CONTINUUM MECHANICS

(Lecture Notes)

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Preface

This text is suitable for a two-semester course on *Continuum Mechanics*. It is based on notes from undergraduate courses that I have taught over the last decade. The material is intended for use by undergraduate students of physics with a year or more of college calculus behind them.

I would like to thank Erik Grafarend, Ctirad Matyska, Detlef Wolf and Jiří Zahradník, whose interest encouraged me to write this text. I would also like to thank my oldest son Zdeněk who plotted most of figures embedded in the text. I am grateful to many students for helping me to reveal typing misprints. I would like to acknowledge my indebtedness to Kevin Fleming, whose through proofreading of the entire text is very much appreciated.

Readers of this text are encouraged to contact me with their comments, suggestions, and questions. I would be very happy to hear what you think I did well and I could do better. My e-mail address is *zm@karel.troja.mff.cuni.cz* and a full mailing address is found on the title page.

Zdeněk Martinec

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Notation

(not completed yet)

b	entropy source per unit mass
$\vec{f}$	body force per unit mass
$\vec{F}$	Lagrangian description of $\vec{f}$
h	heat source per unit mass
$\vec{n}$	normal at κ
$\vec{N}$	normal at κ_0
$\vec{q}$	heat flux at κ
$\vec{Q}$	heat flux at κ_0
s	bounding surface at κ
S	bounding surface at κ_0
$\vec{s}$	flux of entropy at κ
$\vec{S}$	flux of entropy at κ_0
t	time
$\vec{t}_{(\vec{n})}$	stress vector on a surface with the external normal $\vec{n}$
$\boldsymbol{t}(\vec{x},t)$	Eulerian Cauchy stress tensor
$\boldsymbol{t}(\vec{X},t)$	Lagrangian Cauchy stress tensor
$\boldsymbol{T}^{(1)}$	first Piola-Kirchhoff stress tensor
$\boldsymbol{T}^{(2)}$	second Piola-Kirchhoff stress tensor
v	volume at κ
V	volume at κ_0
$\vec{v}$	velocity at κ
$\vec{V}$	Lagrangian description of $\vec{v}$
$\vec{x}$	Eulerian Cartesian coordinates
$\vec{X}$	Lagrangian Cartesian coordinates
$\mathcal{E}$	total internal energy
$\mathcal{H}$	total entropy
$\mathcal{K}$	total kinetic energy
$\mathcal{W}$	total mechanical power
ε	internal energy density per unit mass
η	entropy density per unit mass
ϱ	mass density at κ
$\boldsymbol{\varrho}$	Lagrangian description of ϱ
ϱ_0	mass density at κ_0
κ	present configuration
κ_0	reference configuration
σ	singular surface at κ
Σ	singular surface at κ_0
θ	temperature
$\vec{\nu}$	the speed of singular surface σ
Γ	total entropy production

1. GEOMETRY OF DEFORMATION

1.1 Body, configurations, and motion

The subject of all studies in continuum mechanics, and the domain of all physical quantities, is the material body. A *material body* $\mathcal{B} = \{\mathcal{X}\}$ is a compact measurable set of an infinite number of material elements $\mathcal{X}$, called the *material particles* or *material points*, that can be placed in a one-to-one correspondence with triplets of real numbers. Such triplets are sometimes called the intrinsic coordinates of the particles. Note that whereas a "particle" in classical mechanics has an assigned mass, a "continuum particle" is essentially a material point for which a density is defined.

A material body $\mathcal{B}$ is available to us only by its configuration. The *configuration* κ of $\mathcal{B}$ is the specification of the position of all particles of $\mathcal{B}$ in the physical space E^3 (usually the Euclidean space). Often it is convenient to select one particular configuration, the *reference configuration* κ_0, and refer everything concerning the body to that configuration. Mathematically, the definition of the reference configuration κ_0 is expressed by mapping

$$\begin{aligned} \vec{\gamma}_0 : \quad & \mathcal{B} \to \mathrm{E}^3 \\ & \mathcal{X} \to \vec{X} = \vec{\gamma}_0(\mathcal{X}) \ , \end{aligned} \tag{1.1}$$

where $\vec{X}$ is the position occupied by the particle $\mathcal{X}$ in the reference configuration κ_0, as shown in Figure 1.1.

The choice of reference configuration is arbitrary. It may be any smooth image of the body $\mathcal{B}$, and need not even be a configuration ever occupied by the body. For some choice of κ_0, we may obtain a relatively simple description, just as in geometry one choice of coordinates may lead to a simple equation for a particular figure. However, the reference configuration itself has nothing to do with the motion it is used to describe, just as the coordinate system has nothing to do with geometrical figures themselves. A reference configuration is introduced so as to allow us to employ the mathematical apparatus of Euclidean geometry.

Under the influence of external loads, the body $\mathcal{B}$ deforms, moves and changes its configuration. The configuration of body $\mathcal{B}$ at the present time t is called the *present configuration* κ_t and is defined by mapping

$$\begin{aligned} \vec{\gamma}_t : \quad & \mathcal{B} \to \mathrm{E}^3 \\ & \mathcal{X} \to \vec{x} = \vec{\gamma}_t(\mathcal{X}, t) \ , \end{aligned} \tag{1.2}$$

where $\vec{x}$ is the position occupied by the particle $\mathcal{X}$ in the present configuration κ_t.

A *motion* of body $\mathcal{B}$ is a sequence of mappings $\vec{\chi}$ between the reference configuration κ_0 and the present configuration κ_t:

$$\begin{aligned} \vec{\chi} : \quad & \mathrm{E}^3 \to \mathrm{E}^3 \\ & \vec{X} \to \vec{x} = \vec{\chi}(\vec{X}, t) \ . \end{aligned} \tag{1.3}$$

This equation states that the motion takes a material point $\mathcal{X}$ from its position $\vec{X}$ in the reference configuration κ_0 to a position $\vec{x}$ in the present configuration κ_t. We assume that the motion $\vec{\chi}$ is continuously differentiable in finite regions of the body or in the entire body so that the mapping (1.3) is invertible such that

$$\vec{X} = \vec{\chi}^{-1}(\vec{x}, t) \tag{1.4}$$

holds.

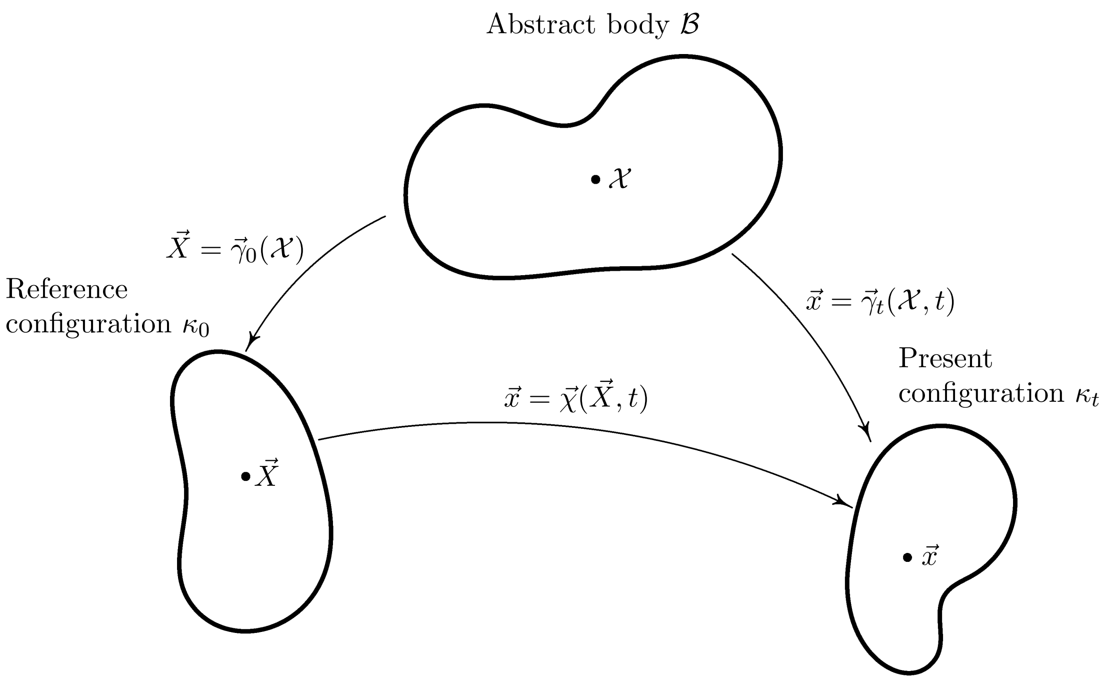

Figure 1.1. Body, its reference and present configurations.

The functional form for a given motion as described by (1.3) depends on the choice of the reference configuration. If more than one reference configuration is used in the discussion, it is necessary to label the function accordingly. For example, relative to two reference configurations κ_1 and κ_2 the same motion could be represented symbolically by the two equations $\vec{x} = \vec{\chi}_{\kappa_1}(\vec{X}, t) = \vec{\chi}_{\kappa_2}(\vec{X}, t)$.

It is often convenient to change the reference configuration in the description of motion. To see how the motion is described in a new reference configuration, consider two different configurations κ_τ and κ_t of body $\mathcal{B}$ at two different times τ and t:

$$\vec{\xi} = \vec{\chi}(\vec{X}, \tau) \ , \qquad \vec{x} = \vec{\chi}(\vec{X}, t) \ , \tag{1.5}$$

that is, $\vec{\xi}$ is the place occupied at time τ by the particle that occupies $\vec{x}$ at time t. Since the function $\vec{\chi}$ is invertible, that is,

$$\vec{X} = \vec{\chi}^{\,-1}(\vec{\xi}, \tau) = \vec{\chi}^{\,-1}(\vec{x}, t) \ , \tag{1.6}$$

we have either

$$\vec{\xi} = \vec{\chi}(\vec{\chi}^{\,-1}(\vec{x}, t), \tau) =: \vec{\chi}_t(\vec{x}, \tau) \ , \tag{1.7}$$

or

$$\vec{x} = \vec{\chi}(\vec{\chi}^{\,-1}(\vec{\xi}, \tau), t) =: \vec{\chi}_\tau(\vec{\xi}, t) \ . \tag{1.8}$$

The map $\vec{\chi}_t(\vec{x}, \tau)$ defines the deformation of the new configuration κ_τ of the body $\mathcal{B}$ relative to the present configuration κ_t, which is considered as reference. On the other hand, the map $\vec{\chi}_\tau(\vec{\xi}, t)$ defines the deformation of the new reference configuration κ_τ of the body $\mathcal{B}$ onto the configuration κ_t. Evidently, it holds

$$\vec{\chi}_t(\vec{x}, \tau) = \vec{\chi}_\tau^{\,-1}(\vec{x}, t) \ . \tag{1.9}$$

The functions $\vec{\chi}_t(\vec{x}, \tau)$ and $\vec{\chi}_\tau(\vec{\xi}, t)$ are called the *relative motion functions*. The subscripts t and τ at functions χ are used to recall which configuration is taken as reference.

We assume that the functions $\vec{\gamma}_0$, $\vec{\gamma}_t$, $\vec{\chi}$, $\vec{\chi}_t$ and $\vec{\chi}_\tau$ are single-valued and possess continuous partial derivatives with respect to their arguments for whatever order is desired, except possibly at some singular points, curves, and surfaces. Moreover, each of these function can uniquely be inverted. This assumption is known as the *axiom of continuity*, which refers to the fact that the matter is *indestructible*. This means that a finite volume of matter cannot be deformed into a zero or infinite volume. Another implication of this axiom is that the matter is *impenetrable*, that is, one portion of matter never penetrates into another. In other words, a motion caries every volume into a volume, every surface onto a surface, and every curve onto a curve. In practice, there are cases where this axiom is violated. We cannot describe such processes as the creation of new material surfaces, the cutting, tearing, or the propagation of cracks, etc. Continuum theories dealing with such processes must weaken the continuity assumption in the neighborhood of those parts of the body, where the map (1.3) becomes discontinuous. The axiom of continuity is mathematically ensured by the well-known implicit function theorem.

1.2 Description of motion

Motion can be described in four ways. Under the assumption that the functions $\vec{\kappa}_0$, $\vec{\kappa}_t$, $\vec{\chi}$, $\vec{\chi}_t$ and $\vec{\chi}_\tau$ are differentiable and invertible, all descriptions are equivalent. We refer to them by the following:

- **Material description**, given by the mapping (1.2), whose independent variables are the abstract particle $\mathcal{X}$ and the time t.
- **Referential description**, given by the mapping (1.3), whose independent variables are the position $\vec{X}$ of the particle $\mathcal{X}$ in an arbitrarily chosen reference configuration, and the time t. When the reference configuration is chosen to be the actual initial configuration at $t = 0$, the referential description is often called the **Lagrangian description**, although many authors call it the material description, using the particle position $\vec{X}$ in the reference configuration as a label for the material particle $\mathcal{X}$ in the material description.
- **Spatial description**, whose independent variables are the present position $\vec{x}$ occupied by the particle at the time t and the present time t. It is the description most used in fluid mechanics, often called the **Eulerian description**.
- **Relative description**, given by the mapping (1.7), whose independent variables are the present position $\vec{x}$ of the particle and a variable time τ, being the time when the particle occupied another position $\vec{\xi}$. The motion is then described with $\vec{\xi}$ as a dependent variable. Alternatively, the motion can be described by the mapping (1.8), whose independent variables are the position $\vec{\xi}$ at time τ and the present time t. These two relative descriptions are actually special cases of the referential description, differing from the Lagrangian description in that the reference positions are now denoted by $\vec{x}$ at time t and $\vec{\xi}$ at time τ, respectively, instead of $\vec{X}$ at time $t = 0$.

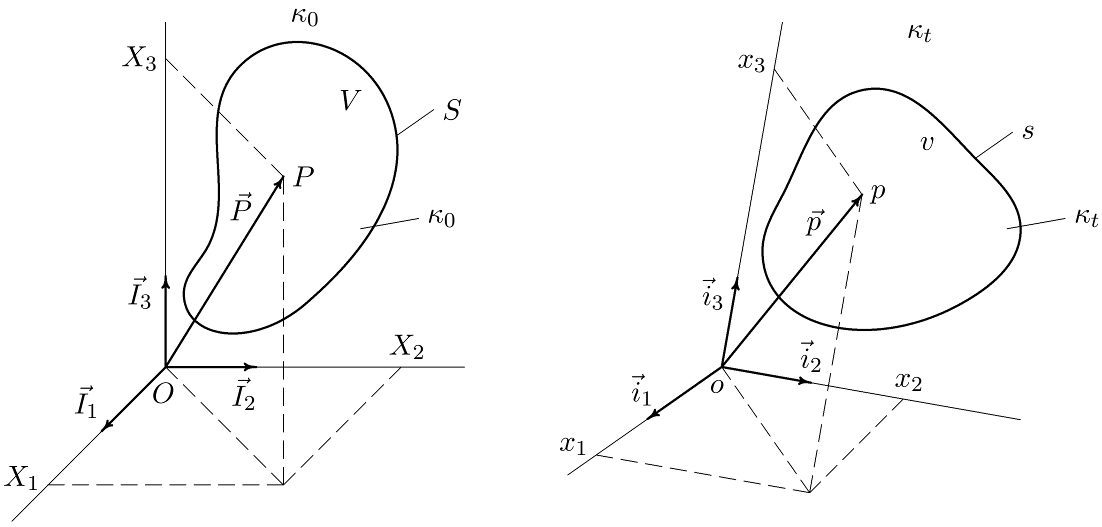

Figure 1.2. Lagrangian and Eulerian coordinates for the reference configuration κ_0 and the present configuration κ_t.

1.3 Lagrangian and Eulerian coordinates

Consider now that the position $\vec{X}$ of the particle $\mathcal{X}$ in the reference configuration κ_0 may be assigned by three Cartesian coordinates X_K, $K = 1, 2, 3$, or by a *position vector* $\vec{P}$ that extends from the origin O of the coordinates to the point P, the place of the particle $\mathcal{X}$ in the reference configuration, as shown in Figure 1.2. The notations $\vec{X}$ and $\vec{P}$ are interchangeable if we use only one reference origin for the position vector $\vec{P}$, but both "place" and "particle" have meaning independent of the choice of origin. The same position $\vec{X}$ (or place P) may have many different position vectors $\vec{P}$ corresponding to different choices of origin, but the relative position vectors $d\vec{X} = d\vec{P}$ of neighboring positions will be the same for all origins.

At the present configuration κ_t, a material particle $\mathcal{X}$ occupies the position $\vec{x}$ or a spatial place p. We may locate place p by a position vector $\vec{p}$ extending from the origin o of a new set of rectangular coordinates x_k, $k = 1, 2, 3$. Following the current terminology, we shall call X_K the *Lagrangian* or *material* coordinates and x_k the *Eulerian* or *spatial* coordinates. In following considerations, we assume that these two coordinate systems, one for the reference configuration κ_0 and one for the present configuration κ_t, are nonidentical.

The reference position $\vec{X}$ of a point P in κ_0 and the present position $\vec{x}$ of p in κ_t, respectively, when referred to the Cartesian coordinates X_K and x_k are given by

$$\vec{X} = X_K \vec{I}_K, \qquad \vec{x} = x_k \vec{i}_k \ . \tag{1.10}$$

where $\vec{I}_K$ and $\vec{i}_k$ are the respective *unit base vectors* in Figure 1.2. (The usual summation convention over repeated indices is employed.) Since Cartesian coordinates are employed, the base vectors are mutually orthogonal,

$$\vec{I}_K \cdot \vec{I}_L = \delta_{KL} \ , \qquad \vec{i}_k \cdot \vec{i}_l = \delta_{kl} \ , \tag{1.11}$$

where δ_{KL} and δ_{kl} are the *Kronecker symbols*, which are equal to 1 when the two indices are equal and zero otherwise; the dot '$\cdot$' stands for the scalar product of vectors. [1]

When the two Cartesian coordinates are not identical, we shall also express a unit base vector in one coordinates in terms of its projection into another coordinates. It can be readily shown that

$$\vec{i}_k = \delta_{kK}\vec{I}_K \ , \qquad \vec{I}_K = \delta_{Kk}\vec{i}_k \ , \tag{1.12}$$

where

$$\delta_{kK} := \vec{i}_k \cdot \vec{I}_K \ , \qquad \delta_{Kk} := \vec{I}_K \cdot \vec{i}_k \tag{1.13}$$

are called *shifters.* They are **not** a Kronecker symbol except when the two coordinate systems are identical. It is clear that (1.13) is simply the cosine directors of the two coordinates x_k and X_K. From the identity

$$\delta_{kl}\,\vec{i}_l = \vec{i}_k = \delta_{kK}\vec{I}_K = \delta_{kK}\delta_{Kl}\,\vec{i}_l \ ,$$

we find that

$$\delta_{kK}\delta_{Kl} = \delta_{kl} \ , \qquad \delta_{Kk}\delta_{kL} = \delta_{KL} \ . \tag{1.14}$$

The notation convention will be such that the quantities associated with the reference configuration κ_0 will be denoted by capital letters, and the quantities associated with the present configuration κ_t by lower case letters. When these quantities are referred to coordinates X_K, their indices will be majuscules; and when they are referred to x_k, their indices will be minuscules. For example, a vector $\vec{V}$ in κ_0 referred to X_K will have the components V_K, while when it is referred to x_k will have the components V_k, such that

$$V_K = \vec{V} \cdot \vec{I}_K \ , \qquad V_k = \vec{V} \cdot \vec{i}_k \ . \tag{1.15}$$

Using (1.12), the components V_K and V_k can be related by

$$V_K = V_k\delta_{kK} \ , \qquad V_k = V_K\delta_{Kk} \ . \tag{1.16}$$

Conversely, considering vector $\vec{v}$ in κ_t that, in general, differs from $\vec{V}$, its components v_K and v_k referred to X_K and x_k, respectively, are

$$v_K = \vec{v} \cdot \vec{I}_K \ , \qquad v_k = \vec{v} \cdot \vec{i}_k \ . \tag{1.17}$$

Again, using (1.12), the components v_K and v_k can be related by

$$v_K = v_k\delta_{kK} \ , \qquad v_k = v_K\delta_{Kk} \ . \tag{1.18}$$

1.4 Lagrangian and Eulerian variables

Every scalar, vector or tensor physical quantity $\mathcal{Q}$ defined for the body $\mathcal{B}$ such as density, temperature, or velocity is defined with respect to a particle $\mathcal{X}$ at a certain time t as

$$\mathcal{Q} = \hat{\mathcal{Q}}(\mathcal{X}, t) \ . \tag{1.19}$$

[1] If a curvilinear coordinate system is employed, the appropriate form of these equations can be obtained by the standard transformation rules. For example, the partial derivatives in Cartesian coordinates must be replaced by the partial covariant derivatives. However, for general considerations, we shall rely on the already introduced Cartesian systems.

Since the particle $\mathcal{X}$ is available to us in the reference or present configurations, the physical quantity $\mathcal{Q}$ is always considered a function of the position of the particle $\mathcal{X}$ in the reference or present configurations. Assuming that the function (1.2) is invertible, that is $\mathcal{X} = \vec{\gamma}_0^{-1}(\vec{X})$, the Lagrangian representation of quantity $\mathcal{Q}$ is

$$\mathcal{Q} = \hat{\mathcal{Q}}(\mathcal{X}, t) = \hat{\mathcal{Q}}(\vec{\gamma}_0^{-1}(\vec{X}), t) =: Q(\vec{X}, t). \tag{1.20}$$

Alternatively, inverting (1.3), that is $\mathcal{X} = \vec{\gamma}_t^{-1}(\vec{x}, t)$, the Eulerian representation of quantity $\mathcal{Q}$ is

$$\mathcal{Q} = \hat{\mathcal{Q}}(\mathcal{X}, t) = \hat{\mathcal{Q}}(\vec{\gamma}_t^{-1}(\vec{x}, t), t) =: q(\vec{x}, t), \tag{1.21}$$

We can see that the *Lagrangian* and *Eulerian variables* $Q(\vec{X}, t)$ and $q(\vec{x}, t)$ are referred to the reference configuration κ_0 and the present configuration κ_t of the body $\mathcal{B}$, respectively. In the Lagrangian description, attention is focused on what is happening to the individual particles during the motion, whereas in the Eulerian description the emphasis is directed to the events taking place at specific points in space. For example, if $\mathcal{Q}$ is temperature, then $Q(\vec{X}, t)$ gives the temperature recorded by a thermometer attached to a moving particle $\vec{X}$, whereas $q(\vec{x}, t)$ gives the temperature recorded at a fixed point $\vec{x}$ in space. The relationship between these two descriptions is

$$Q(\vec{X}, t) = q(\vec{\chi}(\vec{X}, t), t) \ , \qquad q(\vec{x}, t) = Q(\vec{\chi}^{-1}(\vec{x}, t), t) \ , \tag{1.22}$$

where the small and capital letters emphasize different functional forms resulting from the change in variables.

As an example, we define the Lagrangian and Eulerian variables for a vector quantity $\vec{\mathcal{V}}$. Let us assume that $\vec{\mathcal{V}}$ in the Eulerian description is given by

$$\vec{\mathcal{V}} \equiv \vec{v}(\vec{x}, t) \ . \tag{1.23}$$

Vector $\vec{v}$ may be expressed in the Lagrangian or Eulerian components $v_K(\vec{x}, t)$ and $v_k(\vec{x}, t)$ as

$$\vec{v}(\vec{x}, t) = v_K(\vec{x}, t)\vec{I}_K = v_k(\vec{x}, t)\vec{i}_k \ . \tag{1.24}$$

The Lagrangian description of $\vec{\mathcal{V}}$, that is the vector $\vec{V}(\vec{X}, t)$, is defined by $(1.22)_1$:

$$\vec{V}(\vec{X}, t) := \vec{v}(\vec{\chi}(\vec{X}, t), t) \ . \tag{1.25}$$

Representing $\vec{V}(\vec{X}, t)$ in the Lagrangian or Eulerian components $V_K(\vec{X}, t)$ and $V_k(\vec{X}, t)$,

$$\vec{V}(\vec{X}, t) = V_K(\vec{X}, t)\vec{I}_K = V_k(\vec{X}, t)\vec{i}_k \ , \tag{1.26}$$

the definition (1.25) can be interpreted in two possible component forms:

$$V_K(\vec{X}, t) := v_K(\vec{\chi}(\vec{X}, t), t) \ , \qquad V_k(\vec{X}, t) := v_k(\vec{\chi}(\vec{X}, t), t) \ . \tag{1.27}$$

Expressing the Lagrangian components v_K in terms of the Eulerian components v_k according to (1.18) results in

$$V_K(\vec{X}, t) = v_k(\vec{\chi}(\vec{X}, t), t)\delta_{kK} \ , \qquad V_k(\vec{X}, t) = v_K(\vec{\chi}(\vec{X}, t), t)\delta_{Kk} \ . \tag{1.28}$$

An analogous consideration may be carried out for the Eulerian variables $v_K(\vec{x}, t)$ and $v_k(\vec{x}, t)$ in the case where $\vec{\mathcal{V}}$ is given in the Lagrangian description $\vec{\mathcal{V}} \equiv \vec{V}(\vec{X}, t)$.

1.5 Deformation gradient

The coordinate form of the motion (1.3) is

$$x_k = \chi_k(X_1, X_2, X_3, t) \ , \qquad k = 1, 2, 3 \ , \tag{1.29}$$

or, conversely,

$$X_K = \chi_K^{-1}(x_1, x_2, x_3, t) \ , \qquad K = 1, 2, 3 \ . \tag{1.30}$$

According to the implicit function theorem, the mathematical condition that guarantees the existence of such an unique inversion is the non-vanishing of the jacobian determinant J, that is,

$$J(\vec{X}, t) := \det\left(\frac{\partial \chi_k}{\partial X_K}\right) \neq 0 \ . \tag{1.31}$$

The differentials of (1.29) and (1.30), at a fixed time, are

$$dx_k = \chi_{k,K}\, dX_K \ , \qquad dX_K = \chi_{K,k}^{-1}\, dx_k \ , \tag{1.32}$$

where indices following a comma represent partial differentiation with respect to X_K, when they are majuscules, and with respect to x_k when they are minuscules, that is,

$$\chi_{k,K} := \frac{\partial \chi_k}{\partial X_K} \ , \qquad \chi_{K,k}^{-1} := \frac{\partial \chi_K^{-1}}{\partial x_k} \ . \tag{1.33}$$

The two sets of quantities defined by (1.33) are components of the *material and spatial deformation gradient tensors* $\boldsymbol{F}$ and $\boldsymbol{F}^{-1}$, respectively,

$$\boldsymbol{F}(\vec{X}, t) := \chi_{k,K}(\vec{X}, t)(\vec{i}_k \otimes \vec{I}_K) \ , \qquad \boldsymbol{F}^{-1}(\vec{x}, t) := \chi_{K,k}^{-1}(\vec{x}, t)(\vec{I}_K \otimes \vec{i}_k) \ , \tag{1.34}$$

where the symbol $\otimes$ denotes the dyadic product of vectors. Alternatively, (1.34) may be written in symbolic notation as [2]

$$\boldsymbol{F}(\vec{X}, t) := (\mathrm{Grad}\, \vec{\chi})^T \ , \qquad \boldsymbol{F}^{-1}(\vec{x}, t) := (\mathrm{grad}\, \vec{\chi}^{-1})^T \ . \tag{1.35}$$

The deformation gradients $\boldsymbol{F}$ and $\boldsymbol{F}^{-1}$ are two-point tensor fields because they relate a vector $d\vec{x}$ in the present configuration to a vector $d\vec{X}$ in the reference configuration. Their components transform like those of a vector under rotations of only one of two reference axes and like a two-point tensor when the two sets of axes are rotated independently. In symbolic notation, equation (1.32) appears in the form

$$d\vec{x} = dx_k \vec{i}_k = \boldsymbol{F} \cdot d\vec{X} \ , \qquad d\vec{X} = dX_K \vec{I}_K = \boldsymbol{F}^{-1} \cdot d\vec{x} \ . \tag{1.36}$$

The material deformation gradient $\boldsymbol{F}$ can thus be thought of as a mapping of the infinitesimal vector $d\vec{X}$ of the reference configuration onto the infinitesimal vector $d\vec{x}$ of the current configuration; the inverse mapping is performed by the spatial deformation gradient $\boldsymbol{F}^{-1}$.

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[2] The nabla operator $\vec{\nabla}$ is defined as $\vec{\nabla} := \vec{i}_k \dfrac{\partial}{\partial x_k}$. With this operator, the gradient of a vector function $\vec{\phi}$ is defined by the left dyadic product of the nabla operator $\vec{\nabla}$ with $\vec{\phi}$, that is, $\mathrm{grad}\, \vec{\phi} := \vec{\nabla} \otimes \vec{\phi}$. Moreover, the gradients "Grad" and "grad" denote the gradient operator with respect to material and spatial coordinates, respectively, the time t being held constant in each case. Hence, $\boldsymbol{F} = (\vec{\nabla} \otimes \vec{\chi})^T$.

Through the chain rule of partial differentiation it is clear that

$$\chi_{k,K}\chi^{-1}_{K,l} = \delta_{kl} \ , \qquad \chi^{-1}_{K,k}\chi_{k,L} = \delta_{KL} \ , \tag{1.37}$$

or, when written in symbolic notation:

$$\boldsymbol{F}\cdot\boldsymbol{F}^{-1} = \boldsymbol{F}^{-1}\cdot\boldsymbol{F} = \boldsymbol{I} \ , \tag{1.38}$$

where $\boldsymbol{I}$ is the identity tensor. Strictly speaking, there are two identity tensors, one in the Lagrangian coordinates and one in the Eulerian coordinates. However, we shall disregard this subtlety. Equation (1.38) shows that the spatial deformation gradient $\boldsymbol{F}^{-1}$ is the inverse tensor of the material deformation gradient $\boldsymbol{F}$. Each of the two sets of equations (1.37) consists of nine linear equations for the nine unknown $\chi_{k,K}$ or $\chi^{-1}_{K,k}$. Since the jacobian is assumed not to vanish, a unique solution exists and, according to Cramer's rule of determinants, the solution for $\chi^{-1}_{K,k}$ may be written in terms of $\chi_{k,K}$ as

$$\chi^{-1}_{K,k} = \frac{\text{cofactor}\,(\chi_{k,K})}{J} = \frac{1}{2J}\epsilon_{KLM}\epsilon_{klm}\chi_{l,L}\chi_{m,M} \ , \tag{1.39}$$

where ϵ_{KLM} and ϵ_{klm} are the Levi-Cività alternating symbols, and

$$J := \det\,(\chi_{k,K}) = \frac{1}{3!}\epsilon_{KLM}\epsilon_{klm}\chi_{k,K}\chi_{l,L}\chi_{m,M} = \det\boldsymbol{F} \ . \tag{1.40}$$

Note that the jacobian J is identical to the determinant of tensor $\boldsymbol{F}$ only in the case that both the Lagrangian and Eulerian coordinates are of the same type, as in the case here when both are Cartesian coordinates. If the Lagrangian coordinates are of a different type to the Eulerian coordinates, the volume element in these coordinates is not the same and, consequently, the jacobian J will differ from the determinant of $\boldsymbol{F}$.

By differentiating (1.39) and (1.40), we get the following Jacobi identities:

$$\begin{aligned} \frac{d\,J}{d\,\chi_{k,K}} &= J\chi^{-1}_{K,k} \ , \\ (J\chi^{-1}_{K,k})_{,K} &= 0 \ , \qquad \text{or} \qquad (J^{-1}\chi_{k,K})_{,k} = 0 \ . \end{aligned} \tag{1.41}$$

The first identity is proved as follows:

$$\begin{aligned} \frac{d\,J}{d\,\chi_{r,R}} &= \frac{1}{3!}\epsilon_{KLM}\epsilon_{klm}\left[\frac{\partial\chi_{k,K}}{\partial\chi_{r,R}}\chi_{l,L}\chi_{m,M} + \chi_{k,K}\frac{\partial\chi_{l,L}}{\partial\chi_{r,R}}\chi_{m,M} + \chi_{k,K}\chi_{l,L}\frac{\partial\chi_{m,M}}{\partial\chi_{r,R}}\right] \\ &= \frac{1}{3!}\left[\epsilon_{RLM}\epsilon_{rlm}\chi_{l,L}\chi_{m,M} + \epsilon_{KRM}\epsilon_{krm}\chi_{k,K}\chi_{m,M} + \epsilon_{KLR}\epsilon_{klr}\chi_{k,K}\chi_{l,L}\right] \\ &= \frac{1}{2}\epsilon_{RLM}\epsilon_{rlm}\chi_{l,L}\chi_{m,M} = \text{cofactor}\,(\chi_{r,R}) = J\chi^{-1}_{R,r} \ . \end{aligned}$$

Furthermore, differentiating this result with respect to X_K yields

$$\begin{aligned} (J\chi^{-1}_{K,k})_{,K} &= \frac{1}{2}\epsilon_{KLM}\epsilon_{klm}(\chi_{l,LK}\chi_{m,M} + \chi_{l,L}\chi_{m,MK}) \\ &= \epsilon_{KLM}\epsilon_{klm}\chi_{l,LK}\chi_{m,M} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\epsilon_{klm}\chi_{m,M}\Big(\epsilon_{KLM}\chi_{l,LK} + \epsilon_{KLM}\chi_{l,LK}\Big)\\
&= \frac{1}{2}\epsilon_{klm}\chi_{m,M}\Big(\epsilon_{KLM}\chi_{l,LK} - \epsilon_{LKM}\chi_{l,LK}\Big)\\
&= \frac{1}{2}\epsilon_{klm}\chi_{m,M}\Big(\epsilon_{KLM}\chi_{l,LK} - \epsilon_{KLM}\chi_{l,KL}\Big)\\
&= 0\ ,
\end{aligned}
$$

hence proving the second identity in (1.41). In the last step, we have assumed that the order of differentiation with respect to X_K and X_L can be interchanged.

The Jacobi identities can be expressed in symbolic notation: [3]

$$
\begin{aligned}
&\frac{dJ}{d\boldsymbol{F}} = J\boldsymbol{F}^{-T}\ ,\\
&\mathrm{Div}\,(J\boldsymbol{F}^{-1}) = \vec{0}\ , \qquad \text{or} \qquad \mathrm{div}\,(J^{-1}\boldsymbol{F}) = \vec{0}\ .
\end{aligned}
\tag{1.42}
$$

Furthermore, with the help of the basic properties of the Levi-Cività alternating symbols [4], equations (1.39) and (1.40) may be rewritten in the following form:

$$
\begin{aligned}
J\epsilon_{KLM}\chi^{-1}_{K,k} &= \epsilon_{klm}\chi_{l,L}\chi_{m,M}\ ,\\
J\epsilon_{KLM} &= \epsilon_{klm}\chi_{k,K}\chi_{l,L}\chi_{m,M}\ ,
\end{aligned}
\tag{1.43}
$$

or, symbolically,

$$
\begin{aligned}
J\boldsymbol{F}^{-T}\cdot(\vec{A}\times\vec{B}) &= (\boldsymbol{F}\cdot\vec{A})\times(\boldsymbol{F}\cdot\vec{B})\ ,\\
J(\vec{A}\times\vec{B})\cdot\vec{C} &= [(\boldsymbol{F}\cdot\vec{A})\times(\boldsymbol{F}\cdot\vec{B})]\cdot(\boldsymbol{F}\cdot\vec{C})\ ,
\end{aligned}
\tag{1.44}
$$

that is valid for all vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$.

The differential operations in the Eulerian coordinates applied to Eulerian variables can be converted to the differential operations in the Lagrangian coordinates applied to Lagrangian variables by means of the chain rule of differentiation. For example, the following identities can be verified:

$$
\begin{aligned}
\mathrm{grad}\,\bullet &= \boldsymbol{F}^{-T}\cdot\mathrm{Grad}\,\bullet\ ,\\
\mathrm{div}\,\bullet &= \boldsymbol{F}^{-T} : \mathrm{Grad}\,\bullet\ ,\\
\mathrm{div}\,\mathrm{grad}\,\bullet &= \boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-T} : \mathrm{Grad}\,\mathrm{Grad}\ \bullet + \mathrm{div}\,\boldsymbol{F}^{-T}\cdot\mathrm{Grad}\,\bullet\ ,
\end{aligned}
\tag{1.45}
$$

---

[3] Note that $\boldsymbol{F}^{-T} \equiv \left(\boldsymbol{F}^T\right)^{-1} \equiv \left(\boldsymbol{F}^{-1}\right)^T$.

[4] The product of two alternating symbols defines a sixth-order tensor with the components

$$
\epsilon_{KLM}\epsilon_{RST} = \det\begin{pmatrix} \delta_{KR} & \delta_{LR} & \delta_{MR}\\ \delta_{KS} & \delta_{LS} & \delta_{MS}\\ \delta_{KT} & \delta_{LT} & \delta_{MT} \end{pmatrix}\ .
$$

The successive contraction of indeces yields

$$
\begin{aligned}
&\epsilon_{KLM}\epsilon_{KST} = \delta_{LS}\delta_{MT} - \delta_{LT}\delta_{MS}\ ,\\
&\epsilon_{KLM}\epsilon_{KLT} = 2\delta_{MT}\ ,\\
&\epsilon_{KLM}\epsilon_{KLM} = 6\ .
\end{aligned}
$$

where the symbol **:** denotes the double-dot product of tensors. To show it, let us consider a tensor $\mathcal{T}$ represented in the Eulerian variables as $\boldsymbol{t}(\vec{x}, t)$. The corresponding Lagrangian representation of $\mathcal{T}$ is $\boldsymbol{T}(\vec{X}, t) = \boldsymbol{t}(\vec{\chi}(\vec{X}, t), t)$. Then

$$\operatorname{grad} \boldsymbol{t} = \vec{i}_k \otimes \frac{\partial \boldsymbol{t}}{\partial x_k} = \vec{i}_k \otimes \frac{\partial \boldsymbol{T}}{\partial X_K} \frac{\partial X_K}{\partial x_k} = \chi^{-1}_{K,k} \vec{i}_k \otimes \frac{\partial \boldsymbol{T}}{\partial X_K} = \left(\vec{I}_K \cdot \boldsymbol{F}^{-1} \right) \otimes \frac{\partial \boldsymbol{T}}{\partial X_K}$$
$$= \left(\boldsymbol{F}^{-T} \cdot \vec{I}_K \right) \otimes \frac{\partial \boldsymbol{T}}{\partial X_K} = \boldsymbol{F}^{-T} \cdot \left(\vec{I}_K \otimes \frac{\partial \boldsymbol{T}}{\partial X_K} \right) = \boldsymbol{F}^{-T} \cdot \operatorname{Grad} \boldsymbol{T} \ ,$$

hence proving the first identity in (1.45). Replacing dyadic product by scalar product, we have

$$\operatorname{div} \boldsymbol{t} = \vec{i}_k \cdot \frac{\partial \boldsymbol{t}}{\partial x_k} = \vec{i}_k \cdot \frac{\partial \boldsymbol{T}}{\partial X_K} \frac{\partial X_K}{\partial x_k} = \chi^{-1}_{K,k} \vec{i}_k \cdot \frac{\partial \boldsymbol{T}}{\partial X_K} = \chi^{-1}_{K,k} (\vec{i}_k \otimes \vec{I}_K) : \left(\vec{I}_L \otimes \frac{\partial \boldsymbol{T}}{\partial X_L} \right) = \boldsymbol{F}^{-T} : \operatorname{Grad} \boldsymbol{T} \ ,$$

that proves the second identity. The last identity in (1.45) may be verified by using (A.23). The inverse relations are

$$\begin{aligned} \operatorname{Grad} \bullet &= \boldsymbol{F}^T \cdot \operatorname{grad} \bullet \ , \\ \operatorname{Div} \bullet &= \boldsymbol{F}^T : \operatorname{grad} \bullet \ , \\ \operatorname{Div} \operatorname{Grad} \bullet &= \boldsymbol{F} \cdot \boldsymbol{F}^T : \operatorname{grad} \operatorname{grad} \ \bullet + \operatorname{Div} \boldsymbol{F}^T \cdot \operatorname{grad} \ \bullet \ . \end{aligned} \tag{1.46}$$

1.6 Polar decomposition of the deformation gradient

The basic properties of the local behavior of deformation emerge from the possibility of decomposing a deformation into a rotation and a stretch which, roughly speaking, is a change of the shape of a volume element. This decomposition is called the *polar decomposition of the deformation gradient* [5], and is summarized in the following theorem.

A non-singular tensor $\boldsymbol{F}$ ($\det \boldsymbol{F} \neq 0$) permits the polar decomposition in two ways: [6]

$$\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{R} \ , \tag{1.47}$$

where the tensors $\boldsymbol{R}$, $\boldsymbol{U}$ and $\boldsymbol{V}$ have the following properties:

1. The tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ are symmetric and positive definite.
2. The tensor $\boldsymbol{R}$ is orthogonal, $\boldsymbol{R} \cdot \boldsymbol{R}^T = \boldsymbol{R}^T \cdot \boldsymbol{R} = \boldsymbol{I}$, i.e., $\boldsymbol{R}$ is a *rotation tensor*.
3. $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{R}$ are uniquely determined.
4. The eigenvalues of $\boldsymbol{U}$ and $\boldsymbol{V}$ are identical; if $\vec{e}$ is an eigenvector of $\boldsymbol{U}$, then $\boldsymbol{R} \cdot \vec{e}$ is an eigenvector of $\boldsymbol{V}$.

As a preliminary to proving these statements, we note that an arbitrary tensor $\boldsymbol{T}$ is positive definite if $\vec{v} \cdot \boldsymbol{T} \cdot \vec{v} > 0$ for all vectors $\vec{v} \neq \vec{0}$. A necessary and sufficient condition for $\boldsymbol{T}$ to be positive definite is that all of its eigenvalues are positive. In this regard, consider the tensor $\boldsymbol{C}$,

[5] Equation (1.47) is analogous to the polar decomposition of a complex number: $z = re^{i\varphi}$, where $r = (x^2 + y^2)^{1/2}$ and $\varphi = \arctan(y/x)$. For this reason, it is referred as the *polar decomposition*.

[6] The polar decomposition may be applied to every second-order, non-singular tensor as the product of a positive-definite symmetric tensor and an orthogonal tensor.

$\boldsymbol{C} := \boldsymbol{F}^T \cdot \boldsymbol{F}$. Since $\boldsymbol{F}$ is assumed to be non-singular (det $\boldsymbol{F} \neq 0$) and $\boldsymbol{F} \cdot \vec{v} \neq \vec{0}$ if $\vec{v} \neq \vec{0}$, it follows that $(\boldsymbol{F} \cdot \vec{v}) \cdot (\boldsymbol{F} \cdot \vec{v})$ is a sum of squares and hence greater than zero. Thus

$$0 < (\boldsymbol{F} \cdot \vec{v}) \cdot (\boldsymbol{F} \cdot \vec{v}) = \vec{v} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \vec{v} = \vec{v} \cdot \boldsymbol{C} \cdot \vec{v} \ ,$$

and $\boldsymbol{C}$ is positive definite. By the same arguments, we may show that the tensor $\boldsymbol{b}$, $\boldsymbol{b} := \boldsymbol{F} \cdot \boldsymbol{F}^T$, is also positive definite.

