrace{v}_{=0} \frac{\partial}{\partial y}T + \underbrace{w}_{=0} \underbrace{\frac{\partial}{\partial z}}_{=0} T\right) = k\left(\underbrace{\frac{\partial^2}{\partial x^2}}_{=0} T + \frac{\partial^2}{\partial y^2}T + \underbrace{\frac{\partial^2}{\partial z^2}}_{=0} T\right) + 2\mu\partial_{(i}v_{j)}\partial_{(i}v_{j)}, \tag{10.142}$$

$$\underbrace{\rho c\frac{\partial T}{\partial t}}_{\text{energy increase}} = \underbrace{k\frac{\partial^2 T}{\partial y^2}}_{\text{energy diffusion}} + \underbrace{\mu\left(\frac{\partial u}{\partial y}\right)^2}_{\text{viscous work source}}. \tag{10.143}$$

Let us first consider the x momentum equation. Recalling the momentum diffusivity definition, Eq. (6.55), $\nu = \mu/\rho$, we get the following partial differential equation, initial and boundary conditions:

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{10.144}$$

$$u(y,0) = 0, \qquad u(0,t) = U, \qquad u(\infty,t) = 0. \tag{10.145}$$

Now let us scale the equations. Choose

$$u_* = \frac{u}{U}, \qquad t_* = \frac{t}{t_c}, \qquad y_* = \frac{y}{y_c}. \tag{10.146}$$

We have yet to choose characteristic length, (y_c), and time, (t_c), scales. The equations become

$$\frac{U}{t_c}\frac{\partial u_*}{\partial t_*} = \frac{\nu U}{y_c^2}\frac{\partial^2 u_*}{\partial y_*^2}, \tag{10.147}$$

$$\frac{\partial u_*}{\partial t_*} = \frac{\nu t_c}{y_c^2}\frac{\partial^2 u_*}{\partial y_*^2}. \tag{10.148}$$

Wasting no time, we choose

$$y_c \equiv \frac{\nu}{U} = \frac{\mu}{\rho U}. \tag{10.149}$$

Examining the SI units, we see $\mu/(\rho U)$ has units of length: $\frac{\mathrm{N\ s}}{\mathrm{m^2}}\frac{\mathrm{m^3}}{\mathrm{kg}}\frac{\mathrm{s}}{\mathrm{m}} = \frac{\mathrm{kg\ m}}{\mathrm{s^2}}\frac{\mathrm{s}}{\mathrm{m^2}}\frac{\mathrm{m^3}}{\mathrm{kg}}\frac{\mathrm{s}}{\mathrm{m}} = \mathrm{m}$. With this choice, we get

$$\frac{\nu t_c}{y_c^2} = \frac{\nu t_c U^2}{\nu^2} = \frac{t_c U^2}{\nu}. \tag{10.150}$$

This suggests we choose

$$t_c = \frac{\nu}{U^2}. \tag{10.151}$$

With all of these choices the complete system can be written as

$$\frac{\partial u_*}{\partial t_*} = \frac{\partial^2 u_*}{\partial y_*^2}, \tag{10.152}$$

$$u_*(y_*, 0) = 0, \qquad u_*(0, t_*) = 1, \qquad u_*(\infty, t_*) = 0. \tag{10.153}$$

Now for *self-similarity*, we seek a transformation that reduces this partial differential equation, as well as its initial and boundary conditions, into an ordinary differential equation with suitable boundary conditions. If this transformation does not exist, no similarity solution exists. In this, but not all cases, the transformation does exist.

Let us first consider a general transformation from a y_*, t_* coordinate system to a new $\eta_*, \hat{t}_*$ coordinate system. We assume then a general transformation

$$\eta_* = \eta_*(y_*, t_*), \tag{10.154}$$

$$\hat{t}_* = \hat{t}_*(y_*, t_*). \tag{10.155}$$

Following the procedure of Ch. 2.5, this induces

$$\begin{pmatrix} d\eta_* \\ d\hat{t}_* \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial \eta_*}{\partial y_*} & \frac{\partial \eta_*}{\partial t_*} \\ \frac{\partial \hat{t}_*}{\partial y_*} & \frac{\partial \hat{t}_*}{\partial t_*} \end{pmatrix}}_{\mathsf{J}} \begin{pmatrix} dy_* \\ dt_* \end{pmatrix}, \tag{10.156}$$

with Jacobian matrix

$$\mathsf{J} = \begin{pmatrix} \frac{\partial \eta_*}{\partial y_*} & \frac{\partial \eta_*}{\partial t_*} \\ \frac{\partial \hat{t}_*}{\partial y_*} & \frac{\partial \hat{t}_*}{\partial t_*} \end{pmatrix}. \tag{10.157}$$

The chain rule for partial differentiation gives

$$\begin{pmatrix} \frac{\partial}{\partial y_*} \\ \frac{\partial}{\partial t_*} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial \eta_*}{\partial y_*} & \frac{\partial \hat{t}_*}{\partial y_*} \\ \frac{\partial \eta_*}{\partial t_*} & \frac{\partial \hat{t}_*}{\partial t_*} \end{pmatrix}}_{\mathsf{J}^T} \begin{pmatrix} \frac{\partial}{\partial \eta_*} \\ \frac{\partial}{\partial \hat{t}_*} \end{pmatrix}. \tag{10.158}$$

This expands as:

$$\left.\frac{\partial}{\partial y_*}\right|_{t_*} = \left.\frac{\partial \eta_*}{\partial y_*}\right|_{t_*} \left.\frac{\partial}{\partial \eta_*}\right|_{\hat{t}_*} + \left.\frac{\partial \hat{t}_*}{\partial y_*}\right|_{t_*} \left.\frac{\partial}{\partial \hat{t}_*}\right|_{\eta_*}, \tag{10.159}$$

$$\left.\frac{\partial}{\partial t_*}\right|_{y_*} = \left.\frac{\partial \eta_*}{\partial t_*}\right|_{y_*} \left.\frac{\partial}{\partial \eta_*}\right|_{\hat{t}_*} + \left.\frac{\partial \hat{t}_*}{\partial t_*}\right|_{y_*} \left.\frac{\partial}{\partial \hat{t}_*}\right|_{\eta_*}. \tag{10.160}$$

Now we will restrict ourselves to the transformation

$$\hat{t}_* = t_*, \tag{10.161}$$

so we have $\partial \hat{t}_*/\partial t_*|_{y_*} = 1$ and $\partial \hat{t}_*/\partial y_*|_{t_*} = 0$. Thus, our rules for differentiation reduce to

$$\left.\frac{\partial}{\partial y_*}\right|_{t_*} = \left.\frac{\partial \eta_*}{\partial y_*}\right|_{t_*} \left.\frac{\partial}{\partial \eta_*}\right|_{\hat{t}_*}, \tag{10.162}$$

$$\left.\frac{\partial}{\partial t_*}\right|_{y_*} = \left.\frac{\partial \eta_*}{\partial t_*}\right|_{y_*} \left.\frac{\partial}{\partial \eta_*}\right|_{\hat{t}_*} + \left.\frac{\partial}{\partial \hat{t}_*}\right|_{\eta_*}. \tag{10.163}$$

The next assumption is key for a similarity solution to exist. We restrict ourselves to transformations for which all state variables are at most a function of η_*. That is we allow no dependence on $\hat{t}_*$. Hence we must require that $\partial/\partial \hat{t}_*|_{\eta_*} = 0$. Moreover, partial derivatives with respect to η_* become total derivatives, giving us a final form of transformations for the derivatives

$$\left.\frac{\partial}{\partial y_*}\right|_{t_*} = \left.\frac{\partial \eta_*}{\partial y_*}\right|_{t_*} \frac{d}{d\eta_*}, \tag{10.164}$$

$$\left.\frac{\partial}{\partial t_*}\right|_{y_*} = \left.\frac{\partial \eta_*}{\partial t_*}\right|_{y_*} \frac{d}{d\eta_*}. \tag{10.165}$$

Now returning to Stokes' first problem, let us assume that a similarity solution exists of the form $u_*(y_*, t_*) = u_*(\eta_*)$. It is not always possible to find a similarity variable η_*. One of the more robust ways to find a similarity variable, if it exists, comes from group theory,[6] and is explained in detail in the monograph of Cantwell (2002). Group theory, that is too detailed to explicate in full here, relies on a generalized *symmetry* of equations to find

[6]Group theory has a long history in mathematics and physics. Its complicated origins generally include attribution to Évariste Galois, 1811-1832, a somewhat romantic figure, as well as Niels Henrick Abel, 1802-1829, the Norwegian mathematician. Critical developments were formalized by Marius Sophus Lie, 1842-1899, another Norwegian mathematician, in what today is known as Lie group theory. A modern variant, known as "renormalization group" (RNG) theory is an area for active research. The 1982 Nobel prize in physics went to Kenneth Geddes Wilson, 1936-2013, of Cornell University and The Ohio State University, for use of RNG in studying phase transitions, first done in the 1970s. The award citation refers to the possibilities of using RNG in studying the great unsolved problem of turbulence, a modern area of research in which Steven Alan Orszag, 1943-2011, made many contributions.

Quoting from the useful Eric Weisstein's World of Mathematics, available online at simpler forms. In the same sense that a snowflake, subjected to rotations of $\pi/3$, $2\pi/3$, π, $4\pi/3$, $5\pi/3$, or 2π, is transformed into a form that is indistinguishable from its original form, we seek transformations of the variables in our partial differential equation that map the equation into a form that is indistinguishable from the original. When systems are subject to such transformations, known as group operators, they are said to exhibit symmetry.

Let us subject our governing partial differential equation along with initial and boundary conditions to a particularly simple type of transformation, a simple stretching of space, time, and velocity:

$$\tilde{t} = e^a t_*, \qquad \tilde{y} = e^b y_*, \qquad \tilde{u} = e^c u_*. \tag{10.166}$$

Here the "$\sim$" variables are stretched variables, and a, b, and c are constant parameters. The exponential will be seen to be a convenience, that is not absolutely necessary. Note that for $a \in (-\infty, \infty)$, $b \in (-\infty, \infty)$, $c \in (-\infty, \infty)$, that $e^a \in (0, \infty)$, $e^b \in (0, \infty)$, $e^c \in (0, \infty)$. So the stretching does not change the direction of the variable; that is it is not a reflecting transformation. We note that with this stretching, the domain of the problem remains unchanged; that is $t_* \in [0, \infty)$ maps into $\tilde{t} \in [0, \infty)$; $y_* \in [0, \infty)$ maps into $\tilde{y} \in [0, \infty)$. The range is also unchanged if we allow $u_* \in [0, \infty)$, that maps into $\tilde{u} \in [0, \infty)$. Direct substitution of the transformation shows that in the stretched space, the system becomes

$$e^{a-c} \frac{\partial \tilde{u}}{\partial \tilde{t}} = e^{2b-c} \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}, \tag{10.167}$$

$$e^{-c} \tilde{u}(\tilde{y}, 0) = 0, \qquad e^{-c} \tilde{u}(0, \tilde{t}) = 1, \qquad e^{-c} \tilde{u}(\infty, \tilde{t}) = 0. \tag{10.168}$$

In order that the stretching transformation map the system into a form indistinguishable from the original, that is for the transformation to exhibit symmetry, we must take

$$c = 0, \qquad a = 2b. \tag{10.169}$$

`http://mathworld.wolfram.com/Group.html`, "A group G is a finite or infinite set of elements together with a binary operation that together satisfy the four fundamental properties of closure, associativity, the identity property, and the inverse property. The operation with respect to which a group is defined is often called the 'group operation,' and a set is said to be a group 'under' this operation. Elements A, B, $C, \ldots$ with binary operations A and B denoted AB form a group if

1. Closure: If A and B are two elements in G, then the product AB is also in G.
2. Associativity: The defined multiplication is associative, i.e. for all $A, B, C \in G$, $(AB)C = A(BC)$.
3. Identity: There is an identity element I (a.k.a. $\mathbf{1}$, E, or e) such that $IA = AI = A$ for every element $A \in G$.
4. Inverse: There must be an inverse or reciprocal of each element. Therefore, the set must contain an element $B = A^{-1}$ such that $AA^{-1} = A^{-1}A = I$ for each element of G.