The positive roots of $\boldsymbol{C}$ and $\boldsymbol{b}$ define two tensors $\boldsymbol{U}$ and $\boldsymbol{V}$, such that

$$\boldsymbol{U} := \sqrt{\boldsymbol{C}} = \sqrt{\boldsymbol{F}^T \cdot \boldsymbol{F}} \ , \qquad \boldsymbol{V} := \sqrt{\boldsymbol{b}} = \sqrt{\boldsymbol{F} \cdot \boldsymbol{F}^T} \ . \tag{1.48}$$

The tensors $\boldsymbol{U}$ and $\boldsymbol{V}$, called the *right* and *left stretch tensors*, are symmetric, positive definite and are uniquely determined.

Next, two tensors $\boldsymbol{R}$ and $\tilde{\boldsymbol{R}}$ are defined by

$$\boldsymbol{R} := \boldsymbol{F} \cdot \boldsymbol{U}^{-1} \ , \qquad \tilde{\boldsymbol{R}} := \boldsymbol{V}^{-1} \cdot \boldsymbol{F} \ . \tag{1.49}$$

We recognize that both are orthogonal since by definition we have

$$\begin{aligned} \boldsymbol{R} \cdot \boldsymbol{R}^T &= (\boldsymbol{F} \cdot \boldsymbol{U}^{-1}) \cdot (\boldsymbol{F} \cdot \boldsymbol{U}^{-1})^T = \boldsymbol{F} \cdot \boldsymbol{U}^{-1} \cdot \boldsymbol{U}^{-1} \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \boldsymbol{U}^{-2} \cdot \boldsymbol{F}^T = \\ &= \boldsymbol{F} \cdot (\boldsymbol{F}^T \cdot \boldsymbol{F})^{-1} \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-T} \cdot \boldsymbol{F}^T = \boldsymbol{I} \ . \end{aligned}$$

Complementary,

$$\boldsymbol{R}^T \cdot \boldsymbol{R} = (\boldsymbol{F} \cdot \boldsymbol{U}^{-1})^T \cdot (\boldsymbol{F} \cdot \boldsymbol{U}^{-1}) = \boldsymbol{U}^{-1} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \boldsymbol{U}^{-1} = \boldsymbol{U}^{-1} \cdot \boldsymbol{U}^2 \cdot \boldsymbol{U}^{-1} = \boldsymbol{I} \ .$$

Similar proofs hold for $\tilde{\boldsymbol{R}}$.

So far we have demonstrated two decompositions $\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \tilde{\boldsymbol{R}}$, where $\boldsymbol{U}$ and $\boldsymbol{V}$ are symmetric, positive definite and $\boldsymbol{R}$ and $\tilde{\boldsymbol{R}}$ are orthogonal. From

$$\boldsymbol{F} = \boldsymbol{V} \cdot \tilde{\boldsymbol{R}} = (\tilde{\boldsymbol{R}} \cdot \tilde{\boldsymbol{R}}^T) \cdot \boldsymbol{V} \cdot \tilde{\boldsymbol{R}} = \tilde{\boldsymbol{R}} \cdot (\tilde{\boldsymbol{R}}^T \cdot \boldsymbol{V} \cdot \tilde{\boldsymbol{R}}) = \tilde{\boldsymbol{R}} \cdot \tilde{\boldsymbol{U}} \ ,$$

it may be concluded that there may be two decompositions of $\boldsymbol{F}$, namely $\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U}$ and $\boldsymbol{F} = \tilde{\boldsymbol{R}} \cdot \tilde{\boldsymbol{U}}$. However, if this were true we would be forced to conclude that $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F} = \tilde{\boldsymbol{U}}^2 = \boldsymbol{U}^2$, hence concluding that $\tilde{\boldsymbol{U}} = \boldsymbol{U}$ because of the uniqueness of the positive root. This therefore implies that $\tilde{\boldsymbol{R}} = \boldsymbol{R}$, leading to the consequence that $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{R}$ are unique.

Finally, we assume $\vec{e}$ and λ to be an eigenvector and eigenvalue of $\boldsymbol{U}$. Then, we have $\lambda \vec{e} = \boldsymbol{U} \cdot \vec{e}$, as well as $\lambda \boldsymbol{R} \cdot \vec{e} = (\boldsymbol{R} \cdot \boldsymbol{U}) \cdot \vec{e} = (\boldsymbol{V} \cdot \boldsymbol{R}) \cdot \vec{e} = \boldsymbol{V} \cdot (\boldsymbol{R} \cdot \vec{e})$. Thus λ is also eigenvalue of $\boldsymbol{V}$ and $\boldsymbol{R} \cdot \vec{e}$ is an eigenvector. This completes the proof of the theorem.

It is instructive to write the relation (1.47) in componental form as

$$F_{kK} = R_{kL} U_{LK} = V_{kl} R_{lK} \ , \tag{1.50}$$

which means that $\boldsymbol{R}$ is a two-point tensor while $\boldsymbol{U}$ and $\boldsymbol{V}$ are ordinary (one-point) tensors.

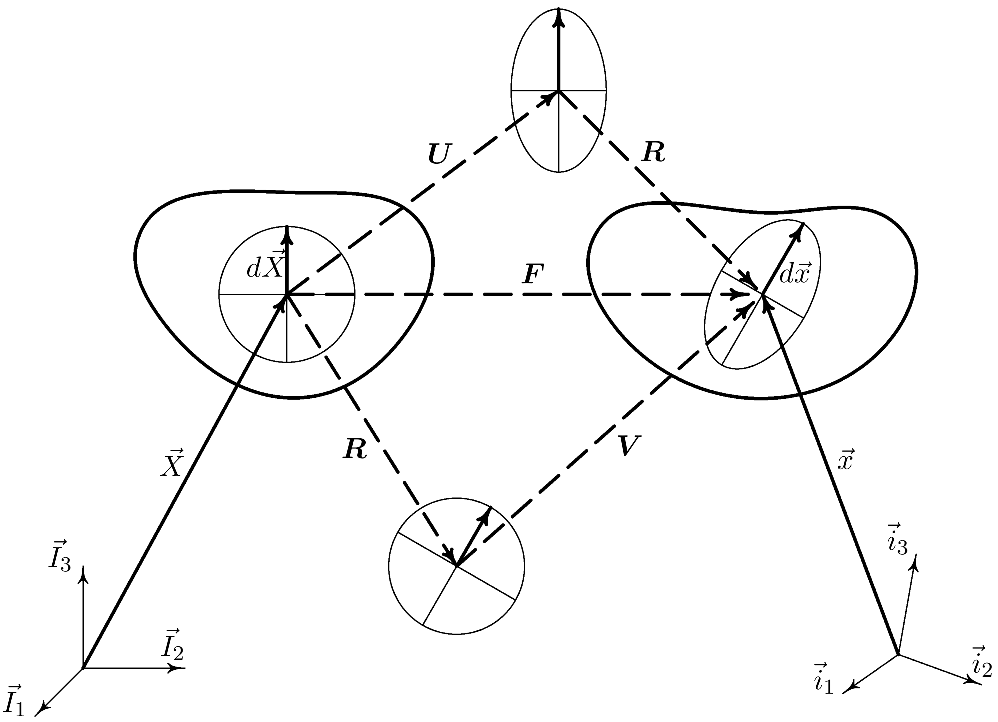

Figure 1.3. Polar decomposition of the deformation gradient.

Equation $d\vec{x} = \boldsymbol{F} \cdot d\vec{X}$ shows that the deformation gradient $\boldsymbol{F}$ can be thought of as a mapping of the infinitesimal vector $d\vec{X}$ of the reference configuration into the infinitesimal vector $d\vec{x}$ of the current configuration. The theorem of polar decomposition replaces the linear transformation $d\vec{x} = \boldsymbol{F} \cdot d\vec{X}$ by two sequential transformations, by rotation and stretching, where the sequence of these two steps may be interchanged, as illustrated in Figure 1.3. The combination of rotation and stretching corresponds to the multiplication of two tensors, namely, $\boldsymbol{R}$ and $\boldsymbol{U}$ or $\boldsymbol{V}$ and $\boldsymbol{R}$,

$$d\vec{x} = (\boldsymbol{R} \cdot \boldsymbol{U}) \cdot d\vec{X} = (\boldsymbol{V} \cdot \boldsymbol{R}) \cdot d\vec{X} \ . \tag{1.51}$$

However, $\boldsymbol{R}$ should not be understood as a rigid body rotation since, in general case, it varies from point to point. Thus the polar decomposition theorem reflects only a local property of motion.

1.7 Measures of deformation

Local changes in the geometry of continuous bodies can be described, as usual in differential geometry, by the changes in the metric tensor. In Euclidean space, it is particularly simple to accomplish. Consider a material point $\vec{X}$ and an infinitesimal material vector $d\vec{X}$. The changes in the length of three such linearly independent vectors describe the local changes in the geometry.

The infinitesimal vector $d\vec{X}$ in κ_0 is mapped onto the infinitesimal vector $d\vec{x}$ in κ_t. The metric properties of the present configuration κ_t can be described by the square of the length of $d\vec{x}$:

$$ds^2 = d\vec{x} \cdot d\vec{x} = (\boldsymbol{F} \cdot d\vec{X}) \cdot (\boldsymbol{F} \cdot d\vec{X}) = d\vec{X} \cdot \boldsymbol{C} \cdot d\vec{X} = \boldsymbol{C} : (d\vec{X} \otimes d\vec{X}) \ , \tag{1.52}$$

where the *Green deformation tensor* $\boldsymbol{C}$ defined by

$$\boldsymbol{C}(\vec{X}, t) := \boldsymbol{F}^T \cdot \boldsymbol{F} \ , \tag{1.53}$$

has already been used in the proof of the polar decomposition theorem. Equation (1.52) describes the local geometric property of the present configuration κ_t with respect to that of the reference configuration κ_0. Alternatively, we can use the inverse tensor expressing the geometry of the reference configuration κ_0 relative to that of the present configuration κ_t. We have

$$dS^2 = d\vec{X} \cdot d\vec{X} = (\boldsymbol{F}^{-1} \cdot d\vec{x}) \cdot (\boldsymbol{F}^{-1} \cdot d\vec{x}) = d\vec{x} \cdot \boldsymbol{c} \cdot d\vec{x} = \boldsymbol{c} : (d\vec{x} \otimes d\vec{x}) \ , \tag{1.54}$$

where $\boldsymbol{c}$ is the *Cauchy deformation tensor* defined by

$$\boldsymbol{c}(\vec{x}, t) := \boldsymbol{F}^{-T} \cdot \boldsymbol{F}^{-1} \ . \tag{1.55}$$

Both tensors $\boldsymbol{c}$ and $\boldsymbol{C}$ are symmetric, that is, $\boldsymbol{c} = \boldsymbol{c}^T$ and $\boldsymbol{C} = \boldsymbol{C}^T$. We see that, in contrast to the deformation gradient tensor $\boldsymbol{F}$ with generally nine independent components, the changes in metric properties, following from the changes of configuration, are described by six independent components of the deformation tensors $\boldsymbol{c}$ or $\boldsymbol{C}$.

Apart from the Cauchy deformation tensor $\boldsymbol{c}$ and the Green deformation tensor $\boldsymbol{C}$, we can introduce other equivalent geometrical measures of deformation. Equation (1.52) and (1.54) yield two different expressions of the squares of element of length, ds^2 and dS^2. The difference $ds^2 - dS^2$ for the same material points is a relative measure of the change of length. When this difference vanishes for any two neighboring points, the deformation has not changed the distance between the pair. When it is zero for all points in the body, the body has undergone only a *rigid displacement*. From (1.52) and (1.54) for this difference we obtain

$$ds^2 - dS^2 = d\boldsymbol{X} \cdot 2\boldsymbol{E} \cdot d\boldsymbol{X} = d\boldsymbol{x} \cdot 2\boldsymbol{e} \cdot d\boldsymbol{x} \ , \tag{1.56}$$

where we have introduced the *Lagrangian* and *Eulerian strain tensors*, respectively,

$$\boldsymbol{E}(\vec{X}, t) := \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) \ , \qquad \boldsymbol{e}(\vec{x}, t) := \frac{1}{2}(\boldsymbol{I} - \boldsymbol{c}) \ . \tag{1.57}$$

Clearly, when either vanishes, $ds^2 = dS^2$. Hence, for a rigid body motion, both strain tensors vanish, but $\boldsymbol{F} = \boldsymbol{R}$. The strain tensors are non-zero only when a strain or stretch arises. Both tensors $\boldsymbol{e}$ and $\boldsymbol{E}$ are symmetric, that is, $\boldsymbol{e} = \boldsymbol{e}^T$ and $\boldsymbol{E} = \boldsymbol{E}^T$. Therefore, in three dimensions there are only six independent components for each of these tensors, for example, E_{11}, E_{22}, E_{33}, $E_{12} = E_{21}$, $E_{13} = E_{31}$, and $E_{23} = E_{32}$. The first three components E_{11}, E_{22}, and E_{33} are called *normal strains* and the last three E_{12}, E_{13}, and E_{23} are called *shear strains*. The reason for this will be discussed later in this chapter.

Two other equivalent measures of deformation are the *reciprocal tensors* $\boldsymbol{b}$ and $\boldsymbol{B}$ (known as the *Finger* and *Piola* deformation tensors, respectively) defined by

$$\boldsymbol{b}(\vec{x}, t) := \boldsymbol{F} \cdot \boldsymbol{F}^T \ , \qquad \boldsymbol{B}(\vec{X}, t) := \boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-T} \ . \tag{1.58}$$

They satisfy the conditions

$$\boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{c} \cdot \boldsymbol{b} = \boldsymbol{I} \ , \qquad \boldsymbol{B} \cdot \boldsymbol{C} = \boldsymbol{C} \cdot \boldsymbol{B} = \boldsymbol{I} \ , \tag{1.59}$$

which can be shown by mere substitution of (1.53) and (1.55).

We have been using the notion *tensor* for quantities such as $\boldsymbol{C}$. Tensor refers to a set of quantities that transform according to a certain definite law upon coordinate transformation. Suppose the Lagrangian Cartesian coordinates X_K are transformed onto X'_K according to

$$X_K = X_K(X'_1, X'_2, X'_3) \ . \tag{1.60}$$

The left-hand side of (1.52) is independent of the coordinate transformations. If we substitute

$$dX_K = \frac{\partial X_K}{\partial X'_M} dX'_M \ ,$$

on the right-hand side of (1.52), we obtain

$$ds^2 = C_{KL} dX_K dX_L = C_{KL} \frac{\partial X_K}{\partial X'_M} \frac{\partial X_L}{\partial X'_N} dX'_M dX'_N \stackrel{!}{=} C'_{MN} dX'_M dX'_N \ .$$

Hence

$$C'_{MN}(\vec{X}', t) = C_{KL}(\vec{X}, t) \frac{\partial X_K}{\partial X'_M} \frac{\partial X_L}{\partial X'_N} \tag{1.61}$$

since dX'_M is arbitrary and $C_{KL} = C_{LK}$. Thus, knowing C_{KL} in one set of coordinates X_K, we can find the corresponding quantities in another set X'_K once the relations (1.60) between X_K and X'_K are given. Quantities that transform according to the law of transformation (1.61) are known as *absolute tensors*.

1.8 Length and angle changes

A geometrical meaning of the normal strains E_{11}, E_{22} and E_{33} is provided by considering the length and angle changes that result from deformation. We consider a material line element $d\vec{X}$ of the length dS that deforms to the element $d\vec{x}$ of the length ds. Let $\vec{K}$ be the unit vector along $d\vec{X}$,

$$\vec{K} := \frac{d\vec{X}}{dS} \ . \tag{1.62}$$

The relative change of length,

$$E_{(\vec{K})} := \frac{ds - dS}{dS} \ , \tag{1.63}$$

is called the *extension* or *elongation* [7] . Dividing $(1.56)_1$ by dS^2, we have

$$\frac{ds^2 - dS^2}{dS^2} = \vec{K} \cdot 2\boldsymbol{E} \cdot \vec{K} \ . \tag{1.64}$$

Expressing the left-hand side by the extension $E_{(\vec{K})}$, we get the quadratic equation for $E_{(\vec{K})}$:

$$E_{(\vec{K})}(E_{(\vec{K})} + 2) - \vec{K} \cdot 2\boldsymbol{E} \cdot \vec{K} = 0 \ . \tag{1.65}$$

From the two possible solutions of this equation, we choose the physically admissible option:

$$E_{(\vec{K})} = -1 + \sqrt{1 + \vec{K} \cdot 2\boldsymbol{E} \cdot \vec{K}} \ . \tag{1.66}$$

[7] Complementary to the Lagrangian extension $E_{(\vec{K})}$, the Eulerian extension $e_{(\vec{K})}$ can be introduced by

$$e_{(\vec{K})} := \frac{ds - dS}{ds} \ .$$

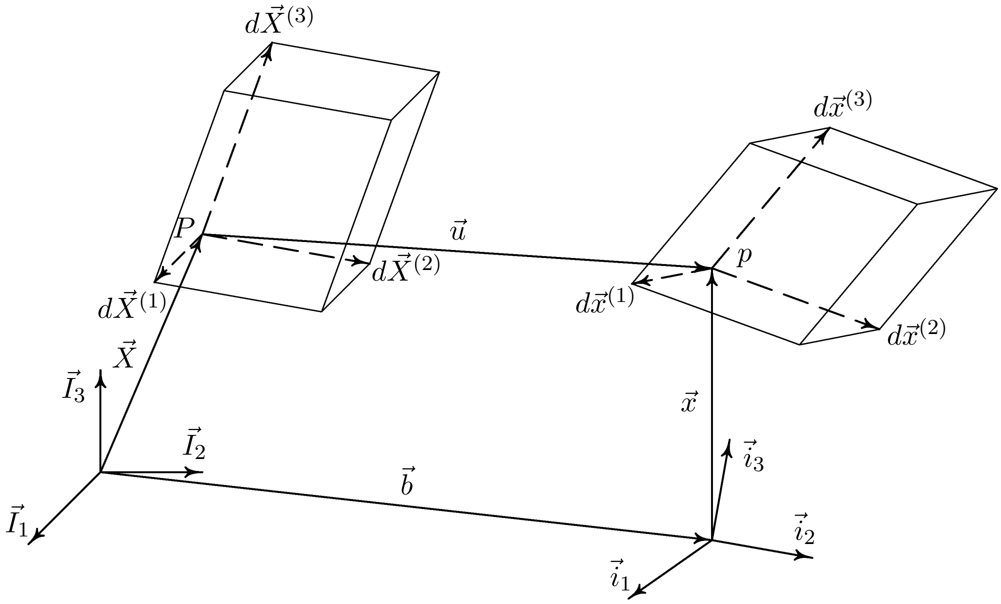

Figure 1.4. Deformation of an infinitesimal rectilinear parallelepiped.

Since $\vec{K}$ is a unit vector, $\vec{K} \cdot 2\boldsymbol{E} \cdot \vec{K} + 1 = \vec{K} \cdot \boldsymbol{C} \cdot \vec{K}$. As proved in Section 1.6, the Green deformation tensor $\boldsymbol{C}$ is positive symmetric, that is, $\vec{K} \cdot \boldsymbol{C} \cdot \vec{K} > 0$. Hence, the argument of the square root in (1.66) is positive, particularly when $\vec{K}$ is taken along the X_1-axis. This gives

$$E_{(1)} = -1 + \sqrt{1 + 2E_{11}} \; . \tag{1.67}$$

The geometrical meaning of the shear strains E_{12}, E_{13}, and E_{23} is found by considering the angles between two directions $\vec{K}^{(1)}$ and $\vec{K}^{(2)}$,

$$\vec{K}^{(1)} := \frac{d\vec{X}^{(1)}}{dS^{(1)}} \; , \qquad \vec{K}^{(2)} := \frac{d\vec{X}^{(2)}}{dS^{(2)}} \; . \tag{1.68}$$

The angle Θ between these vectors in the reference configuration κ_0,

$$\cos\Theta = \frac{d\vec{X}^{(1)}}{dS^{(1)}} \cdot \frac{d\vec{X}^{(2)}}{dS^{(2)}} \; , \tag{1.69}$$

is changed by deformation to

$$\cos\theta = \frac{d\vec{x}^{(1)}}{ds^{(1)}} \cdot \frac{d\vec{x}^{(2)}}{ds^{(2)}} = \frac{\boldsymbol{F} \cdot d\vec{X}^{(1)}}{ds^{(1)}} \cdot \frac{\boldsymbol{F} \cdot d\vec{X}^{(2)}}{ds^{(2)}} = \frac{d\vec{X}^{(1)} \cdot \boldsymbol{C} \cdot d\vec{X}^{(2)}}{ds^{(1)}\, ds^{(2)}} \; . \tag{1.70}$$

From (1.62) and (1.63), we obtain

$$\cos\theta = (\vec{K}^{(1)} \cdot \boldsymbol{C} \cdot \vec{K}^{(2)}) \frac{dS^{(1)}}{ds^{(1)}} \frac{dS^{(2)}}{ds^{(2)}} = \frac{\vec{K}^{(1)} \cdot \boldsymbol{C} \cdot \vec{K}^{(2)}}{(E_{(\vec{K}_1)} + 1)(E_{(\vec{K}_2)} + 1)} \; , \tag{1.71}$$

which can be rewritten in terms of the Lagrangian strain tensor as

$$\cos\theta = \frac{\vec{K}^{(1)} \cdot (\boldsymbol{I} + 2\boldsymbol{E}) \cdot \vec{K}^{(2)}}{\sqrt{1 + \vec{K}^{(1)} \cdot 2\boldsymbol{E} \cdot \vec{K}^{(1)}}\sqrt{1 + \vec{K}^{(2)} \cdot 2\boldsymbol{E} \cdot \vec{K}^{(2)}}} \; . \tag{1.72}$$

When $\vec{K}^{(1)}$ is taken along X_1-axis and $\vec{K}^{(2)}$ along X_2-axis, (1.72) reduces to

$$\cos\theta_{(12)} = \frac{2E_{12}}{\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}} \ . \tag{1.73}$$

1.9 Surface and volume changes

The change in surface and volume with deformation will now be determined. The oriented surface element in the reference configuration built on the edge vectors $d\vec{X}^{(1)}$ and $d\vec{X}^{(2)}$,

$$d\vec{A} := d\vec{X}^{(1)} \times d\vec{X}^{(2)} \ , \tag{1.74}$$

after deformation becomes the oriented surface element with edge vectors $d\vec{x}^{(1)}$ and $d\vec{x}^{(2)}$:

$$d\vec{a} := d\vec{x}^{(1)} \times d\vec{x}^{(2)} = (\boldsymbol{F}\cdot d\vec{X}^{(1)}) \times (\boldsymbol{F}\cdot d\vec{X}^{(2)}) = J\boldsymbol{F}^{-T}\cdot(d\vec{X}^{(1)} \times d\vec{X}^{(2)}) \ ,$$

where we have used the identity $(1.44)_1$. Substituting for $d\vec{A}$, the transformation of a surface element from the reference configuration to the present configuration is

$$d\vec{a} = J\boldsymbol{F}^{-T}\cdot d\vec{A} \ , \qquad \text{or} \qquad da_k = JX_{K,k}dA_K \ . \tag{1.75}$$

To express the interface conditions at discontinuity surfaces in the Lagrangian description, we need to find the relation between the unit normals $\vec{n}$ and $\vec{N}$ to the deformed discontinuity σ and the undeformed discontinuity Σ. Considering the transformation (1.75) between the spatial and referential surface elements, and writing $d\vec{a} = \vec{n}da$ and $d\vec{A} = \vec{N}dA$, we readily find that

$$da = J\sqrt{\vec{N}\cdot\boldsymbol{B}\cdot\vec{N}}\,dA \ , \tag{1.76}$$

where $\boldsymbol{B}$ is the Piola deformation tensor, $\boldsymbol{B} = \boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-T}$. Combining (1.75) with (1.76) results in

$$\vec{n} = \frac{\vec{N}\cdot\boldsymbol{F}^{-1}}{\sqrt{\vec{N}\cdot\boldsymbol{B}\cdot\vec{N}}} \ . \tag{1.77}$$

To calculate the deformed volume element, an infinitesimal rectilinear parallelepiped in the reference configuration spanned by the vectors $d\vec{X}^{(1)}$, $d\vec{X}^{(2)}$ and $d\vec{X}^{(3)}$ is considered (see Figure 1.4). Its volume is given by the scalar product of $d\vec{X}^{(3)}$ with $d\vec{X}^{(1)} \times d\vec{X}^{(2)}$:

$$dV := (d\vec{X}^{(1)} \times d\vec{X}^{(2)})\cdot d\vec{X}^{(3)} \ . \tag{1.78}$$

In the present configuration, the volume of the parallelepiped is

$$dv := (d\vec{x}^{(1)} \times d\vec{x}^{(2)})\cdot d\vec{x}^{(3)} = [(\boldsymbol{F}\cdot d\vec{X}^{(1)}) \times (\boldsymbol{F}\cdot d\vec{X}^{(2)})]\cdot(\boldsymbol{F}\cdot d\vec{X}^{(3)}) = J(d\vec{X}^{(1)} \times d\vec{X}^{(2)})\cdot d\vec{X}^{(3)} \ ,$$

where we have used the identity $(1.44)_2$. Substituting for dV, the transformation of volume element from the reference configuration to the present configuration is

$$dv = JdV \ . \tag{1.79}$$

Consequently, the determinant J of the deformation gradient $\boldsymbol{F}$ measures the volume changes of infinitesimal elements. For this reason, J must be **positive** for material media.

1.10 Strain invariants, principal strains

In this section, a brief account of the invariants for a second-order symmetric tensor is given. The Lagrangian strain tensor $\boldsymbol{E}$ is considered as a typical example of this group. It is of interest to determine, at a given point $\vec{X}$, the directions $\vec{V}$ for which the expression $\vec{V} \cdot \boldsymbol{E} \cdot \vec{V}$ takes extremum values. For this we must differentiate $\vec{V} \cdot \boldsymbol{E} \cdot \vec{V}$ with respect to $\vec{V}$ subject to the condition $\vec{V} \cdot \vec{V} = 1$. Using Lagrange's method of multipliers, we set

$$\frac{\partial}{\partial V_K} \left[\vec{V} \cdot \boldsymbol{E} \cdot \vec{V} - \lambda(\vec{V} \cdot \vec{V} - 1) \right] = 0 \,, \tag{1.80}$$

where λ is the unknown Lagrange multiplier. This gives

$$(E_{KL} - \lambda \delta_{KL})\, V_L = 0 \,. \tag{1.81}$$

A nontrivial solution of the homogeneous equations (1.81) exists only if the *characteristic determinant* vanishes,

$$\det\left(\boldsymbol{E} - \lambda \boldsymbol{I}\right) = 0 \,. \tag{1.82}$$

Upon expanding this determinant, we obtain a cubic algebraic equation in λ, known as the *characteristic equation* of the tensor $\boldsymbol{E}$:

$$-\lambda^3 + I_E\, \lambda^2 - II_E\, \lambda + III_E = 0 \,, \tag{1.83}$$

where

$$\begin{aligned} I_E &:= E_{11} + E_{22} + E_{33} \equiv \operatorname{tr} \boldsymbol{E} \,, \\ II_E &:= E_{11}E_{22} + E_{11}E_{33} + E_{22}E_{33} - E_{12}^2 - E_{13}^2 - E_{23}^2 \equiv \frac{1}{2} \left[(\operatorname{tr} \boldsymbol{E})^2 - \operatorname{tr}(\boldsymbol{E}^2) \right] \,, \\ III_E &:= \det \boldsymbol{E} \,. \end{aligned} \tag{1.84}$$

The quantities I_E, II_E and III_E are known as the *principal invariants* of tensor $\boldsymbol{E}$. These quantities remain invariant upon any orthogonal transformation of $\boldsymbol{E}$, $\boldsymbol{E}^* = \boldsymbol{Q} \cdot \boldsymbol{E} \cdot \boldsymbol{Q}^T$, where $\boldsymbol{Q}$ is an orthogonal tensor. This can be deduced from the equivalence of the characteristic equations for $\boldsymbol{E}^*$ and $\boldsymbol{E}$:

$$\begin{aligned} 0 &= \det\left(\boldsymbol{E}^* - \lambda \boldsymbol{I}\right) = \det\left(\boldsymbol{Q} \cdot \boldsymbol{E} \cdot \boldsymbol{Q}^T - \lambda \boldsymbol{I}\right) = \det \left[\boldsymbol{Q} \cdot (\boldsymbol{E} - \lambda \boldsymbol{I}) \cdot \boldsymbol{Q}^T \right] \\ &= \det \boldsymbol{Q} \det\left(\boldsymbol{E} - \lambda \boldsymbol{I}\right) \det \boldsymbol{Q}^T = (\det \boldsymbol{Q})^2 \det\left(\boldsymbol{E} - \lambda \boldsymbol{I}\right) = \det\left(\boldsymbol{E} - \lambda \boldsymbol{I}\right) \,. \end{aligned}$$

A second-order tensor $\boldsymbol{E}$ in three dimensions possesses only three independent invariants. That is, all other invariants of $\boldsymbol{E}$ can be shown to be functions of the above three invariants. For instance, three other invariants are

$$\tilde{I}_E := \operatorname{tr} \boldsymbol{E} \,, \quad \tilde{II}_E := \operatorname{tr} \boldsymbol{E}^2 \,, \quad \tilde{III}_E := \operatorname{tr} \boldsymbol{E}^3 \,. \tag{1.85}$$

The relationships between these to I_E, II_E and III_E are

$$\begin{aligned} I_E &= \tilde{I}_E \,, & \tilde{I}_E &= I_E \,, \\ II_E &= \tfrac{1}{2} \left(\tilde{I}_E^2 - \tilde{II}_E \right) \,, & \tilde{II}_E &= I_E^2 - 2II_E \,, \\ III_E &= \tfrac{1}{3} \left(\tilde{III}_E - \tfrac{3}{2} \tilde{I}_E\, \tilde{II}_E + \tfrac{1}{2} \tilde{I}_E^3 \right) \,, & \tilde{III}_E &= I_E^3 - 3I_E\, II_E + 3III_E \,. \end{aligned} \tag{1.86}$$

The roots λ_α, $\alpha = 1, 2, 3$, of the characteristic equation (1.83) are called the *characteristic roots*. If $\boldsymbol{E}$ is the Lagrangian or Eulerian strain tensor, λ_α are called the *principal strains*. With each of the characteristic roots, we can determine a *principal direction* $\vec{V}_\alpha$, $\alpha = 1, 2, 3$, solving the equation

$$\boldsymbol{E} \cdot \vec{V}_\alpha = \lambda_\alpha \vec{V}_\alpha \; , \tag{1.87}$$

together with the normalizing condition $\vec{V}_\alpha \cdot \vec{V}_\alpha = 1$. For a symmetric tensor $\boldsymbol{E}$, it is not difficult to show that (i) all characteristic roots are real, and (ii) the principal directions corresponding to two distinct characteristic roots are unique and mutually orthogonal. If, however, there is a pair of equal roots, say $\lambda_1 = \lambda_2$, then only the direction associated with λ_3 will be unique. In this case, any other two directions which are orthogonal to $\vec{V}_3$, and to one another so as to form a right-handed system, may be taken as principal directions. If $\lambda_1 = \lambda_2 = \lambda_3$, every set of right-handed orthogonal axes qualifies as principal axes, and every direction is said to be a principal direction. Thus we can see that it is always possible to find at point $\vec{X}$, at least three mutually orthogonal directions for which the expression $\vec{V} \cdot \boldsymbol{E} \cdot \vec{V}$ takes the stationary values.

The tensor $\boldsymbol{E}$ takes a particularly simple form when the reference coordinate system is selected to coincide with the principal directions. Let the component of tensor $\boldsymbol{E}$ be given initially with respect to arbitrary Cartesian axes X_K with the base vectors $\vec{I}_K$, and let the principal axes of $\boldsymbol{E}$ be designated by X_α with the base vectors $\vec{I}_\alpha \equiv \vec{V}_\alpha$. In invariant notation, the tensor $\boldsymbol{E}$ is represented in the form

$$\boldsymbol{E} = E_{KL}(\vec{I}_K \otimes \vec{I}_L) = E_{\alpha\beta}(\vec{V}_\alpha \otimes \vec{V}_\beta) \; , \tag{1.88}$$

where the diagonal elements $E_{\alpha\alpha}$ are equal to the principal values λ_α, while the off-diagonal elements $E_{\alpha\beta} = 0$, $\alpha \neq \beta$, are zero. The projection of the vector $\vec{I}_K$ on the base of vectors $\vec{V}_\alpha$ is

$$\vec{I}_K = (\vec{I}_K \cdot \vec{V}_\alpha)\vec{V}_\alpha \; , \tag{1.89}$$

where $\vec{I}_K \cdot \vec{V}_\alpha$ are the direction cosines between the two established sets of axes X_K and X_α. By carrying (1.89) into (1.88), we find that

$$E_{\alpha\beta} = (\vec{I}_M \cdot \vec{V}_\alpha)(\vec{I}_N \cdot \vec{V}_\beta)E_{MN} \; . \tag{1.90}$$

Hence, the determination of the principal directions $\vec{V}_\alpha$ and the characteristic roots λ_α of a tensor $\boldsymbol{E}$ is equivalent to finding a rectangular coordinate system of reference in which the matrix $||E_{KL}||$ takes the diagonal form.

1.11 Displacement vector

The geometrical measures of deformation can also be expressed in terms of the *displacement vector* $\vec{u}$ that extends from a material points $\vec{X}$ in the reference configuration κ_0 to its spatial position $\vec{x}$ in the present configuration κ_t, as illustrated in Figure 1.5:

$$\vec{u} := \vec{x} - \vec{X} + \vec{b} \; . \tag{1.91}$$

This definition may be interpreted in either the Lagrangian or Eulerian descriptions of $\vec{u}$:

$$\vec{U}(\vec{X}, t) = \vec{\chi}(\vec{X}, t) - \vec{X} + \vec{b} \; , \qquad \vec{u}(\vec{x}, t) = \vec{x} - \vec{\chi}^{-1}(\vec{x}, t) + \vec{b} \; . \tag{1.92}$$

Taking the scalar product of both sides of $(1.92)_1$ and $(1.92)_2$ by $\vec{i}_k$ and $\vec{I}_K$ respectively, we obtain

$$U_k(\vec{X}, t) = \chi_k - \delta_{kL}X_L + b_k \; , \qquad u_K(\vec{x}, t) = \delta_{Kl}x_l - \chi_K^{-1} + B_K \; , \tag{1.93}$$

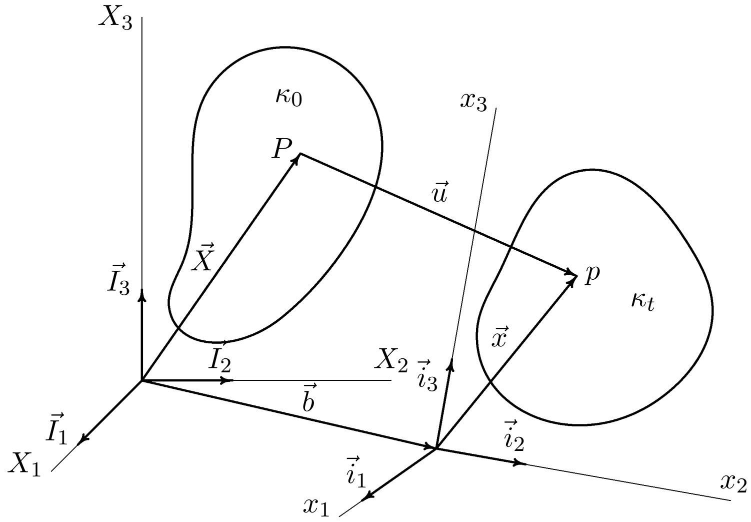

Figure 1.5. Displacement vector.

where $b_k = \vec{b}\cdot\vec{i}_k$ and $B_K = \vec{b}\cdot\vec{I}_K$. Here again we can see the appearance of shifters. Differentiating $(1.93)_1$ with respect to X_K and $(1.93)_2$ with respect to x_k, we find

$$U_{k,K}(\vec{X},t) = \chi_{k,K} - \delta_{kL}\delta_{LK}\ , \qquad\qquad u_{K,k}(\vec{x},t) = \delta_{Kl}\delta_{lk} - \overset{-1}{\chi}_{K,k}\ .$$

In view of (1.14), this can be simplified as

$$U_{k,K}(\vec{X},t) = \chi_{k,K} - \delta_{kK}\ , \qquad\qquad u_{K,k}(\vec{x},t) = \delta_{Kk} - \overset{-1}{\chi}_{K,k}\ . \tag{1.94}$$

In principle, all physical quantities representing the measures of deformation can be expressed in terms of the displacement vector and its gradient. For instance, the Lagrangian strain tensor may be written in indicial notation as

$$2E_{KL} = C_{KL} - \delta_{KL} = \chi_{k,K}\chi_{k,L} - \delta_{KL} = (\delta_{kK} + U_{k,K})(\delta_{kL} + U_{k,L}) - \delta_{KL}\ ,$$

which reduces to

$$2E_{KL} = U_{K,L} + U_{L,K} + U_{k,K}U_{k,L}\ . \tag{1.95}$$

Likewise, the Eulerian strain tensor may be written in the form

$$2e_{kl} = \delta_{kl} - c_{kl} = \delta_{kl} - \overset{-1}{\chi}_{K,k}\overset{-1}{\chi}_{K,l} = \delta_{kl} - (\delta_{Kk} - u_{K,k})(\delta_{Kl} - u_{K,l})\ ,$$

which reduces to

$$2e_{kl} = u_{k,l} + u_{l,k} - u_{K,k}u_{K,l}\ . \tag{1.96}$$

It is often convenient not to distinguish between the Lagrangian coordinates X_K and the Eulerian coordinates x_k. In such a case, the shifter symbol δ_{Kl} reduces to the Kronecker delta δ_{kl} and the displacement vector is defined by

$$\vec{U}(\vec{X},t) = \vec{\chi}(\vec{X},t) - \vec{X}\ . \tag{1.97}$$

Consequently, equation $(1.94)_1$ can be written in symbolic form:

$$\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{H}^T \ , \tag{1.98}$$

where $\boldsymbol{H}$ is the *displacement gradient tensor* defined by

$$\boldsymbol{H}(\vec{X}, t) := \operatorname{Grad} \vec{U}(\vec{X}, t) \ . \tag{1.99}$$

Furthermore, the indicial notation (1.95) of the Lagrangian strain tensor is simplified to symbolic notation to have

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{H} + \boldsymbol{H}^T + \boldsymbol{H} \cdot \boldsymbol{H}^T) \ . \tag{1.100}$$

1.12 Geometrical linearization

If the numerical values of all components of the displacement gradient tensor are very small compared to one, we may neglect the squares and products of these quantities in comparison to the gradients themselves. A convenient measure of smallness of deformations is the magnitude of the displacement gradient,

$$|\boldsymbol{H}| \ll 1 \ , \tag{1.101}$$

where

$$|\boldsymbol{H}|^2 = H_{kl} H_{kl} = \boldsymbol{H} : \boldsymbol{H}^T. \tag{1.102}$$

In the following, the term *small deformation* will correspond to the case of small displacement gradients. *Geometrical linearization* is the process of developing all kinematic variables correct to the first order in $|\boldsymbol{H}|$ and neglecting all terms of orders higher than $O(|\boldsymbol{H}|)$. In its geometrical interpretation, a small value of $|\boldsymbol{H}|$ implies small strains as well as small rotations.

1.12.1 Linearized analysis of deformation

Let us decompose the transposed displacement gradient into the symmetric and skew-symmetric parts:

$$\boldsymbol{H}^T = \tilde{\boldsymbol{E}} + \tilde{\boldsymbol{R}} \ , \tag{1.103}$$

where

$$\tilde{\boldsymbol{E}} := \frac{1}{2}(\boldsymbol{H} + \boldsymbol{H}^T) \ , \qquad \tilde{\boldsymbol{R}} := \frac{1}{2}(\boldsymbol{H}^T - \boldsymbol{H}) \ . \tag{1.104}$$

The symmetric part $\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}^T$ is the *linearized Lagrangian strain tensor*, and the skew-symmetric part $\tilde{\boldsymbol{R}} = -\tilde{\boldsymbol{R}}^T$ is the *linearized Lagrangian rotation tensor.* Carrying this decomposition into (1.100), we obtain

$$\boldsymbol{E} = \tilde{\boldsymbol{E}} + \frac{1}{2}(\tilde{\boldsymbol{E}} - \tilde{\boldsymbol{R}}) \cdot (\tilde{\boldsymbol{E}} + \tilde{\boldsymbol{R}}) \ . \tag{1.105}$$

It is now clear that for $\boldsymbol{E} \approx \tilde{\boldsymbol{E}}$, not only the strains $\tilde{\boldsymbol{E}}$ must be small, but also the rotations $\tilde{\boldsymbol{R}}$ so that products such as $\tilde{\boldsymbol{E}}^T \cdot \tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{E}}^T \cdot \tilde{\boldsymbol{R}}$, and $\tilde{\boldsymbol{R}}^T \cdot \tilde{\boldsymbol{R}}$ will be negligible compared to $\tilde{\boldsymbol{E}}$.

In view of the decomposition (1.103), the deformation gradient may be written in the form

$$\boldsymbol{F} = \boldsymbol{I} + \tilde{\boldsymbol{E}} + \tilde{\boldsymbol{R}} \ , \tag{1.106}$$

Thus, we can see that for small deformations, the multiplicative decomposition of the deformation gradient into orthogonal and positive definite factors is approximated by the additive decomposition into symmetric and skew-symmetric parts.

In the geometrical linearization, other deformation and rotation tensors take the form:

$$\begin{aligned}
\boldsymbol{F}^{-1} &= \boldsymbol{I} - \boldsymbol{H}^T + O(|\boldsymbol{H}|^2) \ , \\
\det \boldsymbol{F} &= 1 + \operatorname{tr} \boldsymbol{H} + O(|\boldsymbol{H}|^2) \ , \\
\boldsymbol{C} &= \boldsymbol{I} + \boldsymbol{H} + \boldsymbol{H}^T + O(|\boldsymbol{H}|^2) \ , \\
\boldsymbol{B} &= \boldsymbol{I} - \boldsymbol{H} - \boldsymbol{H}^T + O(|\boldsymbol{H}|^2) \ , \\
\boldsymbol{U} &= \boldsymbol{I} + \frac{1}{2}(\boldsymbol{H} + \boldsymbol{H}^T) + O(|\boldsymbol{H}|^2) \ , \\
\boldsymbol{V} &= \boldsymbol{I} + \frac{1}{2}(\boldsymbol{H} + \boldsymbol{H}^T) + O(|\boldsymbol{H}|^2) \ , \\
\boldsymbol{R} &= \boldsymbol{I} + \frac{1}{2}(\boldsymbol{H}^T - \boldsymbol{H}) + O(|\boldsymbol{H}|^2) \ .
\end{aligned} \tag{1.107}$$

The transformation (1.45) between the differential operations applied to the Eulerian variables and those applied to the Lagrangian variables can be linearized by the following:

$$\begin{aligned}
\operatorname{grad} \bullet &= \operatorname{Grad} \bullet - \boldsymbol{H} \cdot \operatorname{Grad} \bullet + O(|\boldsymbol{H}|^2) \ , \\
\operatorname{div} \bullet &= \operatorname{Div} \bullet - \boldsymbol{H} : \operatorname{Grad} \bullet + O(|\boldsymbol{H}|^2) \ , \\
\operatorname{div} \operatorname{grad} \bullet &= \operatorname{Div} \operatorname{Grad} \bullet - 2\boldsymbol{H} : \operatorname{Grad} \operatorname{Grad} \bullet - \operatorname{Div} \boldsymbol{H} \cdot \operatorname{Grad} \bullet + O(|\boldsymbol{H}|^2) \ ,
\end{aligned} \tag{1.108}$$

or, conversely,

$$\begin{aligned}
\operatorname{Grad} \bullet &= \operatorname{grad} \bullet + \boldsymbol{H} \cdot \operatorname{grad} \bullet + O(|\boldsymbol{H}|^2) \ , \\
\operatorname{Div} \bullet &= \operatorname{div} \bullet + \boldsymbol{H} : \operatorname{grad} \bullet + O(|\boldsymbol{H}|^2) \ , \\
\operatorname{Div} \operatorname{Grad} \bullet &= \operatorname{div} \operatorname{grad} \bullet + 2\boldsymbol{H} : \operatorname{grad} \operatorname{grad} \bullet + \operatorname{Div} \boldsymbol{H} \cdot \operatorname{grad} \bullet + O(|\boldsymbol{H}|^2) \ .
\end{aligned} \tag{1.109}$$

Equation (1.96) shows that the geometrical linearization of the Eulerian strain tensor $\boldsymbol{e}$ results in the *linearized Eulerian strain tensor* $\tilde{\boldsymbol{e}}$:

$$\boldsymbol{e} \approx \tilde{\boldsymbol{e}} = \frac{1}{2}\Big[\operatorname{grad} \vec{u} + (\operatorname{grad} \vec{u})^T\Big] \ . \tag{1.110}$$

Because of $(1.108)_1$, the tensor $\tilde{\boldsymbol{e}}$ is, correct to the first order of $\|\boldsymbol{H}\|$, equal to the tensor $\tilde{\boldsymbol{E}}$,

$$\tilde{\boldsymbol{e}} = \tilde{\boldsymbol{E}} + O(|\boldsymbol{H}|^2) \ . \tag{1.111}$$

Thus, in the linearized theory, the distinction between the Lagrangian and Eulerian strain tensors disappears.

1.12.2 Length and angle changes

Expanding the square root in (1.67) by the binomial theorem and neglecting the square and higher powers of E_{11}, we obtain

$$E_{(1)} \approx E_{11} \approx \tilde{E}_{11} \ . \tag{1.112}$$

Similar results are of course valid for E_{22} and E_{33}, which indicates that the infinitesimal normal strains are approximately the extensions of the fibers along the coordinate axes in the reference configuration. Likewise, the geometrical linearization of (1.73) yields

$$\cos\theta_{(12)} \approx 2E_{12} \approx 2\tilde{E}_{12} \ . \tag{1.113}$$

Hence, writing $\cos\theta_{(12)} = \sin\Gamma_{(12)} \approx \gamma_{(12)}$, we have

$$\gamma_{(12)} \approx 2E_{12} \approx 2\tilde{E}_{12} \ . \tag{1.114}$$

Similar results are valid for E_{13} and E_{23}. This provides geometrical meaning for shear strains. The infinitesimal shear strains are approximately one half of the angle change between the coordinate axes in the reference configuration.

1.12.3 Surface and volume changes

Substituting the linearized form (1.107) for the spatial deformation gradient $\boldsymbol{F}^{-1}$ and the jacobian J into (1.75), we obtain the linearized relation between $d\vec{a}$ and $d\vec{A}$:

$$d\vec{a} = [\boldsymbol{I} + (\operatorname{tr}\boldsymbol{H})\boldsymbol{I} - \boldsymbol{H}] \cdot d\vec{A} + O(|\boldsymbol{H}|^2) \ . \tag{1.115}$$

We may also linearize the separate contributions to $d\vec{a}$. Using the linearized form (1.107) for the Piola deformation tensor $\boldsymbol{B}$, we can write

$$\frac{1}{\sqrt{\vec{N} \cdot \boldsymbol{B} \cdot \vec{N}}} = \frac{1}{\sqrt{1 - 2\vec{N} \cdot \boldsymbol{H} \cdot \vec{N}}} = 1 + \vec{N} \cdot \boldsymbol{H} \cdot \vec{N} + O(|\boldsymbol{H}|^2) \ . \tag{1.116}$$

Employing this and the linearized forms of the spatial deformation gradient $\boldsymbol{F}^{-1}$ and the jacobian J, the unit normal $\vec{n}$ to the deformed surface σ and the surface element da of σ may, within the framework of linear approximation, be written as

$$da = (1 + \operatorname{tr}\boldsymbol{H} - \vec{N} \cdot \boldsymbol{H} \cdot \vec{N})dA + O(|\boldsymbol{H}|^2) \ , \tag{1.117}$$

$$\vec{n} = (1 + \vec{N} \cdot \boldsymbol{H} \cdot \vec{N})\vec{N} - \boldsymbol{H} \cdot \vec{N} + O(|\boldsymbol{H}|^2) \ . \tag{1.118}$$

where $\vec{N}$ is the unit normal to the undeformed surface Σ, the Lagrangian description of the deformed surface σ. The first equation accounts for the change of surface element, the second gives the deflection of the unit normal.

The linearized expressions for the deformed surface element can also be expressed in terms of the *surface-gradient operator* $\operatorname{Grad}_\Sigma$ and the *surface-divergence operator* $\operatorname{Div}_\Sigma$ defined in Appendix B. For example, the surface gradient and the surface divergence of the displacement vector $\vec{U}$ are given by (B.30) and (B.31):

$$\operatorname{Grad}_\Sigma \vec{U} = \boldsymbol{H} - \vec{N} \otimes (\vec{N} \cdot \boldsymbol{H}) \ , \tag{1.119}$$

$$\operatorname{Div}_\Sigma \vec{U} = \operatorname{tr}\boldsymbol{H} - \vec{N} \cdot \boldsymbol{H} \cdot \vec{N} \ , \tag{1.120}$$

where $\boldsymbol{H} = \operatorname{Grad}\vec{U}$ and $\operatorname{tr}\boldsymbol{H} = \operatorname{Div}\vec{U}$. In terms of the surface displacement gradient and divergence, equations (1.115), (1.117) and (1.118) have the form

$$d\vec{a} = [\boldsymbol{I} + (\operatorname{Div}_\Sigma \vec{U})\boldsymbol{I} - \operatorname{Grad}_\Sigma \vec{U}] \cdot d\vec{A} + O(|\boldsymbol{H}|^2) \ , \tag{1.121}$$

$$da = (1 + \operatorname{Div}_\Sigma \vec{U})dA + O(|\boldsymbol{H}|^2) \ , \tag{1.122}$$

$$\vec{n} = (\boldsymbol{I} - \operatorname{Grad}_\Sigma \vec{U}) \cdot \vec{N} + O(|\boldsymbol{H}|^2) \ . \tag{1.123}$$

Equation (1.122) is the surface analogue of the volumetric relation (1.79); the deformed and undeformed volume elements dv and dV are related by

$$dv = (1 + \mathrm{Div}\,\vec{U})dV + O(|\boldsymbol{H}|^2) \ . \qquad (1.124)$$

2. KINEMATICS

2.1 Material and spatial time derivatives

In the previous chapter we discussed the geometrical properties of the present configuration κ_t under the assumption that the parameter t describing time changes of the body is kept fixed. That is, the function $\vec{\chi}$ was considered as a deformation map $\vec{\chi}(\cdot, t)$. Now we turn our attention to problems for the case of motion, that is where the function $\vec{\chi}$ is treated as the map $\vec{\chi}(\vec{X}, \cdot)$ for a chosen point $\vec{X}$. We assume that $\vec{\chi}(\vec{X}, \cdot)$ is twice differentiable.

If we focus our attention on a specific particle X^P having the material position vector $\vec{X}^P$, (1.3) takes the form

$$\vec{x}^P = \vec{\chi}(\vec{X}^P, t) \tag{2.1}$$

and describes the *path* or *trajectory* of that particle as a function of time. The *velocity* $\vec{v}^P$ of the particle along this path is defined as the time rate of change of position, or

$$\vec{V}^P := \frac{d\vec{x}^P}{dt} = \left(\frac{\partial \vec{\chi}}{\partial t}\right)\bigg|_{\vec{X}=\vec{X}^P} , \tag{2.2}$$

where the subscript $\vec{X}$ accompanying a vertical bar indicates that $\vec{X}$ is held constant (equal to $\vec{X}^P$) in the differentiation of $\vec{\chi}$. In an obvious generalization, we may define the *velocity* of the total body as the derivative

$$\vec{V}(\vec{X}, t) := \frac{d\vec{x}}{dt}\bigg|_{\vec{X}} = \left(\frac{\partial \vec{\chi}}{\partial t}\right)\bigg|_{\vec{X}} . \tag{2.3}$$

This is the Lagrangian representation of velocity. The time rate of change of a physical quantity with respect to a fixed, but moving material particle is called the *material time derivative.* Hence, the velocity of the material particle is defined as the material time derivative of its position $\vec{x}$. Similarly, the material time derivative of $\vec{v}$ defines the Lagrangian representation of acceleration,

$$\vec{A}(\vec{X}, t) := \frac{d\vec{v}}{dt}\bigg|_{\vec{X}} = \left(\frac{\partial \vec{V}}{\partial t}\right)\bigg|_{\vec{X}} = \left(\frac{\partial^2 \vec{\chi}(\vec{X}, t)}{\partial^2 t}\right)\bigg|_{\vec{X}} . \tag{2.4}$$

By using $(1.92)_1$ we may also write

$$\vec{V}(\vec{X}, t) = \left(\frac{\partial \vec{U}}{\partial t}\right)\bigg|_{\vec{X}} , \qquad \vec{A}(\vec{X}, t) = \left(\frac{\partial^2 \vec{U}}{\partial^2 t}\right)\bigg|_{\vec{X}} . \tag{2.5}$$

The material particle with a given velocity or acceleration is, in the Lagrangian representation, identifiable, such that, both the velocity and acceleration fields are defined with respect to the reference configuration κ_0. This is frequently not convenient, for instance, in classical fluid mechanics. When the present configuration κ_t is chosen as the reference configuration, the function $\vec{\chi}(\vec{X}, t)$ cannot be specified. Thus, in the Eulerian description, the velocity and acceleration at time t at a spatial point are known, but the particle occupying this point is not known.

Since the fundamental laws of continuum dynamics involve the acceleration of particles and since the Lagrangian formulation of velocity may not be available, the acceleration must be calculated from the Eulerian formulation of velocity. To accomplish this, only the *existence* of the unknown trajectories, $\vec{x} = \chi(\vec{X}, t)$, must be assumed. By the substitution of (1.4) for $\vec{X}$ in (2.3), we have

$$\vec{v}(\vec{x}, t) = \vec{V}(\chi^{-1}(\vec{x}, t), t) \ , \tag{2.6}$$

where the particle, which at time t occupies the position $\vec{x}$, is held fixed. This relation gives the velocity field at each spatial point $\vec{x}$ at time t with no specification of its relationship to the material point $\vec{X}$. This is the Eulerian representation of velocity.

Based on the same assumption of the existence of the trajectories $\vec{x} = \chi(\vec{X}, t)$, the Eulerian representation of velocity (2.6) may be considered in the form $\vec{v} = \vec{v}(\vec{x}(\vec{X}, t), t)$. By the chain rule of calculus, we obtain

$$\left.\frac{d\vec{v}}{dt}\right|_{\vec{X}} = \left.\frac{\partial\vec{v}}{\partial t}\right|_{\vec{x}} + \left.\frac{\partial\vec{v}}{\partial x_k}\frac{dx_k}{dt}\right|_{\vec{X}} + \left.\frac{\partial\vec{v}}{\partial x_k}\frac{\partial x_k}{\partial X_K}\frac{dX_K}{dt}\right|_{\vec{X}} .$$

Since $dX_K/dt|_{\vec{X}} = 0$, the last expression reduces to

$$\left.\frac{d\vec{v}}{dt}\right|_{\vec{X}} = \left.\frac{\partial\vec{v}}{\partial t}\right|_{\vec{x}} + \vec{v}\cdot\operatorname{grad}\vec{v} \ , \tag{2.7}$$

where differentiation in the grad-operator is taken with respect to the spatial variables. [8] This is the desired equation for the Eulerian representation of acceleration, which is expressed in terms of the Eulerian representation of velocity,

$$\vec{a}(\vec{x}, t) = \left.\frac{\partial\vec{v}}{\partial t}\right|_{\vec{x}} + \vec{v}\cdot\operatorname{grad}\vec{v} \ . \tag{2.8}$$

In this equation, the first term on the right-hand side gives the time rate of change of velocity at a fixed position $\vec{x}$, known as the *local rate of change* or *spatial time derivatives.* The second term results from the particles changing position in space and is referred to as the *convective term.* Note that the convective term can equivalently be represented in the form

$$\vec{v}\cdot\operatorname{grad}\vec{v} = \operatorname{grad}\frac{v^2}{2} - \vec{v}\times\operatorname{rot}\vec{v} \ . \tag{2.9}$$

The material time derivative of any other field quantity can be calculated in the same way if its Lagrangian or Eulerian representation is known. This allows the introduction of the *material time derivative operator*

$$\frac{D}{Dt} \equiv (\dot{\ }) := \left.\frac{d}{dt}\right|_{\vec{X}} = \begin{cases} \left.\dfrac{\partial}{\partial t}\right|_{\vec{X}} & \text{for a field in the Lagrangian representation ,} \\ \left.\dfrac{\partial}{\partial t}\right|_{\vec{x}} + \vec{v}\cdot\operatorname{grad} & \text{for a field in the Eulerian representation ,} \end{cases} \tag{2.10}$$

which can be applied to any field quantity given in the Lagrangian or Eulerian representation.

2.2 Time changes of some geometric objects

[8]The components of $\operatorname{grad}\vec{v}$ in the Cartesian coordinates x_k are $(\operatorname{grad}\vec{v})_{kl} = v_{l,k}$.

All of the geometric objects that we discussed in Chapter 1 can be calculated from the deformation gradient $\boldsymbol{F}$. For this reason, we begin with an investigation of the material time derivative of $\boldsymbol{F}$. In indicial notation, we can write

$$\frac{D\chi_{k,K}}{Dt} = \frac{D}{Dt}\Big(\frac{\partial \chi_k}{\partial X_K}\Big) = \frac{\partial}{\partial t}\Big(\frac{\partial \chi_k}{\partial X_K}\Big)\Big|_{\vec{X}} = \frac{\partial}{\partial X_K}\Big(\frac{\partial \chi_k}{\partial t}\Big)\Big|_{\vec{X}} = \frac{\partial V_k}{\partial X_K} = V_{k,K} = v_{k,l}\chi_{l,K} \ , \qquad (2.11)$$

where we have used the fact that X_K are kept fixed in material time derivative so that material time derivative D/Dt and material gradient $\partial/\partial X_K$ commute. In addition, we have employed the Lagrangian and Eulerian representations of velocity (2.3) and (2.6), respectively. In symbolic notation, we have

$$\dot{\boldsymbol{F}} = \boldsymbol{l}\cdot\boldsymbol{F} \ , \qquad \text{or} \qquad \boldsymbol{l} = \dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1} \ , \qquad (2.12)$$

where $\boldsymbol{l}$ is the transpose *spatial velocity gradient*,

$$\boldsymbol{l}(\vec{x},t) := \operatorname{grad}^T \vec{v}(\vec{x},t), \qquad \text{or} \qquad l_{kl}(\vec{x},t) := \frac{\partial v_k(\vec{x},t)}{\partial x_l} \ . \qquad (2.13)$$

A corollary of this lemma is

$$(\boldsymbol{F}^{-1})^{\bullet} = -\boldsymbol{F}^{-1}\cdot\boldsymbol{l} \ . \qquad (2.14)$$

To prove it, we take the material time derivative of $\boldsymbol{F}\cdot\boldsymbol{F}^{-1} = \boldsymbol{I}$. Hence,

$$\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1} + \boldsymbol{F}\cdot(\boldsymbol{F}^{-1})^{\bullet} = \boldsymbol{0} \ .$$

Using (2.12), we obtain (2.14).

The material time derivative of the Jacobian is given by

$$\dot{J} = J \operatorname{div}\vec{v} = J \operatorname{tr}\boldsymbol{l} \ . \qquad (2.15)$$

To show this, we have

$$\dot{J} = (\det \chi_{k,K})^{\bullet} = \frac{\partial J}{\partial \chi_{k,K}}(\chi_{k,K})^{\bullet} = \frac{\partial J}{\partial \chi_{k,K}}\, v_{k,l}x_{l,K} \ .$$

Using $(1.41)_1$, this gives (2.15).

The velocity gradient $\boldsymbol{l}$ determines the material time derivatives of the material line element $d\vec{x}$, surface element $d\vec{a}$ and volume element dv according to the formulae:

$$(d\vec{x})^{\bullet} = \boldsymbol{l}\cdot d\vec{x} \ , \qquad (2.16)$$

$$(d\vec{a})^{\bullet} = \Big[(\operatorname{div}\vec{v})\boldsymbol{I} - \boldsymbol{l}^T\Big]\cdot d\vec{a} \ , \qquad (2.17)$$

$$(dv)^{\bullet} = \operatorname{div}\vec{v}\, dv \ . \qquad (2.18)$$

The proof is immediate. We take the material time derivative of $(1.36)_1$ and substitute from (2.12):

$$(d\vec{x})^{\bullet} = \dot{\boldsymbol{F}}\cdot d\vec{X} = \boldsymbol{l}\cdot\boldsymbol{F}\cdot d\vec{X} \ .$$

Replacing $d\vec{X}$ by $d\vec{x}$ proves (2.16). By differentiating (1.75) and using (2.14) and (2.15), we have

$$(d\vec{a})^{\bullet} = \Big[\dot{J}\,\boldsymbol{F}^{-T} + J(\boldsymbol{F}^{-T})^{\bullet}\Big]\cdot d\vec{A} = \Big[J\operatorname{div}\vec{v}\,\boldsymbol{F}^{-T} - J\,\boldsymbol{l}^T\cdot\boldsymbol{F}^{-T}\Big]\cdot d\vec{A} \ .$$

Replacing $d\vec{A}$ by $d\vec{a}$ we obtain (2.17). The third statement can be verified by differentiating (1.79) and using (2.15):

$$(dv)^{\bullet} = \dot{J}\, dV = J\, \mathrm{div}\vec{v}\, dV = \mathrm{div}\vec{v}\, dv \ ,$$

which completes the proofs of these statements.

As with any 2nd order tensor, the spatial velocity gradient $\boldsymbol{l}$ can be uniquely decomposed into symmetric and skew-symmetric parts

$$\boldsymbol{l} = \boldsymbol{d} + \boldsymbol{w} \ , \tag{2.19}$$

where

$$\boldsymbol{d} := \frac{1}{2}(\boldsymbol{l} + \boldsymbol{l}^T) \ , \qquad \boldsymbol{w} := \frac{1}{2}(\boldsymbol{l} - \boldsymbol{l}^T) \ . \tag{2.20}$$

The symmetric tensor $\boldsymbol{d}$, $\boldsymbol{d} = \boldsymbol{d}^T$, is called the *strain-rate* or *stretching tensor* and the skew-symmetric tensor $\boldsymbol{w}$, $\boldsymbol{w} = -\boldsymbol{w}^T$, is the *spin* or *vorticity tensor.*

To highlight the meaning of the spin tensor $\boldsymbol{w}$, we readily see that any skew-symmetric 2nd-order tensor $\boldsymbol{w}$,

$$\boldsymbol{w} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} (\vec{i}_k \otimes \vec{i}_l) \ ,$$

can be represented in terms of the so-called *vorticity vector* $\vec{w}$ in the form:

$$w_{kl} = \varepsilon_{lkm} w_m \ , \qquad \text{or} \qquad w_k = \frac{1}{2}\varepsilon_{klm} w_{ml} \ .$$

Since $w_{kl} = \frac{1}{2}(v_{k,l} - v_{l,k})$, we find that the vorticity vector $\vec{w}$ is equal to one-half of the curl of the velocity vector $\vec{v}$:

$$\vec{w} = \frac{1}{2}\mathrm{rot}\, \vec{v} \ . \tag{2.21}$$

Hence the name *spin*, or *vorticity*, given to the tensor $\boldsymbol{w}$. The scalar product of $\boldsymbol{w}$ with a vector $\vec{a}$ is given by

$$\boldsymbol{w} \cdot \vec{a} = \vec{w} \times \vec{a}.$$

Equation (2.16) can now be written in the form

$$(d\vec{x})^{\bullet} = \boldsymbol{d} \cdot d\vec{x} + \vec{w} \times d\vec{x} \ . \tag{2.22}$$

The physical significance of the spin tensor may be seen by considering the case when the strain-rate tensor is zero, $\boldsymbol{d} = \boldsymbol{0}$. Then $(d\vec{x})^{\bullet} = \vec{w} \times d\vec{x}$, according to which the spin tensor $\boldsymbol{w}$ describes an instantaneous local rigid-body rotation about an axis passing through a point $\vec{x}$; the corresponding angular velocity, the direction and the sense of this rotation is described by the spin vector $\vec{w}$. If, on the other hand, the spin tensor is zero in a region, $\boldsymbol{w} = \boldsymbol{0}$, the velocity field is said to be *irrotational* in the region.

The material time derivative of the Lagrangian strain tensor is given by

$$\dot{\boldsymbol{E}} = \frac{1}{2}\, \dot{\boldsymbol{C}} = \boldsymbol{F}^T \cdot \boldsymbol{d} \cdot \boldsymbol{F} \ . \tag{2.23}$$

To show this, we have

$$\dot{\boldsymbol{C}} = (\boldsymbol{F}^T \cdot \boldsymbol{F})^{\bullet} = (\boldsymbol{F}^T)^{\bullet} \cdot \boldsymbol{F} + \boldsymbol{F}^T \cdot \dot{\boldsymbol{F}} = \boldsymbol{F}^T \cdot \boldsymbol{l}^T \cdot \boldsymbol{F} + \boldsymbol{F}^T \cdot \boldsymbol{l} \cdot \boldsymbol{F} = \boldsymbol{F}^T \cdot (\boldsymbol{l} + \boldsymbol{l}^T) \cdot \boldsymbol{F} = 2\boldsymbol{F}^T \cdot \boldsymbol{d} \cdot \boldsymbol{F} \ .$$

The material time derivative of strain tensor $\boldsymbol{E}$ is determined by the tensor $\boldsymbol{d}$, but not by the tensor $\boldsymbol{w}$. This is why $\boldsymbol{d}$ is called the *strain-rate* tensor.

The material time derivative of the Eulerian strain tensor is given by

$$\dot{\boldsymbol{e}} = -\frac{1}{2}\,\dot{\boldsymbol{c}} = \boldsymbol{d} - (\boldsymbol{e}\cdot\boldsymbol{l} + \boldsymbol{l}^T\cdot\boldsymbol{e}) \ . \tag{2.24}$$

To show it, we have

$$\dot{\boldsymbol{c}} = (\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1})^{\bullet} = (\boldsymbol{F}^{-T})^{\bullet}\cdot\boldsymbol{F}^{-1} + \boldsymbol{F}^{-T}\cdot(\boldsymbol{F}^{-1})^{\bullet} = -\boldsymbol{l}^T\cdot\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1} - \boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{l} = -\boldsymbol{l}^T\cdot\boldsymbol{c} - \boldsymbol{c}\cdot\boldsymbol{l} \ .$$

By substituting from $(1.57)_2$, we obtain (2.24). Note that $\dot{\boldsymbol{e}}$ does not equal to $\boldsymbol{d}$.

Likewise, it can readily be shown that the material time derivative of the Piola deformation tensor $\boldsymbol{B} = \boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-T}$ is

$$\dot{\boldsymbol{B}} = -\boldsymbol{F}^{-1}\cdot 2\boldsymbol{d}\cdot\boldsymbol{F}^{-T} \ . \tag{2.25}$$

In view of this, the material time of the unit normal $\vec{n}$ to surface σ in the present configuration is

$$(\vec{n})^{\bullet} = -\vec{n}\cdot\boldsymbol{l} + (\vec{n}\cdot\boldsymbol{l}\cdot\vec{n})\vec{n} \ . \tag{2.26}$$

To show this, we take the material time derivative of (1.77) and use (2.14) and (2.25):

$$\begin{aligned}(\vec{n})^{\bullet} &= \frac{\vec{N}\cdot(F^{-1})^{\bullet}}{(\vec{N}\cdot\boldsymbol{B}\cdot\vec{N})^{1/2}} - \frac{\vec{N}\cdot\boldsymbol{F}^{-1}}{2(\vec{N}\cdot\boldsymbol{B}\cdot\vec{N})^{3/2}}(\vec{N}\cdot\dot{\boldsymbol{B}}\cdot\vec{N}) \\ &= -\frac{\vec{N}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{l}}{(\vec{N}\cdot\boldsymbol{B}\cdot\vec{N})^{1/2}} + \frac{\vec{N}\cdot\boldsymbol{F}^{-1}}{(\vec{N}\cdot\boldsymbol{B}\cdot\vec{N})^{3/2}}(\vec{N}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{d}\cdot\boldsymbol{F}^{-T}\cdot\vec{N}) \ .\end{aligned}$$

By substituting from (1.77), we obtain (2.25).

The material time derivative of the square of the arc length is given by

$$(ds^2)^{\bullet} = 2\,d\vec{x}\cdot\boldsymbol{d}\cdot d\vec{x} \ . \tag{2.27}$$

The proof follows from (1.52):

$$\begin{aligned}(ds^2)^{\bullet} &= (d\vec{x}\cdot d\vec{x})^{\bullet} = (d\vec{X}\cdot\boldsymbol{C}\cdot d\vec{X})^{\bullet} = d\vec{X}\cdot\dot{\boldsymbol{C}}\cdot d\vec{X} = (\boldsymbol{F}^{-1}\cdot d\vec{x})\cdot\dot{\boldsymbol{C}}\cdot(\boldsymbol{F}^{-1}\cdot d\vec{x}) \\ &= d\vec{x}\cdot\boldsymbol{F}^{-T}\cdot\dot{\boldsymbol{C}}\cdot\boldsymbol{F}^{-1}\cdot d\vec{x} \ .\end{aligned}$$

Using (2.23), this gives (2.27).

Higher-order material time derivatives can be carried out by repeated differentiation. Therefore, the nth material time derivative of ds^2 can be expressed as

$$(ds^2)^{(n)} = d\vec{x}\cdot\boldsymbol{a}_n\cdot d\vec{x} \ , \tag{2.28}$$

where

$$\boldsymbol{a}_n(\vec{x},t) := \boldsymbol{F}^{-T}\cdot\overset{(n)}{\boldsymbol{C}}\cdot\boldsymbol{F}^{-1} \tag{2.29}$$

are known as the *Rivlin-Ericksen* tensors of order n. The proof of (2.28) is similar to that of (2.27). Note that $\overset{(n)}{\boldsymbol{C}}$ is the nth material time derivative of the Green deformation tensor. The first two Rivlin-Ericksen tensors are

$$\boldsymbol{a}_0 = \boldsymbol{I} \ , \qquad \boldsymbol{a}_1 = 2\boldsymbol{d} \ . \tag{2.30}$$

The Rivlin-Ericksen tensors are used in the formulation of non-linear viscoelasticity, and, in particular, in the description of non-Newtonian fluids (see Chapter 5).

2.3 Reynolds's transport theorem

In this section we will prove that the material time derivative of a volume integral of any scalar or vector field ϕ over the spatial volume $v(t)$ is given by

$$\frac{D}{Dt}\int_{v(t)} \phi dv = \int_{v(t)} \left(\frac{D\phi}{Dt} + \phi \operatorname{div} \vec{v} \right) dv \ . \tag{2.31}$$

To prove this, we firstly transform the integral over the spatial volume to an integral over the material volume V. Under the assumption of existence of the mapping (1.3), and by using (1.79), we have

$$\frac{D}{Dt}\int_{v(t)} \phi dv = \frac{D}{Dt}\int_{V} \Phi J dV \ ,$$

where $\Phi(\vec{X}, t) = \phi(\vec{x}(\vec{X}, t), t)$. Since V is a fixed volume in the Lagrangian configuration, the differentiation D/Dt and the integration over V commute and the differentiation D/Dt can be performed inside the integral sign,

$$\frac{D}{Dt}\int_{V} \Phi J dV = \int_{V} \frac{D}{Dt} (\Phi J)\, dV = \int_{V} \left(\frac{D\Phi}{Dt} J + \Phi \frac{DJ}{Dt} \right) dV = \int_{V} \left(\frac{D\Phi}{Dt} + \Phi \operatorname{div} \vec{v} \right) J dV \ .$$

By converting this back to the spatial formulation by (1.79), we prove (2.31). Equation (2.31) is often referred to as the *Reynolds transport theorem.*

This theorem may be expressed in an alternative form. We firstly determine the material time derivative of a scalar field ϕ,

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \vec{v} \cdot \operatorname{grad} \phi \ , \tag{2.32}$$

and substitute this back into (2.31). With the product rule

$$\phi \operatorname{div} \vec{v} + \vec{v} \cdot \operatorname{grad} \phi = \operatorname{div} (\vec{v}\phi) \ , \tag{2.33}$$

which is valid for a scalar field ϕ, we then arrive at

$$\frac{D}{Dt}\int_{v(t)} \phi dv = \int_{v(t)} \left(\frac{\partial \phi}{\partial t} + \operatorname{div} (\vec{v}\phi) \right) dv \ .$$

Arranging the second term on the right-hand side according to the Gauss theorem

$$\int_{v} \operatorname{div} \vec{u}\ dv = \oint_{s} \vec{n} \cdot \vec{u}\ da \ , \tag{2.34}$$

where $\vec{u}$ is a vector-valued function continuously differentiable in v, s is the surface bounding volume v and $\vec{n}$ is the outward unit normal to s, we arrive at the equivalent form of the Reynolds transport theorem

$$\frac{D}{Dt}\int_{v(t)} \phi dv = \int_{v(t)} \frac{\partial \phi}{\partial t} dv + \oint_{s(t)} (\vec{n} \cdot \vec{v}) \phi\, da \ , \tag{2.35}$$

where both ϕ and $\vec{v}$ are again required to be continuously differentiable in v.

The same form (2.35) holds also for a vector-valued function ϕ. The only difference is that the product rule

$$\phi \operatorname{div} \vec{v} + \vec{v} \cdot \operatorname{grad} \phi = \operatorname{div} (\vec{v} \otimes \phi) \ , \tag{2.36}$$

must be applied instead of (2.33) and the Gauss theorem must be used for a 2nd-order tensor $\boldsymbol{A}$ in the form

$$\int_v \operatorname{div} \boldsymbol{A} \, dv = \oint_s \vec{n} \cdot \boldsymbol{A} \, da \ . \tag{2.37}$$

To prove it, we apply the Gauss theorem (2.34) to vector $\boldsymbol{A} \cdot \vec{c}$, where $\vec{c}$ is a vector with constant magnitude and constant but arbitrary direction:

$$\int_v \operatorname{div} (\boldsymbol{A} \cdot \vec{c}) \, dv = \oint_s \vec{n} \cdot \boldsymbol{A} \cdot \vec{c} \, da \ .$$

Using the identity $\operatorname{div} (\boldsymbol{A} \cdot \vec{c}) = \operatorname{div} \boldsymbol{A} \cdot \vec{c}$, valid for a constant vector $\vec{c}$, we may further write

$$\left[\int_v \operatorname{div} \boldsymbol{A} \, dv - \oint_s \vec{n} \cdot \boldsymbol{A} \, da \right] \cdot \vec{c} = 0 \ .$$

Since $|\vec{c}| \neq 0$ and its direction is arbitrary, meaning that the cosine of the included angle cannot *always* vanish, the term in brackets must vanish, which verifies (2.37).

The material time derivative of a flux of a vector $\vec{q}$ across a surface $s(t)$ is

$$\frac{D}{Dt} \int_{s(t)} \vec{q} \cdot d\vec{a} = \int_{s(t)} \left(\frac{D\vec{q}}{Dt} + (\operatorname{div} \vec{v}) \vec{q} - \vec{q} \cdot \operatorname{grad} \vec{v} \right) \cdot d\vec{a} \ . \tag{2.38}$$

To prove this transport theorem, the integral over the spatial surface $s(t)$ is transformed into the referential, time-independent surface S, the derivative D/DT is carried inside of the integral, (1.75) is applied and then it is proceed in a similar manner as in the proof of (2.17).

2.4 Modified Reynolds's transport theorem

In the preceding section, we derived the Reynolds's transport theorem under the assumption that field quantities ϕ and $\vec{v}$ are continuously differentiable within volume $v(t)$. This assumption is also implemented in the Gauss theorems (2.34) and (2.37) for field variables $\vec{u}$ and $\boldsymbol{A}$, respectively. When a field variable does not satisfy the continuity conditions at a surface intersected volume $v(t)$, the two integral theorems must be modified.

The surface within a material body across which a physical quantity undergoes a discontinuity is called a *singular surface*. In particular, if the singular surface within a body is formed by the same material elements or particles at all times, it is called the *material surface*.

Let $\sigma(t)$ be a singular surface, not necessarily material, across which a physical variable may be discontinuous, but in the remaining part of a material body the variable is supposed to be continuously differentiable. Let us assume that $\sigma(t)$ moves with velocity $\vec{w}$, not necessarily equal to the material velocity $\vec{v}$. We will derive a condition for time evolution of $\sigma(t)$.

The moving singular surface $\sigma(t)$ can be defined implicitly by the equation

$$f_\sigma(\vec{x}, t) = 0 \qquad \vec{x} \in \sigma(t) \ , \tag{2.39}$$

where f_σ is differentiable. Let us assume that there is a surface Σ in the reference configuration κ_0 such that surfaces $\sigma(t)$ and Σ are related by an one-to-one mapping of the form

$$\vec{x} = \vec{\chi}_\sigma(\vec{X}, t) \qquad \vec{x} \in \sigma(t), \ \vec{X} \in \Sigma \ . \tag{2.40}$$

Since $\sigma(t)$ moves with velocity $\vec{w}$, in general different from the material velocity $\vec{v}$, the mapping $\vec{x} = \vec{\chi}_\sigma(\vec{X}, t)$ may differ from the motion $\vec{x} = \vec{\chi}(\vec{X}, t)$. The Lagrangian and Eulerian representations of the velocity of points of $\sigma(t)$ are given by (2.3) and (2.6), respectively, but now applied to the mapping (2.40):

$$\vec{W}(\vec{X}, t) = \left(\frac{\partial \vec{\chi}_\sigma}{\partial t} \right)\bigg|_{\vec{X}} , \qquad \vec{w}(\vec{x}, t) = \vec{W}(\vec{\chi}_\sigma^{-1}(\vec{x}, t), t) . \tag{2.41}$$

In view of (2.10), the material time derivative of the implicit equation (2.39) is

$$\frac{\partial f_\sigma}{\partial t} + \vec{w} \cdot \operatorname{grad} f_\sigma = 0 , \tag{2.42}$$

where the differentiation in gradient is with respect to $\vec{x}$. Introducing the unit normal $\vec{n}$ to $\sigma(t)$ by

$$\vec{n}(\vec{x}, t) = \frac{\operatorname{grad} f_\sigma}{|\operatorname{grad} f_\sigma|} \qquad \vec{x} \in \sigma(t) , \tag{2.43}$$

we can alternatively write

$$\frac{\partial f_\sigma}{\partial t} + |\operatorname{grad} f_\sigma|(\vec{n} \cdot \vec{w}) = 0 . \tag{2.44}$$

The condition (2.42) or (2.44) is called the *kinematic condition* for time evolution of moving surface $\sigma(t)$. The velocities $\vec{w}$ and $\vec{W}$ of singular surface $\sigma(t)$ in the reference and present configurations are known as the *displacement velocity* and the *propagation velocity*, respectively.

Consider a material volume v which is intersected by a moving singular surface $\sigma(t)$ across which a tensor-valued function $\boldsymbol{A}$ undergoes a jump. The surface $\sigma(t)$ divides the material volume v in two parts, namely v^+ into which the normal $\vec{n}$ is directed and v^- on the other. Figure 2.1 demonstrates this concept and the rule of sign convection. The Gauss theorem (2.37) is then modified to become

$$\int_{v-\sigma} \operatorname{div} \boldsymbol{A} \, dv = \oint_{s-\sigma} \vec{n} \cdot \boldsymbol{A} \, da - \int_\sigma \vec{n} \cdot [\boldsymbol{A}]_-^+ \, da . \tag{2.45}$$

The volume integral over $v - \sigma$ refers to the volume v of the body excluding the material points located on the singular surface σ. Similarly, the integral over the surface $s - \sigma$ excludes the line of intersection of σ with s, that is,

$$v - \sigma := v^+ + v^- , \qquad s - \sigma := s^+ + s^- . \tag{2.46}$$

The brackets indicate the jump of the enclosed quantity across $\sigma(t)$,

$$[\boldsymbol{A}]_-^+ := \boldsymbol{A}^+ - \boldsymbol{A}^- . \tag{2.47}$$

To prove (2.45), we apply the Gauss theorem (2.37) to the two volumes v^+ and v^- bounded by $s^+ + \sigma^+$ and $s^- + \sigma^-$, respectively. Hence

$$\int_{v^+} \operatorname{div} \boldsymbol{A} \, dv = \int_{s^+} \vec{n} \cdot \boldsymbol{A} \, da + \int_{\sigma^+} \vec{n}^+ \cdot \boldsymbol{A}^+ \, da ,$$

$$\int_{v^-} \operatorname{div} \boldsymbol{A} \, dv = \int_{s^-} \vec{n} \cdot \boldsymbol{A} \, da + \int_{\sigma^-} \vec{n}^- \cdot \boldsymbol{A}^- \, da ,$$

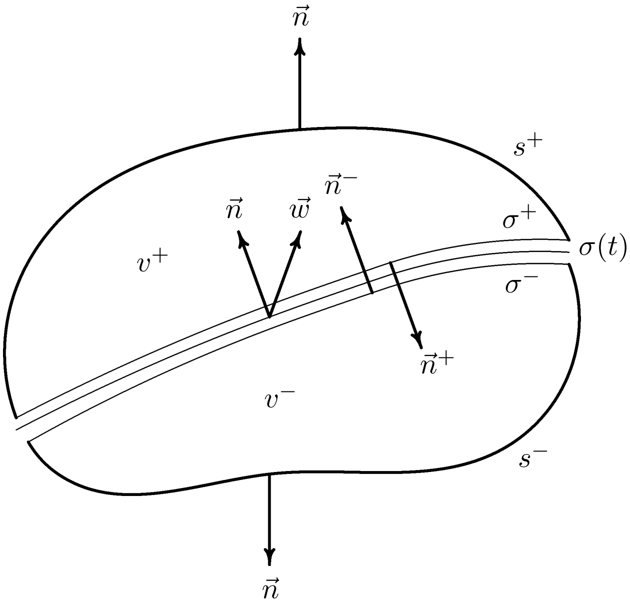

Figure 2.1. The moving singular surface $\sigma(t)$.

where $\vec{n}^+$ and $\vec{n}^-$ are the exterior normals to σ^+ and σ^-, respectively. Adding these two equations, we find

$$\int_{v^+ + v^-} \operatorname{div} \boldsymbol{A} \; dv = \oint_{s^+ + s^-} \vec{n} \cdot \boldsymbol{A} \; da + \int_{\sigma^+} \vec{n}^+ \cdot \boldsymbol{A}^+ \; da + \int_{\sigma^-} \vec{n}^- \cdot \boldsymbol{A}^- \; da \; .$$

From Figure 2.1 we can deduce that

$$\vec{n}^+ = -\vec{n}^- = -\vec{n} \; .$$

The negative sign at $\vec{n}$ appears since the normal vector at the singular surface is directed to its positive side. Finally, letting σ^+ and σ^- approach σ, we obtain

$$\int_{\sigma^+} \vec{n}^+ \cdot \boldsymbol{A}^+ \; da + \int_{\sigma^-} \vec{n}^- \cdot \boldsymbol{A}^- \; da = \int_{\sigma} \vec{n} \cdot \left(\boldsymbol{A}^- - \boldsymbol{A}^+ \right) \; da = - \int_{\sigma} \vec{n} \cdot [\boldsymbol{A}]_-^+ \; da \; .$$

The Reynolds transport theorem (2.31) has also to be modified once the singular surface $\sigma(t)$ moves with velocity $\vec{w}$ which differs from the material velocity $\vec{v}$. The modification of (2.31) reads

$$\frac{D}{Dt} \int_{v-\sigma} \phi dv = \int_{v-\sigma} \left(\frac{D\phi}{Dt} + \phi \operatorname{div} \vec{v} \right) dv + \int_{\sigma} \vec{n} \cdot [(\vec{v} - \vec{w}) \otimes \phi]_-^+ \, da \; . \tag{2.48}$$

Both ϕ and $\vec{v}$ are required to be continuously differentiable in $v - \sigma$. To prove (2.48), we apply (2.35) to the two volumes v^+ and v^- bounded by $s^+ + \sigma^+$ and $s^- + \sigma^-$, respectively. Hence

$$\frac{D}{Dt} \int_{v^+} \phi dv = \int_{v^+} \frac{\partial \phi}{\partial t} dv + \int_{s^+} (\vec{n} \cdot \vec{v}) \, \phi \, da + \int_{\sigma^+} (\vec{n}^+ \cdot \vec{w}) \, \phi^+ \, da \; ,$$

$$\frac{D}{Dt} \int_{v^-} \phi dv = \int_{v^-} \frac{\partial \phi}{\partial t} dv + \int_{s^-} (\vec{n} \cdot \vec{v}) \, \phi \, da + \int_{\sigma^-} (\vec{n}^- \cdot \vec{w}) \, \phi^- \, da \; .$$

Adding these two equations, letting σ^+ and σ^- approach σ and realizing that $\vec{n}^+ = -\vec{n}^- = -\vec{n}$, we obtain

$$\frac{D}{Dt}\int_{v^+ + v^-} \phi dv = \int_{v^+ + v^-} \frac{\partial \phi}{\partial t} dv + \oint_{s^+ + s^-} (\vec{n} \cdot \vec{v})\, \phi \, da - \int_\sigma (\vec{n} \cdot \vec{w})\, [\phi]_-^+ \, da \ .$$

Replacing the second term on the right-hand side from the Gauss theorem (2.45) applied to $\boldsymbol{A} = \vec{v} \otimes \phi$, we get

$$\frac{D}{Dt}\int_{v^+ + v^-} \phi dv = \int_{v^+ + v^-} \left(\frac{\partial \phi}{\partial t} + \text{div}\ (\vec{v} \otimes \phi) \right) dv + \int_\sigma \vec{n} \cdot [(\vec{v} - \vec{w}) \otimes \phi]_-^+ \, da \ .$$

To complete the proof, the first term on the right-hand side needs to be arranged by making use of (2.36).