$\ldots$, A map between two groups that preserves the identity and the group operation is called a homomorphism. If a homomorphism has an inverse that is also a homomorphism, then it is called an isomorphism and the two groups are called isomorphic. Two groups that are isomorphic to each other are considered to be 'the same' when viewed as abstract groups." For example, the group of 90 degree rotations of a square are isomorphic.

So our symmetry transformation is

$$\tilde{t} = e^{2b} t_*, \qquad \tilde{y} = e^{b} y_*, \qquad \tilde{u} = u_*, \tag{10.170}$$

giving in transformed space

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}, \tag{10.171}$$

$$\tilde{u}(\tilde{y}, 0) = 0, \qquad \tilde{u}(0, \tilde{t}) = 1, \qquad \tilde{u}(\infty, \tilde{t}) = 0. \tag{10.172}$$

Now both the original and transformed systems are the same, and the remaining stretching parameter b does not enter directly into either formulation, so we cannot expect it in the solution of either form. That is we expect a solution to be independent of the stretching parameter b. This can be achieved if we take both u_* and $\tilde{u}$ to be functions of special combinations of the independent variables, combinations that are formed such that b does not appear. Eliminating b via

$$e^b = \frac{\tilde{y}}{y_*}, \tag{10.173}$$

we get

$$\frac{\tilde{t}}{t_*} = \left(\frac{\tilde{y}}{y_*}\right)^2, \tag{10.174}$$

or after rearrangement

$$\frac{y_*}{\sqrt{t_*}} = \frac{\tilde{y}}{\sqrt{\tilde{t}}}. \tag{10.175}$$

We thus expect $u_* = u_*\left(y_*/\sqrt{t_*}\right)$ or equivalently $\tilde{u} = \tilde{u}\left(\tilde{y}/\sqrt{\tilde{t}}\,\right)$. This form also allows $u_* = u_*\left(\alpha y_*/\sqrt{t_*}\,\right)$, where α is any constant. Let us then define our similarity variable η_* as

$$\eta_* = \frac{y_*}{2\sqrt{t_*}}. \tag{10.176}$$

Here the factor of 1/2 is simply a convenience adopted so that the solution takes on a traditional form. We would find that any constant in the similarity transformation would induce a self-similar result.

Let us rewrite the differential equation, boundary, and initial conditions ($\partial u_*/\partial t_* = \partial^2 u_*/\partial y_*^2$, $u_*(y_*, 0) = 0$, $u_*(0, t_*) = 1$, $u_*(\infty, t_*) = 0$), in terms of the similarity variable η_*. We first must use the chain rule to get expressions for the derivatives. Applying the general results just developed, we get

$$\frac{\partial u_*}{\partial t_*} = \frac{\partial \eta_*}{\partial t_*}\frac{du_*}{d\eta_*} = -\frac{1}{2}\frac{y_*}{2} t_*^{-3/2}\frac{du_*}{d\eta_*} = -\frac{\eta_*}{2t_*}\frac{du_*}{d\eta_*}, \tag{10.177}$$

$$\frac{\partial u_*}{\partial y_*} = \frac{\partial \eta_*}{\partial y_*}\frac{du_*}{d\eta_*} = \frac{1}{2\sqrt{t_*}}\frac{du_*}{d\eta_*}, \tag{10.178}$$

$$\frac{\partial^2 u_*}{\partial y_*^2} = \frac{\partial}{\partial y_*}\left(\frac{\partial u_*}{\partial y_*}\right) = \frac{\partial}{\partial y_*}\left(\frac{1}{2\sqrt{t_*}}\frac{du_*}{d\eta_*}\right), \tag{10.179}$$

$$= \frac{1}{2\sqrt{t_*}}\frac{\partial}{\partial y_*}\left(\frac{du_*}{d\eta_*}\right) = \frac{1}{2\sqrt{t_*}}\left(\frac{1}{2\sqrt{t_*}}\frac{d^2u_*}{d\eta_*^2}\right) = \frac{1}{4t_*}\frac{d^2u_*}{d\eta_*^2}. \quad (10.180)$$

Thus, applying these rules to our governing linear momenta equation, we recover

$$-\frac{\eta_*}{2t_*}\frac{du_*}{d\eta_*} = \frac{1}{4t_*}\frac{d^2u_*}{d\eta_*^2}, \quad (10.181)$$

$$\underbrace{-2\eta_*\frac{du_*}{d\eta_*}}_{\text{acceleration}} = \underbrace{\frac{d^2u_*}{d\eta_*^2}}_{\text{viscous force imbalance}}, \quad (10.182)$$

$$\frac{d^2u_*}{d\eta_*^2} + 2\eta_*\frac{du_*}{d\eta_*} = 0. \quad (10.183)$$

Our governing equation has a singularity at $t_* = 0$. As it appears on both sides of the equation, we cancel it on both sides, but we shall see that this point is associated with special behavior of the similarity solution. The important result is that the reduced equation has dependency on η_* only. If this did not occur, we could not have a similarity solution.

Now consider the initial and boundary conditions. They transform as follows:

$$y_* = 0, \Longrightarrow \eta_* = 0, \quad (10.184)$$

$$y_* \to \infty, \Longrightarrow \eta_* \to \infty, \quad (10.185)$$

$$t_* \to 0, \Longrightarrow \eta_* \to \infty. \quad (10.186)$$

The three important points for t_* and y_* collapse into two corresponding points in η_*. This is also necessary for the similarity solution to exist. Consequently, our conditions in η_* space reduce to

$$u_*(0) = 1, \qquad \text{no-slip}, \quad (10.187)$$

$$u_*(\infty) = 0, \qquad \text{initial and far-field}. \quad (10.188)$$

We solve the second order differential equation by the method of reduction of order, noticing that it is really two first order equations in disguise:

$$\frac{d}{d\eta_*}\left(\frac{du_*}{d\eta_*}\right) + 2\eta_*\left(\frac{du_*}{d\eta_*}\right) = 0. \quad (10.189)$$

$$\text{multiplying by the integrating factor} \qquad e^{\eta_*^2}, \quad (10.190)$$

$$e^{\eta_*^2}\frac{d}{d\eta_*}\left(\frac{du_*}{d\eta_*}\right) + 2\eta_* e^{\eta_*^2}\left(\frac{du_*}{d\eta_*}\right) = 0. \quad (10.191)$$

$$\frac{d}{d\eta_*}\left(e^{\eta_*^2}\frac{du_*}{d\eta_*}\right) = 0, \quad (10.192)$$

$$e^{\eta_*^2}\frac{du_*}{d\eta_*} = A, \quad (10.193)$$

$$\frac{du_*}{d\eta_*} = Ae^{-\eta_*^2}, \tag{10.194}$$

$$u_* = B + A\int_0^{\eta_*} e^{-s^2}\,ds. \tag{10.195}$$

Now applying the condition $u_* = 1$ at $\eta_* = 0$ gives

$$1 = B + A\underbrace{\int_0^{0} e^{-s^2}\,ds}_{=0}, \tag{10.196}$$

$$B = 1. \tag{10.197}$$

So we have

$$u_* = 1 + A\int_0^{\eta_*} e^{-s^2}\,ds. \tag{10.198}$$

Now applying the condition $u_* = 0$ at $\eta_* \to \infty$, we get

$$0 = 1 + A\underbrace{\int_0^{\infty} e^{-s^2}\,ds}_{=\sqrt{\pi}/2}, \tag{10.199}$$

$$0 = 1 + A\frac{\sqrt{\pi}}{2}, \tag{10.200}$$

$$A = -\frac{2}{\sqrt{\pi}}. \tag{10.201}$$

Though not immediately obvious, it can be shown by a simple variable transformation to a polar coordinate system that the integral from 0 to ∞ has the value $\sqrt{\pi}/2$. It is not surprising that this integral has finite value over the semi-infinite domain as the integrand is bounded between zero and one, and decays rapidly to zero as $s \to \infty$.

Let us divert to evaluate this integral. To do so, consider the related integral I_2 defined over the first quadrant in $s-t$ space, where

$$I_2 \equiv \int_0^{\infty}\int_0^{\infty} e^{-s^2-t^2}\,ds\,dt, \tag{10.202}$$

$$= \int_0^{\infty} e^{-t^2}\int_0^{\infty} e^{-s^2}\,ds\,dt, \tag{10.203}$$

$$= \left(\int_0^{\infty} e^{-s^2}\,ds\right)\left(\int_0^{\infty} e^{-t^2}\,dt\right), \tag{10.204}$$

$$= \left(\int_0^{\infty} e^{-s^2}\,ds\right)^2, \tag{10.205}$$

$$\sqrt{I_2} = \int_0^{\infty} e^{-s^2}\,ds. \tag{10.206}$$

Now transform to polar coordinates with $s = r\cos\theta$, $t = r\sin\theta$. With this, we can easily show $ds\ dt = r\ dr\ d\theta$ and $s^2 + t^2 = r^2$. Substituting this into Eq. (10.202) and changing the limits of integration appropriately, we get

$$I_2 = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r\ dr\ d\theta, \tag{10.207}$$

$$= \int_0^{\pi/2}\left(-\frac{1}{2}e^{-r^2}\right)_0^{\infty} d\theta, \tag{10.208}$$

$$= \int_0^{\pi/2}\left(\frac{1}{2}\right) d\theta, \tag{10.209}$$

$$= \frac{\pi}{4}. \tag{10.210}$$

Comparing with Eq. (10.206), we deduce

$$\sqrt{I_2} = \int_0^{\infty} e^{-s^2}\ ds = \frac{\sqrt{\pi}}{2}. \tag{10.211}$$

With this verified, we can return to our original analysis and say that the velocity profile can be written as

$$u_*(\eta_*) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta_*} e^{-s^2}\ ds, \tag{10.212}$$

$$u_*(y_*, t_*) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{y_*}{2\sqrt{t_*}}} e^{-s^2}\ ds, \tag{10.213}$$

$$u_*(y_*, t_*) = \text{erfc}\left(\frac{y_*}{2\sqrt{t_*}}\right). \tag{10.214}$$

In the last form here, we have introduced the so-called error function complement, "erfc." Plots for the velocity profile in terms of both η_* and y_*, t_* are given in Fig. 10.9. We see that in similarity space, the curve is a single curve that in which u_* has a value of unity at $\eta_* = 0$ and has nearly relaxed to zero when $\eta_* = 1$. In dimensionless physical space, we see that at early time, there is a thin momentum layer near the surface. At later time more momentum is present in the fluid. We can say in fact that momentum is diffusing into the fluid.

We define the momentum diffusion length as the length for which significant momentum has diffused into the fluid. This is well estimated by taking $\eta_* = 1$. In terms of physical variables, we have

$$\frac{y_*}{2\sqrt{t_*}} = 1, \tag{10.215}$$

$$y_* = 2\sqrt{t_*}, \tag{10.216}$$

$$\frac{y}{\frac{\nu}{U}} = 2\sqrt{\frac{t}{\frac{\nu}{U^2}}}, \tag{10.217}$$

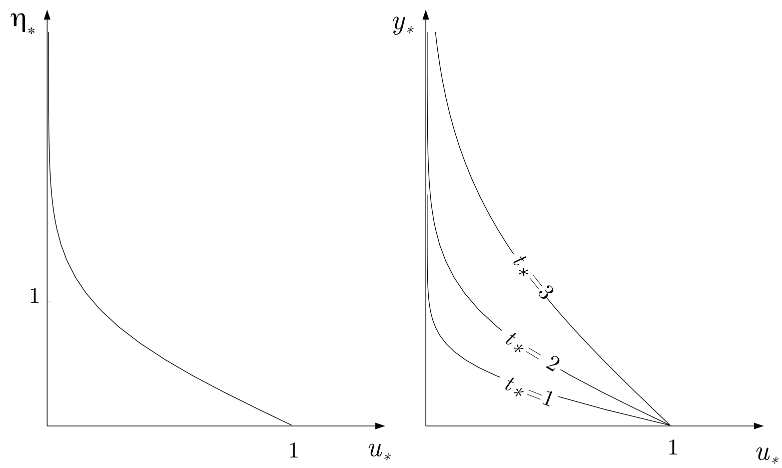

Figure 10.9: Sketch of velocity field solution for Stokes' first problem in both similarity coordinate η_* and primitive coordinates y_*, t_*.

$$y = \frac{2\nu}{U}\sqrt{\frac{U^2 t}{\nu}}, \tag{10.218}$$

$$y = 2\sqrt{\nu t}. \tag{10.219}$$

We can in fact define this as a boundary layer thickness. That is to say the momentum boundary layer thickness in Stokes' first problem grows at a rate proportional to the square root of momentum diffusivity and time. This class of result is a hallmark of all diffusion processes, be it mass, momentum, or energy. Taking standard properties of air, we find after one minute that its boundary layer thickness is 0.01 m. For oil after one minute, we get a thickness of 0.002 m.

We next consider the shear stress field. For this problem, the shear stress reduces to simply

$$\tau = \mu \frac{\partial u}{\partial y}. \tag{10.220}$$

Scaling as before by a characteristic stress τ_c, we get

$$\tau_* \tau_c = \frac{\mu U}{\frac{\nu}{U}} \frac{\partial u_*}{\partial y_*}, \tag{10.221}$$

$$\tau_* = \frac{\mu U^2}{\nu} \frac{1}{\tau_c} \frac{\partial u_*}{\partial y_*}. \tag{10.222}$$

Taking $\tau_c = \mu U^2/\nu = \mu U^2/(\mu/\rho) = \rho U^2$, we get

$$\tau_* = \frac{\partial u_*}{\partial y_*} = \frac{1}{2\sqrt{t_*}} \frac{du_*}{d\eta_*}, \tag{10.223}$$

$$= \frac{1}{2\sqrt{t_*}} \left(-\frac{2}{\sqrt{\pi}} e^{-\eta_*^2} \right), \tag{10.224}$$

$$= -\frac{1}{\sqrt{\pi t_*}} e^{-\eta_*^2}, \tag{10.225}$$

$$= -\frac{1}{\sqrt{\pi t_*}} \exp\left(-\frac{y_*}{2\sqrt{t_*}}\right)^2. \tag{10.226}$$

Now at the wall, $y_* = 0$, and we get

$$\tau_*|_{y_*=0} = -\frac{1}{\sqrt{\pi t_*}}. \tag{10.227}$$

So the shear stress does not have a similarity solution, but is directly related to time variation. The equation holds that the stress is infinite at $t_* = 0$, and decreases as time increases. This is because the velocity gradient flattens as time progresses. It can also be shown that while the stress is unbounded at a single point in time, that the impulse over a finite time span is finite, even when the time span includes $t_* = 0$. It can also be shown that the flow corresponds to a pulse of vorticity being introduced at the wall, that subsequently diffuses into the fluid.

In dimensional terms, we can say

$$\frac{\tau}{\rho U^2} = -\frac{1}{\sqrt{\pi \frac{U^2 t}{\nu}}}, \tag{10.228}$$

$$\tau = -\frac{\rho U^2}{\sqrt{\pi}\frac{U}{\sqrt{\nu}}\sqrt{t}}, \tag{10.229}$$

$$= -\frac{\rho U \sqrt{\frac{\mu}{\rho}}}{\sqrt{\pi t}}, \tag{10.230}$$

$$= -\frac{U\sqrt{\rho\mu}}{\sqrt{\pi t}}. \tag{10.231}$$

For this two-dimensional flow, the dimensionless vorticity vector is confined to the z_* direction and has magnitude

$$\omega_{z_*} = \underbrace{\frac{\partial v_*}{\partial x_*}}_{=0} - \frac{\partial u_*}{\partial y_*} = \frac{1}{\sqrt{\pi t_*}} \exp\left(-\frac{y_*}{2\sqrt{t_*}}\right)^2. \tag{10.232}$$

The dimensionless acceleration vector is confined to the x_* direction and has magnitude

$$a_{x_*} = \frac{\partial u_*}{\partial t_*} = \frac{1}{2\sqrt{\pi} t_*^{3/2}} \exp\left(-\frac{y_*}{2\sqrt{t_*}}\right)^2. \tag{10.233}$$

The dimensionless deformation tensor reduces to

$$\mathsf{D}_* = \begin{pmatrix} 0 & \frac{1}{2}\frac{\partial u_*}{\partial y_*} \\ \frac{1}{2}\frac{\partial u_*}{\partial y_*} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2\sqrt{\pi t_*}} \exp\left(-\frac{y_*}{2\sqrt{t_*}}\right)^2 \\ -\frac{1}{2\sqrt{\pi t_*}} \exp\left(-\frac{y_*}{2\sqrt{t_*}}\right)^2 & 0 \end{pmatrix}. \tag{10.234}$$

The deformation tensor has eigenvalues that represent the extreme values of extensional strain, and they are

$$\mathcal{D}_* = \pm \frac{1}{2\sqrt{\pi t_*}} \exp\left(-\frac{y_*}{2\sqrt{t_*}}\right)^2. \tag{10.235}$$

It is straightforward to show that the eigenvectors of $\mathcal{D}_*$ are aligned with coordinates axes that have been rotated through an angle of $\pi/4$.

Now let us consider the heat transfer problem. Recall the governing equation, initial and boundary conditions are

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{\partial u}{\partial y}\right)^2, \tag{10.236}$$

$$T(y,0) = T_o, \qquad T(0,t) = T_o, \qquad T(\infty,t) = T_o. \tag{10.237}$$

We will adopt the same time t_c and length y_c scales as before. Take the dimensionless temperature to be

$$T_* = \frac{T - T_o}{T_o}. \tag{10.238}$$

So we get

$$\frac{\rho c T_o}{t_c} \frac{\partial T_*}{\partial t_*} = \frac{k T_o}{y_c^2} \frac{\partial^2 T_*}{\partial y_*^2} + \frac{\mu U^2}{y_c^2} \left(\frac{\partial u_*}{\partial y_*}\right)^2, \tag{10.239}$$

$$\frac{\partial T_*}{\partial t_*} = \frac{k T_o}{y_c^2} \frac{t_c}{\rho c T_o} \frac{\partial^2 T_*}{\partial y_*^2} + \frac{\mu U^2}{y_c^2} \frac{t_c}{\rho c T_o} \left(\frac{\partial u_*}{\partial y_*}\right)^2, \tag{10.240}$$

$$\text{now} \quad \frac{k}{y_c^2} \frac{t_c}{\rho c} = \frac{k U^2}{\nu^2} \frac{\nu}{U^2} \frac{1}{\rho c} = \frac{k}{\rho c \nu} = \frac{k}{\mu c} = \frac{1}{Pr}, \tag{10.241}$$

$$\frac{\mu T^2}{y_c^2} \frac{t_c}{\rho c T_o} = \frac{\mu U^2 U^2}{\nu^2} \frac{\nu}{U^2} \frac{1}{\rho c T_o} = \frac{\mu U^2}{\frac{\mu}{\rho} \rho c T_o} = \frac{U^2}{c T_o} = Ec. \tag{10.242}$$

So we have in dimensionless form

$$\frac{\partial T_*}{\partial t_*} = \frac{1}{Pr} \frac{\partial^2 T_*}{\partial y_*^2} + Ec \left(\frac{\partial u_*}{\partial y_*}\right)^2, \tag{10.243}$$

$$T_*(y_*,0) = 0, \qquad T_*(0,t_*) = 0, \qquad T_*(\infty,t_*) = 0. \tag{10.244}$$

Notice that the only driving inhomogeneity is the viscous work. Now we know from our solution of the linear momentum equation that

$$\frac{\partial u_*}{\partial y*} = -\frac{1}{\sqrt{\pi t_*}} \exp\left(-\frac{y_*^2}{4t_*}\right). \tag{10.245}$$

So we can rewrite the equation for temperature variation as

$$\frac{\partial T_*}{\partial t_*} = \frac{1}{Pr}\frac{\partial^2 T_*}{\partial y_*^2} + \frac{Ec}{\pi t_*}\exp\left(-\frac{y_*^2}{2t_*}\right), \tag{10.246}$$

$$T_*(y_*,0) = 0, \qquad T_*(0,t_*) = 0, \qquad T_*(\infty,t_*) = 0. \tag{10.247}$$

Before considering the general solution, let us consider some limiting cases.

- $Ec \to 0$

  In the limit as $Ec \to 0$, we get a trivial solution, $T_*(y_*,t_*) = 0$.

- $Pr \to \infty$

  Recalling that the Prandtl number is the ratio of momentum diffusivity to thermal diffusivity, Eq. (6.123), this limit corresponds to materials for which momentum diffusivity is much greater than thermal diffusivity. For example for SAE 30 oil, the Prandtl number is around 3500. Naïvely assuming that we can simply neglect conduction, we write the energy equation in this limit as

$$\frac{\partial T_*}{\partial t_*} = \frac{Ec}{\pi t_*}\exp\left(-\frac{y_*^2}{2t_*}\right). \tag{10.248}$$

  and with $T_* = T_*(\eta_*)$ and $\eta_* = y_*/(2\sqrt{t_*})$, we get the transformed partial time derivative to be

$$\frac{\partial T_*}{\partial t_*} = -\frac{\eta_*}{2t_*}\frac{dT_*}{d\eta_*}. \tag{10.249}$$

  So the governing equation reduces to

$$-\frac{\eta_*}{2t_*}\frac{dT_*}{d\eta_*} = \frac{Ec}{\pi t_*}e^{-2\eta_*^2}, \tag{10.250}$$

$$\frac{dT_*}{d\eta_*} = -\frac{2Ec}{\pi}\frac{1}{\eta_*}e^{-2\eta_*^2}, \tag{10.251}$$

$$T_* = \frac{2Ec}{\pi}\int_{\eta_*}^{\infty}\frac{1}{s}e^{-2s^2}\,ds. \tag{10.252}$$

  We cannot satisfy both boundary conditions; the equation has been solved so as to satisfy the boundary condition in the far field of $T_*(\infty) = 0$.

  Unfortunately, we notice that we cannot satisfy the boundary condition at $\eta_* = 0$. We simply do not have enough degrees of freedom. In actuality, what we have found is an outer solution, and to match the boundary condition at 0, we would have to reintroduce conduction, that has a higher derivative.

  First let us see how the outer solution behaves near $\eta_* = 0$. Expanding the differential equation in a Taylor series about $\eta_* = 0$ and solving gives

$$\frac{dT_*}{d\eta_*} = -\frac{2Ec}{\pi}\left(\frac{1}{\eta_*} - 2\eta_* + 2\eta_*^3 + \dots\right), \tag{10.253}$$

$$T_* = -\frac{2Ec}{\pi}\left(\ln\eta_* - \eta_*^2 + \frac{1}{2}\eta_*^4 + \dots\right). \tag{10.254}$$

It turns out that solving the inner layer problem and the matching is of about the same difficulty as solving the full general problem, so we will defer this until later in this section.

- $Pr \to 0$

  In this limit, we get

$$\frac{\partial^2 T_*}{\partial y_*^2} = 0. \tag{10.255}$$

  The solution that satisfies the boundary conditions is

$$T_* = 0. \tag{10.256}$$

  In this limit, momentum diffuses slowly relative to energy. So we can interpret the results as follows. In the boundary layer, momentum is generated in a thin layer. Viscous dissipation in this layer gives rise to a local change in temperature in the layer that rapidly diffuses throughout the entire flow. The effect of smearing a localized finite thermal energy input over a semi-infinite domain has a negligible influence on the temperature of the global domain.