3. MEASURES OF STRESS

3.1 Mass and density

Mass is a physical variable associated with a body. At the intuitive level, mass is perceived to be a measure of the amount of material contained in an arbitrary portion of body. As such it is *non-negative* scalar quantity independent of the time. Mass is *additive*, that is the mass of a body is the sum of the masses of its parts. These statetments imply the existence of a scalar field ϱ, assigned to each particle $\mathcal{X}$ such that the mass of the body $\mathcal{B}$ currently occupying finite volume $v(\mathcal{B})$ is determined by

$$m(\mathcal{B}) = \int_{v(\mathcal{B})} \varrho dv \ . \tag{3.1}$$

ϱ is called the *density* or the *mass density* of the material composing $\mathcal{B}$. As introduced, ϱ defines the mass per unit volume. If the mass is not continuous in $\mathcal{B}$, then instead of (3.1) we write

$$m(\mathcal{B}) = \int_{v_1(\mathcal{B})} \varrho dv + \sum_\alpha m_\alpha \ , \tag{3.2}$$

where the summation is taken over all *discrete* masses contained in the body. We shall be dealing with a continuous mass medium in which (3.1) is valid, which implies that $m(\mathcal{B}) \to 0$ as $v(\mathcal{B}) \to 0$. We therefore have

$$0 \le \varrho < \infty \ . \tag{3.3}$$

3.2 Volume and surface forces

The forces that act on a continuum or between portions of it may be divided into *long-range forces* and *short-range forces.*

Long-range forces are comprised of *gravitational, electromagnetic* and *inertial* forces. These forces decrease very gradually with increasing distance between interacting particles. As a result, long-range forces act uniformly on all matter contained within a sufficiently small volume, so that, they are proportional to the volume size involved. In continuum mechanics, long-range forces are referred to as *volume* or *body forces.*

The body force acting on $\mathcal{B}$ is specified by vector field $\vec{f}$ defined on the configuration $\mathcal{B}$. This field is taken as measured per unit mass and is assumed to be continuous. The total body force acting on the body $\mathcal{B}$ currently occupying finite volume $v(\mathcal{B})$ is expressed as

$$\vec{F}(\mathcal{B}) = \int_{v(\mathcal{B})} \varrho \vec{f} dv \ . \tag{3.4}$$

Short-range forces comprise several types of *molecular* forces. Their characteristic feature is that they decrease extremely abruptly with increasing distance between the interacting particles. Hence, they are of consequence only when this distance does not exceed molecular dimensions. As a result, if matter inside a volume is acted upon by short-range forces originating from interactions with matter outside this volume, these forces only act upon a thin layer immediately below its surface. In continuum mechanics, short-range forces are called *surface* or *contact* forces and are specified more closely by constitutive equations (Chapter 5).

3.3 Cauchy traction principle

A mathematical description of surface forces stems from the following Cauchy traction principle.

We consider a material body $b(t)$ which is subject to body forces $\vec{f}$ and surface forces $\vec{g}$. Let p be an interior point of $b(t)$ and imagine a plane surface a^* passing through point p (sometimes referred to as a *cutting plane*) so as to partition the body into two portions, designated I and II (Figure 3.1). Point p is lying in the area element Δa^* of the cutting plane, which is defined by the unit normal $\vec{n}$ pointing in the direction from Portion I into Portion II, as shown in Figure 3.1. The internal forces being transmitted across the cutting plane due to the action of Portion II upon Portion I will give rise to a force distribution on Δa^* equivalent to a resultant surface force $\Delta\vec{g}$, as also shown in Figure 3.1. (For simplicity, body forces and surface forces acting on the body as a whole are not drawn in Figure 3.1.) Notice that $\Delta\vec{g}$ are not necessarily in the direction of the unit normal vector $\vec{n}$. The *Cauchy traction principle* postulates that the limit when the area Δa^* shrinks to zero, with p remaining an interior point, exists and is given by

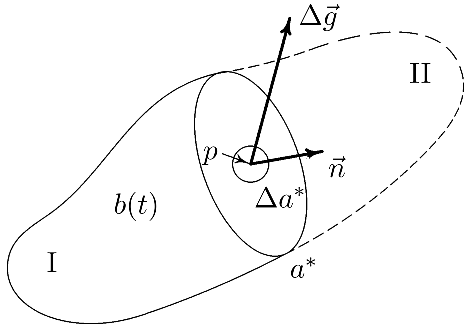

Figure 3.1.
Surface force on surface element Δa^*.

$$\vec{t}_{(\vec{n})} = \lim_{\Delta a^* \to 0} \frac{\Delta\vec{g}}{\Delta a^*} \ . \tag{3.5}$$

Obviously, this limit is meaningful only if Δa^* degenerates not into a curve but into a point p. The vector $\vec{t}_{(\vec{n})}$ is called the *Cauchy stress vector* or the *Cauchy traction vector* (force per unit area). It is important to note that, in general, $\vec{t}_{(\vec{n})}$ depends not only on the position of p on Δa^* but also the orientation of surface Δa^*, i.e., on its external normal $\vec{n}$. This dependence is therefore indicated by the subscript $\vec{n}$. [9] Thus, for the infinity of cutting planes imaginable through point p, each identified by a specific $\vec{n}$, there is also an infinity of associated stress vectors $\vec{t}_{(\vec{n})}$ for a given loading of the body.

We incidentally mention that a continuous distribution of surface forces acting across some surface is, in general, equivalent to a resultant force and a resultant torque. In (3.5) we have made the assumption that, in the limit at p, the torque per unit area vanishes and therefore there is no remaining concentrated torque, or *couple stress*. This material is called the *non-polar continuum.* For a discussion of couple stresses and polar media, the reader is referred to Eringen, 1967.

3.4 Cauchy lemma

To determine the dependence of the stress vector on the exterior normal, we next apply the principle of balance of linear momentum to a small tetrahedron of volume Δv having its vertex at p, three coordinate surfaces Δa_k, and the base Δa on a with an oriented normal $\vec{n}$ (Figure 3.2). The stress vector [10] on the coordinate surface $x_k =$ const. is denoted by $-\vec{t}_k$.

[9] The assumption that the stress vector $\vec{t}_{(\vec{n})}$ depends only on the outer normal vector $\vec{n}$ and not on differential geometric property of the surface such as the curvature, has been introduced by Cauchy and is referred to as the Cauchy assumption.

[10] Since the exterior normal of a coordinate surface x_k = const. is in the direction of $-x_k$, without loss in generality, we denote the stress vector acting on this coordinate surface by $-\vec{t}_k$ rather than $\vec{t}_k$.

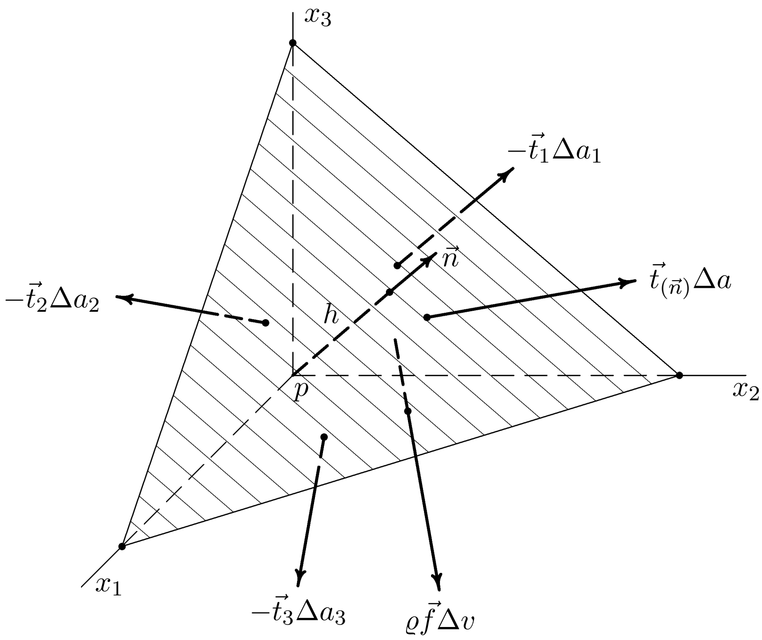

Figure 3.2. Equilibrium of an infinitesimal tetrahedron.

We now apply the equation of balance of linear momentum (Sect.4.1) to this tetrahedron,

$$\int_{\Delta v} \varrho \vec{f} dv - \int_{\Delta a_k} \vec{t}_k da_k + \int_{\Delta a} \vec{t}_{(\vec{n})} da = \frac{D}{Dt} \int_{\Delta v} \varrho \vec{v} dv \ .$$

The surface and volume integrals may be evaluated by the mean value theorem:

$$\varrho^* \vec{f}^* \Delta v - \vec{t}_k^* \Delta a_k + \vec{t}_{(\vec{n})}^* \Delta a = \frac{D}{Dt} \left(\varrho^* \vec{v}^* \Delta v \right) \ , \tag{3.6}$$

where ϱ^*, $\vec{f}^*$, and $\vec{v}^*$ are, respectively, the values of ϱ, $\vec{f}$, and $\vec{v}$ at some interior points of the tetrahedron and $\vec{t}_{(\vec{n})}^*$ and $\vec{t}_k^*$ are the values of $\vec{t}_{(\vec{n})}$ and $\vec{t}_k$ on the surface Δa and on coordinate surfaces Δa_k. The volume of the tetrahedron is given by

$$\Delta v = \frac{1}{3} h \Delta a \ , \tag{3.7}$$

where h is the perpendicular distance from point p to the base Δa. Moreover, the area vector $\Delta \vec{a}$ is equal to the sum of coordinate area vectors, that is,

$$\Delta \vec{a} = \vec{n} \Delta a = \Delta a_k \vec{i}_k \ . \tag{3.8}$$

Thus

$$\Delta a_k = n_k \Delta a \ . \tag{3.9}$$

Inserting (3.7) and (3.9) into (3.6) and canceling the common factor Δa, we obtain

$$\frac{1}{3} \varrho^* \vec{f}^* h - \vec{t}_k^* n_k + \vec{t}_{(\vec{n})}^* = \frac{1}{3} \frac{D}{Dt} \left(\varrho^* \vec{v}^* h \right) \ . \tag{3.10}$$

Now, letting the tetrahedron shrink to point p by taking the limit $h \to 0$ and noting that in this limiting process the starred quantities take on the actual values of those same quantities at point p, we have

$$\vec{t}_{(\vec{n})} = \vec{t}_k n_k \ , \tag{3.11}$$

which is the *Cauchy stress formula*. Equation (3.11) allows the determination of the Cauchy stress vector at some point acting across an arbitrarily inclined plane, if the Cauchy stress vectors acting across the three coordinate surfaces through that point are known.

The stress vectors $\vec{t}_k$ are, by definition, independent of $\vec{n}$. From (3.11) it therefore follows that

$$-\vec{t}_{(-\vec{n})} = \vec{t}_{(\vec{n})} \ . \tag{3.12}$$

The stress vector acting on a surface with the unit normal $\vec{n}$ is equal to the negative stress vector acting on the corresponding surface with the unit normal $-\vec{n}$. In Newtonian mechanics this statement is known as Newton's third law. The calculations show that this statement is valid for stress vector.

We now introduce the definition of the Cauchy stress tensor. The t_{kl} component of the *Cauchy stress tensor* $\boldsymbol{t}$ is given by the lth component of the stress vector $\vec{t}_k$ acting on the positive side of the kth coordinate surface:

$$\vec{t}_k = t_{kl}\vec{i}_l \qquad \text{or} \qquad t_{kl} = \vec{t}_k \cdot \vec{i}_l \ . \tag{3.13}$$

The first subscript in t_{kl} indicates the coordinate surface x_k = const. on which the stress vector $\vec{t}_k$ acts, while the second subscript indicates the direction of the component of $\vec{t}_k$. For example, t_{23} is the x_3-components of the stress vector $\vec{t}_2$ acting on the coordinate surface x_2 = const.. Now, if the exterior normal of x_2 = const. points in the positive direction of the x_2-axis, t_{23} points in the positive direction of the x_3-axis. If the exterior normal of x_2 = const. is in the negative direction of the x_2-axis, t_{23} is directed in the negative direction of the x_3-axis. The positive stress components on the faces of a parallelepiped built on the coordinate surfaces are shown in Figure 3.3. The nine components t_{kl} of the Cauchy stress tensor $\boldsymbol{t}$ may be arranged in a matrix form

$$\boldsymbol{t} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} (\vec{i}_k \otimes \vec{i}_l) \ . \tag{3.14}$$

Considering (3.13), the Cauchy stress formula (3.11) reads

$$\vec{t}_{(\vec{n})} = \vec{n} \cdot \boldsymbol{t} \ , \tag{3.15}$$

which says that the Cauchy stress vector acting on any plane through a point is fully characterized as a linear function of the stress tensor at that point. The normal component of stress vector,

$$t_n = \vec{t}_{(\vec{n})} \cdot \vec{n} = \vec{n} \cdot \boldsymbol{t} \cdot \vec{n} \ , \tag{3.16}$$

is called the *normal stress* and is said to be *tensile* when positive and *compressive* when negative. The stress vector directed tangentially to surface has the form

$$\vec{t}_t = \vec{t}_{(\vec{n})} - t_n\vec{n} = \vec{n} \cdot \boldsymbol{t} - (\vec{n} \cdot \boldsymbol{t} \cdot \vec{n})\vec{n} \ . \tag{3.17}$$

The size of $\vec{t}_t$ is known as the *shear stress*. For example, the components t_{11}, t_{22} and t_{23} in Figure 3.3 are the normal stresses and the mixed components t_{12}, t_{13}, etc. are the shear stresses.

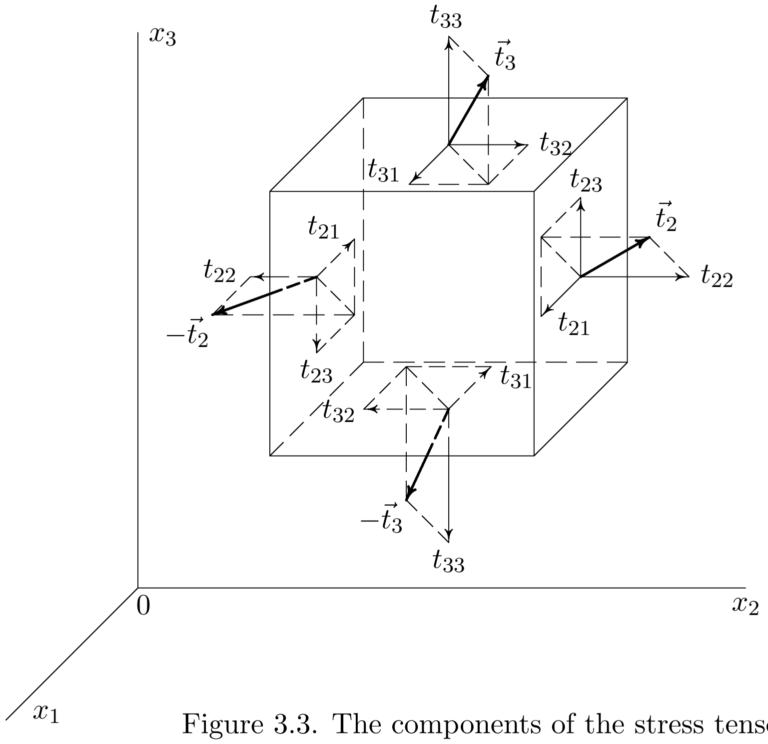

Figure 3.3. The components of the stress tensor.

If, in some configuration, the shear stress is identically zero and the normal stress is independent of $\vec{n}$, the stress is said to be *spherical.* In this case, there is a scalar field p, called the *pressure*, such that

$$\vec{t}_{(\vec{n})} = -p\vec{n} \qquad \text{and} \qquad \boldsymbol{t} = -p\boldsymbol{I} \ . \tag{3.18}$$

3.5 Other measures of stress

So far, we have represented short-range intermolecular forces in terms of the Cauchy stress vector $\vec{t}_k$ or tensor $\boldsymbol{t}$. There are, however, two other ways of measuring or representing these forces, each of which plays a certain role in the theory of continuum mechanics. The Eulerian Cauchy stress tensor gives the surface force acting on the deformed elementary area da in the form

$$d\vec{g} = \vec{t}_{(\vec{n})}da = (\vec{n}\cdot\boldsymbol{t})da = d\vec{a}\cdot\boldsymbol{t} \ . \tag{3.19}$$

The Cauchy stress tensor, like any other variable, has both an Eulerian and a Lagrangian description; the corresponding Lagrangian Cauchy stress tensor is defined by $\boldsymbol{T}(\vec{X},t) := \boldsymbol{t}(\vec{x}(\vec{X},t),t)$. We make, however, an exception in the notation and use $\boldsymbol{t}(\vec{X},t)$ for the Lagrangian description of the Cauchy stress tensor. The Eulerian Cauchy stress tensor $\boldsymbol{t}(\vec{x},t)$ arises naturally in the Eulerian form of the balance of linear momentum; the corresponding Lagrangian form of this principle cannot, however, be readily expressed in terms of the Lagrangian Cauchy stress tensor $\boldsymbol{t}(\vec{X},t)$.

A simple Lagrangian form of the balance of linear momentum can be obtained if a stress measure is referred to a surface in the reference configuration. This can be achieved by introducing the so-called *first Piola-Kirchhoff stress tensor* $\boldsymbol{T}^{(1)}$ as a stress measure referred to the referential area element $d\vec{A}$:

$$d\vec{g} = d\vec{a} \cdot \boldsymbol{t} =: d\vec{A} \cdot \boldsymbol{T}^{(1)} \; . \tag{3.20}$$

Here, the tensor $\boldsymbol{T}^{(1)}$ gives the surface force acting on the deformed area $d\vec{a}$ at $\vec{x}$ in terms of the corresponding referential element $d\vec{A}$ at the point $\vec{X}$. Thus, $\boldsymbol{T}^{(1)}$ is a measure of the force per unit referential area, whereas both the Eulerian and Lagrangian Cauchy stresses $\boldsymbol{t}(\vec{x}, t)$ and $\boldsymbol{t}(\vec{X}, t)$ are measures of the force per unit spatial area. The relationship between $\boldsymbol{T}^{(1)}$ and $\boldsymbol{t}$ is found using the transformation (1.75) between the deformed and undeformed elementary areas. The result can be expressed in either of the two equivalent forms

$$\boldsymbol{T}^{(1)}(\vec{X}, t) = J\boldsymbol{F}^{-1} \cdot \boldsymbol{t}(\vec{X}, t) \; , \qquad \boldsymbol{t}(\vec{X}, t) = J^{-1}\boldsymbol{F} \cdot \boldsymbol{T}^{(1)}(\vec{X}, t) \; . \tag{3.21}$$

The surface-force vector $d\vec{g}$ in (3.20) acts upon the displaced point $\vec{x}$, whereas the surface-element vector $d\vec{A}$ is referred to the reference point $\vec{X}$. The first Piola-Kirchhoff stress tensor $\boldsymbol{T}^{(1)}$ is therefore a two-point tensor. This can also be observed from the componental form of (3.21):

$$T^{(1)}_{Kl}(\vec{X}, t) = JX_{K,k}t_{kl}(\vec{X}, t) \; , \qquad t_{kl}(\vec{X}, t) = J^{-1}x_{k,K}T^{(1)}_{Kl}(\vec{X}, t) \; . \tag{3.22}$$

The constitutive equations for a simple materials (see equation (5.37) in Chapter 5) are expressed most conveniently in terms of another measure of stress, known as the *second Piola-Kirchhoff stress tensor*. This quantity, denoted by $\boldsymbol{T}^{(2)}$, gives, instead of the actual surface force $d\vec{g}$ acting on the deformed area element $d\vec{a}$, a force $d\vec{G}$ related to $d\vec{g}$ in the same way as the referential differential $d\vec{X}$ is related to the spatial differential $d\vec{x}$. That is

$$d\vec{G} = \boldsymbol{F}^{-1} \cdot d\vec{g} \; , \tag{3.23}$$

in the same manner as $d\vec{X} = \boldsymbol{F}^{-1} \cdot d\vec{x}$. Defining $\boldsymbol{T}^{(2)}$ by

$$d\vec{G} =: d\vec{A} \cdot \boldsymbol{T}^{(2)} \; , \tag{3.24}$$

we find the first and second Piola-Kirchhoff stresses are related by

$$\boldsymbol{T}^{(2)} = \boldsymbol{T}^{(1)} \cdot \boldsymbol{F}^{-T} \; , \qquad \boldsymbol{T}^{(1)} = \boldsymbol{T}^{(2)} \cdot \boldsymbol{F}^{T} \; . \tag{3.25}$$

Comparing this result with (3.21), we obtain the corresponding relationship between the second Piola-Kirchhoff stress tensor and the Lagrangian Cauchy stress tensor:

$$\boldsymbol{T}^{(2)}(\vec{X}, t) = J\boldsymbol{F}^{-1} \cdot \boldsymbol{t}(\vec{X}, t) \cdot \boldsymbol{F}^{-T} \; , \qquad \boldsymbol{t}(\vec{X}, t) = J^{-1}\boldsymbol{F} \cdot \boldsymbol{T}^{(2)}(\vec{X}, t) \cdot \boldsymbol{F}^{T} \; . \tag{3.26}$$

Since the transformed surface force $d\vec{G}$ may be considered to act at the referential position $\vec{X}$ rather than at the spatial position $\vec{x}$, the second Piola-Kirchhoff stress tensor is an ordinary (a one-point) rather than a two-point tensor. This can also be seen from the componental form of (3.25):

$$T^{(2)}_{KL}(\vec{X}, t) = JX_{K,k}X_{L,l}t_{kl}(\vec{X}, t) \; , \qquad t_{kl}(\vec{X}, t) = J^{-1}x_{k,K}x_{l,L}T^{(2)}_{KL}(\vec{X}, t) \; . \tag{3.27}$$

The foregoing expressions may be used as a source for the linearized theory in which the displacement gradient $\boldsymbol{H}$ is much smaller when compared to unity, hence justifying linearization. To this end, we carry the linearized forms $(1.106)_{1,2}$ into $(3.21)_1$ and $(3.26)_1$ and obtain

$$\begin{aligned} \boldsymbol{T}^{(1)} &= (1 + \operatorname{tr} \boldsymbol{H})\, \boldsymbol{t} - \boldsymbol{H}^T \cdot \boldsymbol{t} + O(\|\boldsymbol{H}\|^2) \; , \\ \boldsymbol{T}^{(2)} &= (1 + \operatorname{tr} \boldsymbol{H})\, \boldsymbol{t} - \boldsymbol{H}^T \cdot \boldsymbol{t} - \boldsymbol{t} \cdot \boldsymbol{H} + O(\|\boldsymbol{H}\|^2) \; . \end{aligned} \tag{3.28}$$

The last equation demonstrates that the symmetry of tensor $\boldsymbol{T}^{(2)}$ has not been violated by linearization process. Conversely,

$$\begin{aligned} \boldsymbol{t} &= (1 - \operatorname{tr} \boldsymbol{H})\, \boldsymbol{T}^{(1)} + \boldsymbol{H}^T \cdot \boldsymbol{T}^{(1)} + O(\|\boldsymbol{H}\|^2) \\ &= (1 - \operatorname{tr} \boldsymbol{H})\, \boldsymbol{T}^{(2)} + \boldsymbol{H}^T \cdot \boldsymbol{T}^{(2)} + \boldsymbol{T}^{(2)} \cdot \boldsymbol{H} + O(\|\boldsymbol{H}\|^2) \ . \end{aligned} \tag{3.29}$$

Supposing, in addition, that stresses are small compared to unity (the *infinitesimal deformation and stress theory*), then

$$\boldsymbol{T}^{(1)} \cong \boldsymbol{T}^{(2)} \cong \boldsymbol{t} \ , \tag{3.30}$$

showing that, when considering infinitesimal deformation and stress, a distinction between the Cauchy and the Piola-Kirchhoff stresses is not necessary.

4. FUNDAMENTAL BALANCE LAWS

4.1 Global balance laws

The fundamental laws of continuum mechanics are principles dealing with the conservation of some physical quantity. These *balance laws*, as they are often called, are postulated for all material continua, irrespective of material constitution and geometry, and result in equations that must always be satisfied. These conservations laws deal with mass, linear and angular momentum, energy and entropy. They are valid for all bodies subject to thermomechanical effects.

The balance laws are usually formulated in global (integral) form derived by a consideration of the conservation of some property of the body as a whole. The global equations then may be used to develop associated *field equations* that are valid at all points within the body and on its boundary.

Fundamental Principle 1 *(Conservation of Mass). The total mass of a body is unchanged with motion.*

This principle assumes that the mass production and supply is zero. Hence, the mass of a body is invariant under motion and remains constant in every configuration:

$$\int_V \varrho_0 \, dV = \int_{v(t)} \varrho \, dv \ , \tag{4.1}$$

where V and $v(t)$ are the reference and the current volumes of the body, respectively, $\varrho_0(\vec{X})$ is the mass density of the body in the reference configuration and $\varrho(\vec{x}, t)$ is the mass density of the body in the present configuration. Applying the material derivative to (4.1) results in the alternative form

$$\frac{D}{Dt} \int_{v(t)} \varrho \, dv = 0 \ . \tag{4.2}$$

Fundamental Principle 2 *(Balance of Linear Momentum). The time rate of change of the total linear momentum of a body is equal to the resultant force acting on the body.*

Let a body having a current volume $v(t)$ and bounding surface $s(t)$ with exterior unit normal $\vec{n}$ be subject to surface traction $\vec{t}_{(\vec{n})}$ and body force $\vec{f}$ (body force per unit mass of the body). The resultant force acting on the body is

$$\oint_{s(t)} \vec{t}_{(\vec{n})} \, da + \int_{v(t)} \varrho \vec{f} \, dv \ .$$

In addition, let the body be in motion under the velocity field $\vec{v}(\vec{x}, t)$. The *linear momentum* of the body is defined by the vector

$$\int_{v(t)} \varrho \vec{v} \, dv \ .$$

Thus the balance of linear momentum states that

$$\frac{D}{Dt} \int_{v(t)} \varrho \vec{v} \, dv = \oint_{s(t)} \vec{t}_{(\vec{n})} \, da + \int_{v(t)} \varrho \vec{f} \, dv \ . \tag{4.3}$$

Fundamental Principle 3 *(Balance of Angular Momentum). The time rate of change of the total angular momentum of a body is equal to the resultant moment of all forces acting on the body.*

Mathematically, this principle is expressed as

$$\frac{D}{Dt}\int_{v(t)} \vec{x}\times\varrho\vec{v}\,dv = \oint_{s(t)} \vec{x}\times\vec{t}_{(\vec{n})}\,da + \int_{v(t)} \vec{x}\times\varrho\vec{f}\,dv \ , \tag{4.4}$$

where the left-hand side is the time rate of change of the total *angular momentum* about the origin, which is also frequently called the *moment of momentum.* On the right-hand side the surface integral is the moment of the surface tractions about the origin, and the volume integral is the total moment of body forces about the origin.

Fundamental Principle 4 *(Conservation of Energy). The time rate of change of the sum of kinetic energy $\mathcal{K}$ and internal energy $\mathcal{E}$ is equal to the sum of the rate of work $\mathcal{W}$ of the surface and body forces and all other energies $\mathcal{U}_\alpha$ that enter and leave body per unit time.*

Mathematically, this principle is expressed as

$$\frac{D}{Dt}(\mathcal{K}+\mathcal{E}) = \mathcal{W} + \sum_\alpha \mathcal{U}_\alpha \ . \tag{4.5}$$

The total *kinetic energy* of the body is given by

$$\mathcal{K} = \frac{1}{2}\int_{v(t)} \varrho\vec{v}\cdot\vec{v}\,dv \ . \tag{4.6}$$

In continuum mechanics, the existence of the internal energy density ε is postulated such that

$$\mathcal{E} = \int_{v(t)} \varrho\varepsilon\,dv \ . \tag{4.7}$$

The *mechanical power*, or *rate of work* of the surface traction $\vec{t}_{(\vec{n})}$ and body forces $\vec{f}$ is given by

$$\mathcal{W} = \oint_{s(t)} \vec{t}_{(\vec{n})}\cdot\vec{v}\,da + \int_{v(t)} \varrho\vec{f}\cdot\vec{v}\,dv \ . \tag{4.8}$$

Other energies $\mathcal{U}_\alpha$ ($\alpha = 1, 2, ..., n$) that enter and leave the body may be of thermal, electromagnetic, chemical, or some other origin. In this text, we consider that the energy transfer in continuum is *thermo-mechanical* and thus only due to work or heat. The *heat energy* consists of the *heat flux* per unit area $\vec{q}$ that enters or leaves through the surface of the body and the *heat source*, or *heat supply*, h per unit mass produced by internal sources (for example, radioactive decay). Thus we set $\mathcal{U}_\alpha = 0$ except for

$$\mathcal{U}_1 := -\oint_{s(t)} \vec{q}\cdot\vec{n}\,da + \int_{v(t)} \varrho h\,dv \ , \tag{4.9}$$

where the unit normal $\vec{n}$ is directed outward from the surface of the body. The negative sign at the surface integral is needed because the heat flux vector $\vec{q}$ is pointing from high temperature toward lower temperature, so that $\oint_{s(t)} \vec{q} \cdot \vec{n}\, da$ is the total outward heat flux. Thus the principle of conservation of energy states that

$$\frac{D}{Dt} \int_{v(t)} (\varrho\varepsilon + \frac{1}{2}\varrho\vec{v} \cdot \vec{v})\, dv = \oint_{s(t)} (\vec{t}_{(\vec{n})} \cdot \vec{v} - \vec{q} \cdot \vec{n})\, da + \int_{v(t)} (\varrho\vec{f} \cdot \vec{v} + \varrho h)\, dv \ , \tag{4.10}$$

which is the statement of the *first law of thermodynamics.*

Fundamental Principle 5 *(Entropy inequality). The time rate of change of the total entropy H is never less than the sum of the flux of entropy $\vec{s}$ through the surface of the body and the entropy B supplied by the body forces. This law is postulated to hold for all independent processes.*

Mathematically, this principle is expressed as

$$\Gamma := \frac{DH}{Dt} - B + \oint_{s(t)} \vec{s} \cdot \vec{n}\, da \geq 0 \ , \tag{4.11}$$

where Γ is the *total entropy production.* Note that $\vec{s}$ is the outward entropy flux vector. In classical continuum mechanics, the entropy density η per unit mass and the entropy source b per unit mass and unit time are postulated to exist such that

$$H = \int_{v(t)} \varrho\eta\, dv \ , \qquad B = \int_{v(t)} \varrho b\, dv \ .$$

The entropy inequality then becomes

$$\frac{D}{Dt} \int_{v(t)} \varrho\eta\, dv - \int_{v(t)} \varrho b\, dv + \oint_{s(t)} \vec{s} \cdot \vec{n}\, da \geq 0 \ . \tag{4.12}$$

The inequality implies internal entropy production in an irreversible process; the equality holds for a reversible process.

At the first moment is seems unclear why we consider an additional variable - the entropy - for a complete description of thermomechanical phenomena. Experience tell us, however, that the real physical processes are directional, that is, they can proceed only in a certain chronology but not in the reverse of this. This principle of *irreversibility* can be accounted for by the introduction of the balance statement for entropy, in which it is required that its specific production can always have only one sign for all realistic thermomechanical processes. More precise specifications of this phenomenological idea will be given later. Hence, we consider the entropy (or its density) and temperature as primitive variables the existence of which is unquestioned.

The five laws are postulated to hold for all bodies, irrespective of their geometries and constitutions. To obtain local equations, further restrictions are necessary, as will be discussed in the next section.

4.2 Local balance laws in the spatial description

4.2.1 Continuity equation

The application of the Reynolds transport theorem (2.48) with $\phi = \varrho$ to (4.2) results in

$$\int_{v(t)-\sigma(t)} \left(\frac{D\varrho}{Dt} + \varrho \operatorname{div} \vec{v}\right)\, dv + \int_{\sigma(t)} \vec{n} \cdot [\varrho(\vec{v} - \vec{w})]^{+}_{-}\, da = 0 \ . \tag{4.13}$$

We now assume that density $\varrho(\vec{x}, t)$, velocity $\vec{v}$ and $[\varrho(\vec{v} - \vec{w})]_-^+$ are continuously differentiable functions of the spatial variables x_k and time t in the volume $v(t) - \sigma(t)$ and across the discontinuity $\sigma(t)$, respectively. This implies that the integrands of the volume and surface integrals in (4.13) are continuous in x_k and t. Moreover, we postulate that *all global balance laws are valid for an arbitrary part of the volume and of the discontinuity surface* (*the additive principle*). [11] Applied to (4.13), this implies that integrands of each of the integral must vanish identically. Thus

$$\begin{aligned} \frac{D\varrho}{Dt} + \varrho \operatorname{div} \vec{v} &= 0 \qquad \text{in } \ v(t) - \sigma(t) \ , && (4.14) \\ \vec{n} \cdot [\varrho(\vec{v} - \vec{w})]_-^+ &= 0 \qquad \text{on } \ \sigma(t) \ . && (4.15) \end{aligned}$$

These are the equations of local conservation of mass and the interface condition in spatial form. Equation (4.14) is often called the *continuity equation*. Writing the material time derivative of ϱ as

$$\frac{D\varrho}{Dt} = \frac{\partial \varrho}{\partial t} + \vec{v} \cdot \operatorname{grad} \varrho$$

allows (4.14) to be expressed in the alternative form

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \vec{v}) = 0 \qquad \text{in } \ v(t) - \sigma(t) \ . \tag{4.16}$$

Equation (4.15) means that the mass flux (the amount of mass per unit surface and unit time) entering the discontinuity surface must leave it on the other side, that is, there is no mass cummulation on the discontinuity. In other words, the quantity in square brackets in (4.15) is the amount of mass swept through a running discontinuity in relation to the motion of body behind and ahead of the surface. According to (4.15), they must be equal.

4.2.2 Equation of motion

By substituting for the Cauchy stress vector $\vec{t}_{(\vec{n})}$ from (3.15), the equation of global balance of linear momentum (4.3) reads

$$\frac{D}{Dt} \int_{v(t)-\sigma(t)} \varrho \vec{v} \, dv = \oint_{s(t)-\sigma(t)} \vec{n} \cdot \boldsymbol{t} \, da + \int_{v(t)-\sigma(t)} \varrho \vec{f} \, dv \ . \tag{4.17}$$

By applying the modified Gauss's theorem (2.45) to the surface integral on the right-hand side, we obtain

$$\frac{D}{Dt} \int_{v(t)-\sigma(t)} \varrho \vec{v} \, dv = \int_{v(t)-\sigma(t)} (\operatorname{div} \boldsymbol{t} + \varrho \vec{f}) \, dv + \int_{\sigma(t)} \vec{n} \cdot [\boldsymbol{t}]_-^+ \, da \ , \tag{4.18}$$

which upon using Reynolds's transport theorem (2.48) with $\phi = \varrho \vec{v}$ yields

$$\int_{v(t)-\sigma(t)} \left(\frac{D(\varrho \vec{v})}{Dt} + \varrho \vec{v} \operatorname{div} \vec{v} - \operatorname{div} \boldsymbol{t} - \varrho \vec{f} \right) dv + \int_{\sigma(t)} \vec{n} \cdot [\varrho(\vec{v} - \vec{w}) \otimes \vec{v} - \boldsymbol{t}]_-^+ \, da = \vec{0} \ . \tag{4.19}$$

[11] For nonlocal continuum theories this postulate is revoked, and only the global balance laws (valid for the entire body) are considered to be valid.

This is postulated to be valid for all parts of the body (satisfying the additive principle). Thus, the integrands vanish separately. By using (4.14), this is simplified to

$$\begin{aligned} \operatorname{div} \boldsymbol{t} + \varrho \vec{f} &= \varrho \frac{D\vec{v}}{Dt} && \text{in } \; v(t) - \sigma(t) \; , && (4.20) \\ \vec{n} \cdot [\varrho(\vec{v} - \vec{w}) \otimes \vec{v} - \boldsymbol{t}]^+_- &= \vec{0} && \text{on } \; \sigma(t) \; . && (4.21) \end{aligned}$$

Equation (4.20) is known as *Cauchy's equation of motion*, expressing the local balance of linear momentum in spatial form, and (4.21) is the associated interface condition on the singular surface σ.

4.2.3 Symmetry of the Cauchy stress tensor

The angular momentum of the surface tractions about the origin occurring in the global law of balance of the angular momentum (4.4), can be rewritten using the Cauchy stress formula (3.15), the tensor identity

$$\vec{v} \times (\vec{w} \cdot \boldsymbol{A}) = -\vec{w} \cdot (\boldsymbol{A} \times \vec{v}) \; , \tag{4.22}$$

where $\vec{v}$, $\vec{w}$ are vectors, $\boldsymbol{A}$ is a second-order tensor, and the modified Gauss theorem (2.45), to the form

$$\oint_{s(t)-\sigma(t)} \vec{x} \times \vec{t}_{(\vec{n})} \, da = -\int_{v(t)-\sigma(t)} \operatorname{div} (\boldsymbol{t} \times \vec{x}) \, dv - \int_{\sigma(t)} \vec{n} \cdot [\boldsymbol{t} \times \vec{x}]^+_- \, da \; . \tag{4.23}$$

By making use of the two differential identities,

$$\operatorname{div} (\boldsymbol{t} \times \vec{v}) = \operatorname{div} \boldsymbol{t} \times \vec{v} + \boldsymbol{t}^T \stackrel{\bullet}{\times} \operatorname{grad} \vec{v} \; , \tag{4.24}$$

$$\operatorname{grad} \vec{x} = \boldsymbol{I} \; , \tag{4.25}$$

where the superscript T at tensor $\boldsymbol{t}$ stands for transposition, $\stackrel{\bullet}{\times}$ denotes the dot-cross product of the 2nd order tensors, and $\boldsymbol{I}$ is the second-order identity tensor, we can write

$$\int_{v(t)-\sigma(t)} \operatorname{div} (\boldsymbol{t} \times \vec{x}) \, dv = \int_{v(t)-\sigma(t)} (\operatorname{div} \boldsymbol{t} \times \vec{x} + \boldsymbol{t}^T \stackrel{\bullet}{\times} \boldsymbol{I}) \, dv \; .$$

Upon carrying this and (4.23) into the equation of balance of the angular momentum (4.3) and using the Reynolds transport theorem (2.48) with $\phi = \vec{x} \times \varrho \vec{v}$, we obtain

$$\begin{aligned} &\int_{v(t)-\sigma(t)} \left(\frac{D(\varrho \vec{x} \times \vec{v})}{Dt} + (\varrho \vec{x} \times \vec{v}) \operatorname{div} \vec{v} + \operatorname{div} \boldsymbol{t} \times \vec{x} + \boldsymbol{t}^T \stackrel{\bullet}{\times} \boldsymbol{I} - \varrho \vec{x} \times \vec{f} \right) dv \\ &\quad + \int_{\sigma(t)} \vec{n} \cdot [(\vec{v} - \vec{w}) \otimes (\vec{x} \times \varrho \vec{v})]^+_- \, da + \int_{\sigma(t)} \vec{n} \cdot [\boldsymbol{t} \times \vec{x}]^+_- \, da = \vec{0} \; , \end{aligned}$$

which can be arranged to the form

$$\begin{aligned} &\int_{v(t)-\sigma(t)} \left[(\vec{x} \times \vec{v}) \left(\frac{D\varrho}{Dt} + \varrho \operatorname{div} \vec{v} \right) + \varrho \frac{D\vec{x}}{Dt} \times \vec{v} + \vec{x} \times \left(\varrho \frac{D\vec{v}}{Dt} - \operatorname{div} \boldsymbol{t} - \varrho \vec{f} \right) + \boldsymbol{t}^T \stackrel{\bullet}{\times} \boldsymbol{I} \right] dv \\ &\quad - \int_{\sigma(t)} \vec{n} \cdot [\varrho(\vec{v} - \vec{w}) \otimes \vec{v} - \boldsymbol{t}]^+_- \times \vec{x} \, da = \vec{0} \; . \end{aligned} \tag{4.26}$$

Considering

$$\frac{D\vec{x}}{Dt} \times \vec{v} = \vec{v} \times \vec{v} = \vec{0}$$

along with the local laws of conservation of mass (4.14), the balance of linear momentum (4.20), and the associated interface condition (4.21), equation (4.26) reduces to

$$\int_{v(t)-\sigma(t)} \boldsymbol{t}^T \overset{\bullet}{\times} \boldsymbol{I}\, dv = \vec{0}\ . \tag{4.27}$$

Again, postulating that this to be valid for all parts of $v(t) - \sigma(t)$, the integrand must vanish, such that

$$\boldsymbol{t}^T \overset{\bullet}{\times} \boldsymbol{I} = \vec{0} \qquad \text{or} \qquad \boldsymbol{t}^T = \boldsymbol{t} \qquad \text{in}\ \ v(t) - \sigma(t)\ . \tag{4.28}$$

Thus, the necessary and sufficient condition for the satisfaction of the local balance of angular momentum is the symmetry of the Cauchy stress tensor $\boldsymbol{t}$. We have seen that the associated interface condition for the angular momentum is satisfied identically.

Note that in formulating the angular principle by (4.4) we have assumed that no body or surface couples act on the body. If any such concentrated moments do act, the material is said to be a *polar* material and the symmetry property of $\boldsymbol{t}$ no longer holds. However, this is a rather specialized situation, and we will not consider it here.

4.2.4 Energy equation

The same methodology may be applied for the equation of energy balance (4.10). The integrand of the surface integral on the right-hand side of (4.10) is expressed by the Cauchy stress formula (3.15). The surface integral is then converted to the volume integral by the Gauss theorem (2.45):

$$\oint_{s(t)-\sigma(t)} (\vec{t}_{(\vec{n})} \cdot \vec{v} - \vec{q} \cdot \vec{n})\, da = \int_{v(t)-\sigma(t)} (\operatorname{div}(\boldsymbol{t} \cdot \vec{v}) - \operatorname{div}\vec{q})\, dv + \int_{\sigma(t)} \vec{n} \cdot [\boldsymbol{t} \cdot \vec{v} - \vec{q}]_-^+\, da\ . \tag{4.29}$$

The divergence of vector $\boldsymbol{t} \cdot \vec{v}$ will be arranged by making use of the identity

$$\operatorname{div}(\boldsymbol{t} \cdot \vec{v}) = \operatorname{div}\boldsymbol{t} \cdot \vec{v} + \boldsymbol{t}^T : \operatorname{grad}\vec{v}\ , \tag{4.30}$$

where $:$ denotes the double-dot product of tensors. The left-hand side of the equation of energy balance (4.10) can be arranged by Reynolds's transport theorem (2.48) with $\phi = \varrho\varepsilon + \frac{1}{2}\varrho\vec{v} \cdot \vec{v}$ as

$$\frac{D}{Dt} \int_{v(t)-\sigma(t)} (\varrho\varepsilon + \frac{1}{2}\varrho\vec{v} \cdot \vec{v})\, dv$$

$$= \int_{v(t)-\sigma(t)} \left[\frac{D}{Dt}(\varrho\varepsilon + \frac{1}{2}\varrho\vec{v} \cdot \vec{v}) + (\varrho\varepsilon + \frac{1}{2}\varrho\vec{v} \cdot \vec{v}) \operatorname{div}\vec{v}\right] dv + \int_{\sigma(t)} \vec{n} \cdot \left[(\vec{v} - \vec{w})(\varrho\varepsilon + \frac{1}{2}\varrho\vec{v} \cdot \vec{v})\right]_-^+ da$$

$$= \int_{v(t)-\sigma(t)} \left[(\varepsilon + \frac{1}{2}\varrho\vec{v} \cdot \vec{v})\left(\frac{D\varrho}{Dt} + \varrho \operatorname{div}\vec{v}\right) + \varrho\frac{D\varepsilon}{Dt} + \varrho\frac{D\vec{v}}{Dt} \cdot \vec{v}\right] dv + \int_{\sigma(t)} \vec{n} \cdot \left[(\vec{v} - \vec{w})(\varrho\varepsilon + \frac{1}{2}\varrho\vec{v} \cdot \vec{v})\right]_-^+ da$$

which, by the law of mass conservation (4.14), reduces to

$$\frac{D}{Dt} \int_{v(t)-\sigma(t)} (\varrho\varepsilon + \frac{1}{2}\varrho\vec{v} \cdot \vec{v}) dv = \int_{v(t)-\sigma(t)} \varrho \left(\frac{D\varepsilon}{Dt} + \frac{D\vec{v}}{Dt} \cdot \vec{v}\right) dv + \int_{\sigma(t)} \vec{n} \cdot \left[(\vec{v} - \vec{w})(\varrho\varepsilon + \frac{1}{2}\varrho\vec{v} \cdot \vec{v})\right]_-^+ da\ . \tag{4.31}$$

In view of (4.29)–(4.31), the equation of motion (4.20), the symmetry of the Cauchy stress tensor (4.28), and upon setting the integrand of the result equal to zero, we obtain

$$\varrho \frac{D\varepsilon}{Dt} = \boldsymbol{t} : \boldsymbol{l} - \operatorname{div} \vec{q} + \varrho h \qquad \text{in} \quad v(t) - \sigma(t) \ , \tag{4.32}$$

$$\vec{n} \cdot \left[(\vec{v} - \vec{w})(\varrho\varepsilon + \frac{1}{2}\varrho\vec{v}\cdot\vec{v}) - \boldsymbol{t}\cdot\vec{v} + \vec{q} \right]_-^+ = 0 \qquad \text{on} \quad \sigma(t) \ , \tag{4.33}$$

where $\boldsymbol{l}$ is the transposed velocity gradient tensor introduced by (2.13). Because of the symmetry of the Cauchy stress tensor $\boldsymbol{t}$, equation (4.32) can be written in the alternative form

$$\varrho \frac{D\varepsilon}{Dt} = \boldsymbol{t} : \boldsymbol{d} - \operatorname{div} \vec{q} + \varrho h \qquad \text{in} \ \ v(t) - \sigma(t) \ , \tag{4.34}$$

where $\boldsymbol{d}$ is the strain-rate tensor introduced by $(2.20)_1$. Equation (4.34) is the energy equation for a *thermomechanical continuum* in spatial form and (4.33) is the associated interface condition on the discontinuity σ.