So let us bring back diffusion and study solutions for finite Prandtl number. Our governing equation in similarity variables then becomes

$$-\frac{\eta_*}{2t_*}\frac{dT_*}{d\eta_*} = \frac{1}{Pr}\frac{1}{4t_*}\frac{d^2T_*}{d\eta_*^2} + \frac{Ec}{\pi t_*}e^{-2\eta_*^2}, \tag{10.257}$$

$$\underbrace{-2\eta_*\frac{dT_*}{d\eta_*}}_{\text{temperature evolution}} = \underbrace{\frac{1}{Pr}\frac{d^2T_*}{d\eta_*^2}}_{\text{energy diffusion}} + \underbrace{\frac{4Ec}{\pi}e^{-2\eta_*^2}}_{\text{dissipation}}, \tag{10.258}$$

$$\frac{d^2T_*}{d\eta_*^2} + 2Pr\ \eta_*\frac{dT_*}{d\eta_*} = -\frac{4}{\pi}EcPr\ e^{-2\eta_*^2}, \tag{10.259}$$

$$T_*(0) = 0, \qquad T_*(\infty) = 0. \tag{10.260}$$

The second order differential equation is really two first order differential equations in disguise. There is an integrating factor of $e^{Pr\ \eta_*^2}$. Multiplying by the integrating factor and operating on the system, we find

$$e^{Pr\ \eta_*^2}\frac{d^2T_*}{d\eta_*^2} + 2Pr\ \eta_* e^{Pr\ \eta_*^2}\frac{dT_*}{d\eta_*} = -\frac{4}{\pi}EcPr\ e^{(Pr-2)\eta_*^2}, \tag{10.261}$$

$$\frac{d}{d\eta_*}\left(e^{Pr\ \eta_*^2}\frac{dT_*}{d\eta_*}\right) = -\frac{4}{\pi}EcPr\ e^{(Pr-2)\eta_*^2}, \tag{10.262}$$

$$e^{Pr\ \eta_*^2}\frac{dT_*}{d\eta_*} = -\frac{4}{\pi}EcPr\int_0^{\eta_*} e^{(Pr-2)s^2}\ ds + C_1, \tag{10.263}$$

$$\frac{dT_*}{d\eta_*} = -\frac{4}{\pi}EcPr\ e^{-Pr\ \eta_*^2}\int_0^{\eta_*} e^{(Pr-2)s^2}\ ds + C_1 e^{-Pr\eta_*^2},$$

(10.264)

$$
\begin{aligned}
T_* &= -\frac{4}{\pi} Ec Pr \int_0^{\eta_*} e^{-Pr\ p^2} \int_0^p e^{(Pr-2)s^2}\ ds\ dp \\
&\quad + C_1 \int_0^{\eta_*} e^{-Pr\ s^2}\ ds + C_2.
\end{aligned} \tag{10.265}
$$

The boundary condition $T_*(0) = 0$ gives us $C_2 = 0$. The boundary condition at ∞ gives us then

$$
0 = -\frac{4}{\pi} Ec Pr \int_0^{\infty} e^{-Pr\ p^2} \int_0^p e^{(Pr-2)s^2}\ ds\ dp + C_1 \underbrace{\int_0^{\infty} e^{-Pr\ s^2}\ ds}_{\frac{1}{2}\sqrt{\frac{\pi}{Pr}}}. \tag{10.266}
$$

(10.267)

Therefore, we get

$$
\frac{4}{\pi} Ec Pr \int_0^{\infty} e^{-Pr\ p^2} \int_0^p e^{(Pr-2)s^2}\ ds\ dp = \frac{C_1}{2}\sqrt{\frac{\pi}{Pr}}, \tag{10.268}
$$

$$
C_1 = \frac{8}{\pi^{3/2}} Ec Pr^{3/2} \int_0^{\infty} e^{-Pr\ p^2} \int_0^p e^{(Pr-2)s^2}\ ds\ dp. \tag{10.269}
$$

So finally, we have for the temperature profile

$$
\begin{aligned}
T_*(\eta_*) &= -\frac{4}{\pi} Ec Pr \int_0^{\eta_*} e^{-Pr\ p^2} \int_0^p e^{(Pr-2)s^2}\ ds\ dp \\
&\quad + \left(\frac{8}{\pi^{3/2}} Ec Pr^{3/2} \int_0^{\infty} e^{-Pr\ p^2} \int_0^p e^{(Pr-2)s^2}\ ds\ dp \right) \int_0^{\eta_*} e^{-Pr\ s^2}\ ds.
\end{aligned} \tag{10.270}
$$

This simplifies somewhat to

$$
-\frac{2 Ec Pr}{\sqrt{\pi(2-Pr)}} \left(\int_0^{\eta} e^{-Pr\ p^2} \text{erf}\left(\sqrt{2-Pr}\ p\right)\ dp - \text{erf}\left(\sqrt{Pr}\ \eta\right) \int_0^{\infty} e^{-Pr\ p^2} \text{erf}\left(\sqrt{2-Pr}\ p\right)\ dp \right). \tag{10.271}
$$

This analysis simplifies considerably in the limit of $Pr = 1$, that is when momentum and energy diffuse at the same rate. This is a close to reality for many gases. In this case, the temperature profile becomes

$$
T_*(\eta_*) = -\frac{4}{\pi} Ec \int_0^{\eta_*} e^{-p^2} \int_0^p e^{-s^2}\ ds\ dp + C_1 \int_0^{\eta_*} e^{-s^2}\ ds. \tag{10.272}
$$

Now if $h(p) = \int_0^p e^{-s^2}\ ds$, we get $dh/dp = e^{-p^2}$. Using this, we can rewrite the temperature profile as

$$
T_*(\eta_*) = -\frac{4}{\pi} Ec \int_0^{\eta_*} h(p) \frac{dh}{dp}\ dp + C_1 \int_0^{\eta_*} e^{-s^2}\ ds, \tag{10.273}
$$

$$= -\frac{4Ec}{\pi}\int_0^{\eta_*} d\left(\frac{h^2}{2}\right) + C_1\int_0^{\eta_*} e^{-s^2}\,ds, \tag{10.274}$$

$$= -\frac{4Ec}{\pi}\left(\frac{1}{2}\right)\left(\int_0^{\eta_*} e^{-s^2}\,ds\right)^2 + C_1\int_0^{\eta_*} e^{-s^2}\,ds, \tag{10.275}$$

$$= \left(-\frac{2Ec}{\pi}\int_0^{\eta_*} e^{-s^2}ds + C_1\right)\int_0^{\eta_*} e^{-s^2}\,ds. \tag{10.276}$$

Now for $T(\infty) = 0$, we get

$$0 = \left(-\frac{2Ec}{\pi}\int_0^{\infty} e^{-s^2}ds + C_1\right)\int_0^{\infty} e^{-s^2}\,ds, \tag{10.277}$$

$$0 = \left(-\frac{2Ec}{\pi}\frac{\sqrt{\pi}}{2} + C_1\right)\frac{\sqrt{\pi}}{2}, \tag{10.278}$$

$$C_1 = \frac{Ec}{\sqrt{\pi}}. \tag{10.279}$$

So the temperature profile can be expressed as

$$T_*(\eta_*) = \frac{Ec}{\sqrt{\pi}}\left(\int_0^{\eta_*} e^{-s^2}\,ds\right)\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta_*} e^{-s^2}\,ds\right). \tag{10.280}$$

We notice that we can write this directly in terms of the velocity as

$$T_*(\eta_*) = \frac{Ec}{2}u_*(\eta_*)\left(1 - u_*(\eta_*)\right). \tag{10.281}$$

This is a consequence of what is known as *Reynolds' analogy* that holds for $Pr = 1$ that the temperature field can be directly related to the velocity field. The temperature field for Stokes' first problem for $Pr = 1$, $Ec = 1$ is plotted in Fig. 10.10.

10.3.2 Blasius boundary layer

We next consider the well known problem of the flow of a viscous fluid over a flat plate. This problem forms the foundation for a variety of viscous flows over more complicated geometries. It also illustrates some important features of viscous flow physics, as well as giving the original motivating problem for the mathematical technique of matched asymptotic expansions. Here we will consider, as sketched in Fig. 10.11, the incompressible flow of viscous fluid of constant viscosity and thermal conductivity over a flat plate. In the far field, the fluid will be a uniform stream with constant velocity. At the plate surface, the no-slip condition must be enforced, that will give rise to a zone of adjustment where the fluid's velocity changes from zero at the plate surface to its freestream value. This zone is called the *boundary layer*. The sketch has a small inaccuracy as it only gives the u velocity component variation. Actually there is also a v component of the velocity vector, and that induces streamline curvature, not apparent in Fig. 10.11.

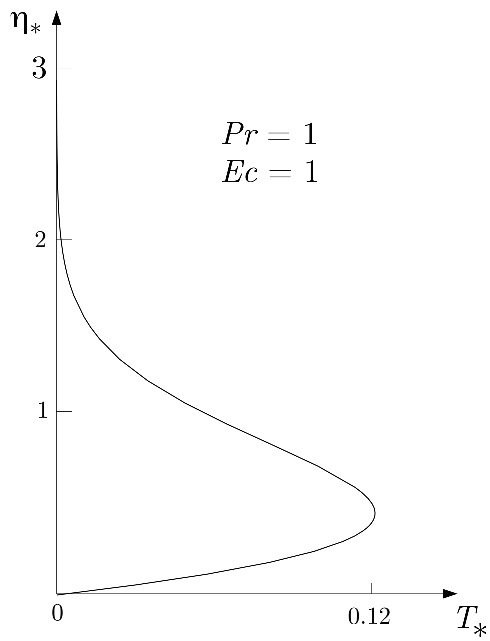

Figure 10.10: Plot of temperature field for Stokes' first problem for $Pr = 1$, $Ec = 1$.

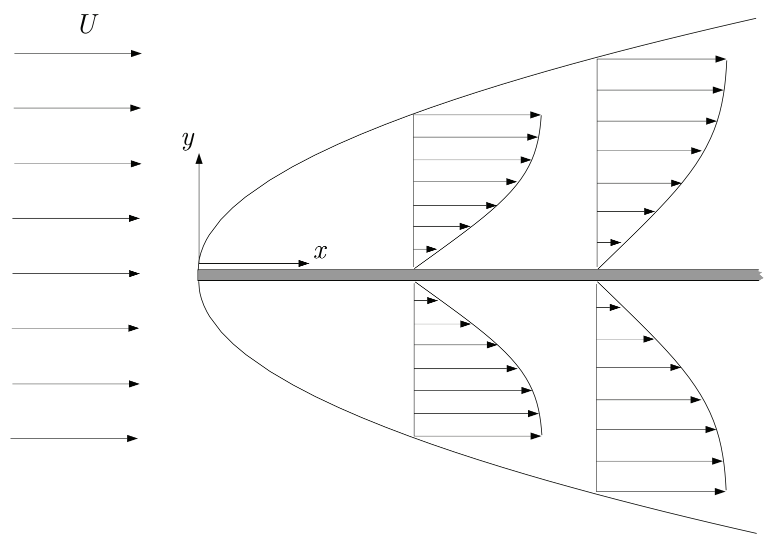

Figure 10.11: Schematic for flat plate boundary layer problem.

Considering first the velocity field, we find, assuming the flow is steady as well, that the dimensionless two-dimensional Navier-Stokes equations are as follows

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{10.282}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{10.283}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right). \tag{10.284}$$

The dimensionless boundary conditions are

$$u(x, y\to\infty) = 1, \tag{10.285}$$

$$p(x, y\to\infty) = 0, \tag{10.286}$$

$$u(x,0) = 0, \tag{10.287}$$

$$v(x,0) = 0. \tag{10.288}$$

In this section, we are dispensing with the $*$'s and assuming all variables are dimensionless. In fact we have assumed a scaling of the following form, where *dim* is a subscript denoting a dimensional variable.

$$u=\frac{u_{dim}}{U},\quad v=\frac{v_{dim}}{U},\quad x=\frac{x_{dim}}{L},\quad y=\frac{y_{dim}}{L},\quad p=\frac{p_{dim}-p_o}{\rho U^2}. \tag{10.289}$$

For our flat plate of semi-infinite extent, we do not have a natural length scale. This suggests that we may find a similarity solution that removes the effect of L. It also may be appropriate to think of making the requirement that

$$L\gg\frac{\nu}{U}. \tag{10.290}$$

This will insure that $Re=UL/\nu\gg 1$. This is not entirely satisfying as we really want our domain to be semi-infinite with $L\to\infty$.

Now let us consider that for $Re\to\infty$, we have an outer solution of $u=1$ to be valid for most of the flow field sufficiently far away from the plate surface. In fact the solution $u=1$, $v=0$, $p=0$, satisfies all of the governing equations and boundary conditions except for the no-slip condition at $y=0$. Because in the limit as $Re\to\infty$, we effectively ignore the high order derivatives found in the viscous terms, we cannot expect to satisfy all boundary conditions for the full problem. We call this the *outer solution*, that is also an inviscid solution to the equations, allowing for a slip condition at the boundary.

Let us *rescale* our equations near the plate surface $y=0$ to

- bring back the effect of the viscous terms,
- bring back the no-slip condition, and

- match our inviscid outer solution to a viscous inner solution.

This is *the* first example of the use of the method of matched asymptotic expansions as introduced by Prandtl and his student Blasius in the early twentieth century.

With some difficulty, we could show how to choose the scaling, let us simply adopt a scaling and show that it indeed achieves our desired end. So let us take a scaled y distance and velocity, denoted by a ˜ superscript, to be

$$\tilde{v} = \sqrt{Re}\ v, \qquad \tilde{y} = \sqrt{Re}\ y. \tag{10.291}$$

With this scaling, assuming the Reynolds number is large, when we examine small y or v, we are examining an order unity $\tilde{y}$ or $\tilde{v}$. Our equations rescale as

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{1/\sqrt{Re}}{1/\sqrt{Re}}\frac{\partial \tilde{v}}{\partial \tilde{y}} &= 0, && (10.292)\\
u\frac{\partial u}{\partial x} + \frac{1/\sqrt{Re}}{1/\sqrt{Re}}\tilde{v}\frac{\partial u}{\partial \tilde{y}} &= -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + Re\frac{\partial^2 u}{\partial \tilde{y}^2}\right), && (10.293)\\
\frac{1}{\sqrt{Re}}u\frac{\partial \tilde{v}}{\partial x} + \frac{(1/\sqrt{Re})(1/\sqrt{Re})}{1/\sqrt{Re}}\tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} &= -\frac{1}{1/\sqrt{Re}}\frac{\partial p}{\partial \tilde{y}}\\
&\quad + \frac{1}{Re}\left(\frac{1}{\sqrt{Re}}\frac{\partial^2 \tilde{v}}{\partial x^2} + \frac{1/\sqrt{Re}}{1/Re}\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}\right). && (10.294)
\end{aligned}$$

Simplifying, this reduces to

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial \tilde{v}}{\partial \tilde{y}} &= 0, && (10.295)\\
u\frac{\partial u}{\partial x} + \tilde{v}\frac{\partial u}{\partial \tilde{y}} &= -\frac{\partial p}{\partial x} + \frac{1}{Re}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial \tilde{y}^2}, && (10.296)\\
u\frac{\partial \tilde{v}}{\partial x} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} &= -Re\frac{\partial p}{\partial \tilde{y}} + \frac{1}{Re}\frac{\partial^2 \tilde{v}}{\partial x^2} + \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}. && (10.297)
\end{aligned}$$

Now in the limit as $Re \to \infty$, the rescaled equations reduce to the well known *boundary layer equations*:

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial \tilde{v}}{\partial \tilde{y}} &= 0, && (10.298)\\
u\frac{\partial u}{\partial x} + \tilde{v}\frac{\partial u}{\partial \tilde{y}} &= -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial \tilde{y}^2}, && (10.299)\\
0 &= \frac{\partial p}{\partial \tilde{y}}. && (10.300)
\end{aligned}$$

To match the outer solution, we need the boundary conditions that are

$$u(x, \tilde{y} \to \infty) = 1, \tag{10.301}$$

$$p(x, \tilde{y} \to \infty) = 0, \tag{10.302}$$
$$u(x, 0) = 0, \tag{10.303}$$
$$\tilde{v}(x, 0) = 0. \tag{10.304}$$

The $\tilde{y}$ momentum equation gives us

$$p = p(x). \tag{10.305}$$

In general, we can consider this to be an imposed pressure gradient that is supplied by the outer inviscid solution. For general flows, that pressure gradient dp/dx will be non-zero. For the Blasius problem, we will choose to study problems for which there is *no pressure gradient*. That is we take

$$p(x) = 0, \qquad \text{for Blasius flat plate boundary layer.} \tag{10.306}$$

So called Falkner-Skan solutions consider flows over curved plates, for which the outer inviscid solution does not have a constant pressure. This ultimately affects the behavior of the fluid in the boundary layer, giving results that differ in important features from our Blasius problem.

With our assumptions, the Blasius problem reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0, \tag{10.307}$$
$$u\frac{\partial u}{\partial x} + \tilde{v}\frac{\partial u}{\partial \tilde{y}} = \frac{\partial^2 u}{\partial \tilde{y}^2}. \tag{10.308}$$

The boundary conditions are now only on velocity and are

$$u(x, \tilde{y} \to \infty) = 1, \tag{10.309}$$
$$u(x, 0) = 0, \tag{10.310}$$
$$\tilde{v}(x, 0) = 0. \tag{10.311}$$

Now to simplify, we invoke the stream function ψ, that allows us to satisfy continuity automatically and eliminate u and $\tilde{v}$ at the expense of raising the order of the differential equation. So taking

$$u = \frac{\partial \psi}{\partial \tilde{y}}, \qquad \tilde{v} = -\frac{\partial \psi}{\partial x}, \tag{10.312}$$

we find that mass conservation reduces to $\partial^2\psi/\partial x\partial\tilde{y} - \partial^2\psi/\partial\tilde{y}\partial x = 0$. The x momentum equation and associated boundary conditions become

$$\frac{\partial \psi}{\partial \tilde{y}}\frac{\partial^2 \psi}{\partial x \partial \tilde{y}} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial \tilde{y}^2} = \frac{\partial^3 \psi}{\partial \tilde{y}^3}, \tag{10.313}$$
$$\frac{\partial \psi}{\partial \tilde{y}}(x, \tilde{y} \to \infty) = 1, \tag{10.314}$$
$$\frac{\partial \psi}{\partial \tilde{y}}(x, 0) = 0, \tag{10.315}$$
$$\frac{\partial \psi}{\partial x}(x, 0) = 0. \tag{10.316}$$

Let us try stretching all the variables of this system to see if there are stretching transformations under which the system exhibits symmetry; that is we seek a stretching transformation under which the system is invariant. Take

$$\hat{x} = e^a x, \qquad \hat{y} = e^b \tilde{y}, \qquad \hat{\psi} = e^c \psi. \tag{10.317}$$

Under this transformation, the x momentum equation and boundary conditions transform to

$$e^{a+2b-2c}\frac{\partial \hat{\psi}}{\partial \hat{y}}\frac{\partial^2 \hat{\psi}}{\partial \hat{x}\partial \hat{y}} - e^{a+2b-2c}\frac{\partial \hat{\psi}}{\partial \hat{x}}\frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} = e^{3b-c}\frac{\partial^3 \hat{\psi}}{\partial \hat{y}^3}, \tag{10.318}$$

$$e^{b-c}\frac{\partial \hat{\psi}}{\partial \hat{y}}(\hat{x}, \hat{y} \to \infty) = 1, \tag{10.319}$$

$$e^{b-c}\frac{\partial \hat{\psi}}{\partial \hat{y}}(\hat{x}, 0) = 0, \tag{10.320}$$

$$e^{a-c}\frac{\partial \hat{\psi}}{\partial \hat{x}}(\hat{x}, 0) = 0. \tag{10.321}$$

If we demand $b = c$ and $a = 2c$, then the transformation is invariant, yielding

$$\frac{\partial \hat{\psi}}{\partial \hat{y}}\frac{\partial^2 \hat{\psi}}{\partial \hat{x}\partial \hat{y}} - \frac{\partial \hat{\psi}}{\partial \hat{x}}\frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} = \frac{\partial^3 \hat{\psi}}{\partial \hat{y}^3}, \tag{10.322}$$

$$\frac{\partial \hat{\psi}}{\partial \hat{y}}(\hat{x}, \hat{y} \to \infty) = 1, \tag{10.323}$$

$$\frac{\partial \hat{\psi}}{\partial \hat{y}}(\hat{x}, 0) = 0, \tag{10.324}$$

$$\frac{\partial \hat{\psi}}{\partial \hat{x}}(\hat{x}, 0) = 0. \tag{10.325}$$

Now our transformation is reduced to

$$\hat{x} = e^{2c} x, \qquad \hat{y} = e^c \tilde{y}, \qquad \hat{\psi} = e^c \psi. \tag{10.326}$$

Because c does not appear explicitly in either the original equation set nor the transformed equation set, the solution must not depend on this stretching. Eliminating c from the transformation by $e^c = \sqrt{\hat{x}/x}$ we find that

$$\frac{\hat{y}}{\tilde{y}} = \sqrt{\frac{\hat{x}}{x}}, \qquad \frac{\hat{\psi}}{\psi} = \sqrt{\frac{\hat{x}}{x}}, \tag{10.327}$$

or

$$\frac{\hat{y}}{\sqrt{\hat{x}}} = \frac{\tilde{y}}{\sqrt{x}}, \qquad \frac{\hat{\psi}}{\sqrt{\hat{x}}} = \frac{\psi}{\sqrt{x}}. \tag{10.328}$$

Thus motivated, let us seek solutions of the form

$$\frac{\psi}{\sqrt{x}} = f\left(\frac{\tilde{y}}{\sqrt{x}}\right). \tag{10.329}$$

That is taking

$$\eta = \frac{\tilde{y}}{\sqrt{x}}, \tag{10.330}$$

we seek

$$\psi = \sqrt{x} f(\eta). \tag{10.331}$$

Let us check that our similarity variable is independent of L our unknown length scale.

$$\eta = \frac{\tilde{y}}{\sqrt{x}} = \frac{\sqrt{Re}\ y}{\sqrt{x}} = \frac{\sqrt{Re}\ y_{dim}/L}{\sqrt{x_{dim}/L}} = \sqrt{\frac{UL}{\nu}} \frac{y_{dim}}{L} \frac{\sqrt{L}}{\sqrt{x_{dim}}} = \sqrt{\frac{U}{\nu}} \frac{y_{dim}}{\sqrt{x_{dim}}}. \tag{10.332}$$

So indeed, our similarity variable is independent of any arbitrary length scale we happen to have chosen.