4.2.5 Entropy inequality

The same methodology may be applied for the global law of entropy to derive its local form. Using again the Reynolds transport theorem, the Cauchy stress formula and the Gauss theorem, and assuming that the global law of entropy (4.12) is valid for any part of the body, we obtain the local form of the entropy inequality

$$\varrho \frac{D\eta}{Dt} + \operatorname{div} \vec{s} - \varrho b \ \geq \ 0 \qquad \text{in} \ \ v(t) - \sigma(t) \ , \tag{4.35}$$

$$\vec{n} \cdot [\varrho\eta(\vec{v} - \vec{w}) + \vec{s}]_-^+ \ \geq \ 0 \qquad \text{on} \ \ \sigma(t) \ . \tag{4.36}$$

4.2.6 Résumé of local balance laws

In summary, all local balance laws may be expressed in the spatial description by:

(i) *Conservation of mass*

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho\vec{v}) = 0 \qquad \text{in} \ \ v(t) - \sigma(t) \ , \tag{4.37}$$

$$\vec{n} \cdot [\varrho(\vec{v} - \vec{w})]_-^+ = 0 \qquad \text{on} \ \ \sigma(t) \ . \tag{4.38}$$

(ii) *Balance of linear momentum*

$$\operatorname{div} \boldsymbol{t} + \varrho\vec{f} = \varrho \frac{D\vec{v}}{Dt} \qquad \text{in} \ \ v(t) - \sigma(t) \ , \tag{4.39}$$

$$\vec{n} \cdot [\varrho(\vec{v} - \vec{w}) \otimes \vec{v} - \boldsymbol{t}]_-^+ = \vec{0} \qquad \text{on} \ \ \sigma(t) \ . \tag{4.40}$$

(iii) *Balance of angular momentum*

$$\boldsymbol{t} = \boldsymbol{t}^T \qquad \text{in} \ \ v(t) - \sigma(t) \ . \tag{4.41}$$

(iv) *Conservation of energy*

$$\varrho\,\frac{D\varepsilon}{Dt} = \boldsymbol{t} : \boldsymbol{d} - \operatorname{div}\vec{q} + \varrho h \qquad \text{in} \quad v(t) - \sigma(t)\ , \tag{4.42}$$

$$\vec{n}\cdot\Big[(\vec{v}-\vec{w})(\varrho\varepsilon + \frac{1}{2}\varrho\vec{v}\cdot\vec{v}) - \boldsymbol{t}\cdot\vec{v} + \vec{q}\Big]_-^+ = 0 \qquad \text{on} \quad \sigma(t)\ . \tag{4.43}$$

(v) *Entropy inequality*

$$\varrho\,\frac{D\eta}{Dt} + \operatorname{div}\vec{s} - \varrho b \;\geq\; 0 \qquad \text{in} \quad v(t) - \sigma(t)\ , \tag{4.44}$$

$$\vec{n}\cdot[\varrho\eta(\vec{v}-\vec{w}) + \vec{s}]_-^+ \;\geq\; 0 \qquad \text{on} \quad \sigma(t)\ . \tag{4.45}$$

4.3 Interface conditions in special cases

If there is a moving discontinuity surface $\sigma(t)$ sweeping the body with a velocity $\vec{w}$ in the direction of the unit normal $\vec{n}$ of $\sigma(t)$, then the interface conditions (4.38), (4.40) (4.43) and (4.45) must be satisfied on the surface $\sigma(t)$. Some of these interface conditions will now be applied to two special cases:

(i) The discontinuity surface is a **material surface**, that is, the surface frozen in the body (i.e. always containing the same material points). In this case, $\vec{w} = \vec{v}$, (4.38) is satisfied identically, (4.40) and (4.43) reduce to

$$\vec{n}\cdot[\boldsymbol{t}]_-^+ = \vec{0}\ , \tag{4.46}$$

$$\vec{n}\cdot[\boldsymbol{t}\cdot\vec{v} - \vec{q}]_-^+ = 0\ . \tag{4.47}$$

With the help of (4.46), the interface condition (4.47) may also be written in the form

$$\vec{n}\cdot\boldsymbol{t}\cdot[\vec{v}]_-^+ - \vec{n}\cdot[\vec{q}]_-^+ = 0\ . \tag{4.48}$$

Hence, on a material interface between two media the surface traction $\vec{n}\cdot\boldsymbol{t}$ is continuous, and the jump on the energy of tractions across this interface is balanced with that of the normal component of the heat vector.

The *dynamic* interface conditions (4.46) and (4.47) must be supplemented by the *kinematic* interface conditions. At a *welded interface*, like on a discontinuity between two solids, there is no tangential slip across the interface and the spatial velocity must be continuous:

$$[\vec{v}]_-^+ = \vec{0}\ . \tag{4.49}$$

Consequently, the interface condition (4.48) is further reduces to

$$\vec{n}\cdot[\vec{q}]_-^+ = 0\ , \tag{4.50}$$

which states the continuity in the normal component of the heat vector across σ. Tangential slip is allowed at an interface between a solid and an inviscid fluid or it may also occur on an idealized fault surface separating two solids. At such a *slipping interface*, equation (4.49) is replaced by

$$[\vec{n}\cdot\vec{v}]_-^+ = 0\ . \tag{4.51}$$

The last condition guarantees that there is no separation or interpenetration of the two materials at the interface.

A *frictionless interface* of two materials is the material discontinuity across which the motion from one of its side runs without friction. This means that the shear stresses of the Cauchy stress tensor $\boldsymbol{t}$ are equal to zero from one of discontinuity side (e.g., with superscript '$^-$'),

$$\vec{n}^- \cdot \boldsymbol{t}^- \cdot (\boldsymbol{I} - \vec{n}^- \otimes \vec{n}^-) = \vec{0} \qquad \text{or} \qquad \vec{n}^- \cdot \boldsymbol{t}^- = (\vec{n}^- \cdot \boldsymbol{t}^- \cdot \vec{n}^-)\, \vec{n}^- \; . \tag{4.52}$$

Carrying this into (4.46) and considering that $\vec{n} = \vec{n}^- = -\vec{n}^+$ across σ, the stress vector $\vec{n} \cdot \boldsymbol{t}$ from both side of a frictionless discontinuity is of the form

$$\vec{n} \cdot \boldsymbol{t} = -p\vec{n} \; , \tag{4.53}$$

where p is the negative normal component of the stress vector, $p = -(\vec{n} \cdot \boldsymbol{t} \cdot \vec{n})$. The condition (4.46) reduces to the condition of the continuity p across σ,

$$[p]_-^+ = 0 \; . \tag{4.54}$$

(ii) The discontinuity surface coincides with the **surface of the body.** In this case $\varrho^+ = 0$, $\vec{v}^- = \vec{w}$. Again (4.38) gives an identity and the others reduce to

$$\vec{n} \cdot [\boldsymbol{t}]_-^+ \;=\; \vec{0} \; , \tag{4.55}$$

$$\vec{n} \cdot [\boldsymbol{t} \cdot \vec{v} - \vec{q}]_-^+ \;=\; 0 \; , \tag{4.56}$$

where $\vec{n} \cdot \boldsymbol{t}^+$ is interpreted as the external surface load and $\boldsymbol{t}^+ \cdot \vec{v}^+$ as the energy of this load. If the external surface load is equal to zero, $\boldsymbol{t}^+ = 0$, then $\vec{n} \cdot \boldsymbol{t}^- = \vec{0}$, and the first term on the left of (4.56) is equal to zero. Hence, we obtain the boundary condition

$$\vec{n} \cdot [\vec{q}]_-^+ = 0 \tag{4.57}$$

involving the heat alone.

4.4 Local balance laws in the referential description

In the previous section, the local balance laws have been expressed in spatial form. These equations may also be cast in the referential form, which we now introduce.

4.4.1 Continuity equation

The referential form of continuity equation can be derived from (4.1) by using the transformation law $dv = JdV$:

$$\int_V (\varrho_0 - \varrho J)\, dV = 0 \; , \tag{4.58}$$

where V is the volume of the body in the reference configuration and $\varrho(\vec{X}, t)$ is the Lagrangian description of the density,

$$\varrho(\vec{X}, t) := \varrho(\vec{\chi}(\vec{X}, t), t) \; . \tag{4.59}$$

But V is arbitrary, so that,

$$\varrho_0 = \varrho J \qquad\qquad \text{in} \;\; V - \Sigma \; , \tag{4.60}$$

which is equivalent to

$$\frac{D\varrho_0}{Dt} = 0 \qquad \text{in} \;\; V - \Sigma \; , \tag{4.61}$$

Equations (4.60) and (4.61) are called the referential form of the continuity equation.

The relation (4.60) can be considered as the general solution to the continuity equation (4.14). To show it, let a body occupies the reference configuration κ_0 and the present configuration κ_t at time $t = 0$ and at time t, respectively. Making use of the relation for the material time derivative of the jacobian, $\dot{J} = J \operatorname{div} \vec{v}$, the continuity equation (4.14) can be written in the form

$$\frac{\dot{\varrho}}{\varrho} + \frac{\dot{J}}{J} = 0 \; .$$

Integrating this equation with respect to time from $t = 0$ to t and realizing that $J = 1$ for $t = 0$, we find that

$$\ln \frac{\varrho}{\varrho_0} + \ln J = 0.$$

This gives (4.60) after short algebraic manipulation. Hence, in the case that the deformation gradient $\boldsymbol{F}$ is given as the solution to the other field equations, the mass density ϱ can be calculated from (4.60) after solving these field equations. It means that equation (4.60) does not have to be included in the set of the governing equations. Such a situation appears in the Lagrangian description of solids. This is, however, not the case if we employ the Eulerian description. Since the initial configuration is not the reference configuration, neither $\boldsymbol{F}$ nor J follow from the field equations, and the continuity equation (4.14) must be included in the set of the governing equations in order to specify ϱ.

To express the spatial interface conditions (4.15) in the referential form, we must first modify (4.15) such that

$$[\vec{n} da \cdot \varrho(\vec{v} - \vec{w})]_-^+ = 0 \qquad \text{on} \;\; \sigma(t) \; , \tag{4.62}$$

where we have inserted the spatial surface element da into equation (4.15) since it appears as the surface element in the surface integral in (4.13). Moreover, we have involved the unit normal $\vec{n}$ into the square brackets assuming that $\vec{n}$ points towards the + side of the discontinuity $\sigma(t)$ on both sides of $\sigma(t)$. Since the spatial surface element da of $\sigma(t)$ changes continuously across $\sigma(t)$, that is, $da^+ = da^-$, the interface condition (4.62) is equivalent to the form (4.15). Considering the transformation (1.75) between the spatial and referential surface elements, $d\vec{a} = J d\vec{A} \cdot \boldsymbol{F}^{-1}$, (4.62) transforms to

$$\Big[\varrho_0 (\vec{N} dA \cdot \vec{W})\Big]_-^+ = 0 \qquad \text{on} \;\; \Sigma \; , \tag{4.63}$$

where Σ is the surface discontinuity in the reference configuration and

$$\vec{W}(\vec{X}, t) := \boldsymbol{F}^{-1} \cdot (\vec{v} - \vec{w}) \; . \tag{4.64}$$

In contrast to the spatial surface element da, the referential surface element dA may change discontinuously on Σ, $dA^+ \neq dA^-$, if there is a tangential slip on this discontinuity, like on a fluid-solid discontinuity (see Figure 8.1). On the other hand, at a discontinuity with no tangential slip, like on a solid-solid discontinuity, $dA^+ = dA^-$, and this factor can be dropped from the interface condition (4.63).

4.4.2 Equation of motion

A simple form of the Lagrangian equation of motion can be obtained in terms of the first Piola-Kirchhoff stress tensor that is related to the Lagrangian Cauchy stress tensor by (3.26). Applying the divergence operation on $(3.26)_2$ and making use of the following differential identity

$$\mathrm{div}\,(\boldsymbol{A}\cdot\boldsymbol{B}) = (\mathrm{div}\,\boldsymbol{A})\cdot\boldsymbol{B} + \boldsymbol{A}^T : \mathrm{grad}\,\boldsymbol{B}\ , \tag{4.65}$$

which is valid for all differentiable tensors $\boldsymbol{A}$ and $\boldsymbol{B}$, the divergence of the Cauchy stress tensor can be arranged as follows

$$\mathrm{div}\,\boldsymbol{t} = \mathrm{div}\,(J^{-1}\boldsymbol{F}\cdot\boldsymbol{T}^{(1)}) = \mathrm{div}\,(J^{-1}\boldsymbol{F})\cdot\boldsymbol{T}^{(1)} + J^{-1}\boldsymbol{F}^T : \mathrm{grad}\,\boldsymbol{T}^{(1)}\ .$$

The first term is equal to zero because of the Jacobi identity $(1.42)_2$ and the second term can be arranged according to the differential identity $(1.46)_2$. Hence

$$\mathrm{div}\,\boldsymbol{t} = J^{-1}\mathrm{Div}\,\boldsymbol{T}^{(1)}\ . \tag{4.66}$$

Using (4.60), the equation of motion (4.20) can be expressed in the referential form as

$$\mathrm{Div}\,\boldsymbol{T}^{(1)} + \varrho_0\vec{F} = \varrho_0\frac{D\vec{v}}{Dt} \qquad \text{in}\ \ V-\Sigma\ , \tag{4.67}$$

where $\vec{F}(\vec{X},t)$ is the Lagrangian description of the body force,

$$\vec{F}(\vec{X},t) := \vec{f}(\vec{\chi}(\vec{X},t),t)\ . \tag{4.68}$$

Note that the divergence of the Eulerian Cauchy stress tensor $\mathrm{div}\,\boldsymbol{t}(\vec{X},t)$ in (4.20) is tranformed into the divergence of the first Piola-Kirchhoff stress tensor $\mathrm{Div}\,\boldsymbol{T}^{(1)}(\vec{X},t)$ in (4.67). In fact, the original definition (3.25) of $\boldsymbol{T}^{(1)}$ was motivated by this simple transformation.

In similar way as for the conservation of mass, the interface condition (4.21) can be expressed in the referential form. Using (1.75), (4.64) and $(3.26)_2$, we obtain

$$\Big[\varrho_0(\vec{N}dA\cdot\vec{W})\vec{v} - \vec{N}dA\cdot\boldsymbol{T}^{(1)}\Big]_-^+ = \vec{0} \qquad \text{on}\ \ \Sigma\ . \tag{4.69}$$

Note that the Lagrangian equation of motion and the interface conditions can also be written in terms of the second Piola-Kirchhoff stress tensor rather than in terms of the first Piola-Kirchhoff stress tensor by substituting $\boldsymbol{T}^{(1)} = \boldsymbol{T}^{(2)}\cdot\boldsymbol{F}^T$.

4.4.3 Symmetries of the Piola-Kirchhoff stress tensors

Upon $(3.21)_2$ and $(3.26)_2$, the symmetry of the Cauchy stress tensor $\boldsymbol{t} = \boldsymbol{t}^T$ has two different forms

$$\left(\boldsymbol{T}^{(1)}\right)^T = \boldsymbol{F}\cdot\boldsymbol{T}^{(1)}\cdot\boldsymbol{F}^{-T}\ , \qquad \left(\boldsymbol{T}^{(2)}\right)^T = \boldsymbol{T}^{(2)}\ . \tag{4.70}$$

It shows that $\boldsymbol{T}^{(2)}$ is symmetric whenever $\boldsymbol{t}$ is symmetric (nonpolar case), but $\boldsymbol{T}^{(1)}$ is in general not symmetric.

4.4.4 Energy equation

Let $\vec{Q}$ be defined as the heat flux with respect to the surface element $d\vec{A}$ in the reference configuration:

$$\vec{q}\cdot d\vec{a} = \vec{Q}\cdot d\vec{A}\ , \tag{4.71}$$

where $\vec{q}$ is the heat flux with respect to the surface element $d\vec{a}$ in the present configuration. By using the transformation rule for surface elements between the present and reference configurations, $d\vec{a} = Jd\vec{A} \cdot \boldsymbol{F}^{-1}$, the referential and spatial heat fluxes are related by

$$\vec{q} = J^{-1}\boldsymbol{F} \cdot \vec{Q} \ , \qquad \vec{Q} = J\boldsymbol{F}^{-1} \cdot \vec{q} \ . \tag{4.72}$$

The referential heat flux and the Piola-Kirchhoff stress tensors will be employed to transform the energy equation from the spatial form to the referential form. First, the divergence of heat flux $\vec{q}$ can be arranged by making use of the differential identity (4.30) as follows

$$\operatorname{div} \vec{q} = \operatorname{div} (J^{-1}\boldsymbol{F} \cdot \vec{Q}) = \operatorname{div} (J^{-1}\boldsymbol{F}) \cdot \vec{Q} + J^{-1}\boldsymbol{F}^T : \operatorname{grad} \vec{Q} \ .$$

The first term is equal to zero because of the Jacobi identity $(1.42)_2$ and the second term can be arranged according to the differential identity $(1.45)_2$. Hence

$$\operatorname{div} \vec{q} = J^{-1}\operatorname{Div} \vec{Q} \ . \tag{4.73}$$

Next, the stress power per unit volume in the present configuration, $\boldsymbol{t} : \boldsymbol{d}$, can be referred back to the reference configuration. Using two tensor identities

$$(\boldsymbol{A} \cdot \boldsymbol{B}) : \boldsymbol{C} = \boldsymbol{A} : (\boldsymbol{B} \cdot \boldsymbol{C}) \ , \qquad (\boldsymbol{A} \cdot \boldsymbol{B}) : (\boldsymbol{C} \cdot \boldsymbol{D}) = (\boldsymbol{D} \cdot \boldsymbol{A}) : (\boldsymbol{B} \cdot \boldsymbol{C}) \ , \tag{4.74}$$

valid for the 2nd order tensors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$, equations $(2.12)_2$ and $(3.26)_2$, we have

$$\begin{aligned} \boldsymbol{t} : \boldsymbol{d} &= \boldsymbol{t} : \boldsymbol{l} = J^{-1}(\boldsymbol{F} \cdot \boldsymbol{T}^{(1)}) : (\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}) = J^{-1}(\boldsymbol{F}^{-1} \cdot \boldsymbol{F}) : (\boldsymbol{T}^{(1)} \cdot \dot{\boldsymbol{F}}) = J^{-1}\boldsymbol{I} : (\boldsymbol{T}^{(1)} \cdot \dot{\boldsymbol{F}}) \\ &= J^{-1}(\boldsymbol{I} \cdot \boldsymbol{T}^{(1)}) : \dot{\boldsymbol{F}} \ , \end{aligned}$$

or,

$$\boldsymbol{t} : \boldsymbol{d} = J^{-1}\boldsymbol{T}^{(1)} : \dot{\boldsymbol{F}} \ , \tag{4.75}$$

where $\dot{\boldsymbol{F}}$ is the material time derivative of the deformation gradient tensor. Introducing the second Piola-Kirchhoff stress tensor $\boldsymbol{T}^{(2)}$ instead of $\boldsymbol{T}^{(1)}$, the stress power can alternatively be expressed as

$$\boldsymbol{t} : \boldsymbol{d} = J^{-1}(\boldsymbol{T}^{(2)} \cdot \boldsymbol{F}^T) : \dot{\boldsymbol{F}} = J^{-1}\boldsymbol{T}^{(2)} : (\boldsymbol{F}^T \cdot \dot{\boldsymbol{F}}) = J^{-1}\boldsymbol{T}^{(2)} : (\boldsymbol{F}^T \cdot \boldsymbol{l} \cdot \boldsymbol{F}) = J^{-1}\boldsymbol{T}^{(2)} : (\boldsymbol{F}^T \cdot \boldsymbol{d} \cdot \boldsymbol{F}),$$

or,

$$\boldsymbol{t} : \boldsymbol{d} = J^{-1}\boldsymbol{T}^{(2)} : \dot{\boldsymbol{E}} \ , \tag{4.76}$$

where $\dot{\boldsymbol{E}}$ is the material time derivative of the Lagrangian strain tensor $\boldsymbol{E}$ given by (2.23).

We are now ready to express the conservation of energy (4.34) in the referential form. Using (4.73) and (4.76), along with (4.60), we obtain

$$\varrho_0 \frac{D\varepsilon}{Dt} = \boldsymbol{T}^{(2)} : \dot{\boldsymbol{E}} - \operatorname{Div} \vec{Q} + \varrho_0 h \qquad \text{in} \ \ V - \Sigma \ . \tag{4.77}$$

The dissipative term is also equal to $\boldsymbol{T}^{(1)} : \dot{\boldsymbol{F}}$.

The referential form of energy interface condition (4.33) can be derived by substituting (4.62) and (4.64) into (4.33) and considering (4.71):

$$\Big[\varrho_0(\vec{N}dA\cdot\vec{W})(\varepsilon+\frac{1}{2}\varrho\vec{v}\cdot\vec{v})-\vec{N}dA\cdot\boldsymbol{T}^{(1)}\cdot\vec{v}+\vec{N}dA\cdot\vec{Q}\Big]_-^+=0 \qquad \text{on } \Sigma\,. \tag{4.78}$$

4.4.5 Entropy inequality

The same program can be applied to the entropy inequality to carry out it to the referential form. Let $\vec{S}$ be defined as the entropy influx with respect to the surface element $d\vec{A}$ in the reference configuration:

$$\vec{s}\cdot d\vec{a}=\vec{S}\cdot d\vec{A}\,, \tag{4.79}$$

where $\vec{s}$ is the heat flux with respect to the surface element $d\vec{a}$ in the present configuration. Considering the transformation between the spatial and referential surface elements, $d\vec{a}=Jd\vec{A}\cdot\boldsymbol{F}^{-1}$, the referential and spatial entropy influxes are related by

$$\vec{s}=J^{-1}\boldsymbol{F}\cdot\vec{S}\,,\qquad \vec{S}=J\boldsymbol{F}^{-1}\cdot\vec{s}\,. \tag{4.80}$$

In an analogous way as in the preceding section, the entropy inequality (4.35) and the entropy interface condition (4.36) may then be expressed in the referential form as

$$\varrho_0\frac{D\eta}{Dt}+\mathrm{Div}\,\vec{S}-\varrho_0 b \;\geq\; 0 \qquad \text{in } V-\Sigma\,, \tag{4.81}$$

$$\Big[\varrho_0\eta(\vec{N}dA\cdot\vec{W})+\vec{N}dA\cdot\vec{S}\Big]_-^+ \;\geq\; 0 \qquad \text{on } \Sigma\,. \tag{4.82}$$

5. MOVING SPATIAL FRAME

5.1 Observer transformation

In Chapter 6, we will require that the form of the constitutive equations is independent of the movement of an observer. The notion of frame helps us to formulate this requirement mathematically. A *frame* can be understood as an observer who is equipped to measure position in Euclidean space. To every frame belongs a reference point, the so-called *origin*, from which an observer measures distances or defines position vectors in space.

Let us consider two different frames, one fixed (unstarred) and the other one in motion (starred). Both are later considered to describe the present configuration of a material body and are, therefore, called the spatial frames. Figure 5.1 shows two such frames and the relationship between the position vectors of the same observer measured in both frame. Let $\vec{x}$ be a position vector of the observer P in the present configuration relative to the fixed frame and $\vec{x}^{**}$ the position vector of the same observer in the moving frame. They are hence connected by the relationship

$$\vec{x}^{**} = \vec{x} + \vec{b}^{**} , \tag{5.1}$$

where $\vec{b}^{**}$ gives the displacement between the unstarred and the starred reference points (origins). Notice that this relation is independent of the choice of *coordinate system.* However, the observer may refer his position vector not only to the origin, but also to a coordinate system, which is attached to the origin. The coordinate system can be chosen arbitrarily or matched to a realistic situation. In Figure 5.1, these coordinate systems are Cartesian.

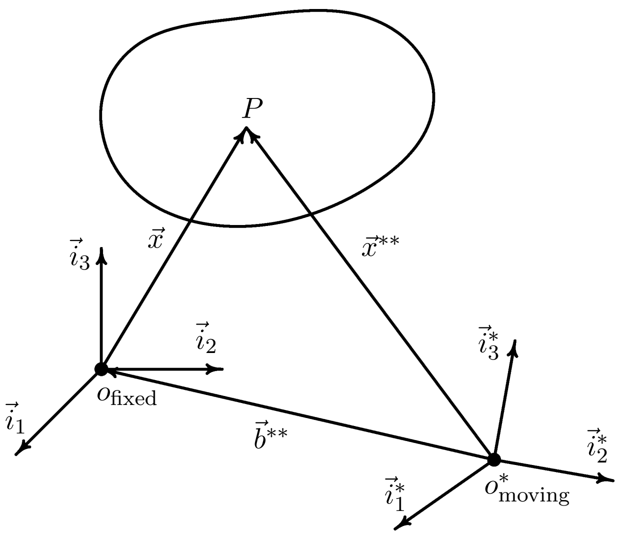

Figure 5.1. Observer transformation.

The component form of (5.1) is

$$x^*_{k^*}\vec{i}^*_{k^*} = x_k\vec{i}_k + b^*_{k^*}\vec{i}^*_{k^*} . \tag{5.2}$$

The Cartesian base vectors $\vec{i}_k$ and $\vec{i}^*_{k^*}$ associated with the unstarred and starred spatial frames are related by (1.12):

$$\vec{i}_k = \delta_{kk^*}\vec{i}^*_{k^*} , \quad \vec{i}^*_{k^*} = \delta_{k^*k}\vec{i}_k , \tag{5.3}$$

where δ_{kk^*} and δ_{k^*k} are the shifters between the two spatial frames. Substituting for $\vec{i}_k$ from $(5.3)_1$ into (5.2) and comparing the components at $\vec{i}^*_{k^*}$, we obtain

$$x^*_{k^*} = \delta_{kk^*}x_k + b^*_{k^*} . \tag{5.4}$$

Multiplying by $\vec{i}_{k^*}$ and defining two vectors

$$\vec{x}^* := x^*_{k^*}\vec{i}_{k^*} , \qquad \vec{b}^* := b^*_{k^*}\vec{i}_{k^*} , \tag{5.5}$$

we obtain [12]

$$\vec{x}^* = \delta_{kk^*} x_k \vec{i}_{k^*} + \vec{b}^* \ . \tag{5.6}$$

Making use of $x_k = \vec{i}_k \cdot \vec{x}$, the identity $\vec{i}_{k^*}(\vec{i}_k \cdot \vec{x}) = (\vec{i}_{k^*} \otimes \vec{i}_k) \cdot \vec{x}$ and introducing tensor $\boldsymbol{O}$,

$$\boldsymbol{O} := O_{k^*k}(\vec{i}_{k^*} \otimes \vec{i}_k) \ , \qquad O_{k^*k} = \delta_{kk^*} \ , \tag{5.7}$$

equation (5.6) may be rewritten in invariant notation as

$$\vec{x}^* = \boldsymbol{O}(t) \cdot \vec{x} + \vec{b}^*(t) \ . \tag{5.8}$$

This equation shows that the same point can be represented by its components x^*_k in the moving coordinate system as well as by x_k in the fixed coordinate system. This expresses a rigid motion of the starred spatial frame. In fact, $\vec{b}^*(t)$ corresponds to the translation and $\boldsymbol{O}(t)$ to the rotation of this frame. As indicated, both $\vec{b}^*(t)$ and $\boldsymbol{O}(t)$ can be time-dependent. The transformation (5.8) is often referred to as the *observer transformation* or the *Euclidean transformation* $(x, t) \to (x^*, t)$. [13]

In view of $(5.3)_2$, the tensor $\boldsymbol{O}(t)$ may be expressed as the tensor product of the starred and unstarred base vectors:

$$\boldsymbol{O}(t) = \vec{i}^*_k(t) \otimes \vec{i}_k \ . \tag{5.9}$$

The transposed tensor to $\boldsymbol{O}(t)$ is

$$\boldsymbol{O}^T(t) = \delta_{k^*k}(\vec{i}_{k^*} \otimes \vec{i}_k) = \vec{i}_k \otimes \vec{i}^*_k(t) \ . \tag{5.10}$$

This implies that $\boldsymbol{O}(t)$ is an orthogonal tensor since

$$\boldsymbol{O}(t) \cdot \boldsymbol{O}^T(t) = \boldsymbol{O}^T(t) \cdot \boldsymbol{O}(t) = \boldsymbol{I} \ , \tag{5.11}$$

where $\boldsymbol{I}$ is the identity tensor. Strictly speaking, there are two identity tensors, one in the unstarred frame, $\boldsymbol{I} = \vec{i}_k \otimes \vec{i}_k$, and one in the starred frame $\boldsymbol{I}^* = \vec{i}^*_k \otimes \vec{i}^*_k$; we shall, however, disregard this subtlety.

We say that a scalar-, vector- and tensor-valued quantity ϕ is *objective* or *frame indifferent* if it is invariant under all observer transformation (5.8), that is, if $\phi^{**} = \phi$. For instance, tensor $\boldsymbol{a}$ is objective if its components transform under the observer transformation (5.8) according to the relation

$$a^*_{k^*l^*} = O_{k^*k}(t) a_{kl} O_{l^*l}(t) \ , \tag{5.12}$$

where a_{kl} and $a^*_{k^*l^*}$ are components of $\boldsymbol{a}$ relative to the unstarred and starred frames, respectively. To see it, let us rewrite the transformation relation (5.3) for the base vectors in terms of the components of the tensor $\boldsymbol{O}$. By $(5.7)_2$, the relation (5.3) may be rewritten in the form

$$\vec{i}_k = O_{k^*k} \vec{i}^*_{k^*} \ , \qquad \vec{i}^*_{k^*} = O_{kk^*} \vec{i}_k \ . \tag{5.13}$$

[12] Note that we distinguish between three different vectors, $\vec{x} = x_k \vec{i}_k$, $\vec{x}^* = x^*_k \vec{i}_k$ and $\vec{x}^{**} = x^*_k \vec{i}^*_k$. Vector notation becomes ambiguous if the vectors $\vec{x}^*$ and $\vec{x}^{**}$ are denoted by the same symbol $\vec{x}^*$. Compare (5.1) and (5.8) in this case.

[13] The most general change of frame $(x, t) \to (x^*, t^*)$ is, in addition, characterized by a shift in time:

$$t^* = t - a \ ,$$

where a is a particular time.

Then

$$\boldsymbol{a} = a_{kl}(\vec{i}_k \otimes \vec{i}_l) = a_{kl}O_{k^*k}O_{l^*l}(\vec{i}^*_{k^*} \otimes \vec{i}^*_{l^*}) \stackrel{!}{=} \boldsymbol{a}^{**} = a^*_{k^*l^*}(\vec{i}^*_{k^*} \otimes \vec{i}^*_{l^*}) \ ,$$

which yields (5.12). Introducing tensor $\boldsymbol{a}^*$,

$$\boldsymbol{a}^* := a^*_{k^*l^*}(\vec{i}_{k^*} \otimes \vec{i}_{l^*}) \ , \tag{5.14}$$

the component form (5.12) may be written in invariant form [14]

$$\boldsymbol{a}^* = \boldsymbol{O}(t) \cdot \boldsymbol{a} \cdot \boldsymbol{O}^T(t) \ . \tag{5.15}$$

In an analogous way, a scalar- and vector-valued physical quantities λ and $\vec{u}$ are called objective if they transform under a rigid motion of spatial frame according to

$$\begin{aligned} \lambda^* &= \lambda \ , \\ u^*_{k^*} &= O_{k^*k}(t)u_k \ , \qquad \text{or, invariantly,} \qquad \vec{u}^{\,*} = \boldsymbol{O}(t) \cdot \vec{u} \ . \end{aligned} \tag{5.16}$$

5.2 Objectivity of some geometric objects

Let us now examine the objectivity property of different geometric objects. We begin with the Eulerian velocity $\vec{v}$ and the Eulerian acceleration $\vec{a}$. Suppose that the motion is represented in the unstarred frame by (1.29), $x_k = \chi_k(X_K, t)$. Then, in view of (5.8), it is given in the starred frame by

$$\vec{\chi}^*(\vec{X}, t) = \boldsymbol{O}(t) \cdot \vec{\chi}(\vec{X}, t) + \vec{b}^*(t) \ . \tag{5.17}$$

where $\vec{\chi}^*(\vec{X}, t) := \chi^*_{k^*}(\vec{X}, t)\,\vec{i}_{k^*}$. Differentiation of (5.17) with respect to t yields the following connection between the velocities and accelerations in the starred and unstarred frames:

$$\vec{v}^*(\vec{x}^*, t) = \boldsymbol{O}(t) \cdot \vec{v}(\vec{x}, t) + \dot{\boldsymbol{O}}(t) \cdot \vec{x} + \dot{\vec{b}}^*(t) \ , \tag{5.18}$$

$$\vec{a}^*(\vec{x}^*, t) = \boldsymbol{O}(t) \cdot \vec{a}(\vec{x}, t) + 2\, \dot{\boldsymbol{O}}(t) \cdot \vec{v}(\vec{x}, t) + \ddot{\boldsymbol{O}}(t) \cdot \vec{x} + \ddot{\vec{b}}^*(t) \ . \tag{5.19}$$

Let us introduce the angular velocity tensor $\boldsymbol{\Omega}$ which represents the spin of the starred frame with respect to the unstarred frame:

$$\boldsymbol{\Omega}(t) := \dot{\boldsymbol{O}}(t) \cdot \boldsymbol{O}^T(t) \ . \tag{5.20}$$

The relation

$$\boldsymbol{0} = (\boldsymbol{O} \cdot \boldsymbol{O}^T)^{\bullet} = \dot{\boldsymbol{O}} \cdot \boldsymbol{O}^T + \boldsymbol{O} \cdot \dot{\boldsymbol{O}}^T = \dot{\boldsymbol{O}} \cdot \boldsymbol{O}^T + (\dot{\boldsymbol{O}} \cdot \boldsymbol{O}^T)^T = \boldsymbol{\Omega} + \boldsymbol{\Omega}^T$$

shows that $\boldsymbol{\Omega}$ is a skew-symmetric tensor. Moreover,

$$\dot{\boldsymbol{\Omega}} = (\dot{\boldsymbol{O}} \cdot \boldsymbol{O}^T)^{\bullet} = \ddot{\boldsymbol{O}} \cdot \boldsymbol{O}^T + \dot{\boldsymbol{O}} \cdot \dot{\boldsymbol{O}}^T = \ddot{\boldsymbol{O}} \cdot \boldsymbol{O}^T + \dot{\boldsymbol{O}} \cdot (\boldsymbol{O}^T \cdot \boldsymbol{O}) \cdot \dot{\boldsymbol{O}}^T = \ddot{\boldsymbol{O}} \cdot \boldsymbol{O}^T + \boldsymbol{\Omega} \cdot \boldsymbol{\Omega}^T = \ddot{\boldsymbol{O}} \cdot \boldsymbol{O}^T - \boldsymbol{\Omega} \cdot \boldsymbol{\Omega} \ ,$$

which yields

$$\ddot{\boldsymbol{O}} \cdot \boldsymbol{O}^T = \dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \cdot \boldsymbol{\Omega} \ . \tag{5.21}$$

[14] Note again that we distinguish between three different tensors $\boldsymbol{a}$, $\boldsymbol{a}^*$ and $\boldsymbol{a}^{**}$.

With the aid of (5.8), (5.20) and (5.21), the transformation formulae for the velocity and acceleration can be expressed in the forms

$$\begin{aligned} \vec{v}^* &= \boldsymbol{O}\cdot\vec{v} + \boldsymbol{\Omega}\cdot(\vec{x}^* - \vec{b}^*) + \dot{\vec{b}}^* \,, && (5.22)\\ \vec{a}^* &= \boldsymbol{O}\cdot\vec{a} + 2\boldsymbol{\Omega}\cdot(\vec{v}^* - \dot{\vec{b}}^*) - \boldsymbol{\Omega}\cdot\boldsymbol{\Omega}\cdot(\vec{x}^* - \vec{b}^*) + \dot{\boldsymbol{\Omega}}\cdot(\vec{x}^* - \vec{b}^*) + \ddot{\vec{b}}^* \,. && (5.23)\end{aligned}$$

Inspection of these equations shows that both the velocity and the acceleration are not objective vectors. The additional terms causing the failure of objectivity have the following names:

$\boldsymbol{\Omega}\cdot(\vec{x}^* - \vec{b}^*)$ - relative angular velocity of the starred frame with respect to unstarred frame,
$\dot{\vec{b}}^*$ - relative translational velocity of these two frames,
$2\boldsymbol{\Omega}\cdot(\vec{v}^* - \dot{\vec{b}}^*)$ - Coriolis acceleration,
$-\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}\cdot(\vec{x}^* - \vec{b}^*)$ - centrifugal acceleration,
$\dot{\boldsymbol{\Omega}}\cdot(\vec{x}^* - \vec{b}^*)$ - Euler acceleration,
$\ddot{\vec{b}}^*$ - relative translational acceleration.

Among all Euclidean transformations, we can choose transformations that transform the acceleration in the objective way. In such a case, we have

$$\vec{a}^* = \boldsymbol{O}\cdot\vec{a} \quad\Leftrightarrow\quad \boldsymbol{\Omega} = \boldsymbol{0}\,,\ \ddot{\vec{b}}^* = \vec{0} \quad\Leftrightarrow\quad \vec{b}^*(t) = \vec{V}t + \vec{b}^*_0\,,\ \boldsymbol{O}(t) = \boldsymbol{O}\,, \tag{5.24}$$

where $\vec{V}$, $\vec{b}^*_0$ and $\boldsymbol{O}$ are time-independent. The change of frame defined by such constants,

$$\vec{x}^* = \boldsymbol{O}\cdot\vec{x} + \vec{V}t + \vec{b}^*_0\,, \tag{5.25}$$

is called the *Galilean transformation*. It means that the starred frame moved with a constant velocity with respect to the unstarred frame. Certainly, the acceleration is objective with respect to the Galilean transformation, whereas the velocity is not.

In contrast to the velocity field which is frame dependent (non-objective), the divergence of the velocity field is an objective scalar,

$$\operatorname{div}^*\vec{v}^* = \operatorname{div}\vec{v}\,. \tag{5.26}$$

To show it, we have

$$\begin{aligned}\operatorname{div}^*\vec{v}^* &= \frac{\partial v^*_{k^*}}{\partial x^*_{k^*}} = \frac{\partial}{\partial x^*_{k^*}}\Big[O_{k^*k}v_k + \Omega_{k^*k}(x^*_k - b^*_k) + \dot{b}^*_k\Big] = O_{k^*k}\frac{\partial v_k}{\partial x^*_{k^*}} + \Omega_{k^*k}\frac{\partial x^*_k}{\partial x^*_{k^*}} = O_{k^*k}\frac{\partial v_k}{\partial x^*_{k^*}} + \Omega_{kk}\\ &= O_{k^*k}\frac{\partial v_k}{\partial x^*_{k^*}} = O_{k^*k}\frac{\partial v_k}{\partial x_l}\frac{\partial x_l}{\partial x^*_{k^*}} = O_{k^*k}\frac{\partial v_k}{\partial x_l}O_{k^*l} = \delta_{kl}\frac{\partial v_k}{\partial x_l} = \frac{\partial v_k}{\partial x_k} = \operatorname{div}\vec{v}\,.\end{aligned}$$

To study the effect of an observer transformation on the basic balance equations derived in Chapter 3, let us show that (i) the spatial gradient of an objective scalar is an objective vector, (ii) the spatial divergence of an objective vector is an objective scalar, and (iii) the spatial divergence of an objective tensor is an objective vector.

(i) Using (5.8) and $(5.16)_1$, we have

$$(\operatorname{grad}^*\lambda^*)_{k*} = \frac{\partial\lambda^*}{\partial x^*_{k^*}} = \frac{\partial\lambda}{\partial x_k}\frac{\partial x_k}{\partial x^*_{k^*}} = O_{k^*k}\frac{\partial\lambda}{\partial x_k} = O_{k^*k}\,(\operatorname{grad}\lambda)_k\,.$$

Multiplying by $\vec{i}_{k^*}$ and introducing a new vector $\text{grad}\,\lambda^* := \dfrac{\partial \lambda^*}{\partial x^*_{k^*}}\vec{i}_{k^*}$, we obtain

$$\text{grad}\,\lambda^* = \boldsymbol{O}(t)\cdot \text{grad}\,\lambda\ . \tag{5.27}$$

(ii) Next, from (5.8) and $(5.16)_2$, we have

$$\text{div}^*\,\vec{u}^* = \frac{\partial u^*_{k^*}}{\partial x^*_{k^*}} = \frac{\partial (O_{k^*l}u_l)}{\partial x_k}\frac{\partial x_k}{\partial x^*_{k^*}} = O_{k^*l}\frac{\partial u_l}{\partial x_k}O_{k^*k} = \delta_{kl}\frac{\partial u_l}{\partial x_k} = \frac{\partial u_k}{\partial x_k} = \text{div}\,\vec{u}\ . \tag{5.28}$$

(iii) And lastly, (5.8) and (5.12) give

$$(\text{div}^*\,\boldsymbol{a}^*)_{l^*} = \frac{\partial a^*_{k^*l^*}}{\partial x^*_{k^*}} = \frac{\partial (O_{k^*k}a_{kl}O_{l^*l})}{\partial x_m}\frac{\partial x_m}{\partial x^*_{k^*}} = O_{k^*k}O_{l^*l}\frac{\partial a_{kl}}{\partial x_m}O_{k^*m} = \delta_{km}O_{l^*l}\frac{\partial a_{kl}}{\partial x_m} = O_{l^*l}(\text{div}\,\boldsymbol{a})_l\ .$$

Multiplying by $\vec{i}_{l^*}$ and introducing a new vector $\text{div}\,\boldsymbol{a}^* := \dfrac{\partial a^*_{k^*l^*}}{\partial x^*_{k^*}}\vec{i}_{l^*}$, we obtain

$$\text{div}\,\boldsymbol{a}^* = \boldsymbol{O}(t)\cdot \text{div}\,\boldsymbol{a}\ . \tag{5.29}$$

The transformation rule for the deformation gradient is given by

$$\boldsymbol{F}^*(\vec{X},t) = \boldsymbol{O}(t)\cdot \boldsymbol{F}(\vec{X},t)\ , \tag{5.30}$$

where $\boldsymbol{F}^*(\vec{X},t) := \chi^*_{k^*,K}(\vec{i}_{k^*}\otimes \vec{I}_K)$. To show it, we express the deformation gradient in the starred frame according to $(1.34)_1$ and substitute from (5.17):

$$F^*_{k^*K} = \frac{\partial \chi^*_{k^*}}{\partial X_K} = \frac{\partial}{\partial X_K}\Big(O_{k^*k}\,\chi_k + b^*_{k^*}\Big) = O_{k^*k}\frac{\partial \chi_k}{\partial X_K} = O_{k^*k}F_{kK}\ .$$

Multiplying by the tensor product $\vec{i}_{k^*}\otimes \vec{I}_K$, we obtain (5.30). Thus, the two-point deformation gradient tensor $\boldsymbol{F}$ is not an objective tensor. However, three columns of $\boldsymbol{F}$ (for $K = 1,2,3$) are objective vectors.