With our similarity transformation, we have

$$\frac{\partial \eta}{\partial x} = -\frac{1}{2}\tilde{y}x^{-3/2} = -\frac{1}{2}\frac{\eta}{x}, \tag{10.333}$$

$$\frac{\partial \eta}{\partial \tilde{y}} = \frac{1}{\sqrt{x}}. \tag{10.334}$$

Now we need expressions for $\partial\psi/\partial x$, $\partial\psi/\partial\tilde{y}$, $\partial^2\psi/\partial x\partial\tilde{y}$, $\partial^2\psi/\partial\tilde{y}^2$, and $\partial^3\psi/\partial\tilde{y}^3$. First, consider the partial derivatives of the stream function ψ. Operating on each partial derivative, we find

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x} f(\eta)\right), \tag{10.335}$$

$$= \sqrt{x}\frac{df}{d\eta}\frac{\partial \eta}{\partial x} + \frac{1}{2}\frac{1}{\sqrt{x}} f, \tag{10.336}$$

$$= \sqrt{x}\left(-\frac{1}{2}\right)\frac{\eta}{x}\frac{df}{d\eta} + \frac{1}{2}\frac{1}{\sqrt{x}} f, \tag{10.337}$$

$$= \frac{1}{2\sqrt{x}}\left(f - \eta\frac{df}{d\eta}\right). \tag{10.338}$$

So we get for $\tilde{v}$ that

$$\tilde{v} = -\frac{\partial \psi}{\partial x} = \frac{1}{2\sqrt{x}}\left(\eta\frac{df}{d\eta} - f\right). \tag{10.339}$$

And then we find

$$\frac{\partial \psi}{\partial \tilde{y}} = \frac{\partial}{\partial \tilde{y}}\left(\sqrt{x} f(\eta)\right), \tag{10.340}$$

$$= \sqrt{x}\frac{\partial}{\partial \tilde{y}}(f(\eta)), \tag{10.341}$$

$$= \sqrt{x}\frac{df}{d\eta}\frac{\partial \eta}{\partial \tilde{y}}, \tag{10.342}$$

$$= \sqrt{x}\frac{df}{d\eta}\frac{1}{\sqrt{x}}, \tag{10.343}$$

$$= \frac{df}{d\eta}. \tag{10.344}$$

Thus for u, we get

$$u = \frac{\partial \psi}{\partial \tilde{y}} = \frac{df}{d\eta}. \tag{10.345}$$

So

$$\frac{\partial^2 \psi}{\partial x \partial \tilde{y}} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial \tilde{y}}\right) = \frac{\partial}{\partial x}\left(\frac{df}{d\eta}\right) = \frac{d^2 f}{d\eta^2}\frac{\partial \eta}{\partial x} = -\frac{1}{2x}\eta\frac{d^2 f}{d\eta^2}. \tag{10.346}$$

$$\frac{\partial^2 \psi}{\partial \tilde{y}^2} = \frac{\partial}{\partial \tilde{y}}\left(\frac{\partial \psi}{\partial \tilde{y}}\right) = \frac{\partial}{\partial \tilde{y}}\left(\frac{df}{d\eta}\right) = \frac{d^2 f}{d\eta^2}\frac{\partial \eta}{\partial \tilde{y}} = \frac{1}{\sqrt{x}}\frac{d^2 f}{d\eta^2}. \tag{10.347}$$

$$\frac{\partial^3 \psi}{\partial \tilde{y}^3} = \frac{\partial}{\partial \tilde{y}}\left(\frac{\partial^2 \psi}{\partial \tilde{y}^2}\right) = \frac{\partial}{\partial \tilde{y}}\left(\frac{1}{\sqrt{x}}\frac{d^2 f}{d\eta^2}\right) = \frac{1}{\sqrt{x}}\frac{\partial}{\partial \tilde{y}}\left(\frac{d^2 f}{d\eta^2}\right) = \frac{1}{\sqrt{x}}\frac{d^3 f}{d\eta^3}\frac{\partial \eta}{\partial \tilde{y}}, \tag{10.348}$$

$$= \frac{1}{x}\frac{d^3 f}{d\eta^3}. \tag{10.349}$$

Now we substitute each of these expressions into the x momentum equation and get

$$\underbrace{\frac{df}{d\eta}}_{u}\underbrace{\left(-\frac{1}{2x}\eta\frac{d^2 f}{d\eta^2}\right)}_{\frac{\partial u}{\partial x}} + \underbrace{\frac{1}{2\sqrt{x}}\left(\eta\frac{df}{d\eta} - f\right)}_{\tilde{v}}\underbrace{\frac{1}{\sqrt{x}}\frac{d^2 f}{d\eta^2}}_{\frac{\partial u}{\partial \tilde{y}}} = \underbrace{\frac{1}{x}\frac{d^3 f}{d\eta^3}}_{\frac{\partial^2 u}{\partial \tilde{y}^2}}, \tag{10.350}$$

$$-\eta\frac{df}{d\eta}\frac{d^2 f}{d\eta^2} + \left(\eta\frac{df}{d\eta} - f\right)\frac{d^2 f}{d\eta^2} = 2\frac{d^3 f}{d\eta^3}, \tag{10.351}$$

$$\underbrace{-f\frac{d^2 f}{d\eta^2}}_{\text{advection}} = \underbrace{2\frac{d^3 f}{d\eta^3}}_{\text{momentum diffusion}}, \tag{10.352}$$

$$\frac{d^3 f}{d\eta^3} + \frac{1}{2}f\frac{d^2 f}{d\eta^2} = 0. \tag{10.353}$$

This is a third order non-linear ordinary differential equation for $f(\eta)$. We need three boundary conditions. Now at the surface $\tilde{y} = 0$, we have $\eta = 0$. And as $\tilde{y} \to \infty$, we have $\eta \to \infty$. To satisfy the no-slip condition on u at the plate surface, we require

$$\left.\frac{df}{d\eta}\right|_{\eta=0} = 0. \tag{10.354}$$

For no-slip on $\tilde{v}$, we require

$$\begin{aligned}
\tilde{v}(0) = 0 &= \frac{1}{2\sqrt{x}}\left(\eta\frac{df}{d\eta} - f\right), &(10.355)\\
0 &= \underbrace{0\left.\frac{df}{d\eta}\right|_{\eta=0}}_{=0} - f(0), &(10.356)\\
f(0) &= 0. &(10.357)
\end{aligned}$$

And to satisfy the freestream condition on u as $\eta \to \infty$, we need

$$\left.\frac{df}{d\eta}\right|_{\eta\to\infty} = 1. \tag{10.358}$$

The most standard way to solve non-linear ordinary differential equations of this type is to reduce them to systems of first order ordinary differential equations and use some numerical technique, such as a Runge[7]-Kutta integration. We recall that Runge-Kutta techniques, as well as most other common techniques, require a well-defined set of initial conditions to predict the final state. To achieve the desired form, we define

$$g \equiv \frac{df}{d\eta}, \qquad h \equiv \frac{d^2 f}{d\eta^2}. \tag{10.359}$$

Thus the x momentum equation becomes

$$\frac{dh}{d\eta} + \frac{1}{2}fh = 0. \tag{10.360}$$

But this is one equation in three unknowns. We need to write our equations as a system of three first order equations, along with associated initial conditions. They are

$$\begin{aligned}
\frac{df}{d\eta} &= g, \qquad f(0) = 0, &(10.361)\\
\frac{dg}{d\eta} &= h, \qquad g(0) = 0, &(10.362)\\
\frac{dh}{d\eta} &= -\frac{1}{2}fh, \qquad h(0) = ?. &(10.363)
\end{aligned}$$

[7]Carl David Tolmè Runge, 1856-1927, German mathematician and physicist, close friend of Max Planck, studied spectral line elements of non-Hydrogen molecules, held chairs at Hanover and Göttingen, entertained grandchildren at age 70 by doing handstands.

Everything is well-defined except we do not have an initial condition on h. We do however have a far-field condition on g that is $g(\infty) = 1$. One viable option we have for getting a final solution is to use a numerical trial and error procedure, guessing $h(0)$ until we find that $g(\infty) \to 1$. We will use a slightly more efficient method here, that only requires one guess.

To do this, let us first demonstrate the following lemma: If $F(\eta)$ is a solution to $d^3f/d\eta^3 + \frac{1}{2}f(d^2f/d\eta^2) = 0$, then $aF(a\eta)$ is also a solution. The proof is as follows. Take $w(\eta) = aF(a\eta)$. Then we have

$$w = aF(a\eta), \tag{10.364}$$

$$\frac{dw}{d\eta} = a^2\frac{dF(a\eta)}{d\eta}, \tag{10.365}$$

$$\frac{d^2w}{d\eta^2} = a^3\frac{d^2F(a\eta)}{d\eta^2}, \tag{10.366}$$

$$\frac{d^3w}{d\eta^3} = a^4\frac{d^3F(a\eta)}{d\eta^3}. \tag{10.367}$$

Substituting these expressions into the x momentum equation, we find

$$a^4\frac{d^3F(a\eta)}{d\eta^3} + \frac{1}{2}a^4F(a\eta)\frac{d^2F(a\eta)}{d\eta^2} = 0, \tag{10.368}$$

$$\frac{d^3F(a\eta)}{d\eta^3} + \frac{1}{2}F(a\eta)\frac{d^2F(a\eta)}{d\eta^2} = 0. \tag{10.369}$$

But we know this to be true as $F(a\eta)$ is a solution. Hence $aF(a\eta)$ is also a solution.

So to solve our non-linear system, let us first solve the following related system:

$$\frac{dF}{d\eta} = G, \qquad F(0) = 0, \tag{10.370}$$

$$\frac{dG}{d\eta} = H, \qquad G(0) = 0, \tag{10.371}$$

$$\frac{dH}{d\eta} = -\frac{1}{2}FH, \qquad H(0) = 1. \tag{10.372}$$

After one numerical integration, we find that with this guess for $H(0)$ that

$$G(\infty) = 2.08540918... \tag{10.373}$$

Now our numerical solution also gives us F, and so we know that $f = aF(a\eta)$ is also a solution. Moreover

$$\frac{df}{d\eta} = a^2\frac{dF(a\eta)}{d\eta}, \qquad \text{that is} \tag{10.374}$$

$$g(\eta) = a^2G(a\eta). \tag{10.375}$$

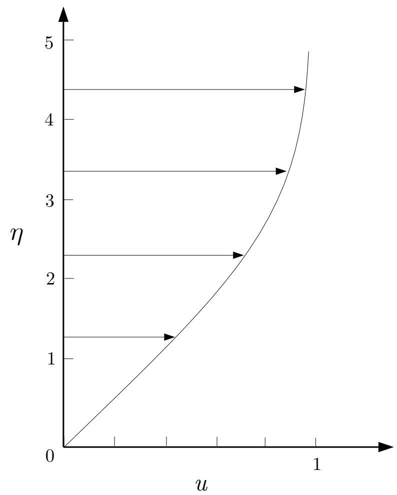

Figure 10.12: Velocity component profile for u for a Blasius boundary layer.

Now we want $g(\infty) = 1$, so take $1 = a^2 G(\infty)$, so $a^2 = 1/G(\infty)$. So

$$a = \frac{1}{\sqrt{G(\infty)}}. \tag{10.376}$$

Now

$$\frac{d^2 f}{d\eta^2} = a^3 \frac{d^2 F(a\eta)}{d\eta^2}, \tag{10.377}$$

$$\left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0} = a^3 \left.\frac{d^2 F(a\eta)}{d\eta^2}\right|_{\eta=0}, \tag{10.378}$$

$$h(0) = a^3 H(0), \tag{10.379}$$

$$= a^3(1), \tag{10.380}$$

$$= a^3 = G^{-3/2}(\infty), \tag{10.381}$$

$$= (2.08540918...)^{-3/2}, \tag{10.382}$$

$$= 0.332057335... \tag{10.383}$$

This is the proper choice for the initial condition on h. Numerically integrating once more, we get the behavior of f, g, and h as functions of η that indeed satisfies the condition at ∞. A plot of $u = df/d\eta$ as a function of η is shown in Fig. 10.12. From Fig. 10.12, we see that when $\eta = 5$, the velocity has nearly acquired the freestream value of $u = 1$. We can plot streamlines and the velocity vector field as well using the transformations to acquire ψ, u and $\tilde{v}$ as functions of x and $\tilde{y}$. They are plotted in in Fig. 10.13. Notice that the streamlines have curvature, consistent with the velocity vector having non-zero components in the x and $\tilde{y}$ directions.