Let us verify that the jacobian J, the Green deformation tensor $\boldsymbol{C}$, the right stretch tensor $\boldsymbol{U}$ are all objective scalars, the rotation tensor $\boldsymbol{R}$ is an objective vector and the left stretch tensor $\boldsymbol{V}$, the Finger deformation tensor $\boldsymbol{b}$ and strain-rate tensor $\boldsymbol{d}$ are all objective tensors:

$$J^* = J, \qquad \boldsymbol{C}^* = \boldsymbol{C}, \qquad \boldsymbol{U}^* = \boldsymbol{U}, \tag{5.31}$$

$$\boldsymbol{R}^* = \boldsymbol{O}(t)\cdot \boldsymbol{R}, \tag{5.32}$$

$$\boldsymbol{V}^* = \boldsymbol{O}(t)\cdot \boldsymbol{V}\cdot \boldsymbol{O}^T(t), \qquad \boldsymbol{b}^* = \boldsymbol{O}(t)\cdot \boldsymbol{b}\cdot \boldsymbol{O}^T(t), \qquad \boldsymbol{d}^* = \boldsymbol{O}(t)\cdot \boldsymbol{d}\cdot \boldsymbol{O}^T(t)\ . \tag{5.33}$$

On contrary, the spatial velocity gradient $\boldsymbol{l}$ is not an objective tensor:

$$\boldsymbol{l}^* = \boldsymbol{O}(t)\cdot \boldsymbol{l}\cdot \boldsymbol{O}^T(t) + \boldsymbol{\Omega}(t)\ . \tag{5.34}$$

The proof is immediate by making use of (1.40), (1.48), (1.49), (1.53), (1.58), (2.12), (2.20), (5.11) and (5.30):

$$J^* = \det\boldsymbol{F}^* = \det(\boldsymbol{O}\cdot\boldsymbol{F}) = \det\boldsymbol{O}\,\det\boldsymbol{F} = \det\boldsymbol{F} = J,$$

$$
\begin{aligned}
&\boldsymbol{C}^* = (\boldsymbol{F}^*)^T \cdot \boldsymbol{F}^* = \boldsymbol{F}^T \cdot \boldsymbol{O}^T \cdot \boldsymbol{O} \cdot \boldsymbol{F} = \boldsymbol{F}^T \cdot \boldsymbol{F} = \boldsymbol{C},\\
&\boldsymbol{U}^* = \sqrt{\boldsymbol{C}^*} = \sqrt{\boldsymbol{C}} = \boldsymbol{U},\\
&\boldsymbol{R}^* = \boldsymbol{F}^* \cdot (\boldsymbol{U}^*)^{-1} = \boldsymbol{O} \cdot \boldsymbol{F} \cdot \boldsymbol{U}^{-1} = \boldsymbol{O} \cdot \boldsymbol{R},\\
&\boldsymbol{V}^* = \boldsymbol{R}^* \cdot \boldsymbol{U}^* \cdot (\boldsymbol{R}^*)^T = \boldsymbol{O} \cdot \boldsymbol{R} \cdot \boldsymbol{U} \cdot \boldsymbol{R}^T \cdot \boldsymbol{O}^T = \boldsymbol{O} \cdot \boldsymbol{V} \cdot \boldsymbol{O}^T,\\
&\boldsymbol{b}^* = \boldsymbol{F}^* \cdot (\boldsymbol{F}^*)^T = \boldsymbol{O} \cdot \boldsymbol{F} \cdot \boldsymbol{F}^T \cdot \boldsymbol{O}^T = \boldsymbol{O} \cdot \boldsymbol{b} \cdot \boldsymbol{O}^T,\\
&\boldsymbol{l}^* = (\boldsymbol{F}^*)^{\bullet} \cdot (\boldsymbol{F}^*)^{-1} = (\boldsymbol{O} \cdot \dot{\boldsymbol{F}} + \dot{\boldsymbol{O}} \cdot \boldsymbol{F})(\boldsymbol{O} \cdot \boldsymbol{F})^{-1} = \boldsymbol{O} \cdot \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{O}^T + \dot{\boldsymbol{O}} \cdot \boldsymbol{O}^T = \boldsymbol{O} \cdot \boldsymbol{l} \cdot \boldsymbol{O}^T + \boldsymbol{\Omega},\\
&\boldsymbol{d}^* = \frac{1}{2}(\boldsymbol{l}^* + \boldsymbol{l}^{*T}) = \frac{1}{2}(\boldsymbol{O} \cdot \boldsymbol{l} \cdot \boldsymbol{O}^T + \boldsymbol{O} \cdot \boldsymbol{l}^T \cdot \boldsymbol{O}^T + \boldsymbol{\Omega} + \boldsymbol{\Omega}^T) = \boldsymbol{O} \cdot \frac{1}{2}(\boldsymbol{l} + \boldsymbol{l}^T) \cdot \boldsymbol{O}^T = \boldsymbol{O} \cdot \boldsymbol{d} \cdot \boldsymbol{O}^T.
\end{aligned}
$$

5.3 Objective material time derivative

Let us now deal with the material time derivative of an objective scalar, an objective vector and an objective tensor. For an objective scalar λ, for which $\lambda^{**} = \lambda^* = \lambda$, it trivially holds that

$$\dot{\lambda^*} = \dot{\lambda} \; , \tag{5.35}$$

that is, the material time derivative of an objective scalar is again an objective scalar.

For an objective vector $\vec{u}$, for which $\vec{u}^* = \boldsymbol{O}(t) \cdot \vec{u}$, the material time derivative is

$$\dot{\vec{u}^*} = \boldsymbol{O} \cdot \dot{\vec{u}} + \dot{\boldsymbol{O}} \cdot \vec{u} = \boldsymbol{O} \cdot \dot{\vec{u}} + \dot{\boldsymbol{O}} \cdot \boldsymbol{O}^T \cdot \vec{u}^* \; ,$$

or, with the help of $\boldsymbol{\Omega} = \dot{\boldsymbol{O}} \cdot \boldsymbol{O}^T$, we have

$$\dot{\vec{u}^*} = \boldsymbol{O} \cdot \dot{\vec{u}} + \boldsymbol{\Omega} \cdot \vec{u}^* \; . \tag{5.36}$$

This means that the material time derivative of an objective vector is not an objective vector. There are a few possibilities where one can define time derivative of an objective vector to be again objective, and to obey a property of time derivative.

For example, the *Jaumann-Zaremba* or *corotational* time derivative of a vector $\vec{v}$ is defined as:

$$
\begin{aligned}
\frac{D_{\text{Jau}}\vec{u}}{Dt} &:= \dot{\vec{u}} \; ,\\
\frac{D_{\text{Jau}}\vec{u}^*}{Dt} &:= \dot{\vec{u}^*} - \boldsymbol{\Omega} \cdot \vec{u}^* \; .
\end{aligned}
\tag{5.37}
$$

An immediate consequence of (5.36) is that

$$\frac{D_{\text{Jau}}\vec{u}^*}{Dt} = \boldsymbol{O}(t) \cdot \frac{D_{\text{Jau}}\vec{u}}{Dt} \; , \tag{5.38}$$

that is, the Jaumann time derivative of an objective vector is again objective vector.

Likewise, the material time derivative of an objective tensor $\boldsymbol{a}$, for which $\boldsymbol{a}^* = \boldsymbol{O}(t) \cdot \boldsymbol{a} \cdot \boldsymbol{O}^T(t)$, is

$$\dot{\boldsymbol{a}^*} = \boldsymbol{O} \cdot \dot{\boldsymbol{a}} \cdot \boldsymbol{O}^T + \dot{\boldsymbol{O}} \cdot \boldsymbol{a} \cdot \boldsymbol{O}^T + \boldsymbol{O} \cdot \boldsymbol{a} \cdot \dot{\boldsymbol{O}}^T = \boldsymbol{O} \cdot \dot{\boldsymbol{a}} \cdot \boldsymbol{O}^T + \dot{\boldsymbol{O}} \cdot \boldsymbol{O}^T \cdot \boldsymbol{a}^* \cdot \boldsymbol{O} \cdot \boldsymbol{O}^T + \boldsymbol{O} \cdot \boldsymbol{O}^T \cdot \boldsymbol{a}^* \cdot \boldsymbol{O} \cdot \dot{\boldsymbol{O}}^T \; ,$$

or, with the help of $\boldsymbol{\Omega} = \dot{\boldsymbol{O}} \cdot \boldsymbol{O}^T$, we have

$$\dot{\boldsymbol{a}}^* = \boldsymbol{O} \cdot \dot{\boldsymbol{a}} \cdot \boldsymbol{O}^T + \boldsymbol{\Omega} \cdot \boldsymbol{a}^* - \boldsymbol{a}^* \cdot \boldsymbol{\Omega} \ . \tag{5.39}$$

Hence, the material time derivative of an objective tensor is not objective.

The *Jaumann-Zaremba* or *corotational* time derivative of tensor $\boldsymbol{a}$ is defined as follows:

$$\begin{aligned} \frac{D_{\text{Jau}}\boldsymbol{a}}{Dt} &:= \dot{\boldsymbol{a}} \ , \\ \frac{D_{\text{Jau}}\boldsymbol{a}^*}{Dt} &:= \dot{\boldsymbol{a}}^* - \boldsymbol{\Omega} \cdot \boldsymbol{a}^* + \boldsymbol{a}^* \cdot \boldsymbol{\Omega} \ . \end{aligned} \tag{5.40}$$

An immediate consequence of (5.39) is

$$\frac{D_{\text{Jau}}\boldsymbol{a}^*}{Dt} = \boldsymbol{O}(t) \cdot \frac{D_{\text{Jau}}\boldsymbol{a}}{Dt} \cdot \boldsymbol{O}^T(t) \ , \tag{5.41}$$

that is, the Jaumann time derivative of an objective tensor is again an objective tensor.

The Oldroyd derivative is another possibility to introduce the objective time derivative of vectors and tensors. Let $\vec{u}$ and $\boldsymbol{a}$ be an objective vector and tensor, respectively. The *Oldroyd derivative* of $\vec{u}$ and $\boldsymbol{a}$ is defined by the respective formulae

$$\frac{D_{\text{Old}}\vec{u}}{Dt} := \dot{\vec{u}} - \boldsymbol{l} \cdot \vec{u} \ , \tag{5.42}$$

$$\frac{D_{\text{Old}}\boldsymbol{a}}{Dt} := \dot{\boldsymbol{a}} - \boldsymbol{l} \cdot \boldsymbol{a} - \boldsymbol{a} \cdot \boldsymbol{l}^T \ , \tag{5.43}$$

where $\boldsymbol{l}$ is the spatial velocity gradient defined by (2.13). The objectivity of the Oldroyd derivative of an objective vector follows from (5.20), (5.34), (5.36):

$$\begin{aligned} \frac{D_{\text{Old}}\vec{u}^*}{Dt} &= \dot{\vec{u}}^* - \boldsymbol{l}^* \cdot \vec{u}^* = \boldsymbol{O} \cdot \dot{\vec{u}} + \dot{\boldsymbol{O}} \cdot \vec{u} - (\boldsymbol{O} \cdot \boldsymbol{l} \cdot \boldsymbol{O}^T + \boldsymbol{\Omega}) \cdot \boldsymbol{O} \cdot \vec{u} \\ &= \boldsymbol{O} \cdot \dot{\vec{u}} + \dot{\boldsymbol{O}} \cdot \vec{u} - \boldsymbol{O} \cdot \boldsymbol{l} \cdot \boldsymbol{O}^T \cdot \boldsymbol{O} \cdot \vec{u} - \boldsymbol{\Omega} \cdot \boldsymbol{O} \cdot \vec{u} = \boldsymbol{O} \cdot \dot{\vec{u}} - \boldsymbol{O} \cdot \boldsymbol{l} \cdot \vec{u} \ . \end{aligned}$$

Hence,

$$\frac{D_{\text{Old}}\vec{u}^*}{Dt} = \boldsymbol{O}(t) \cdot \frac{D_{\text{Old}}\vec{u}}{Dt} \ . \tag{5.44}$$

Likewise, making use of (5.20), (5.34), (5.39), we have

$$\begin{aligned} \frac{D_{\text{Old}}\boldsymbol{a}^*}{Dt} &= \dot{\boldsymbol{a}}^* - \boldsymbol{l}^* \cdot \boldsymbol{a}^* - \boldsymbol{a}^* \cdot \boldsymbol{l}^{*T} \\ &= \boldsymbol{O} \cdot \dot{\boldsymbol{a}} \cdot \boldsymbol{O}^T + \dot{\boldsymbol{O}} \cdot \boldsymbol{a} \cdot \boldsymbol{O}^T + \boldsymbol{O} \cdot \boldsymbol{a} \cdot \dot{\boldsymbol{O}}^T \\ &- (\boldsymbol{O} \cdot \boldsymbol{l} \cdot \boldsymbol{O}^T + \boldsymbol{\Omega}) \cdot \boldsymbol{O} \cdot \vec{a} \cdot \boldsymbol{O}^T - \boldsymbol{O} \cdot \vec{a} \cdot \boldsymbol{O}^T \cdot (\boldsymbol{O} \cdot \boldsymbol{l}^T \cdot \boldsymbol{O}^T + \boldsymbol{\Omega}^T) \\ &= \boldsymbol{O} \cdot \dot{\boldsymbol{a}} \cdot \boldsymbol{O}^T - \boldsymbol{O} \cdot \boldsymbol{l} \cdot \boldsymbol{a} \cdot \boldsymbol{O}^T - \boldsymbol{O} \cdot \boldsymbol{a} \cdot \boldsymbol{l}^T \cdot \boldsymbol{O}^T \ . \end{aligned}$$

Hence,

$$\frac{D_{\text{Old}}\boldsymbol{a}^*}{Dt} = \boldsymbol{O}(t) \cdot \frac{D_{\text{Old}}\boldsymbol{a}}{Dt} \cdot \boldsymbol{O}^T(t) \ . \tag{5.45}$$

6. CONSTITUTIVE EQUATIONS

6.1 The need for constitutive equations

Basic principles of continuum mechanics, namely, conservation of mass, balance of momenta, and conservation of energy, discussed in Chapter 4, lead to the fundamental equations:

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \vec{v}) = 0 \ , \tag{6.1}$$

$$\operatorname{div} \boldsymbol{t} + \varrho \vec{f} = \varrho \frac{D\vec{v}}{Dt} \ , \qquad \boldsymbol{t} = \boldsymbol{t}^T \ , \tag{6.2}$$

$$\varrho \frac{D\varepsilon}{Dt} = \boldsymbol{t} : \boldsymbol{d} - \operatorname{div} \vec{q} + \varrho h \ . \tag{6.3}$$

In total, they constitute 5 independent equations (one for mass, three for linear momentum and one for energy) for 15 unknown field variables, namely,

- mass density ϱ,
- velocity $\vec{v}$,
- Cauchy's stress tensor $\boldsymbol{t}$,
- internal energy ε,
- heat flux $\vec{q}$,
- temperature θ

provided that body forces $\vec{f}$ and distribution of heat sources h are given. Clearly, the foregoing basic equations are not adequate for the determination of these unknowns except for some trivial situations, for example, rigid body motions in the absence of heat conduction. Hence, 10 additional equations must be supplied to make the problem well-posed.

In the derivation of the equations (6.1) to (6.3) no differentiation has been made between various types materials. It is therefore not surprising that the foregoing equations are not sufficient to explain fully the motions of materials having various type of physical properties. The character of the material is brought into the formulation through the so-called *constitutive equations*, which specify the mechanical and thermal properties of particular materials based upon their internal constitution. Mathematically, the usefulness of these constitutive equations is to describe the relationships among the kinematic, mechanical, and thermal field variables and to permit the formulations of well-posed problems of continuum mechanics. Physically, the constitutive equations define various idealized materials which serve as models for the behavior of real materials. However, it is not possible to write one equation capable of representing a given material over its entire range of application, since many materials behave quite differently under changing levels of loading, such as elastic-plastic response due to increasing stress. Thus, in this sense it is perhaps better to think of constitutive equations as representative of a particular *behavior* rather than of a particular *material.*

6.2 Formulation of thermomechanical constitutive equations

In this text we deal with the constitutive equations of *thermomechanical materials.* The study of the chemical changes and electromagnetic effects are excluded. A large class of materials does not undergo chemical transition or produce appreciable electromagnetic effects when deformed. However, the deformation and motion generally produce heat. Conversely, materials subjected to thermal changes deform and flow. The effect of thermal changes on the material behavior depends on the range and severity of such changes.

The thermomechanical constitutive equations are relations between a set of thermomechanical variables. They may be expressed as an implicit tensor-valued functional $\mathcal{R}$ of 15 unknown field variables:

$$\mathop{\mathcal{R}}_{\substack{\vec{X}' \in \mathcal{B} \\ \tau \leq t}} \left[\varrho(\vec{X}', \tau), \vec{\chi}(\vec{X}', \tau), \theta(\vec{X}', \tau), \boldsymbol{t}(\vec{X}', \tau), \vec{q}(\vec{X}', \tau), \varepsilon(\vec{X}', \tau), \vec{X} \right] = 0 , \tag{6.4}$$

where τ are all past times and t is the present time. The constraints $\vec{X}' \in \mathcal{B}$ and $\tau \leq t$ express the **principle of determinism** postulating that the present state of the thermomechanical variables at a material point $\vec{X}$ of the body $\mathcal{B}$ at time t is uniquely determined by the past history of the motion and the temperature of all material points of the body $\mathcal{B}$. The principle of determinism is a principle of exclusions. It excludes the dependence of the material behavior on any point outside the body and any future events.

We shall restrict the functional in (6.4) to be of a type that does not change with time, that is, that does not depend on the present time t explicitly but only implicitly via thermomechanical variables. Such a functional is invariant with respect to translation in time. The materials described by (6.4) possess time-independent thermomechanical property.

The constitutive functional $\mathcal{R}$ describes the material property of a given material particle $\mathcal{X}$ with the position $\vec{X}$. The functional form may, in general, be different for different particles and $\mathcal{R}$ may thus change with the change of position within the body $\mathcal{B}$; such a material is called *heterogeneous*. If functional $\mathcal{R}$ is independent of $\vec{X}$, material is *homogeneous*.

For a simple material (see the next section), the implicit functional equation (6.4) is supposed to be solved uniquely for the present values of thermomechanical variables. In this case, the implicit functional equation (6.4) is replaced by a set of explicit functional equations:

$$\begin{aligned} \boldsymbol{t}(\vec{X}, t) &= \mathop{\mathcal{F}}_{\substack{\vec{X}' \in \mathcal{B} \\ \tau \leq t}} \left[\varrho(\vec{X}', \tau), \vec{\chi}(\vec{X}', \tau), \theta(\vec{X}', \tau), \vec{X} \right] , \\ \vec{q}(\vec{X}, t) &= \mathop{\mathcal{Q}}_{\substack{\vec{X}' \in \mathcal{B} \\ \tau \leq t}} \left[\varrho(\vec{X}', \tau), \vec{\chi}(\vec{X}', \tau), \theta(\vec{X}', \tau), \vec{X} \right] , \\ \varepsilon(\vec{X}, t) &= \mathop{\mathcal{E}}_{\substack{\vec{X}' \in \mathcal{B} \\ \tau \leq t}} \left[\varrho(\vec{X}', \tau), \vec{\chi}(\vec{X}', \tau), \theta(\vec{X}', \tau), \vec{X} \right] , \end{aligned} \tag{6.5}$$

where $\mathcal{F}$, $\mathcal{Q}$ and $\mathcal{E}$ are respectively tensor-valued, vector-valued and scalar-valued functionals. Note that all constitutive functionals $\mathcal{F}$, $\mathcal{Q}$ and $\mathcal{E}$ are assumed to depend on the same set of variables $\varrho(\vec{X}', \tau)$, $\vec{\chi}(\vec{X}', \tau)$, $\theta(\vec{X}', \tau)$ and $\vec{X}$. This is known as the **principle of equipresence**.

However, the implicit functional equation (6.4) need not be of such a nature as to determine $\boldsymbol{t}$, $\vec{q}$ and ε at $(\vec{X}, t)$ explicitly. For instance, the stress at $(\vec{X}, t)$ may depend not only on the motion and temperature at all other points of the body but also on the histories of the stress, the heat flux and the internal energy. Various types of approximations of (6.4) exist in which the dependence on $\boldsymbol{t}(\vec{X}', \tau)$ is replaced by the history of various order of stress rates, heat rates, etc.

For example, in a special case the constitutive equation (6.4) may be written explicitly for the stress rates $\dot{\boldsymbol{t}}$ at $(\vec{X},t)$:

$$\dot{\boldsymbol{t}}\,(\vec{X},t) = \underset{\substack{\vec{X}'\in\mathcal{B}\\ \tau\le t}}{\mathcal{F}}\left[\boldsymbol{t}(\vec{X},t), \varrho(\vec{X}',\tau), \vec{\chi}(\vec{X}',\tau), \theta(\vec{X}',\tau), \vec{X}\right] . \tag{6.6}$$

More generally, we may have

$$\boldsymbol{t}^{(p)}(\vec{X},t) = \underset{\substack{\vec{X}'\in\mathcal{B}\\ \tau\le t}}{\mathcal{F}}\left[\boldsymbol{t}^{(p-1)}(\vec{X},t), \boldsymbol{t}^{(p-2)}(\vec{X},t), \ldots, \boldsymbol{t}(\vec{X},t), \varrho(\vec{X}',\tau), \vec{\chi}(\vec{X}',\tau), \theta(\vec{X}',\tau), \vec{X}\right] , \tag{6.7}$$

which involves the stress rates up to the pth order at $(\vec{X},t)$. This type of generalization is, for instance, needed to interpret creep data.

Except the Maxwell viscoelastic solid, we shall consider the explicit constitutive equations (6.5). Together with 5 basic balance equations (6.1)–(6.3) they form 15 equations for 15 unknowns. Since $\boldsymbol{t}$, $\vec{q}$, and ε are expressed explicitly in (6.5), it is, in principle, possible to eliminate these variables in (6.1)–(6.3). Then we obtain 5 equations (the so called *field equations of thermomechanics*) for 5 unknown field variables: mass density ϱ, velocity $\vec{v}$ and temperature θ.

We now proceed to deduce the consequence of additional restrictions on the functionals $\mathcal{F}$, $\mathcal{Q}$ and $\mathcal{E}$. Since the procedure is similar for all functionals, for the sake of brevity, we carry out the analysis only for stress functional $\mathcal{F}$. The results for $\mathcal{Q}$ and $\mathcal{E}$ are then written down immediately by analogy.

6.3 Simple materials

The constitutive functionals $\mathcal{F}$, $\mathcal{Q}$ and $\mathcal{E}$ are subject to another fundamental principle, the **principle of local action** postulating that the motion and the temperature at distant material points from $\vec{X}$ does not affect appreciably the stress, the heat flux and the internal energy at $\vec{X}$. Suppose that the functions $\varrho(\vec{X}',\tau)$, $\vec{\chi}(\vec{X}',\tau)$ and $\theta(\vec{X}',\tau)$ admit Taylor series expansion about $\vec{X}$ for all $\tau < t$. According to the principle of local action, a relative deformation of the neighborhood of material point $\vec{X}$ is permissible to approximate only by the first-order gradient

$$\begin{aligned}
\varrho(\vec{X}',\tau) &\approx \varrho(\vec{X},\tau) + \mathrm{Grad}\,\varrho(\vec{X},\tau)\cdot d\vec{X}\ ,\\
\vec{\chi}(\vec{X}',\tau) &\approx \vec{\chi}(\vec{X},\tau) + \boldsymbol{F}(\vec{X},\tau)\cdot d\vec{X}\ ,\\
\theta(\vec{X}',\tau) &\approx \theta(\vec{X},\tau) + \mathrm{Grad}\,\theta(\vec{X},\tau)\cdot d\vec{X}\ ,
\end{aligned} \tag{6.8}$$

where $\boldsymbol{F}(\vec{X},\tau)$ is the deformation gradient tensor at $\vec{X}$ and time τ, and $d\vec{X} = \vec{X}' - \vec{X}$. Since the relative motion and temperature history of an infinitesimal neighborhood of $\vec{X}$ is completely determined by the history of density, deformation and temperature gradients at $\vec{X}$, the stress $\boldsymbol{t}(\vec{X},t)$ must be determined by the history of $\mathrm{Grad}\,\varrho(\vec{X},\tau)$, $\boldsymbol{F}(\vec{X},\tau)$ and $\mathrm{Grad}\,\theta(\vec{X},\tau)$ for $\tau \le t$. Such materials are called *simple materials.* We also note that if we retain higher-order gradients in (6.8), then we obtain nonsimple materials of various classes. For example, by including the second-order gradients into argument of $\mathcal{F}$ we get the theory of *couple stress.* In other words, the behavior of the material point $\vec{X}$ within a simple material is not affected by the histories of the distant points from $\vec{X}$. To any desired degree of accuracy, the whole configuration of a sufficiently small neighborhood of the material point $\vec{X}$ is determined by the history of $\mathrm{Grad}\,\varrho(\vec{X},\tau)$, $\boldsymbol{F}(\vec{X},\tau)$

and $\mathrm{Grad}\,\theta(\vec{X},\tau)$, and we may say that the stress $\boldsymbol{t}(\vec{X},t)$, which was assumed to be determined by the local configuration, is completely determined by $\mathrm{Grad}\,\varrho(\vec{X},\tau)$, $\boldsymbol{F}(\vec{X},\tau)$ and $\mathrm{Grad}\,\theta(\vec{X},\tau)$. That is, the general constitutive equation $(6.5)_1$ reduces to the form

$$\boldsymbol{t}(\vec{X},t) = \mathop{\mathcal{F}}_{\tau\le t}\left[\varrho(\vec{X},\tau), \mathrm{Grad}\,\varrho(\vec{X},\tau), \vec{\chi}(\vec{X},\tau), \boldsymbol{F}(\vec{X},\tau), \theta(\vec{X},\tau), \mathrm{Grad}\,\theta(\vec{X},\tau), \vec{X}\right] . \tag{6.9}$$

Given the deformation gradient $\boldsymbol{F}$, the density in the reference configuration is expressed through the continuity equation (4.60) as

$$\varrho(\vec{X},\tau) = \frac{\varrho_0(\vec{X})}{\det \boldsymbol{F}(\vec{X},\tau)} . \tag{6.10}$$

We can thus drop the argument $\varrho(\vec{X},\tau)$ in the functional $\mathcal{F}$ since $\varrho(\vec{X},\tau)$ is expressible in terms of $\boldsymbol{F}(\vec{X},\tau)$ (the factor $\varrho_0(\vec{X})$ is a fixed expression – not dependent on time – for a given reference configuration). Moreover, by applying the gradient operator on (6.10), the term $\mathrm{Grad}\,\varrho(\vec{X},\tau)$ can be expressed in term of $\mathrm{Grad}\,\boldsymbol{F}(\vec{X},\tau)$. For a simple material, however, $\mathrm{Grad}\,\boldsymbol{F}$ can be neglected with respect to $\boldsymbol{F}$. In summary, the principle of local action applied for a simple material leads to the following constitutive equation:

$$\boldsymbol{t}(\vec{X},t) = \mathop{\mathcal{F}}_{\tau\le t}\left[\vec{\chi}(\vec{X},\tau), \boldsymbol{F}(\vec{X},\tau), \theta(\vec{X},\tau), \vec{G}_\theta(\vec{X},\tau), \vec{X}\right] , \tag{6.11}$$

where $\vec{G}_\theta$ stands for the temperature gradient, $\vec{G}_\theta := \mathrm{Grad}\,\theta$.

6.4 Material objectivity

The change of an observer frame to another observer frame was studied in Chapter 5, where the quantities such as velocity, acceleration and other kinematic quantities were tranformed from a fixed observer frame to a moving observer frame. The most general form of a transformation between two observer frames moving against each other is the observer transformation discussed in Section 5.1. [15]

We now will specify the behavior under the observer transformation of the fields which represent the primitive concepts of mass, force, internal energy and heating. Of these the density ϱ, the stress vector $\vec{t}_{(\vec{n})}$, the internal energy ε and the heat flux $\vec{h}_{(\vec{n})} := \vec{q}\cdot\vec{n}$, being associated with the internal circumstances of a material body, are expected to appear the same to equivalent observers. They are accordingly taken to be objective, an assumption which may properly be viewed as a part of the principle of objectivity. From this assumption, we will show that the Cauchy stress tensor $\boldsymbol{t}$ and the heat flux vector $\vec{q}$ are objective. The objectivity of $\vec{t}_{(\vec{n})}$ means that

$$[\vec{t}^{\,*}_{(\vec{n}^*)}]_{k^*} = O_{k^*k}[\vec{t}_{(\vec{n})}]_k , \tag{6.12}$$

or, by the Cauchy stress formula (3.15), $[\vec{t}_{(\vec{n})}]_k = n_l t_{lk}$, it holds

$$n^*_{l^*} t^*_{l^*k^*} = O_{k^*k} n_l t_{lk} . \tag{6.13}$$

[15] Note that an observer frame is a spatial frame of reference.

Making use of the objectivity of the normal vector $\vec{n}$, [16] that is $n^*_{l^*} = O_{l^*l} n_l$ leads to

$$O_{l^*l} n_l t^*_{l^*k^*} = O_{k^*k} n_l t_{lk} \ . \tag{6.14}$$

Introducing a new tensor $\boldsymbol{t}^* := t^*_{k^*l^*}(\vec{i}_{k^*} \otimes \vec{i}_{l^*})$, we have

$$\vec{n} \cdot (\boldsymbol{O}^T \boldsymbol{t}^* - \boldsymbol{t} \cdot \boldsymbol{O}^T) = \vec{0} \ , \tag{6.15}$$

which must hold for all surface passing through a material point. Hence $\boldsymbol{O}^T \boldsymbol{t}^* - \boldsymbol{t} \cdot \boldsymbol{O}^T = \boldsymbol{0}$. With the orthogonality property of $\boldsymbol{O}$, we finally obtain

$$\boldsymbol{t}^* = \boldsymbol{O}(t) \cdot \boldsymbol{t} \cdot \boldsymbol{O}^T(t) \ , \tag{6.16}$$

which shows that the Cauchy stress tensor is an objective tensor. Likewise, the objectivity of $\vec{q} \cdot \vec{n}$ implies that the heat flux vector $\vec{q}$ is an objective vector,

$$\vec{q}^* = \boldsymbol{O}(t) \cdot \vec{q} \ . \tag{6.17}$$

Let us verify that the referential heat flux vector $\vec{G}_\theta$ and the second Piola-Kirchhoff stress tensor $\boldsymbol{T}^{(2)}$ are both objective scalars:

$$\vec{G}^*_{\theta^*} = \vec{G}_\theta, \qquad \boldsymbol{T}^{(2)^*} = \boldsymbol{T}^{(2)}. \tag{6.18}$$

The proof is immediate by making use of $(3.26)_1$, (5.30), $(5.31)_1$ and (6.16):

$$\vec{G}^*_{\theta^*} = \operatorname{Grad} \theta^* = \operatorname{Grad} \theta = \vec{G}_\theta,$$
$$\boldsymbol{T}^{(2)^*} = J^* (\boldsymbol{F}^*)^{-1} \cdot \boldsymbol{t}^* \cdot (\boldsymbol{F}^*)^{-T} = J \boldsymbol{F}^{-1} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{Q} \cdot \boldsymbol{t} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{Q} \cdot \boldsymbol{F}^{-T} = J \boldsymbol{F}^{-1} \cdot \boldsymbol{t} \cdot \boldsymbol{F}^{-T} = \boldsymbol{T}^{(2)}.$$

The constitutive functionals are subject to yet another fundamental principle. It has become practical evidence that there are no known cases in which constitutive equations are frame dependent. The postulate of the indifference of the constitutive equations against observer transformations is called the **principle of material objectivity**. This principle states that a constitutive equation must be form-invariant under rigid motions of the observer frame or, in other words, the material properties cannot depend on the motion of an observer. To express this principle mathematically, let us write the constitutive equation (6.11) in the unstarred and starred frames:

$$\begin{aligned} \boldsymbol{t}(\vec{X}, t) &= \mathop{\mathcal{F}}_{\tau \le t} \left[\vec{\chi}(\vec{X}, \tau), \boldsymbol{F}(\vec{X}, \tau), \theta(\vec{X}, \tau), \vec{G}_\theta(\vec{X}, \tau), \vec{X} \right] \ , \\ \boldsymbol{t}^*(\vec{X}, t) &= \mathop{\mathcal{F}^*}_{\tau \le t} \left[\vec{\chi}^*(\vec{X}, \tau), \boldsymbol{F}^*(\vec{X}, \tau), \theta^*(\vec{X}, \tau), \vec{G}^*_\theta(\vec{X}, \tau), \vec{X} \right] \ , \end{aligned} \tag{6.19}$$

where $\chi(\vec{X}, t)$ and $\chi^*(\vec{X}, t)$ are related by (5.17). In general, the starred and unstarred functionals $\mathcal{F}$ and $\mathcal{F}^*$ may differ, but the principle of material objectivity requires that the form of the constitutive functional $\mathcal{F}$ must be the same under any two rigid motions of observer frame. Mathematically, no star is attached to the functional $\mathcal{F}$:

$$\mathcal{F}[\ \cdot\ ] = \mathcal{F}^*[\ \cdot\ ] \ , \tag{6.20}$$

[16] The normal vector to a surface may be defined as the gradient of a scalar function. From (5.27) it then follows that $\vec{n}$ is an objective vector.

where the arguments are those of (6.19). [17] Written it in the form (6.16), we have

$$\begin{aligned} &\boldsymbol{O}(t)\cdot \underset{\tau\le t}{\mathcal{F}}\left[\vec{\chi}(\vec{X},\tau),\boldsymbol{F}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau),\vec{X}\right]\cdot \boldsymbol{O}^T(t) \\ &= \underset{\tau\le t}{\mathcal{F}}\left[\vec{\chi}^*(\vec{X},\tau),\boldsymbol{F}^*(\vec{X},\tau),\theta^*(\vec{X},\tau),\vec{G}^*_\theta(\vec{X},\tau),\vec{X}\right]\ . \end{aligned}$$

Taking into account the transformation relations (5.17), (5.30) and $(6.18)_1$, the restriction placed on constitutive functional $\mathcal{F}$ is

$$\begin{aligned} &\boldsymbol{O}(t)\cdot \underset{\tau\le t}{\mathcal{F}}\left[\vec{\chi}(\vec{X},\tau),\boldsymbol{F}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau),\vec{X}\right]\cdot \boldsymbol{O}^T(t) \\ &= \underset{\tau\le t}{\mathcal{F}}\left[\boldsymbol{O}(\tau)\cdot\vec{\chi}(\vec{X},\tau)+\vec{b}^*(\tau),\boldsymbol{O}(\tau)\cdot\boldsymbol{F}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau),\vec{X}\right]\ . \end{aligned} \tag{6.21}$$

This relation must hold for any arbitrary orthogonal tensor-valued functions $\boldsymbol{O}(t)$ and any arbitrary vector-valued function $\vec{b}^*(t)$.

In particular, let us consider a rigid translation of the observer frame such that the origin of the observer frame moves with the material point $\vec{X}$:

$$\boldsymbol{O}(\tau)=\boldsymbol{I}\ , \qquad \vec{b}^*(\tau)=-\vec{\chi}(\vec{X},\tau)\ . \tag{6.22}$$

This means that the reference observer frame is translated so that the material point $\vec{X}$ at any time τ remains at the origin of this frame. From (5.17) it follows that $\vec{\chi}^*(\vec{X},\tau)=\vec{0}$ and (6.21) becomes

$$\underset{\tau\le t}{\mathcal{F}}\left[\vec{\chi}(\vec{X},\tau),\boldsymbol{F}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau),\vec{X}\right]=\underset{\tau\le t}{\mathcal{F}}\left[\vec{0},\boldsymbol{F}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau),\vec{X}\right]\ , \tag{6.23}$$

which must hold for all deformation and temperature histories. Thus the stress at the material point $\vec{X}$ and time t cannot depend *explicitly* on the history of motion of this point. It also implies that velocity and acceleration and all other higher time derivatives of the motion have no influence on the material laws. Consequently, the general constitutive equation (6.11) reduces to the form

$$\boldsymbol{t}(\vec{X},t)=\underset{\tau\le t}{\mathcal{F}}\left[\boldsymbol{F}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau),\vec{X}\right]\ . \tag{6.24}$$

The restriction (6.21) is now reduced to the form

$$\boldsymbol{O}(t)\cdot \underset{\tau\le t}{\mathcal{F}}\left[\boldsymbol{F}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau),\vec{X}\right]\cdot\boldsymbol{O}^T(t)=\underset{\tau\le t}{\mathcal{F}}\left[\boldsymbol{O}(\tau)\cdot\boldsymbol{F}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau),\vec{X}\right]\ , \tag{6.25}$$

which must hold for all orthogonal tensor-valued functions $\boldsymbol{O}(t)$, all deformation $\boldsymbol{F}(\vec{X},t)$, temperature $\theta(\vec{X},t)$ and temperature gradient $\vec{G}_\theta(\vec{X},t)$ processes.

6.5 Reduction by polar decomposition

The condition (6.25) will now be used to reduce the constitutive equation (6.24). Let us recall the polar decomposition (1.47) of the deformation gradient $\boldsymbol{F}(\vec{X},t)=\boldsymbol{R}(\vec{X},t)\cdot\boldsymbol{U}(\vec{X},t)$ into a rotation

[17] This is a significant advantage of the Lagrangian description, since material properties are always associated with a given material particle $\mathcal{X}$ with the position vector $\vec{X}$, while, in the Eulerian description, various material particles may pass through a given spatial position $\vec{x}$.

tensor $\boldsymbol{R}$ and the right stretch tensor $\boldsymbol{U} = \sqrt{\boldsymbol{C}} = \sqrt{\boldsymbol{F}^T \cdot \boldsymbol{F}}$. We now make a special choice for the orthogonal tensor $\boldsymbol{O}(t)$ in (6.25). For any fixed reference place $\vec{X}$, we put $\boldsymbol{O}(t) = \boldsymbol{R}^T(\vec{X}, t)$ for all t. This special choice of $\boldsymbol{O}(t)$ in (6.25) yields:

$$\begin{aligned} &\boldsymbol{R}^T(\vec{X}, t) \cdot \underset{\tau \le t}{\mathcal{F}} \left[\boldsymbol{F}(\vec{X}, \tau), \theta(\vec{X}, \tau), \vec{G}_\theta(\vec{X}, \tau), \vec{X} \right] \cdot \boldsymbol{R}(\vec{X}, t) \\ &= \underset{\tau \le t}{\mathcal{F}} \left[\boldsymbol{R}^T(\vec{X}, \tau) \cdot \boldsymbol{F}(\vec{X}, \tau), \theta(\vec{X}, \tau), \vec{G}_\theta(\vec{X}, \tau), \vec{X} \right] , \end{aligned} \tag{6.26}$$

which with $\boldsymbol{U} = \boldsymbol{R}^T \cdot \boldsymbol{F}$ reduces to

$$\underset{\tau \le t}{\mathcal{F}} \left[\boldsymbol{F}(\vec{X}, \tau), \theta(\vec{X}, \tau), \vec{G}_\theta(\vec{X}, \tau), \vec{X} \right] = \boldsymbol{R}(\vec{X}, t) \cdot \underset{\tau \le t}{\mathcal{F}} \left[\boldsymbol{U}(\vec{X}, \tau), \theta(\vec{X}, \tau), \vec{G}_\theta(\vec{X}, \tau), \vec{X} \right] \cdot \boldsymbol{R}^T(\vec{X}, t) \ . \tag{6.27}$$

This reduced form has been obtained for a special choice of $\boldsymbol{O}(t)$ in the principle of material objectivity (6.25). This means that (6.27) is a necessary relation for satisfying the principle of material objectivity. Now, we shall prove that (6.27) is also a sufficient relation for satisfying this principle. Suppose, that $\mathcal{F}$ is of the form (6.27) and consider an arbitrary orthogonal tensor history $\boldsymbol{O}(t)$. Since the polar decomposition of $\boldsymbol{O} \cdot \boldsymbol{F}$ is $(\boldsymbol{O} \cdot \boldsymbol{R}) \cdot \boldsymbol{U}$, (6.27) for $\boldsymbol{O} \cdot \boldsymbol{F}$ reads

$$\begin{aligned} &\underset{\tau \le t}{\mathcal{F}} \left[\boldsymbol{O}(\tau) \cdot \boldsymbol{F}(\vec{X}, \tau), \theta(\vec{X}, \tau), \vec{G}_\theta(\vec{X}, \tau), \vec{X} \right] \\ &= \boldsymbol{O}(t) \cdot \boldsymbol{R}(\vec{X}, t) \cdot \underset{\tau \le t}{\mathcal{F}} \left[\boldsymbol{U}(\vec{X}, \tau), \theta(\vec{X}, \tau), \vec{G}_\theta(\vec{X}, \tau), \vec{X} \right] \cdot \left[\boldsymbol{O}(t) \cdot \boldsymbol{R}(\vec{X}, t) \right]^T , \end{aligned}$$

which in view of (6.27) reduces to (6.25), so that (6.25) is satisfied. Therefore the reduced form (6.27) is necessary and sufficient to satisfy the principle of material objectivity.

We have proved that the constitutive equation of a simple material may be put into the form

$$\boldsymbol{t}(\vec{X}, t) = \boldsymbol{R}(\vec{X}, t) \cdot \underset{\tau \le t}{\mathcal{F}} \left[\boldsymbol{U}(\vec{X}, \tau), \theta(\vec{X}, \tau), \vec{G}_\theta(\vec{X}, \tau), \vec{X} \right] \cdot \boldsymbol{R}^T(\vec{X}, t) \ . \tag{6.28}$$

A constitutive equation of this kind, in which the functionals are not subject to any further restriction, is called a *reduced form*. The result (6.28) shows that while the stretch history of a simple material may affect its present stress, past rotations have no effect at all. The present rotation enters (6.28) explicitly.