The associated kinematic topics of acceleration and vorticity vector fields, deformation tensors, are not straightforward and need to be carefully interpreted in light of the fact that

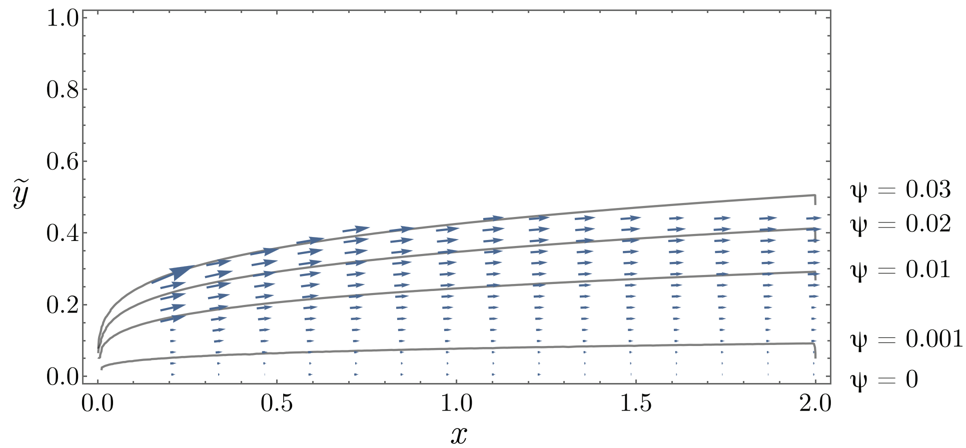

Figure 10.13: Streamlines and velocity vector field for Blasius boundary layer.

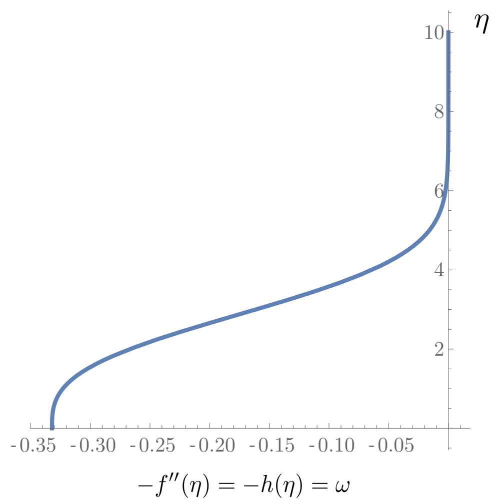

Figure 10.14: Dimensionless vorticity field for Blasius boundary layer.

we have scaled our equations in a particular fashion. Examination of the streamlines and velocity vector field of Fig. 10.13 suggests that the flow is decelerating in the streamwise direction, and that streamline curvature induces a stream-normal component of centripetal acceleration. The only forces available to induce such an acceleration are imbalanced viscous shear forces. Certainly one can visualize that a fluid element is both rotating as well as deforming in a volume-preserving fashion.

Panton, Ch. 20, shows that to leading order, the dimensionless vorticity field is given by

$$\omega = -\frac{d^2 f}{d\eta^2} = -h(\eta). \tag{10.384}$$

A plot is given in in Fig. 10.14. We note the vorticity is maximum at the no-slip boundary at $\eta = 0$. As $\eta \to \infty$, the flow becomes irrotational, consistent with the uniform freestream in the far field.

Examination of the numerical results shows that when $\eta = 4.9$, that the u component of velocity has 0.99 of its freestream value. As the velocity only reaches its freestream value at ∞, we define the *boundary layer thickness*, $\delta_{0.99}$, as that value of y_{dim} for which the velocity has 0.99 of its freestream value. Recalling that

$$\eta = \sqrt{\frac{U}{\nu}} \frac{y_{dim}}{\sqrt{x_{dim}}}, \tag{10.385}$$

we say that

$$4.9 = \sqrt{\frac{U}{\nu}} \frac{\delta_{0.99}}{\sqrt{x_{dim}}}. \tag{10.386}$$

Rearranging, we get

$$\frac{\delta_{0.99}}{x_{dim}} = 4.9 \sqrt{\frac{\nu}{U x_{dim}}}, \tag{10.387}$$

$$= 4.9 Re_{x_{dim}}^{-1/2}. \tag{10.388}$$

Here we have taken a Reynolds number based on local distance to be

$$Re_{x_{dim}} = \frac{U x_{dim}}{\nu}. \tag{10.389}$$

This formula is valid for laminar flows, and has been seen to be valid for $Re_{x_{dim}} < 3 \times 10^6$. For greater lengths, there can be a transition to turbulent flow. For water flowing a 1 m/s and a downstream distance of 1 m, we find $\delta_{0.99} = 0.5$ cm. For air under the same conditions, we find $\delta_{0.99} = 1.9$ cm. We also note that the boundary layer grows with the square root of distance along the plate. We further note that higher kinematic viscosity leads to thicker boundary layers, while lower kinematic viscosity lead to thinner boundary layers.

The velocity $\tilde{v}$ has some non-intuitive behavior. It is plotted in Fig. 10.15. As seen from its definition in Eq. (10.339), scaling by $2\sqrt{x}$ is required to capture the exclusive dependency on η. And the calculation reveals that

$$\lim_{\eta \to \infty} 2\sqrt{x}\; \tilde{v} = 1.72. \tag{10.390}$$

We can unravel the various scalings then to get the following as $\eta \to \infty$:

$$\sqrt{x}\; \tilde{v} = 0.86, \tag{10.391}$$

$$\sqrt{\frac{x_{dim}}{L}} \sqrt{Re}\; v = 0.86, \tag{10.392}$$

$$\sqrt{\frac{x_{dim}}{L}} \sqrt{\frac{UL}{\nu}} \frac{v_{dim}}{U} = 0.86, \tag{10.393}$$

$$\sqrt{\frac{U x_{dim}}{\nu}} \frac{v_{dim}}{U} = 0.86, \tag{10.394}$$

$$\frac{v_{dim}}{U} = 0.86 Re_{x_{dim}}^{-1/2}. \tag{10.395}$$

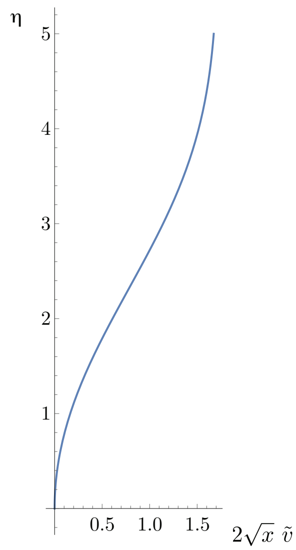

Figure 10.15: Velocity component profile for $2\sqrt{x}\ \tilde{v}$ in Blasius boundary layer.

We might expect our theory to force $v_{dim} \to 0$ as $y_{dim} \to \infty$. This is obviously not the case for our approximation, especially for small x_{dim}. The remedy is a complicated problem in asymptotic analysis as outlined by Van Dyke (1982), Ch. 7.

Now let us determine the shear stress at the wall, and the viscous force acting on the wall. So let us find

$$\tau_w = \mu \left. \frac{\partial u_{dim}}{\partial y_{dim}} \right|_{y_{dim}=0}. \tag{10.396}$$

Consider

$$\begin{aligned}
\frac{\partial u}{\partial \tilde{y}} &= \frac{\partial^2 \psi}{\partial \tilde{y}^2} = \frac{1}{\sqrt{x}} \frac{d^2 f}{d\eta^2}, && (10.397)\\
\frac{\partial \left(\frac{u_{dim}}{U}\right)}{\partial \left(\sqrt{\frac{UL}{\nu}} \frac{y_{dim}}{L}\right)} &= \frac{1}{\sqrt{\frac{x_{dim}}{L}}} \frac{d^2 f}{d\eta^2}, && (10.398)\\
\frac{\partial u_{dim}}{\partial y_{dim}} &= U \sqrt{\frac{\rho U}{\mu}} \frac{1}{\sqrt{x_{dim}}} \frac{d^2 f}{d\eta^2}, && (10.399)\\
\tau &= \mu \frac{\partial u_{dim}}{\partial y_{dim}} = U \sqrt{\frac{\rho U \mu}{x_{dim}}} \frac{d^2 f}{d\eta^2}, && (10.400)\\
\frac{\tau(0)}{\frac{1}{2}\rho U^2} &= C_f = 2\sqrt{\frac{\mu}{\rho U x_{dim}}} \frac{d^2 f}{d\eta^2}(0), && (10.401)\\
C_f &= 2 Re_{x_{dim}}^{-1/2} \frac{d^2 f}{d\eta^2}(0), && (10.402)\\
&= \frac{0.664...}{\sqrt{Re_{x_{dim}}}}. && (10.403)
\end{aligned}$$

We notice that at $x_{dim} = 0$ that the stress is infinite. This seeming problem is seen not to be one when we consider the actual viscous force on a finite length of plate. Consider a plate of length L and width b. Then the viscous force acting on the plate is

$$F = \int_0^L \tau \; dA, \tag{10.404}$$

$$= \int_0^L \tau(x_{dim}, 0) b \; dx_{dim}, \tag{10.405}$$

$$= b \int_0^L f''(0) U \sqrt{\rho U \mu} \frac{1}{\sqrt{x_{dim}}} \; dx_{dim}, \tag{10.406}$$

$$= b f''(0) U \sqrt{\rho U \mu} \int_0^L \frac{dx_{dim}}{\sqrt{x_{dim}}}, \tag{10.407}$$

$$= b f''(0) U \sqrt{\rho U \mu} \left(2 \sqrt{x_{dim}} \right)_0^L, \tag{10.408}$$

$$= 2 b f''(0) U \sqrt{\rho U \mu} \sqrt{L}, \tag{10.409}$$

$$\frac{F}{\frac{1}{2} \rho U^2 L b} = C_D = 4 f''(0) \sqrt{\frac{\mu}{\rho U L}} = 4 f''(0) Re_L^{-1/2} = 1.328 Re_L^{-1/2}. \tag{10.410}$$

Now let us consider the thermal boundary layer. Here we will take the boundary conditions so that the wall and far field are held at a constant fixed temperature $T_{dim} = T_o$. We need to do the scaling on the energy equation, so let us start with the steady incompressible two-dimensional dimensional energy equation:

$$\rho c_p \left(u_{dim} \frac{\partial T_{dim}}{\partial x_{dim}} + v_{dim} \frac{\partial T_{dim}}{\partial y_{dim}} \right) = k \left(\frac{\partial^2 T_{dim}}{\partial x_{dim}^2} + \frac{\partial^2 T_{dim}}{\partial y_{dim}^2} \right) \tag{10.411}$$

$$+ \mu \left(2 \left(\frac{\partial u_{dim}}{\partial x_{dim}} \right)^2 + 2 \left(\frac{\partial v_{dim}}{\partial y_{dim}} \right)^2 + \left(\frac{\partial u_{dim}}{\partial y_{dim}} + \frac{\partial v_{dim}}{\partial x_{dim}} \right)^2 \right).$$

Taking as before,

$$x = \frac{x_{dim}}{L}, \quad y = \frac{y_{dim}}{L}, \quad T = \frac{T_{dim} - T_o}{T_o}, \quad u = \frac{u_{dim}}{U}, \quad v = \frac{v_{dim}}{U}. \tag{10.412}$$

Making these substitutions, we get

$$\frac{\rho c_p U T_o}{L} \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{k T_o}{L^2} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \tag{10.413}$$

$$+ \frac{\mu U^2}{L^2} \left(2 \left(\frac{\partial u}{\partial x} \right)^2 + 2 \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right),$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho c_p U L} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \tag{10.414}$$

$$+\frac{\mu U}{\rho c_p L T_o}\left(2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right). \tag{10.415}$$

Now we have

$$\frac{k}{\rho c_p U L} = \frac{k}{c_p \mu}\frac{\mu}{\rho U L} = \frac{1}{Pr}\frac{1}{Re}, \tag{10.416}$$

$$\frac{\mu U}{\rho c_p L T_o} = \frac{\mu}{\rho U L}\frac{U^2}{c_p T_o} = \frac{Ec}{Re}. \tag{10.417}$$

So the dimensionless energy equation with boundary conditions can be written as

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = \frac{1}{PrRe}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{10.418}$$

$$+\frac{Ec}{Re}\left(2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right),$$

$$T(x,0) = 0, \qquad T(x,\infty)=0. \tag{10.419}$$

Now as $Re \to \infty$, we see that $T = 0$ is a solution that satisfies the energy equation and all boundary conditions. For finite Reynolds number, non-zero velocity gradients generate a temperature field. Once again, we rescale in the boundary layer using $\tilde{v} = \sqrt{Re}\, v$, and $\tilde{y} = \sqrt{Re}\, y$. This gives

$$u\frac{\partial T}{\partial x}+\frac{1}{\sqrt{Re}}\frac{1}{1/\sqrt{Re}}\tilde{v}\frac{\partial T}{\partial \tilde{y}} = \frac{1}{PrRe}\left(\frac{\partial^2 T}{\partial x^2}+Re\frac{\partial^2 T}{\partial \tilde{y}^2}\right) \tag{10.420}$$

$$+\frac{Ec}{Re}\left(2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial \tilde{v}}{\partial \tilde{y}}\right)^2+\left(\sqrt{Re}\frac{\partial u}{\partial \tilde{y}}+\frac{1}{\sqrt{Re}}\frac{\partial \tilde{v}}{\partial x}\right)^2\right).$$

$$u\frac{\partial T}{\partial x}+\tilde{v}\frac{\partial T}{\partial \tilde{y}} = \frac{1}{Pr}\left(\frac{1}{Re}\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial \tilde{y}^2}\right) \tag{10.421}$$

$$+Ec\left(\frac{2}{Re}\left(\frac{\partial u}{\partial x}\right)^2+\frac{2}{Re}\left(\frac{\partial \tilde{v}}{\partial \tilde{y}}\right)^2+\left(\frac{\partial u}{\partial \tilde{y}}+\frac{1}{Re}\frac{\partial \tilde{v}}{\partial x}\right)^2\right). \tag{10.422}$$

Now as $Re \to \infty$,

$$u\frac{\partial T}{\partial x}+\tilde{v}\frac{\partial T}{\partial \tilde{y}} = \frac{1}{Pr}\frac{\partial^2 T}{\partial \tilde{y}^2}+Ec\left(\frac{\partial u}{\partial \tilde{y}}\right)^2. \tag{10.423}$$

Now take $T = T(\eta)$ with $\eta = \tilde{y}/\sqrt{x}$ as well as $u = df/d\eta$, $\tilde{v} = (1/(2\sqrt{x}))\,(\eta(df/d\eta) - f)$ and $\partial u/\partial \tilde{y} = (1/\sqrt{x})(d^2 f/d\eta^2)$. We also have for derivatives, that

$$\frac{\partial T}{\partial x} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial x} = \frac{dT}{d\eta}\left(-\frac{1}{2}\frac{\eta}{x}\right), \tag{10.424}$$

$$\frac{\partial T}{\partial \tilde{y}} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial \tilde{y}} = \frac{dT}{d\eta}\frac{1}{\sqrt{x}}, \tag{10.425}$$

$$\frac{\partial^2 T}{\partial \tilde{y}^2} = \frac{\partial}{\partial \tilde{y}}\left(\frac{\partial T}{\partial \tilde{y}}\right) = \frac{\partial}{\partial \tilde{y}}\left(\frac{1}{\sqrt{x}}\frac{dT}{d\eta}\right) = \frac{1}{\sqrt{x}}\frac{\partial}{\partial \tilde{y}}\frac{dT}{d\eta} = \frac{1}{x}\frac{d^2T}{d\eta^2}. \tag{10.426}$$

The energy equation is then rendered as

$$\underbrace{\frac{df}{d\eta}}_{u}\underbrace{\left(-\frac{1}{2}\frac{\eta}{x}\frac{dT}{d\eta}\right)}_{\frac{\partial T}{\partial x}} + \underbrace{\frac{1}{2\sqrt{x}}\left(\eta\frac{df}{d\eta} - f\right)}_{\tilde{v}}\underbrace{\frac{1}{\sqrt{x}}\frac{dT}{d\eta}}_{\frac{\partial T}{\partial \tilde{y}}} = \underbrace{\frac{1}{Pr}\frac{1}{x}\frac{d^2T}{d\eta^2}}_{\frac{1}{Pr}\frac{\partial^2 T}{\partial \tilde{y}^2}} + \underbrace{\frac{Ec}{x}\left(\frac{d^2f}{d\eta^2}\right)^2}_{Ec\left(\frac{\partial u}{\partial \tilde{y}}\right)^2}, \tag{10.427}$$

$$-\frac{1}{2}\eta\frac{df}{d\eta}\frac{dT}{d\eta} + \frac{1}{2}\left(\eta\frac{df}{d\eta} - f\right)\frac{dT}{d\eta} = \frac{1}{Pr}\frac{d^2T}{d\eta^2} + Ec\left(\frac{d^2f}{d\eta^2}\right)^2, \tag{10.428}$$

$$\underbrace{-\frac{1}{2}f\frac{dT}{d\eta}}_{\text{advection}} = \underbrace{\frac{1}{Pr}\frac{d^2T}{d\eta^2}}_{\text{energy diffusion}} + \underbrace{Ec\left(\frac{d^2f}{d\eta^2}\right)^2}_{\text{dissipation}}, \tag{10.429}$$

$$\frac{d^2T}{d\eta^2} + \frac{1}{2}Pr\, f\frac{dT}{d\eta} = -Pr Ec\left(\frac{d^2f}{d\eta^2}\right)^2, \tag{10.430}$$

$$T(0) = 0, \qquad T(\infty) = 0. \tag{10.431}$$

Now for $Ec \to 0$, we get $T = 0$ as a solution that satisfies the governing differential equation and boundary conditions. Let us consider a solution for non-trivial Ec, but for $Pr = 1$. We could extend this for general values of Pr as well. Here, following Reynolds analogy, when thermal diffusivity equals momentum diffusivity, we expect the temperature field to be directly related to the velocity field. For $Pr = 1$, the energy equation reduces to

$$\underbrace{\frac{d^2T}{d\eta^2}}_{\text{energy diffusion}} + \underbrace{\frac{1}{2}f\frac{dT}{d\eta}}_{\text{advection}} = \underbrace{-Ec\left(\frac{d^2f}{d\eta^2}\right)^2}_{\text{dissipation}}, \tag{10.432}$$

$$T(0) = 0, \qquad T(\infty) = 0. \tag{10.433}$$

Here the integrating factor is

$$e^{\int_0^\eta \frac{1}{2}f(t)\ dt}. \tag{10.434}$$

Multiplying the energy equation by the integrating factor gives

$$e^{\int_0^\eta \frac{1}{2}f(t)\ dt}\frac{d^2T}{d\eta^2} + \frac{1}{2}fe^{\int_0^\eta \frac{1}{2}f(t)\ dt}\frac{dT}{d\eta} = -Ec\ e^{\int_0^\eta \frac{1}{2}f(t)\ dt}\left(\frac{d^2f}{d\eta^2}\right)^2, \tag{10.435}$$

$$\frac{d}{d\eta}\left(e^{\int_0^\eta \frac{1}{2}f(t)\ dt}\frac{dT}{d\eta}\right) = -Ec\ e^{\int_0^\eta \frac{1}{2}f(t)\ dt}\left(\frac{d^2f}{d\eta^2}\right)^2. \tag{10.436}$$

Now from the x momentum equation, $f''' + \frac{1}{2}ff'' = 0$, we have

$$f = -2\frac{f'''}{f''}. \tag{10.437}$$

So we can rewrite the integrating factor as

$$e^{\int_0^\eta \frac{1}{2}f(t)\ dt} = e^{\int_0^\eta \frac{1}{2}\frac{(-2)f'''}{f''}\ dt} = e^{-\ln\left(\frac{f''(\eta)}{f''(0)}\right)} = \frac{f''(0)}{f''(\eta)}. \tag{10.438}$$

So the energy equation can be written as

$$\begin{aligned}
\frac{d}{d\eta}\left(\frac{f''(0)}{f''(\eta)}\frac{dT}{d\eta}\right) &= -Ec\left(\frac{f''(0)}{f''(\eta)}\right)\left(\frac{d^2f}{d\eta^2}\right)^2, && (10.439)\\
&= -Ecf''(0)\frac{d^2f}{d\eta^2}, && (10.440)\\
\frac{f''(0)}{f''(\eta)}\frac{dT}{d\eta} &= -Ecf''(0)\int_0^\eta \frac{d^2f}{ds^2}\ ds + C_1, && (10.441)\\
\frac{dT}{d\eta} &= -Ec\frac{d^2f}{d\eta^2}\int_0^\eta \frac{d^2f}{ds^2}\ ds + C_1\frac{d^2f}{d\eta^2}, && (10.442)\\
&= = -Ec\frac{d^2f}{d\eta^2}\left(\frac{df}{d\eta} - \underbrace{f'(0)}_{=0}\right) + C_1\frac{d^2f}{d\eta^2}, && (10.443)\\
&= -Ec\frac{d^2f}{d\eta^2}\frac{df}{d\eta} + C_1\frac{d^2f}{d\eta^2}, && (10.444)\\
&= -Ec\frac{d}{d\eta}\left(\frac{1}{2}\left(\frac{df}{d\eta}\right)^2\right) + C_1\frac{d^2f}{d\eta^2}, && (10.445)\\
T &= -\frac{Ec}{2}\left(\frac{df}{d\eta}\right)^2 + C_1\frac{df}{d\eta} + C_2, && (10.446)\\
T(0) &= 0 = -\frac{Ec}{2}(\underbrace{f'(0)}_{=0})^2 + C_1\underbrace{f'(0)}_{=0} + C_2, && (10.447)\\
C_2 &= 0, && (10.448)\\
T(\infty) &= 0 = -\frac{Ec}{2}(\underbrace{f'(\infty)}_{=1})^2 + C_1\underbrace{f'(\infty)}_{=1}, && (10.449)\\
C_1 &= \frac{Ec}{2}, && (10.450)\\
T(\eta) &= \frac{Ec}{2}\frac{df}{d\eta}\left(1 - \frac{df}{d\eta}\right), && (10.451)\\
&= \frac{Ec}{2}u(\eta)(1 - u(\eta)). && (10.452)
\end{aligned}$$

A plot of the temperature profile for $Pr = 1$ and $Ec = 1$ is given in Fig. 10.16.

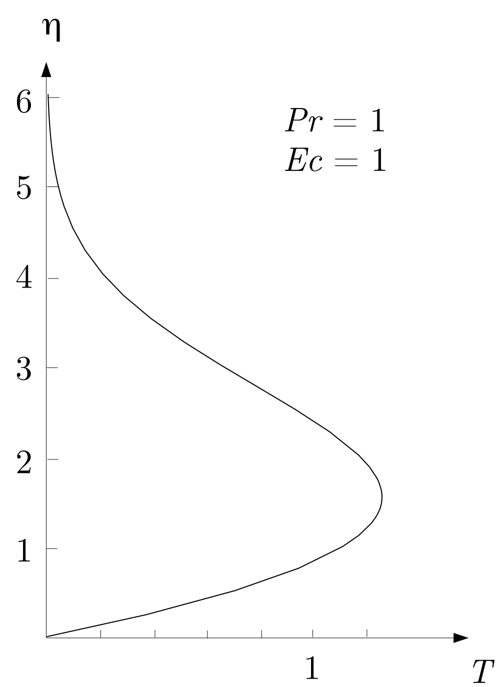

Figure 10.16: Temperature profile for Blasius boundary layer, $Ec = 1$, $Pr = 1$.

Bibliography

This bibliography focuses on books that are closely related to the material presented in this course in classical fluid mechanics, especially with regard to graduate level treatment of continuum mechanical principles applied to fluids, compressible flow, viscous flow, and vortex dynamics. It also has some general works of historic importance. It is by no means a comprehensive survey of works on fluid mechanics. Only a few works are given here that focus on such important topics as low Reynolds number flows, turbulence, bio-fluids, computational fluid dynamics, microfluids, molecular dynamics, magneto-hydrodynamics, geo-physical flows, rheology, astrophysical flows, as well as elementary undergraduate texts. That said, those that are listed are among the best that exist and would be useful to examine.

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