There are many other reduced forms for the constitutive equation of a simple material. Replacing the stretch $\boldsymbol{U}$ by the Green deformation tensor $\boldsymbol{C}$, $\boldsymbol{U} = \sqrt{\boldsymbol{C}}$, and denoting the functional $\mathcal{F}(\sqrt{\boldsymbol{C}}, \cdots)$ again as $\mathcal{F}(\boldsymbol{C}, \cdots)$, we obtain

$$\boldsymbol{t}(\vec{X}, t) = \boldsymbol{R}(\vec{X}, t) \cdot \underset{\tau \le t}{\mathcal{F}} \left[\boldsymbol{C}(\vec{X}, \tau), \theta(\vec{X}, \tau), \vec{G}_\theta(\vec{X}, \tau), \vec{X} \right] \cdot \boldsymbol{R}^T(\vec{X}, t) \ . \tag{6.29}$$

Likewise, expressing rotation $\boldsymbol{R}$ through the deformation gradient, $\boldsymbol{R} = \boldsymbol{F} \cdot \boldsymbol{U}^{-1}$, and replacing the stretch $\boldsymbol{U}$ by Green's deformation tensor $\boldsymbol{C}$, equation (6.29), after introducing a new functional $\tilde{\mathcal{F}}$ of the deformation history $\boldsymbol{C}$, can be put into another reduced form

$$\boldsymbol{t}(\vec{X}, t) = \boldsymbol{F}(\vec{X}, t) \cdot \underset{\tau \le t}{\tilde{\mathcal{F}}} \left[\boldsymbol{C}(\vec{X}, \tau), \theta(\vec{X}, \tau), \vec{G}_\theta(\vec{X}, \tau), \vec{X} \right] \cdot \boldsymbol{F}^T(\vec{X}, t) \ . \tag{6.30}$$

We should emphasize that $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are materially objective functionals.

The principle of material objectivity applied to the constitutive equations (6.5) for the heat flux and the internal energy in analogous way as for the stress results in the following reduced forms:

$$\begin{aligned} \vec{q}(\vec{X},t) &= \boldsymbol{R}(\vec{X},t)\cdot \underset{\tau\le t}{\mathcal{Q}}\left[\boldsymbol{C}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau),\vec{X}\right] , \\ \varepsilon(\vec{X},t) &= \underset{\tau\le t}{\mathcal{E}}\left[\boldsymbol{C}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau),\vec{X}\right] . \end{aligned} \tag{6.31}$$

Another useful reduced form may be obtained if the second Piola-Kirchhoff stress tensor $\boldsymbol{T}^{(2)}$ defined by $(3.26)_1$,

$$\boldsymbol{T}^{(2)} = (\det\boldsymbol{F})\,\boldsymbol{F}^{-1}\cdot\boldsymbol{t}\cdot\boldsymbol{F}^{-T} , \tag{6.32}$$

is used in the constitutive equation instead of the Cauchy stress tensor $\boldsymbol{t}$. With

$$J = \det\boldsymbol{F} = \det(\boldsymbol{R}\cdot\boldsymbol{U}) = \det\boldsymbol{U} = \sqrt{\det\boldsymbol{C}} , \tag{6.33}$$

and defining a new functional $\mathcal{G}$ of the deformation history $\boldsymbol{C}$, the constitutive equation (6.30) can be rewritten for the second Piola-Kirchhoff stress tensor as

$$\boldsymbol{T}^{(2)}(\vec{X},t) = \underset{\tau\le t}{\mathcal{G}}\left[\boldsymbol{C}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau),\vec{X}\right] . \tag{6.34}$$

According to this result, the second Piola-Kirchhoff stress tensor depends only on the Green deformation tensor and not on the rotation. Likewise, the constitutive functional for the heat flux vector in the reference configuration,

$$\vec{Q} = J\boldsymbol{F}^{-1}\cdot\vec{q} , \tag{6.35}$$

is of the form

$$\vec{Q}(\vec{X},t) = \underset{\tau\le t}{\tilde{\mathcal{Q}}}\left[\boldsymbol{C}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau),\vec{X}\right] . \tag{6.36}$$

6.6 Kinematic constraints

A condition of kinematic constraint is a geometric restriction on the set of all motions which are possible for a material body. A condition of constraint can be defined as a restriction on the deformation gradients:

$$\lambda(\boldsymbol{F}(t)) = 0 , \tag{6.37}$$

where to lighten notations in this section we drop the place $\vec{X}$ from the notation as in $\boldsymbol{F}(t) \equiv \boldsymbol{F}(\vec{X},t)$. The requirement that kinematic constraints shall be materially objective implies that equation (6.37) has to be replaced by

$$\lambda(\boldsymbol{C}(t)) = 0 . \tag{6.38}$$

In the context of kinematic constraints, the principle of determinism must be modified; it postulates that only a part of the stress tensor is related to the history of the deformation. Thus the stress tensor (6.24) obtains an additional term

$$\boldsymbol{t}(t) = \boldsymbol{\pi}(t) + \underset{\tau\le t}{\mathcal{F}}\left[\boldsymbol{F}(\tau),\theta(\tau),\vec{G}_\theta(\tau)\right] . \tag{6.39}$$

The tensor $\boldsymbol{t}-\boldsymbol{\pi}$ is called the *determinate stress* because it is uniquely determined by the motion. The tensor $\boldsymbol{\pi}$ is called *indeterminate stress* and represents the *reaction stress* produced by the kinematic constraint (6.38). It is not determined by the motion. In analogy with analytical mechanics, it is assumed that the reaction stress does not perform any work, that is the stress power vanishes for all motions compatible with the constraint condition (6.38):

$$\boldsymbol{\pi}(t) : \boldsymbol{d}(t) = 0 \ , \tag{6.40}$$

where $\boldsymbol{d}$ is the strain-rate tensor defined by $(2.20)_1$.

To derive an alternative form, we modify the constitutive equation (6.34) for the second Piola-Kirchhoff stress tensor as

$$\boldsymbol{T}^{(2)}(t) = \boldsymbol{\Pi}(t) + \underset{\tau\leq t}{\mathcal{G}} \left[\boldsymbol{C}(\tau), \theta(\tau), \vec{G}_\theta(\tau)\right] \ . \tag{6.41}$$

If we take into account (2.23) and $(3.26)_2$, the double-dot product $\boldsymbol{\pi} : \boldsymbol{d}$ can be expressed in terms of the reaction stress $\boldsymbol{\Pi}$ as

$$\boldsymbol{\pi} : \boldsymbol{d} = \frac{1}{J}(\boldsymbol{F}\cdot\boldsymbol{\Pi}\cdot\boldsymbol{F}^T) : \frac{1}{2}(\boldsymbol{F}^{-T}\cdot \dot{\boldsymbol{C}} \cdot\boldsymbol{F}^{-1}) = \frac{1}{2J}(\boldsymbol{\Pi} : \dot{\boldsymbol{C}}) \ ,$$

where (4.74) has been applied in the last equality. Hence, the constraint (6.40) takes the referential form

$$\boldsymbol{\Pi}(t) : \dot{\boldsymbol{C}}\ (t) = 0 \ . \tag{6.42}$$

This says that the reaction stress $\boldsymbol{\Pi}$ has no power to work for all motions compatible with the constraints (6.38). This compatibility can be evaluated in a more specific form. Differentiating (6.38) with respect to time, we obtain

$$\frac{d\lambda(\boldsymbol{C}(t))}{dC_{KL}}\dot{C}_{KL} = 0 \qquad \text{or, symbolically,} \qquad \frac{d\lambda(\boldsymbol{C}(t))}{d\boldsymbol{C}} : \dot{\boldsymbol{C}} = 0 \ . \tag{6.43}$$

This shows that the normal $d\lambda/d\boldsymbol{C}$ to the surface $\lambda(\boldsymbol{C})$=const. is orthogonal to all strain rates $\dot{\boldsymbol{C}}$ which are allowed by the constraint condition (6.38). Equation (6.43) suggests that just the same orthogonality holds for the reaction stress $\boldsymbol{\Pi}$, whence follows that $\boldsymbol{\Pi}$ must be parallel to $d\lambda/d\boldsymbol{C}$:

$$\boldsymbol{\Pi}(t) = \alpha(t)\frac{d\lambda(\boldsymbol{C}(t))}{d\boldsymbol{C}} \ . \tag{6.44}$$

Here, the factor α is left undetermined and must be regarded as an independent field variable in the balance law of linear momentum.

The same arguments apply if simultaneously more constraints are specified. For

$$\lambda_i(\boldsymbol{C}(t)) = 0 \ , \quad i = 1, 2, \ldots, \tag{6.45}$$

we have

$$\boldsymbol{T}^{(2)}(t) = \sum_i \boldsymbol{\Pi}_i(t) + \underset{\tau\leq t}{\mathcal{G}} \left[\boldsymbol{C}(\tau), \theta(\tau), \mathrm{Grad}\,\theta(\tau)\right] \tag{6.46}$$

with

$$\boldsymbol{\Pi}_i(t) = \alpha_i(t)\frac{d\lambda_i(\boldsymbol{C}(t))}{d\boldsymbol{C}} \ . \tag{6.47}$$

As an example, we consider the *volume-preserving* motion for which $J = 1$. Since $J = \varrho_0/\varrho$ due to (4.60), the volume-preserving motion is identical to the *density-preserving* motion for which the mass density of a particle remains unchanged during motion. The volume- or density-preserving constraints are traditionaly regarded as the constraint of *incompressibility*. However, the incompressibility may also mean that the equation of state for density, that is the equation expressing the density as a function of temperature and pressure, $\varrho = \varrho(\theta, p)$, is independent of pressure, that is $\varrho = \varrho(\theta)$. Combining (1.40) and (1.53), the *volume-preserving* constraint becomes

$$J = \sqrt{\det \boldsymbol{C}} = 1 \; . \tag{6.48}$$

This condition places a restriction on $\boldsymbol{C}$, namely, the components of $\boldsymbol{C}$ are not all independent. In this particular case, the constraint (6.38) reads

$$\lambda(\boldsymbol{C}) = \det \boldsymbol{C} - 1 \; . \tag{6.49}$$

Making use of the identity

$$\frac{d}{d\boldsymbol{A}}(\det \boldsymbol{A}) = (\det \boldsymbol{A})\boldsymbol{A}^{-T} \; , \tag{6.50}$$

which is valid for all invertible second-order tensors $\boldsymbol{A}$, we derive

$$\frac{d}{d\boldsymbol{C}}(\det \boldsymbol{C} - 1) = (\det \boldsymbol{C})\boldsymbol{C}^{-1} = \boldsymbol{C}^{-1} \; . \tag{6.51}$$

Equation (6.44) then implies

$$\boldsymbol{\Pi}(t) = \alpha(t)\, \boldsymbol{C}^{-1}(t) \; . \tag{6.52}$$

Instead of α we write $-p$ in order to indicate that p is a *pressure*. Equation (6.41) now reads

$$\boldsymbol{T}^{(2)}(t) = -p(t)\boldsymbol{C}^{-1}(t) + \underset{\tau \le t}{\mathcal{G}} \left[\boldsymbol{C}(\tau), \theta(\tau), \vec{G}_\theta(\tau)\right] \; . \tag{6.53}$$

Complementary, the Cauchy stress tensor is expressed as

$$\boldsymbol{t}(t) = -p(t)\boldsymbol{I} + \boldsymbol{R}(t) \cdot \underset{\tau \le t}{\mathcal{F}} \left[\boldsymbol{C}(\tau), \theta(\tau), \vec{G}_\theta(\tau)\right] \cdot \boldsymbol{R}^T(t) \; . \tag{6.54}$$

This equation shows that for an incompressible simple material the stress is determined by the motion only to within a pressure p.

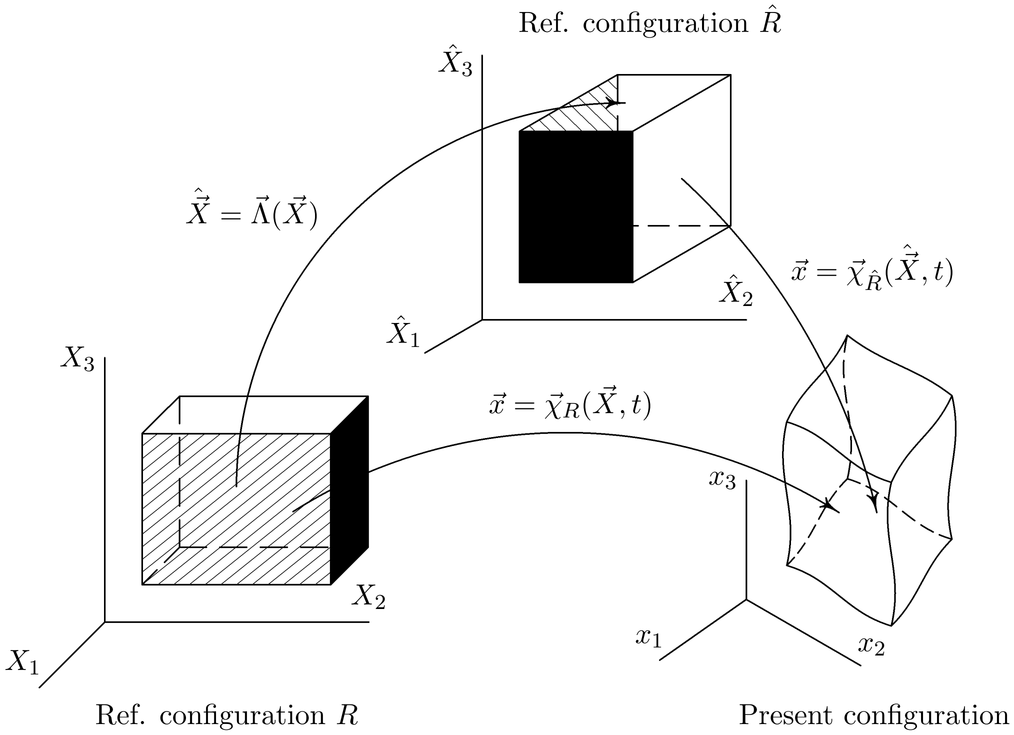

Figure 6.1. Motion of a body with respect to two different reference configurations.

6.7 Material symmetry

In this section, we will confine ourselves to the homogeneous material. We say that the material is *homogeneous* if the constitutive equations do not depend on the translations of the origin the reference configuration. In other words, there is at least one reference configuration in which a constitutive equation of a homogeneous material has the same form for all particles. It means that the explicit dependence of constitutive functionals on the position $\vec{X}$ disappears. For instance, equation (6.24) for a homogeneous material reduces to

$$\boldsymbol{t}(\vec{X}, t) = \underset{\tau \leq t}{\mathcal{F}_R} \left[\boldsymbol{F}(\vec{X}, \tau), \theta(\vec{X}, \tau), \vec{G}_\theta(\vec{X}, \tau) \right] \ . \tag{6.55}$$

Note that the functional $\mathcal{F}$ is labelled by the subscript R since $\mathcal{F}$ depends on the choice of reference configuration.

Material symmetry or material isotropy, if it exists, can be characterized by invariance properties of the constitutive equations with respect to a change of reference configuration. To make this statement more precise and to exploit its consequences, denote by R and $\hat{R}$ two different reference configurations (see Figure 6.1) related by the one-to-one mapping

$$\hat{\vec{X}} = \vec{\Lambda}(\vec{X}) \quad \Longleftrightarrow \quad \vec{X} = \vec{\Lambda}^{-1}(\hat{\vec{X}}) \ , \tag{6.56}$$

where $\vec{X}$ and $\hat{\vec{X}}$ are the positions of the particle $\mathcal{X}$ in the reference configurations R and $\hat{R}$, respectively, $\vec{X} = X_K \vec{I}_K$ and $\hat{\vec{X}} = \hat{X}_{\hat{K}} \hat{\vec{I}}_{\hat{K}}$. Then the motion of the particle in the present configuration $\vec{x} = \vec{\chi}_R(\vec{X}, t)$ takes the form $\vec{x} = \vec{\chi}_{\hat{R}}(\hat{\vec{X}}, t)$, and we obtain the identity

$$\vec{\chi}_R(\vec{X}, t) = \vec{\chi}_{\hat{R}}(\vec{\Lambda}(\vec{X}), t) \ , \tag{6.57}$$

which holds for all $\vec{X}$ and t. Differentiating $\vec{x} = \vec{\chi}_R(\vec{X}, t)$ and $\theta(\vec{X}, t)$ with respect to $\vec{X}$ leads to the transformation rule for the deformation and temperature gradients:

$$\begin{aligned} F_{kK} &= \frac{\partial \chi_k}{\partial X_K} = \frac{\partial \chi_k}{\partial \Lambda_{\hat{L}}} \frac{\partial \Lambda_{\hat{L}}}{\partial X_K} = \hat{F}_{k\hat{L}} P_{\hat{L}K} \ , \\ (\vec{G}_\theta)_K &= \frac{\partial \theta}{\partial X_K} = \frac{\partial \theta}{\partial \Lambda_{\hat{L}}} \frac{\partial \Lambda_{\hat{L}}}{\partial X_K} = (\hat{\vec{G}}_\theta)_{\hat{L}} P_{\hat{L}K} \ , \end{aligned}$$

or, in invariant notation,

$$\begin{aligned} \boldsymbol{F}(\vec{X}, t) &= \hat{\boldsymbol{F}}(\hat{\vec{X}}, t) \cdot \boldsymbol{P}(\vec{X}) \ , \\ \theta(\vec{X}, t) &= \hat{\theta}(\hat{\vec{X}}, t) \ , \\ \vec{G}_\theta(\vec{X}, t) &= \hat{\vec{G}}_{\hat{\theta}}(\hat{\vec{X}}, t) \cdot \boldsymbol{P}(\vec{X}) \ . \end{aligned} \tag{6.58}$$

Here,

$$\hat{\boldsymbol{F}}(\hat{\vec{X}}, t) := (\operatorname{Grad} \vec{\chi}_{\hat{R}} \, (\hat{\vec{X}}, t))^T \tag{6.59}$$

denotes the transposed deformation gradient of the motion $\vec{x} = \vec{\chi}_{\hat{R}}(\hat{\vec{X}}, t)$, and

$$\boldsymbol{P}(\vec{X}) := (\operatorname{Grad} \vec{\Lambda}(\vec{X}))^T = P_{\hat{K}L}(\vec{X})(\hat{\vec{I}}_{\hat{K}} \otimes \vec{I}_L) \ , \qquad P_{\hat{K}L}(\vec{X}) = \frac{\partial \Lambda_{\hat{K}}}{\partial X_L} \ , \tag{6.60}$$

is the transposed gradient of the transformation $\vec{\Lambda}$ that maps the configuration R onto the configuration $\hat{R}$. The deformation and temperature gradients therefore depend on the choice of configuration. Likewise, the Green deformation tensor $\boldsymbol{C}$, $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$, transforms according to the rule:

$$\boldsymbol{C}(\vec{X}, t) = \boldsymbol{P}^T(\vec{X}) \cdot \hat{\boldsymbol{C}}(\hat{\vec{X}}, t) \cdot \boldsymbol{P}(\vec{X}) \ . \tag{6.61}$$

The constitutive equation (6.55) for the reference configuration $\hat{R}$ is of the form

$$\boldsymbol{t}(\hat{\vec{X}}, t) = \underset{\tau \le t}{\mathcal{F}_{\hat{R}}} \left[\hat{\boldsymbol{F}}(\hat{\vec{X}}, \tau), \hat{\theta}(\hat{\vec{X}}, \tau), \hat{\vec{G}}_{\hat{\theta}}(\hat{\vec{X}}, \tau) \right] \ . \tag{6.62}$$

Under the assumption that two different deformation and temperature histories are related by the same mapping as the associated reference configurations, the relation between the two different functionals $\mathcal{F}_R$ and $\mathcal{F}_{\hat{R}}$ is given by the identity

$$\underset{\tau \le t}{\mathcal{F}_R} \left[\boldsymbol{F}(\vec{X}, \tau), \theta(\vec{X}, \tau), \vec{G}_\theta(\vec{X}, \tau) \right] = \underset{\tau \le t}{\mathcal{F}_{\hat{R}}} \left[\hat{\boldsymbol{F}}(\hat{\vec{X}}, \tau), \hat{\theta}(\hat{\vec{X}}, \tau), \hat{\vec{G}}_{\hat{\theta}}(\hat{\vec{X}}, \tau) \right] \ , \tag{6.63}$$

where (6.58) is implicitly considered. Thus, the arguments in configuration R can be expressed in terms of those in configuration $\hat{R}$:

$$\underset{\tau \le t}{\mathcal{F}_R} \left[\hat{\boldsymbol{F}}(\hat{\vec{X}}, \tau) \cdot \boldsymbol{P}(\vec{X}), \hat{\theta}(\hat{\vec{X}}, \tau), \hat{\vec{G}}_{\hat{\theta}}(\hat{\vec{X}}, \tau) \cdot \boldsymbol{P}(\vec{X}) \right] = \underset{\tau \le t}{\mathcal{F}_{\hat{R}}} \left[\hat{\boldsymbol{F}}(\hat{\vec{X}}, \tau), \hat{\theta}(\hat{\vec{X}}, \tau), \hat{\vec{G}}_{\hat{\theta}}(\hat{\vec{X}}, \tau) \right] \ . \tag{6.64}$$

Since (6.64) holds for any deformation and temperature histories, we obtain the identity

$$\underset{\tau\le t}{\mathcal{F}_R}\left[\boldsymbol{F}(\vec{X},\tau)\cdot\boldsymbol{P}(\vec{X}),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau)\cdot\boldsymbol{P}(\vec{X})\right]=\underset{\tau\le t}{\mathcal{F}_{\hat{R}}}\left[\boldsymbol{F}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau)\right]\ . \tag{6.65}$$

This identity can alternatively be arranged by expressing the arguments in configuration $\hat{R}$ in terms of those in configuration R:

$$\underset{\tau\le t}{\mathcal{F}_R}\left[\boldsymbol{F}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau)\right]=\underset{\tau\le t}{\mathcal{F}_{\hat{R}}}\left[\boldsymbol{F}(\vec{X},\tau)\cdot\boldsymbol{P}^{-1}(\vec{X}),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau)\cdot\boldsymbol{P}^{-1}(\vec{X})\right]\ . \tag{6.66}$$

To introduce the concept of material symmetry, consider a particle which at time $t=-\infty$ was in configuration R and subsequently suffered from certain histories of the deformation gradient $\boldsymbol{F}$, the temperature θ and the temperature gradient $\vec{G}_\theta$. At the present time t, the constitutive equation for the stress is

$$\boldsymbol{t}(\vec{X},t)=\underset{\tau\le t}{\mathcal{F}_R}\left[\boldsymbol{F}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau)\right]\ . \tag{6.67}$$

Let the same particle but now in the configuration $\hat{R}$ is experienced the same histories of deformation gradient, temperature and temperature gradient as before. The resulting stress at present time t is

$$\hat{\boldsymbol{t}}(\vec{X},t)=\underset{\tau\le t}{\mathcal{F}_{\hat{R}}}\left[\boldsymbol{F}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau)\right]\ . \tag{6.68}$$

Since $\mathcal{F}_R$ and $\mathcal{F}_{\hat{R}}$ are not, in general, the same functionals, it follows that $\boldsymbol{t}\neq\hat{\boldsymbol{t}}$. However, it may happen that these values coincide, which expresses a certain symmetry of material. The condition for this case is

$$\underset{\tau\le t}{\mathcal{F}_R}\left[\boldsymbol{F}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau)\right]=\underset{\tau\le t}{\mathcal{F}_{\hat{R}}}\left[\boldsymbol{F}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau)\right]\ . \tag{6.69}$$

By expressing the right-hand side according to identity (6.65), we obtain

$$\underset{\tau\le t}{\mathcal{F}_R}\left[\boldsymbol{F}(\vec{X},\tau)\cdot\boldsymbol{P}(\vec{X}),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau)\cdot\boldsymbol{P}(\vec{X})\right]=\underset{\tau\le t}{\mathcal{F}_R}\left[\boldsymbol{F}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau)\right]\ . \tag{6.70}$$

If this relation holds for all deformation and temperature histories, we say that the material at the particle $\mathcal{X}$ is *symmetric* with respect to the transformation $\boldsymbol{P}$: $R\to\hat{R}$.

It shows that any such $\boldsymbol{P}$ as in (6.70) corresponds to a static local deformation (at a given particle $\mathcal{X}$) from R to another reference configuration $\hat{R}$ such that any deformation and temperature histories lead to the same stress in either R or $\hat{R}$ (at a given particle $\mathcal{X}$). Hence $\boldsymbol{P}$ corresponds to a change of reference configuration which cannot be detected by any experiment at a given particle $\mathcal{X}$. The matrices $\boldsymbol{P}$, for which (6.70) applies, are called the *symmetry* transformations. Often the constitutive functionals remain invariant under volume-preserving transformations, that is, the transformations with $\det\boldsymbol{P}=\pm1$. A transformation with this property is called *unimodular*. From now on, we will consider changes of reference configuration for which the associated deformation gradient $\boldsymbol{P}$ is unimodular. We point out that the unimodularity condition on the transformation of reference configuration could, in principle, be omitted. However, no materials are known that satisfy the symmetry condition (6.70) for a non-unimodular transformation.

The properties of material symmetry can be represented in terms of the symmetry group. The set of all unimodular transformations which leave the constitutive equation invariant with respect to R, that is, for which (6.70) holds forms a group. [18] This group is called the *symmetry group* g_R of the material at the particle $\mathcal{X}$ whose place in the reference configuration R is $\vec{X}$:

$$g_R := \Big\{ \boldsymbol{H} | \quad \det \boldsymbol{H} = \pm 1 \quad \text{and} \\ \underset{\tau \le t}{\mathcal{F}_R} \Big[\boldsymbol{F}(\tau) \cdot \boldsymbol{H}, \theta(\tau), \vec{G}_\theta(\tau) \cdot \boldsymbol{H} \Big] = \underset{\tau \le t}{\mathcal{F}_R} \Big[\boldsymbol{F}(\tau), \theta(\tau), \vec{G}_\theta(\tau) \Big] \quad \text{for } \forall \boldsymbol{F}(\tau),\ \forall \theta(\tau),\ \forall \vec{G}_\theta(\tau) \Big\}, \tag{6.71}$$

where we have dropped the place $\vec{X}$ from the notation as, for instance, in $\boldsymbol{F}(\vec{X}, \tau) \equiv \boldsymbol{F}(\tau)$. The operation defined on g_R is the scalar product of tensors and the identity element is the identity tensor.

To show that g_R is a group, let $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ be two elements of g_R. Then we can replace $\boldsymbol{F}$ in (6.71) with $\boldsymbol{F} \cdot \boldsymbol{H}_1$, $\vec{G}_\theta$ with $\vec{G}_\theta \cdot \boldsymbol{H}_1$ and take $\boldsymbol{H} = \boldsymbol{H}_2$ to find

$$\underset{\tau \le t}{\mathcal{F}_R} \Big[\boldsymbol{F}(\tau) \cdot \boldsymbol{H}_1 \cdot \boldsymbol{H}_2, \theta(\tau), \vec{G}_\theta(\tau) \cdot \boldsymbol{H}_1 \cdot \boldsymbol{H}_2 \Big] = \underset{\tau \le t}{\mathcal{F}_R} \Big[\boldsymbol{F}(\tau) \cdot \boldsymbol{H}_1, \theta(\tau), \vec{G}_\theta(\tau) \cdot \boldsymbol{H}_1 \Big] \ .$$

Since $\boldsymbol{H}_1 \in g_R$, we find that

$$\underset{\tau \le t}{\mathcal{F}_R} \Big[\boldsymbol{F}(\tau) \cdot \boldsymbol{H}_1 \cdot \boldsymbol{H}_2, \theta(\tau), \vec{G}_\theta(\tau) \cdot \boldsymbol{H}_1 \cdot \boldsymbol{H}_2 \Big] = \underset{\tau \le t}{\mathcal{F}_R} \Big[\boldsymbol{F}(\tau), \theta(\tau), \vec{G}_\theta(\tau) \Big] \ .$$

This shows that the scalar product $\boldsymbol{H}_1 \cdot \boldsymbol{H}_2 \in g_R$. Furthermore, the identity tensor $\boldsymbol{H}=\boldsymbol{I}$ clearly satisfies (6.71). Finally, if $\boldsymbol{H} \in g_R$ and $\boldsymbol{F}$ is invertible, then $\boldsymbol{F} \cdot \boldsymbol{H}^{-1}$ is an invertible tensor. Hence (6.71) must hold with $\boldsymbol{F}$ replaced by $\boldsymbol{F} \cdot \boldsymbol{H}^{-1}$ (and $\vec{G}_\theta$ with $\vec{G}_\theta \cdot \boldsymbol{H}^{-1}$):

$$\underset{\tau \le t}{\mathcal{F}_R} \Big[\boldsymbol{F}(\tau) \cdot \boldsymbol{H}^{-1} \cdot \boldsymbol{H}, \theta(\tau), \vec{G}_\theta(\tau) \cdot \boldsymbol{H}^{-1} \cdot \boldsymbol{H} \Big] = \underset{\tau \le t}{\mathcal{F}_R} \Big[\boldsymbol{F}(\tau) \cdot \boldsymbol{H}^{-1}, \theta(\tau), \vec{G}_\theta(\tau) \cdot \boldsymbol{H}^{-1} \Big] \ .$$

Hence we find that

$$\underset{\tau \le t}{\mathcal{F}_R} \Big[\boldsymbol{F}(\tau), \theta(\tau), \vec{G}_\theta(\tau) \Big] = \underset{\tau \le t}{\mathcal{F}_R} \Big[\boldsymbol{F}(\tau) \cdot \boldsymbol{H}^{-1}, \theta(\tau), \vec{G}_\theta(\tau) \cdot \boldsymbol{H}^{-1} \Big] \ , \tag{6.72}$$

which shows that $\boldsymbol{H}^{-1} \in g_R$. We have proved that g_R has the structure of a group. Note that (6.72) is an equivalent condition for the material symmetry.

6.8 Material symmetry of reduced-form functionals

We now employ the definition of the symmetry group to find a symmetry criterion for the functionals whose forms are reduced by applying the polar decomposition of the deformation gradient. As an example, let us consider the constitutive equation (6.30) for the Cauchy stress tensor. For

[18] A *group* is a set of abstract elements $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\cdots$, with a defined operation $\boldsymbol{A} \cdot \boldsymbol{B}$ (as, for instance, '*multiplication*') such that: (1) the product $\boldsymbol{A} \cdot \boldsymbol{B}$ is defined for all elements $\boldsymbol{A}$ and $\boldsymbol{B}$ of the set, (2) this product $\boldsymbol{A} \cdot \boldsymbol{B}$ is itself an element of the set for all $\boldsymbol{A}$ and $\boldsymbol{B}$ (the set is *closed* under the operation), (3) the set contains an identity element $\boldsymbol{I}$ such that $\boldsymbol{I} \cdot \boldsymbol{A} = \boldsymbol{A} = \boldsymbol{A} \cdot \boldsymbol{I}$ for all $\boldsymbol{A}$, (4) every element has an inverse $\boldsymbol{A}^{-1}$ in the set such that $\boldsymbol{A} \cdot \boldsymbol{A}^{-1} = \boldsymbol{A}^{-1} \cdot \boldsymbol{A} = \boldsymbol{I}$.

a homogeneous material, this constitutive equation in the reference configurations R and $\hat{R}$ has the form

$$\begin{aligned} \boldsymbol{t}(\vec{X},t) &= \boldsymbol{F}(\vec{X},t)\cdot \underset{\tau\le t}{\tilde{\mathcal{F}}_R}\left[\boldsymbol{C}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau)\right]\cdot \boldsymbol{F}^T(\vec{X},t)\ , \\ \boldsymbol{t}(\hat{\vec{X}},t) &= \hat{\boldsymbol{F}}(\hat{\vec{X}},t)\cdot \underset{\tau\le t}{\tilde{\mathcal{F}}_{\hat{R}}}\left[\hat{\boldsymbol{C}}(\hat{\vec{X}},\tau),\hat{\theta}(\hat{\vec{X}},\tau),\hat{\vec{G}}_{\hat{\theta}}(\hat{\vec{X}},\tau)\right]\cdot \hat{\boldsymbol{F}}^T(\hat{\vec{X}},t)\ , \end{aligned} \tag{6.73}$$

where the subscripts R and $\hat{R}$ refer to the underlying reference configuration. With the help of the transformation relation (6.58) for the deformation gradient, the temperature and the temperature gradient, and (6.61) for the Green deformation tensor, the functional $\tilde{\mathcal{F}}_R$ satisfies the identity analogous to (6.66):

$$\begin{aligned} &\boldsymbol{P}(\vec{X})\cdot \underset{\tau\le t}{\tilde{\mathcal{F}}_R}\left[\boldsymbol{C}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau)\right]\cdot \boldsymbol{P}^T(\vec{X}) \\ =&\underset{\tau\le t}{\tilde{\mathcal{F}}_{\hat{R}}}\left[\boldsymbol{P}^{-T}(\vec{X})\cdot \boldsymbol{C}(\vec{X},\tau)\cdot \boldsymbol{P}^{-1}(\vec{X}),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau)\cdot \boldsymbol{P}^{-1}(\vec{X})\right]\ . \end{aligned} \tag{6.74}$$

Since $\tilde{\mathcal{F}}_R$ and $\tilde{\mathcal{F}}_{\hat{R}}$ are not, in general, the same functionals, the functional $\tilde{\mathcal{F}}_{\hat{R}}$ on the right-hand side of (6.74) cannot be replaced by the functional $\tilde{\mathcal{F}}_R$. However, it may happen, that $\tilde{\mathcal{F}}_R = \tilde{\mathcal{F}}_{\hat{R}}$, which expresses a certain symmetry of material. The condition for this case is

$$\begin{aligned} &\boldsymbol{P}(\vec{X})\cdot \underset{\tau\le t}{\tilde{\mathcal{F}}_R}\left[\boldsymbol{C}(\vec{X},\tau),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau)\right]\cdot \boldsymbol{P}^T(\vec{X}) \\ =&\underset{\tau\le t}{\tilde{\mathcal{F}}_R}\left[\boldsymbol{P}^{-T}(\vec{X})\cdot \boldsymbol{C}(\vec{X},\tau)\cdot \boldsymbol{P}^{-1}(\vec{X}),\theta(\vec{X},\tau),\vec{G}_\theta(\vec{X},\tau)\cdot \boldsymbol{P}^{-1}(\vec{X})\right]\ . \end{aligned} \tag{6.75}$$

We say that a material at the particle $\mathcal{X}$ (whose place in the reference configuration is $\vec{X}$) is *symmetric with respect to the transformation* $\boldsymbol{P} : R \to \hat{R}$ if the last relation is valid for all deformation and temperature histories. The symmetry group g_R of a material characterized by the constitutive functional $\tilde{\mathcal{F}}_R$ is defined as

$$\begin{aligned} g_R := \Big\{\boldsymbol{H}|\ \ &\det \boldsymbol{H} = \pm 1 \quad \text{and} \\ &\boldsymbol{H}\cdot \underset{\tau\le t}{\tilde{\mathcal{F}}_R}\left[\boldsymbol{C}(\tau),\theta(\tau),\vec{G}_\theta(\tau)\right]\cdot \boldsymbol{H}^T = \underset{\tau\le t}{\tilde{\mathcal{F}}_R}\left[\boldsymbol{H}^{-T}\cdot \boldsymbol{C}(\tau)\cdot \boldsymbol{H}^{-1},\theta(\tau),\vec{G}_\theta(\tau)\cdot \boldsymbol{H}^{-1}\right] \\ &\text{for}\ \ \forall \boldsymbol{C}(\tau),\ \forall\theta(\tau),\ \forall\vec{G}_\theta(\tau)\Big\}, \end{aligned} \tag{6.76}$$

where we have dropped the place $\vec{X}$ from the notation as, for instance, in $\boldsymbol{C}(\vec{X},\tau) \equiv \boldsymbol{C}(\tau)$.

By an analogous way, it can be shown that the condition of material symmetry for the materially objective constitutive functional $\mathcal{G}_R$ in constitutive equation (6.34) is similar to that of $\tilde{\mathcal{F}}_R$:

$$\boldsymbol{H}\cdot \underset{\tau\le t}{\mathcal{G}_R}\left[\boldsymbol{C}(\tau),\theta(\tau),\vec{G}_\theta(\tau)\right]\cdot \boldsymbol{H}^T = \underset{\tau\le t}{\mathcal{G}_R}\left[\boldsymbol{H}^{-T}\cdot \boldsymbol{C}(\tau)\cdot \boldsymbol{H}^{-1},\theta(\tau),\vec{G}_\theta(\tau)\cdot \boldsymbol{H}^{-1}\right]\ . \tag{6.77}$$

6.9 Noll's rule

The symmetry group g_R generally depends upon the choice of reference configuration, since the symmetries a body enjoys with respect to one configuration generally differ from those it enjoys with respect to another. However, the symmetry groups g_R and $g_{\hat{R}}$ (for the same particle) relative to two different reference configurations R and $\hat{R}$ are related by Noll's rule:

$$g_{\hat{R}} = \boldsymbol{P} \cdot g_R \cdot \boldsymbol{P}^{-1} \ , \tag{6.78}$$

where $\boldsymbol{P}$ is the fixed local deformation tensor for the deformation from R to $\hat{R}$ given by (6.60). Noll's rule means that every tensor $\hat{\boldsymbol{H}} \in g_{\hat{R}}$ must be of the form $\hat{\boldsymbol{H}} = \boldsymbol{P} \cdot \boldsymbol{H} \cdot \boldsymbol{P}^{-1}$ for some tensor $\boldsymbol{H} \in g_R$, and conversely every tensor $\boldsymbol{H} \in g_R$ is of the form $\boldsymbol{H} = \boldsymbol{P}^{-1} \cdot \hat{\boldsymbol{H}} \cdot \boldsymbol{P}$ for some $\hat{\boldsymbol{H}} \in g_{\hat{R}}$. Note that $\boldsymbol{P}$ not necessarily be unimodular, because $\boldsymbol{P}$ represents a change of reference configuration, not a member of symmetry group. Noll's rule shows that if g_R is known for one configuration, it is known for all. That is, the symmetries of a material in any one configuration determine its symmetries in every other.

To prove Noll's rule, let $\boldsymbol{H} \in g_R$. Then (6.71) holds and it is permissible to replace $\boldsymbol{F}$ by $\boldsymbol{F} \cdot \boldsymbol{P}$ and $\vec{G}_\theta$ by $\vec{G}_\theta \cdot \boldsymbol{P}$:

$$\underset{\tau \le t}{\mathcal{F}_R}\left[\boldsymbol{F}(\tau) \cdot \boldsymbol{P} \cdot \boldsymbol{H}, \theta(\tau), \vec{G}_\theta(\tau) \cdot \boldsymbol{P} \cdot \boldsymbol{H}\right] = \underset{\tau \le t}{\mathcal{F}_R}\left[\boldsymbol{F}(\tau) \cdot \boldsymbol{P}, \theta(\tau), \vec{G}_\theta(\tau) \cdot \boldsymbol{P}\right] \ .$$

By applying identity (6.66) to the both side of this equation, we obtain

$$\underset{\tau \le t}{\mathcal{F}_{\hat{R}}}\left[\boldsymbol{F}(\tau) \cdot \boldsymbol{P} \cdot \boldsymbol{H} \cdot \boldsymbol{P}^{-1}, \theta(\tau), \vec{G}_\theta(\tau) \cdot \boldsymbol{P} \cdot \boldsymbol{H} \cdot \boldsymbol{P}^{-1}\right] = \underset{\tau \le t}{\mathcal{F}_{\hat{R}}}\left[\boldsymbol{F}(\tau), \theta(\tau), \vec{G}_\theta(\tau)\right] \ ,$$

which holds for all deformation and temperature histories. Thus, if $\boldsymbol{H} \in g_R$, then $\boldsymbol{P} \cdot \boldsymbol{H} \cdot \boldsymbol{P}^{-1} \in g_{\hat{R}}$. The argument can be reversed which completes the proof of Noll's rule.

By definition, the symmetry group is a subgroup of the group of all unimodular transformations,

$$g_R \subseteq \mathsf{unim.} \tag{6.79}$$

The 'size' of a specific symmetry group is the measure for the amount of material symmetry. The group of unimodular transformations contains the group of orthogonal transformations (as, for instance, rotations or reflections) but also the group of non-orthogonal volume-preserving transformations (as, for instance, torsions), i.e.,

$$\mathsf{orth.} \subset \mathsf{unim.} \tag{6.80}$$

The smallest symmetry group $\{\boldsymbol{I}, -\boldsymbol{I}\}$, $\{\boldsymbol{I}, -\boldsymbol{I}\} \subset g_R$, corresponds to a material that has no (nontrivial) symmetries. Such a material is called *triclinic*. Obviously,

$$\boldsymbol{P} \cdot \{\boldsymbol{I}, -\boldsymbol{I}\} \cdot \boldsymbol{P}^{-1} = \{\boldsymbol{I}, -\boldsymbol{I}\} \ . \tag{6.81}$$

Hence a triclinic material has no symmetries relative to any reference configuration: no transformation can bring this material into a configuration that has a nontrivial symmetry.

The largest symmetry group is the group of all unimodular transformations. Noll's rule shows that if $\boldsymbol{P}$ is an arbitrary invertible tensor, then $\boldsymbol{P} \cdot \boldsymbol{H} \cdot \boldsymbol{P}^{-1}$ is unimodular for all unimodular $\boldsymbol{H}$, and if $\overline{\boldsymbol{H}}$ is any unimodular tensor, the tensor $\boldsymbol{P}^{-1} \cdot \overline{\boldsymbol{H}} \cdot \boldsymbol{P}$ is a unimodular tensor, and $\overline{\boldsymbol{H}} = \boldsymbol{P} \cdot \boldsymbol{H} \cdot \boldsymbol{P}^{-1}$. Therefore,

$$\boldsymbol{P} \cdot \mathsf{unim.} \cdot \boldsymbol{P}^{-1} = \mathsf{unim.} \tag{6.82}$$

Hence, the maximum symmetry of such a material cannot be destroyed by deformation.

6.10 Classification of the symmetry properties

6.10.1 Isotropic materials

We will now classify material according to the symmetry group it belongs to. We say that a material is *isotropic* if it possesses a configuration R, in which the symmetry condition (6.71) or (6.76) applies at least for all orthogonal transformations. In other words, a material is isotropic if there exists a reference configuration R such that

$$g_R \supseteq \mathsf{orth.} \tag{6.83}$$

where $\mathsf{orth.}$ denotes the group of all orthogonal transformation. For an isotropic material, (6.79) and (6.83) can be combined to give

$$\mathsf{orth.} \subseteq g_R \subseteq \mathsf{unim.} \tag{6.84}$$

All groups of isotropic bodies are thus bounded by the orthogonal and unimodular groups. A question arises of how many groups bounded by $\mathsf{orth.}$ and $\mathsf{unim.}$ may exist. The group theory states that the orthogonal group is a maximal subgroup of the unimodular group. Consequently, there are only two groups satisfying (6.84), either orthogonal or unimodular; no other group exists between them. Hence, there are two kinds of isotropic materials, either those with $g_R = \mathsf{orth.}$ or those with $g_R = \mathsf{unim.}$ All other materials are anisotropic and possess a lower degree of symmetry.

The concept of the symmetry group can be used to define solids and fluids according to Noll.

6.10.2 Fluids

Fluids have the property that can be poured from one container to another and, after some time, no evidence of the previous circumstances remains. Such a change of container can be thought as a change of reference configuration, so that for fluids all reference configurations with the same density are indistinguishable. In other words, every configuration, also the present configuration, can be thought as a reference configuration. According to Noll, a simple material is a *fluid* if its symmetry group has a maximal symmetry, being identical with the set of all unimodular transformations,

$$g_R = \mathsf{unim.} \tag{6.85}$$

Moreover, Noll's rule implies that the condition $g_R = \mathsf{unim.}$ holds for every configuration if it holds for any one reference configuration R, so a fluid has maximal symmetry relative to every reference configuration. A fluid is thus a non-solid material with no preferred configurations. In terms of symmetry groups, $g_{\hat{R}} = g_R$ $(= \mathsf{unim.})$. Moreover, a fluid is isotropic, because of (6.80).

The simple fluid and the simple solid do not exhaust the possible types of simple materials. There is, for instance, a simple material, called *liquid crystal* which is neither a simple fluid nor a simple solid.

6.10.3 Solids

Solids have the property that any change of shape (as represented by a non-orthogonal transformation of the reference configuration) brings the material into a new reference configuration from which its response is different. Hence, according to Noll, a simple material is called a *solid* if there is a reference configuration such that every element of the symmetry group is a rotation,

$$g_R \subseteq \mathsf{orth}. \tag{6.86}$$

A solid whose symmetry group is equal to the full orthogonal group is called *isotropic*,

$$g_R = \mathsf{orth}. \tag{6.87}$$

If the symmetry group of a solid is smaller than the full orthogonal group, the solid is called *anisotropic*,

$$g_R \subset \mathsf{orth}. \tag{6.88}$$

There is only a finite number of symmetry group with this property, and it forms the *32 crystal classes.*

6.11 Constitutive equation for isotropic materials

Every material belonging to a certain symmetry group g_R must also be objective. Thus, the objectivity condition (6.25) and the condition of material symmetry (6.72) must be satisfied simultaneously. Combining them, the material functional must satisfied the following condition

$$\boldsymbol{O}(t) \cdot \underset{\tau \leq t}{\mathcal{F}} \left[\boldsymbol{F}(\tau), \theta(\tau), \vec{G}_\theta(\tau)\right] \cdot \boldsymbol{O}^T(t) = \underset{\tau \leq t}{\mathcal{F}} \left[\boldsymbol{O}(\tau) \cdot \boldsymbol{F}(\tau) \cdot \boldsymbol{H}^{-1}, \theta(\tau), \vec{G}_\theta(\tau) \cdot \boldsymbol{H}^{-1}\right] \tag{6.89}$$

for all orthogonal transformation $\boldsymbol{O}(t)$ of the reference frame and all symmetry transformation $\boldsymbol{H} \in g_R$. Note that the subscript R at the functional $\mathcal{F}$ is omitted since we will not consider more than one reference configuration in the following text. For an isotropic material, the set of unimodular transformations is equal to the set of full orthogonal transformations, $\boldsymbol{H} = \boldsymbol{Q} \in \mathsf{orth}$. In contrast to $\boldsymbol{O}(t)$, the orthogonal tensor $\boldsymbol{Q}$ is time independent. Restricting the last transformation to constant-in-time element $\boldsymbol{Q}$, (6.89) becomes

$$\boldsymbol{Q} \cdot \underset{\tau \leq t}{\mathcal{F}} \left[\boldsymbol{F}(\tau), \theta(\tau), \vec{G}_\theta(\tau)\right] \cdot \boldsymbol{Q}^T = \underset{\tau \leq t}{\mathcal{F}} \left[\boldsymbol{Q} \cdot \boldsymbol{F}(\tau) \cdot \boldsymbol{Q}^T, \theta(\tau), \boldsymbol{Q} \cdot \vec{G}_\theta(\tau)\right] , \tag{6.90}$$

where we have identified $\boldsymbol{Q} \cdot \vec{G}_\theta(\tau) \equiv \vec{G}_\theta(\tau) \cdot \boldsymbol{Q}^T$. Similar constraints can be derived for the constitutive functionals $\mathcal{Q}$ and $\mathcal{E}$ of the heat flux and the internal energy, respectively:

$$\begin{aligned} \boldsymbol{Q} \cdot \underset{\tau \leq t}{\mathcal{Q}} \left[\boldsymbol{F}(\tau), \theta(\tau), \vec{G}_\theta(\tau)\right] &= \underset{\tau \leq t}{\mathcal{Q}} \left[\boldsymbol{Q} \cdot \boldsymbol{F}(\tau) \cdot \boldsymbol{Q}^T, \theta(\tau), \boldsymbol{Q} \cdot \vec{G}_\theta(\tau)\right] , \\ \underset{\tau \leq t}{\mathcal{E}} \left[\boldsymbol{F}(\tau), \theta(\tau), \vec{G}_\theta(\tau)\right] &= \underset{\tau \leq t}{\mathcal{E}} \left[\boldsymbol{Q} \cdot \boldsymbol{F}(\tau) \cdot \boldsymbol{Q}^T, \theta(\tau), \boldsymbol{Q} \cdot \vec{G}_\theta(\tau)\right] . \end{aligned} \tag{6.91}$$

Functionals which satisfy constraints (6.89) and (6.91) are called *tensorial, vectorial* and *scalar isotropic functionals* with respect to orthogonal transformations. All these functionals represent constitutive equations for an isotropic body. Note that the conditions (6.89)–(6.91) are necessary for the material objectivity of isotropic functionals, but not necessarily sufficient, since the principle of the material objectivity is satisfied for a constant, time-independent $\boldsymbol{Q} \in \mathsf{orth}$, but not for a general $\boldsymbol{O}(t) \in \mathsf{orth}$. In many cases, the functionals are also materially objective for all $\boldsymbol{O}(t) \in \mathsf{orth}$, but this must be examined in every individual case.

6.12 Current configuration as reference

So far, we have employed a reference configuration fixed in time, but we can also use a reference configuration varying in time. Thus one motion may be described in terms of any other. The only time-variable reference configuration useful in this way is the present configuration. If we take it as reference, we describe past events as they seem to an observer fixed to the particle $\mathcal{X}$ now at the place $\vec{x}$. The corresponding description is called *relative* and it has been introduced in section 1.2.

To see how such the relative description is constructed, consider the configurations of body $\mathcal{B}$ at the two times τ and t:

$$\begin{aligned} \vec{\xi} &= \vec{\chi}(\vec{X},\tau) , \\ \vec{x} &= \vec{\chi}(\vec{X},t) , \end{aligned} \tag{6.92}$$

that is, ξ is the place occupied at time τ by the particle that occupies $\vec{x}$ at time t. Since the function $\vec{\chi}$ is invertible, that is,

$$\vec{X} = \vec{\chi}^{-1}(\vec{x},t) , \tag{6.93}$$

we have either

$$\vec{\xi} = \vec{\chi}(\vec{\chi}^{-1}(\vec{x},t),\tau) =: \vec{\chi}_t(\vec{x},\tau) , \tag{6.94}$$

where the function $\vec{\chi}_t(\vec{x},\tau)$ is called the *relative motion function.* It describes the deformation of the new configuration κ_τ of the body $\mathcal{B}$ relative to the present configuration κ_t, which is considered as reference. The subscript t at the function $\vec{\chi}$ is used to recall this fact. The relative description, given by the mapping (6.94), is actually a special case of the referential description, differing from the Lagrangian description in that the reference position is now denoted by $\vec{x}$ at time t instead of $\vec{X}$ at time $t = 0$. The variable time τ, being the time when the particle occupied the position ξ, is now considered as an independent variable instead of the time t in the Lagrangian description.

The *relative deformation gradient* $\boldsymbol{F}_t$ is the gradient of the relative motion function:

$$\boldsymbol{F}_t(\vec{x},\tau) := (\operatorname{grad}\vec{\chi}_t(\vec{x},\tau))^T , \qquad (\boldsymbol{F}_t)_{kl} := \frac{\partial(\vec{\chi}_t)_k}{\partial x_l} . \tag{6.95}$$

It can be expressed in terms of the (absolute) deformation gradient $\boldsymbol{F}$. Differentiating the following identity for the motion function,

$$\vec{\xi} = \vec{\chi}(\vec{X},\tau) = \vec{\chi}_t(\vec{x},\tau) = \vec{\chi}_t(\vec{\chi}(\vec{X},t),\tau) ,$$

with respect to X_K and using the chain rule of differentiation, we get

$$\frac{\partial\chi_k(\vec{X},\tau)}{\partial X_K} = \frac{\partial(\vec{\chi}_t)_k(\vec{x},\tau)}{\partial x_l}\frac{\partial\chi_l(\vec{X},t)}{\partial X_K} \qquad \text{or} \qquad (\boldsymbol{F})_{kK}(\vec{X},\tau) = (\boldsymbol{F}_t)_{kl}(\vec{x},\tau)(\boldsymbol{F})_{lK}(\vec{X},t) ,$$

which represents the tensor equation

$$\boldsymbol{F}(\vec{X},\tau) = \boldsymbol{F}_t(\vec{x},\tau)\cdot\boldsymbol{F}(\vec{X},t) , \qquad \text{or} \qquad \boldsymbol{F}_t(\vec{x},\tau) = \boldsymbol{F}(\vec{X},\tau)\cdot\boldsymbol{F}^{-1}(\vec{X},t) . \tag{6.96}$$

In particular,

$$\boldsymbol{F}_t(\vec{x},t) = \boldsymbol{I} . \tag{6.97}$$

The Green deformation tensor $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$ may also be developed in terms of the relative deformation gradient:

$$\boldsymbol{C}(\vec{X},\tau) = \boldsymbol{F}^T(\vec{X},\tau) \cdot \boldsymbol{F}(\vec{X},\tau) = \boldsymbol{F}^T(\vec{X},t) \cdot \boldsymbol{F}_t^T(\vec{x},\tau) \cdot \boldsymbol{F}_t(\vec{x},\tau) \cdot \boldsymbol{F}(\vec{X},t) \ .$$

Defining the *relative Green deformation tensor* by

$$\boldsymbol{c}_t(\vec{x},\tau) := \boldsymbol{F}_t^T(\vec{x},\tau) \cdot \boldsymbol{F}_t(\vec{x},\tau) \ , \tag{6.98}$$

we have

$$\boldsymbol{C}(\vec{X},\tau) = \boldsymbol{F}^T(\vec{X},t) \cdot \boldsymbol{c}_t(\vec{x},\tau) \cdot \boldsymbol{F}(\vec{X},t) \ . \tag{6.99}$$

6.13 Isotropic constitutive functionals in relative representation

Let now the motion be considered in relative representation (6.94). Thus, the present configuration at the current time t serves as reference, while the configuration at past time τ, $(\tau \leq t)$, is taken as present. The relative deformation gradient $\boldsymbol{F}_t(\vec{x},\tau)$ can be expressed in terms of the absolute deformation gradient according to (6.96). Similarly, the relative temperature gradient, $\vec{g}_\theta := \operatorname{grad}\theta$, can be expressed in terms of the absolute temperature gradient, $\vec{G}_\theta := \operatorname{Grad}\theta$, as [19]

$$\begin{aligned}\vec{G}_\theta(\vec{X},\tau) &= \frac{\partial\theta(\vec{X},\tau)}{\partial X_K}\,\vec{I}_K = \frac{\partial\theta(\vec{x},\tau)}{\partial x_k}\frac{\partial\chi_k(\vec{X},t)}{\partial X_K}\,\vec{I}_K = [\vec{g}_\theta(\vec{x},\tau)]_k\,[\boldsymbol{F}(\vec{x},t)]_{kK}\,\vec{I}_K \\ &= \vec{g}_\theta(\vec{x},\tau) \cdot \boldsymbol{F}(\vec{X},t) = \boldsymbol{F}^T(\vec{X},t) \cdot \vec{g}_\theta(\vec{x},\tau) \ .\end{aligned}$$

To abbreviate notations, $(\vec{X},t)$ and $(\vec{x},t)$ will be dropped from notations and $(\vec{X},\tau)$ and $(\vec{x},\tau)$ will be shortened as (τ):

$$\vec{G}_\theta(\tau) = \boldsymbol{F}^T \cdot \vec{g}_\theta(\tau) \ . \tag{6.100}$$

With the help of the polar decomposition $\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U}$, (6.99) and (6.100) can also be written in the form

$$\begin{aligned}\boldsymbol{C}(\tau) &= \boldsymbol{U} \cdot \boldsymbol{R}^T \cdot \boldsymbol{c}_t(\tau) \cdot \boldsymbol{R} \cdot \boldsymbol{U} \ , \\ \vec{G}_\theta(\tau) &= \boldsymbol{U} \cdot \boldsymbol{R}^T \cdot \vec{g}_\theta(\tau) \ .\end{aligned} \tag{6.101}$$

We consider the reduced form (6.29) of the constitutive equation for the Cauchy stress tensor:

$$\boldsymbol{t} = \boldsymbol{R} \cdot \underset{\tau\leq t}{\mathcal{F}} \left[\boldsymbol{C}(\tau), \theta(\tau), \vec{G}_\theta(\tau)\right] \cdot \boldsymbol{R}^T \ . \tag{6.102}$$

Making use of (6.101) together with $\boldsymbol{C} = \boldsymbol{U}^2$, we obtain

$$\boldsymbol{t} = \boldsymbol{R} \cdot \underset{\tau\leq t}{\mathcal{F}} \left[\sqrt{\boldsymbol{C}} \cdot \boldsymbol{R}^T \cdot \boldsymbol{c}_t(\tau) \cdot \boldsymbol{R} \cdot \sqrt{\boldsymbol{C}}, \theta(\tau), \sqrt{\boldsymbol{C}} \cdot \boldsymbol{R}^T \cdot \vec{g}_\theta(\tau)\right] \cdot \boldsymbol{R}^T \ . \tag{6.103}$$

The information contained in the last constitutive equation can alternatively be expressed as

$$\boldsymbol{t} = \boldsymbol{R} \cdot \underset{\tau\leq t}{\mathcal{F}} \left[\boldsymbol{R}^T \cdot \boldsymbol{c}_t(\tau) \cdot \boldsymbol{R}, \theta(\tau), \boldsymbol{R}^T \cdot \vec{g}_\theta(\tau); \boldsymbol{C}\right] \cdot \boldsymbol{R}^T \ , \tag{6.104}$$

[19] The temperature, like any other variable, has both an Eulerian and a Lagrangian description; the corresponding Eulerian temperature is defined by $\vartheta(\vec{x},t) := \theta(\vec{X}(\vec{x},t),t)$. We make, however, an exception in the notation and use $\theta(\vec{x},t)$ for the Eulerian description of temperature.

where the present strain $\boldsymbol{C}(t)$ appears in the functional $\mathcal{F}$ as a parameter. This shows that it is not possible to express the effect of deformation history on the stress entirely by measuring deformation with respect to the present configuration. While the effect of all the past history is accounted for, a fixed reference configuration is required, in general, to allow for the effect of the deformation at the present instant, as indicated by the appearance of $\boldsymbol{C}(t)$ as a parameter in (6.104).

We are searching for a form of functional $\mathcal{F}$ for isotropic materials only. In this case, the functional $\mathcal{F}$ satisfies the isotropy relation analogous to (6.76) (show it):

$$\begin{aligned}&\boldsymbol{Q}\cdot \underset{\tau\leq t}{\mathcal{F}}\left[\boldsymbol{R}^T\cdot\boldsymbol{c}_t(\tau)\cdot\boldsymbol{R},\theta(\tau),\boldsymbol{R}^T\cdot\vec{g}_\theta(\tau);\boldsymbol{C}\right]\cdot\boldsymbol{Q}^T\\ &=\underset{\tau\leq t}{\mathcal{F}}\left[\boldsymbol{Q}\cdot\boldsymbol{R}^T\cdot\boldsymbol{c}_t(\tau)\cdot\boldsymbol{R}\cdot\boldsymbol{Q}^T,\theta(\tau),\boldsymbol{Q}\cdot\boldsymbol{R}^T\cdot\vec{g}_\theta(\tau);\boldsymbol{Q}\cdot\boldsymbol{C}\cdot\boldsymbol{Q}^T\right]\ ,\end{aligned} \tag{6.105}$$

where $\boldsymbol{Q}$ is an orthogonal tensor. Since $\vec{X}$ and t are fixed in the relative description of motion, we may choose $\boldsymbol{Q}(\vec{X})=\boldsymbol{R}(\vec{X},t)$, or more explicitly, $Q_{\hat{K}L}(\vec{X})=Q_{kL}(\vec{X})=R_{kL}(\vec{X},t)$:

$$\underset{\tau\leq t}{\mathcal{F}}\left[\boldsymbol{c}_t(\tau),\theta(\tau),\vec{g}_\theta(\tau);\boldsymbol{b}\right]=\boldsymbol{R}\cdot\underset{\tau\leq t}{\mathcal{F}}\left[\boldsymbol{R}^T\cdot\boldsymbol{c}_t(\tau)\cdot\boldsymbol{R},\theta(\tau),\boldsymbol{R}^T\cdot\vec{g}_\theta(\tau);\boldsymbol{C}\right]\cdot\boldsymbol{R}^T\ , \tag{6.106}$$

where $\boldsymbol{b}$ is the Finger deformation tensor, $\boldsymbol{b}=\boldsymbol{F}\cdot\boldsymbol{F}^T=\boldsymbol{R}\cdot\boldsymbol{C}\cdot\boldsymbol{R}^T$. Equations (6.104) and (6.106) can be combined to give

$$\boldsymbol{t}=\underset{\tau\leq t}{\mathcal{F}}\left[\boldsymbol{c}_t(\tau),\theta(\tau),\vec{g}_\theta(\tau);\boldsymbol{b}\right]\ . \tag{6.107}$$

This is a general form of the constitutive equation for a simple homogeneous isotropic materials expressed in relative representation. It remains to show that this constitutive equation is invariant to any orthogonal transformation $\boldsymbol{Q}$ of reference configuration R, not only to our special choice $\boldsymbol{Q}=\boldsymbol{R}(\vec{X},t)$. Since none of the arguments in (6.107) does depend on the reference configuration R, this requirement is satisfied trivially. For instance, the Finger deformation tensor does not change by the orthogonal transformation of the reference configuration R onto $\hat{R}$:

$$\hat{\boldsymbol{b}}=\hat{\boldsymbol{F}}\cdot\hat{\boldsymbol{F}}^T=(\boldsymbol{F}\cdot\boldsymbol{Q}^{-1})\cdot(\boldsymbol{Q}^{-T}\cdot\boldsymbol{F}^T)=\boldsymbol{F}\cdot\boldsymbol{Q}^{-1}\cdot\boldsymbol{Q}\cdot\boldsymbol{F}^T=\boldsymbol{F}\cdot\boldsymbol{F}^T=\boldsymbol{b}\ .$$

By construction, the constitutive functional $\mathcal{F}$ is not automatically materially objective. The material objectivity (6.16) of the Cauchy stress tensor implies that the functional $\mathcal{F}$ must transform under a rigid motion of (spatial) observer frame as

$$\underset{\tau\leq t}{\mathcal{F}}\left[\boldsymbol{c}_t^*(\tau),\theta^*(\tau),\vec{g}^*_{\theta^*}(\tau);\boldsymbol{b}^*\right]=\boldsymbol{O}(t)\cdot\underset{\tau\leq t}{\mathcal{F}}\left[\boldsymbol{c}_t(\tau),\theta(\tau),\vec{g}_\theta(\tau);\boldsymbol{b}\right]\cdot\boldsymbol{O}^T(t)\ . \tag{6.108}$$

The transformation rules for the relative deformation tensor, the relative Green deformation tensor, the relative temperature gradient and the Finger deformation tensor are

$$\begin{aligned}\boldsymbol{F}_t^*(\tau)&=\boldsymbol{O}(\tau)\cdot\boldsymbol{F}_t(\tau)\cdot\boldsymbol{O}^T(t)\ ,\\ \boldsymbol{c}_t^*(\tau)&=\boldsymbol{O}(t)\cdot\boldsymbol{c}_t(\tau)\cdot\boldsymbol{O}^T(t)\ ,\\ \vec{g}^*_{\theta^*}(\tau)&=\boldsymbol{O}(t)\cdot\vec{g}_\theta(\tau)\ ,\\ \boldsymbol{b}^*(t)&=\boldsymbol{O}(t)\cdot\boldsymbol{b}(t)\cdot\boldsymbol{O}^T(t)\ .\end{aligned} \tag{6.109}$$

The proof is immediate:

$$\boldsymbol{F}^*_t(\tau) = \boldsymbol{F}^*(\tau)\cdot[\boldsymbol{F}^*(t)]^{-1} = \boldsymbol{O}(\tau)\cdot\boldsymbol{F}(\tau)\cdot\boldsymbol{F}^{-1}(t)\cdot\boldsymbol{O}^{-1}(t) = \boldsymbol{O}(\tau)\cdot\boldsymbol{F}_t(\tau)\cdot\boldsymbol{O}^T(t)\ ,$$
$$\boldsymbol{c}^*_t(\tau) = [\boldsymbol{F}^*(t)]^{-T}\cdot\boldsymbol{C}^*(\tau)\cdot[\boldsymbol{F}^*(t)]^{-1} = \boldsymbol{O}(t)\cdot\boldsymbol{F}^{-T}(t)\cdot\boldsymbol{C}(\tau)\cdot\boldsymbol{F}^{-1}(t)\cdot\boldsymbol{O}^{-1}(t) = \boldsymbol{O}(t)\cdot\boldsymbol{c}_t(\tau)\cdot\boldsymbol{O}^T(t)\ ,$$
$$\vec{g}^{\,*}_{\theta^*}(\tau) = [\boldsymbol{F}^*(t)]^{-T}\cdot\vec{G}^*_{\theta^*}(\tau) = \boldsymbol{O}(t)\cdot\boldsymbol{F}^{-T}(t)\cdot\vec{G}_\theta(\tau) = \boldsymbol{O}(t)\cdot\vec{g}_\theta(\tau)\ .$$

By this and realizing that the temperature $\theta(\tau)$ as a scalar quantity is invariant to any transformation of observer frame, the constraint (6.108) reduces to

$$\begin{aligned}\mathop{\mathcal{F}}_{\tau\le t}&\left[\boldsymbol{O}(t)\cdot\boldsymbol{c}_t(\tau)\cdot\boldsymbol{O}^T(t),\theta(\tau),\boldsymbol{O}(t)\cdot\vec{g}_\theta(\tau);\boldsymbol{O}(t)\cdot\boldsymbol{b}\cdot\boldsymbol{O}^T(t)\right]\\ &= \boldsymbol{O}(t)\cdot\mathop{\mathcal{F}}_{\tau\le t}\left[\boldsymbol{c}_t(\tau),\theta(\tau),\vec{g}_\theta(\tau);\boldsymbol{b}\right]\cdot\boldsymbol{O}^T(t)\ .\end{aligned} \tag{6.110}$$

This shows that the functional $\mathcal{F}$ must be a spatially isotropic functional of the variables $\boldsymbol{c}_t(\tau)$, $\theta(\tau)$, $\vec{g}_\theta(\tau)$ and a spatially isotropic function of $\boldsymbol{b}$ for $\mathcal{F}$ to be materially objective. [20]

An analogous arrangement can be carried out for the constitutive equation (6.31) for the heat flux and the internal energy:

$$\begin{aligned}\vec{q} &= \mathop{\mathcal{Q}}_{\tau\le t}\left[\boldsymbol{c}_t(\tau),\theta(\tau),\vec{g}_\theta(\tau);\boldsymbol{b}\right]\ ,\\ \varepsilon &= \mathop{\mathcal{E}}_{\tau\le t}\left[\boldsymbol{c}_t(\tau),\theta(\tau),\vec{g}_\theta(\tau);\boldsymbol{b}\right]\ .\end{aligned} \tag{6.111}$$

Note that the constitutive equations in relative representation will be used to describe material properties of a fluid, while for a solid we will employ the constitutive equation (6.29) in referential representation.

6.14 A general constitutive equation of a fluid

A fluid is characterized through the largest symmetry group, being identical with the set of all unimodular transformations, $g_R = \mathsf{unim}$. Mathematically, the condition for the material symmetry of functional $\mathcal{F}$ in (6.107) under unimodular transformation $\boldsymbol{H}$ is expressed as

$$\mathop{\mathcal{F}}_{\tau\le t}\left[\boldsymbol{c}_t(\tau),\theta(\tau),\vec{g}_\theta(\tau);\boldsymbol{F}\cdot\boldsymbol{F}^T\right] = \mathop{\mathcal{F}}_{\tau\le t}\left[\boldsymbol{c}_t(\tau),\theta(\tau),\vec{g}_\theta(\tau);\boldsymbol{F}\cdot\boldsymbol{H}\cdot\boldsymbol{H}^T\cdot\boldsymbol{F}^T\right]\ , \tag{6.112}$$

since the Finger deformation tensor $\boldsymbol{b} = \boldsymbol{F}\cdot\boldsymbol{F}^T$ transforms under unimodular transformation $\boldsymbol{H}$ as $\boldsymbol{F}\cdot\boldsymbol{H}\cdot\boldsymbol{H}^T\cdot\boldsymbol{F}^T$. This general characterization of a fluid must also hold for the special unimodular transformation

$$\boldsymbol{H} = (\det\boldsymbol{F})^{1/3}\,\boldsymbol{F}^{-1} = \left(\frac{\varrho_0}{\varrho}\right)^{1/3}\boldsymbol{F}^{-1}\ , \tag{6.113}$$

where $\varrho(\vec{x},t)$ and $\varrho_0(\vec{X})$ is the mass density of a body in the present and reference configuration, respectively. The transformation (6.113) is indeed unimodular since

$$\det\boldsymbol{H} = \left[(\det\boldsymbol{F})^{1/3}\right]^3\det\boldsymbol{F}^{-1} = 1\ .$$

[20]Note that this is spatial isotropy, not material isotropy discussed in previous sections.

For such $\boldsymbol{H}$, $\boldsymbol{F} \cdot \boldsymbol{H} \cdot \boldsymbol{H}^T \cdot \boldsymbol{F}^T = (\det \boldsymbol{F})^{2/3} \boldsymbol{I} = (\varrho_0/\varrho)^{2/3} \boldsymbol{I}$, and the constitutive equation (6.107) for an isotropic body can be rewritten with new denotation:

$$\boldsymbol{t} = \mathop{\mathcal{F}_t}_{\tau \leq t} \left[\boldsymbol{c}_t(\tau), \theta(\tau), \vec{g}_\theta(\tau); \varrho \right] \; . \tag{6.114}$$

This is a general form of the constitutive equation for a simple fluid in relative representation. It shows that the Cauchy stress tensor for fluid depends on the relative deformation history, the temperature and temperature gradient histories, and on the present mass density as a parameter. Since none of $\boldsymbol{c}_t(\vec{x}, \tau)$, $\theta(\vec{x}, \tau)$, $\vec{g}_\theta(\vec{x}, \tau)$ and $\varrho(\vec{x}, t)$ does depend on the reference configuration, constitutive equation (6.114) is invariant to any transformation of the reference configuration. Hence, it satisfies trivially the requirement that $g_R = \mathsf{unim}$. In other words, a fluid is a body for which every configuration that leaves the density unchanged can be taken as the reference configuration.

6.15 The principle of bounded memory

This principle states that deformation and temperature events at distant past from the present have a very small influence on the present behavior of material. In other words, the memory of past motions and temperatures of any material point decays rapidly with time. This principle is the counterpart of the principle of smooth neighborhood in the time domain. No unique mathematical formulation can be made of this principle. The following limited interpretation suffices for our purpose.

To express the boundedness of a memory, let $h(\vec{X}, \tau)$ be a function in the argument of constitutive functional. Suppose that $h(\vec{X}, \tau)$ is an analytical function such that it possesses continuous partial derivatives with respect to τ at $\tau = t$. For small $\tau - t$, it can be approximated by the truncated Taylor series expansion at $\tau = t$:

$$h(\vec{X}, \tau) = \sum_{n=0}^{N} \frac{1}{n!} \left. \frac{\partial^n h(\vec{X}, \tau)}{\partial \tau^n} \right|_{\tau = t} (\tau - t)^n \; . \tag{6.115}$$

If a constitutive functional is sufficiently smooth so that the dependence on $h(\vec{X}, \tau)$ can be replaced by the list of functions

$$\overset{(n)}{h} (\vec{X}, t) := \left. \frac{\partial^n h(\vec{X}, \tau)}{\partial \tau^n} \right|_{\tau = t} \qquad \text{for} \;\; n = 0, 1, \ldots, N \; , \tag{6.116}$$

that is, $\overset{(n)}{h}$ is the nth material time derivative of h, $\overset{(n)}{h} = D^n h / D\tau^n$, we say that the material is of the *rate type* of degree N with respect to the variable h. If the material is of the rate type in all its variables, the constitutive equation (6.34) for a homogeneous solid can be approximated by

$$\boldsymbol{T}^{(2)} = \hat{\boldsymbol{T}}^{(2)}(\boldsymbol{C}, \dot{\boldsymbol{C}}, \ldots, \overset{(N_C)}{\boldsymbol{C}}, \theta, \dot{\theta}, \ldots, \overset{(N_\theta)}{\theta}, \vec{G}_\theta, \dot{\vec{G}}_\theta, \ldots, \overset{(N_G)}{\vec{G}_\theta} \,) \; . \tag{6.117}$$

The number of time derivatives for each of the gradients depends on the strength of the memory of these gradients. We note that $\hat{\boldsymbol{T}}^{(2)}$ is no longer functional. It is a tensor-valued function of the arguments listed. They involve time rates of the deformation gradient up to the order N_C and temperature and its gradient up to order N_θ and N_G, respectively. The degrees N_C, N_θ and N_G need not be the same. [21]

The general constitutive equation for a fluid is given by (6.114). We again assume that the relative Green deformation tensor $\boldsymbol{c}_t(\tau)$ can be approximated by the truncated Taylor series expansion of the form (6.115):

$$\boldsymbol{c}_t(\tau) = \boldsymbol{a}_0(t) + \boldsymbol{a}_1(t)(\tau - t) + \boldsymbol{a}_2(t)\frac{(\tau - t)^2}{2} + \ldots \; . \tag{6.118}$$

Using (6.99), the expansion coefficient

$$\boldsymbol{a}_n(\vec{x}, t) := \frac{\partial^n \boldsymbol{c}_t(\vec{x}, \tau)}{\partial \tau^n}\bigg|_{\tau = t} \tag{6.119}$$

can be shown to be the Rivlin-Ericksen tensor of order n defined by (2.29). It is an objective tensor:

$$\boldsymbol{a}_n^*(\vec{x}, t) = \boldsymbol{O}(t) \cdot \boldsymbol{a}_n(\vec{x}, t) \cdot \boldsymbol{O}^T(t) \; . \tag{6.120}$$

To show this, we use $(6.109)_2$:

$$a_n^*(\vec{x}, t) = \frac{\partial^n \boldsymbol{c}_t^*(\vec{x}, \tau)}{\partial \tau^n}\bigg|_{\tau = t} = \frac{\partial^n}{\partial \tau^n}\Big(\boldsymbol{O}(t) \cdot \boldsymbol{c}_t(\vec{x}, \tau) \cdot \boldsymbol{O}^T(t)\Big)\bigg|_{\tau = t} = \boldsymbol{O}(t) \cdot \frac{\partial^n \boldsymbol{c}_t(\vec{x}, \tau)}{\partial \tau^n}\bigg|_{\tau = t} \cdot \boldsymbol{O}^T(t) \; ,$$

which, in view of (6.119), gives (6.120).

We again assume that all the functions in the argument of functional $\mathcal{F}_t$ in (6.114) can be approximated by the truncated Taylor series expansion. Then the functional for a class of materials with bounded memory may be represented by functions involving various time derivative of the argument functions:

$$\boldsymbol{t} = \hat{\boldsymbol{t}}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N_C}, \theta, \dot{\theta}, \ldots, \overset{(N_\theta)}{\theta}, \vec{g}_\theta, \dot{\vec{g}}_\theta, \ldots, \overset{(N_g)}{\vec{g}_\theta}; \varrho) \; , \tag{6.121}$$

where the dependence on $\boldsymbol{a}_0 = \boldsymbol{I}$ has been omitted because of its redundancy. Note that $\hat{\boldsymbol{t}}$ is again no longer functional, but a tensor-valued function. This is an asymptotic approximation of the most general constitutive equation (6.114) for a fluid. Coleman and Noll (1960) showed that this approximation is valid for sufficiently slow deformation processes.

Similar expressions hold for the heat flux vector $\vec{q}$ and the internal energy ε. The number of time derivatives for each of the gradients depends on the strength of the memory of these gradients. According to the axiom of equipresence, however, all constitutive functionals should be expressed in terms of the same list of independent constitutive variables.

[21] The functions in the argument of constitutive functionals may not be so smooth as to admit truncated Taylor series expansion. Nevertheless, the constitutive functionals may be such as to smooth out past discontinuities in these argument functions and/or their derivatives. The principle of bounded memory, in this context also called the *principle of fading memory*, is then a requirement on smoothness of constitutive functionals.

The principle of fading memory, mathematically formulated by Coleman and Noll (1960), starts with the assumption that the so-called Frechét derivatives of constitutive functionals up to an order n exist and are continuous in the neighborhood of histories at time t in the Hilbert space normed by an influence function of order greater than $n + \frac{1}{2}$. Then, the constitutive functionals can be approximated by linear functionals for which explicit mathematical representations are known. The most important result of the fading memory theory is the possibility to approximate asymptotically a sufficiently slow strain and temperature histories by the Taylor series expansion.

6.16 Representation theorems for isotropic functions

We are now looking for the representation of a scalar-, vector- and tensor-valued function which are isotropic, that is, form-invariant for all orthogonal transformations.

6.16.1 Representation theorem for a scalar isotropic function

A scalar-valued function a depending on symmetric tensor $\boldsymbol{S}$ and vector $\vec{v}$ is called isotropic if it satisfies the identity

$$a(\boldsymbol{S},\vec{v}) = a(\boldsymbol{Q}\cdot\boldsymbol{S}\cdot\boldsymbol{Q}^T,\boldsymbol{Q}\cdot\vec{v}) \tag{6.122}$$

for all orthogonal tensors $\boldsymbol{Q}$. This means that a does not depend on components S_{kl} and v_k arbitrarily, but it can only be considered as a function of scalar invariants that are independent of any orthogonal transformation.

Hence, we have to determine the set of all independent invariants of variables $\boldsymbol{S}$ and $\vec{v}$. There are three independent invariants of tensor $\boldsymbol{S}$, namely,

$$I_S = \operatorname{tr}\boldsymbol{S}\ , \qquad II_S = \frac{1}{2}\left[(\operatorname{tr}\boldsymbol{S})^2 - \operatorname{tr}(\boldsymbol{S}^2)\right]\ , \qquad III_S = \det\boldsymbol{S}\ , \tag{6.123}$$

and one invariant of vector $\vec{v}$, the scalar product $\vec{v}\cdot\vec{v}$. Combining $\boldsymbol{S}$ and $\vec{v}$, we can create a number of other invariants,

$$\vec{v}\cdot\boldsymbol{S}\cdot\vec{v}\ ,\quad \vec{v}\cdot\boldsymbol{S}^2\cdot\vec{v}\ ,\quad \vec{v}\cdot\boldsymbol{S}^3\cdot\vec{v}\ ,\ldots,\vec{v}\cdot\boldsymbol{S}^n\cdot\vec{v}\ ,\ldots; \tag{6.124}$$

of course, not all of them are mutually independent. According to the Cayley-Hamilton theorem saying that any matrix satisfies its characteristic polynom, we have

$$\boldsymbol{S}^3 - I_S\,\boldsymbol{S}^2 + II_S\,\boldsymbol{S} - III_S\,\boldsymbol{I} = \boldsymbol{0}\ , \tag{6.125}$$

where $\boldsymbol{I}$ and $\boldsymbol{0}$ are the unit and zero tensors of the same order as $\boldsymbol{S}$. Multiplying this equation from left and right by vector $\vec{v}$, we obtain

$$\vec{v}\cdot\boldsymbol{S}^3\cdot\vec{v} - I_S\,(\vec{v}\cdot\boldsymbol{S}^2\cdot\vec{v}) + II_S\,(\vec{v}\cdot\boldsymbol{S}\cdot\vec{v}) - III_S\,(\vec{v}\cdot\vec{v}) = 0\ . \tag{6.126}$$

We see that $\vec{v}\cdot\boldsymbol{S}^n\cdot\vec{v}$ for $n\geq 3$ can be expressed in terms of $\vec{v}\cdot\boldsymbol{S}^2\cdot\vec{v}$, $\vec{v}\cdot\boldsymbol{S}\cdot\vec{v}$ and $\vec{v}\cdot\vec{v}$ but these cannot be used to express three invariants of $\boldsymbol{S}$. Hence, the set of all independent invariants of variables $\boldsymbol{S}$ and $\vec{v}$ is

$$I_S, II_S, III_S,\ \vec{v}\cdot\vec{v},\ \vec{v}\cdot\boldsymbol{S}\cdot\vec{v},\ \vec{v}\cdot\boldsymbol{S}^2\cdot\vec{v}\ . \tag{6.127}$$

We can conclude that scalar isotropic function a can be represented in the form

$$a(\boldsymbol{S},\vec{v}) = a(I_S, II_S, III_S,\ \vec{v}\cdot\vec{v},\ \vec{v}\cdot\boldsymbol{S}\cdot\vec{v},\ \vec{v}\cdot\boldsymbol{S}^2\cdot\vec{v})\ . \tag{6.128}$$

6.16.2 Representation theorem for a vector isotropic function

A vector-valued function $\vec{b}$ depending on symmetric tensor $\boldsymbol{S}$ and vector $\vec{v}$ is called isotropic if it satisfies the identity

$$\boldsymbol{Q}\cdot\vec{b}(\boldsymbol{S},\vec{v}) = \vec{b}(\boldsymbol{Q}\cdot\boldsymbol{S}\cdot\boldsymbol{Q}^T,\boldsymbol{Q}\cdot\vec{v}) \tag{6.129}$$

for all orthogonal tensors $\boldsymbol{Q}$. For an arbitrary vector $\vec{c}$, let us define function F,

$$F(\boldsymbol{S},\vec{v},\vec{c}) := \vec{c}\cdot\vec{b}(\boldsymbol{S},\vec{v}) \ . \tag{6.130}$$

This function is a scalar isotropic function of variables $\boldsymbol{S}$, $\vec{v}$ and $\vec{c}$ which follows from

$$F(\boldsymbol{Q}\cdot\boldsymbol{S}\cdot\boldsymbol{Q}^T,\boldsymbol{Q}\cdot\vec{v},\boldsymbol{Q}\cdot\vec{c}) = \boldsymbol{Q}\cdot\vec{c}\cdot\vec{b}(\boldsymbol{Q}\cdot\boldsymbol{S}\cdot\boldsymbol{Q}^T,\boldsymbol{Q}\cdot\vec{v}) = \boldsymbol{Q}\cdot\vec{c}\cdot\boldsymbol{Q}\cdot\vec{b}(\boldsymbol{S},\vec{v}) = \vec{c}\cdot\boldsymbol{Q}^T\cdot\boldsymbol{Q}\cdot\vec{b}(\boldsymbol{S},\vec{v}) = F(\boldsymbol{S},\vec{v},\vec{c}) \ .$$

In analogy to the preceding section, the set of all independent invariants of $\boldsymbol{S}$, $\vec{v}$ and $\vec{c}$ is

$$\begin{array}{llllll} I_S, & II_S, & III_S, & \vec{v}\cdot\vec{v}, & \vec{v}\cdot\boldsymbol{S}\cdot\vec{v}, & \vec{v}\cdot\boldsymbol{S}^2\cdot\vec{v}, \\ & & & \vec{c}\cdot\vec{v}, & \vec{c}\cdot\boldsymbol{S}\cdot\vec{v}, & \vec{c}\cdot\boldsymbol{S}^2\cdot\vec{v}, \\ & & & \vec{c}\cdot\vec{c}, & \vec{c}\cdot\boldsymbol{S}\cdot\vec{c}, & \vec{c}\cdot\boldsymbol{S}^2\cdot\vec{c} \ . \end{array} \tag{6.131}$$

If function F was an arbitrary function of $\boldsymbol{S}$, $\vec{v}$ and $\vec{c}$, it could be represented in terms of these 12 invariants in an arbitrary manner. Function F is, however, linearly dependent on $\vec{c}$, so that it can be represented in terms of the invariants that are linear functions of $\vec{c}$ only:

$$F = a_0\,\vec{c}\cdot\vec{v} + a_1\,\vec{c}\cdot\boldsymbol{S}\cdot\vec{v} + a_2\,\vec{c}\cdot\boldsymbol{S}^2\cdot\vec{v} \ , \tag{6.132}$$

where the coefficients a_i depend on the six invariant that are independent of $\vec{c}$. Since $\vec{c}$ was an arbitrary vector in (6.130)–(6.132), we can conclude that vector isotropic function $\vec{b}$ can be represented in the form

$$\vec{b} = a_0\,\vec{v} + a_1\,\boldsymbol{S}\cdot\vec{v} + a_2\,\boldsymbol{S}^2\cdot\vec{v} \ , \tag{6.133}$$

where

$$a_i = a_i\left(I_S, II_S, III_S,\ \vec{v}\cdot\vec{v},\ \vec{v}\cdot\boldsymbol{S}\cdot\vec{v},\ \vec{v}\cdot\boldsymbol{S}^2\cdot\vec{v}\right) \ . \tag{6.134}$$

6.16.3 Representation theorem for a symmetric tensor-valued isotropic function

(a) *Symmetric tensor-valued isotropic function of symmetric tensor and vector*. First, let us consider symmetric tensor-valued function $\boldsymbol{T}$ depending on symmetric tensor $\boldsymbol{S}$ and vector $\vec{v}$. Tensor $\boldsymbol{T}$ is called isotropic if it satisfies the identity

$$\boldsymbol{Q}\cdot\boldsymbol{T}(\boldsymbol{S},\vec{v})\cdot\boldsymbol{Q}^T = \boldsymbol{T}(\boldsymbol{Q}\cdot\boldsymbol{S}\cdot\boldsymbol{Q}^T,\boldsymbol{Q}\cdot\vec{v}) \tag{6.135}$$

for all orthogonal tensors $\boldsymbol{Q}$. For a tensor $\boldsymbol{A}$, let us define function F,

$$F(\boldsymbol{S},\vec{v},\boldsymbol{A}) := \mathrm{tr}(\boldsymbol{A}\cdot\boldsymbol{T}(\boldsymbol{S},\vec{v})) \ . \tag{6.136}$$

This function is a scalar isotropic function of variables $\boldsymbol{S}$, $\vec{v}$ and $\boldsymbol{A}$ which follows from

$$F(\boldsymbol{Q}\cdot\boldsymbol{S}\cdot\boldsymbol{Q}^T,\boldsymbol{Q}\cdot\vec{v},\boldsymbol{Q}\cdot\boldsymbol{A}\cdot\boldsymbol{Q}^T) = \mathrm{tr}(\boldsymbol{Q}\cdot\boldsymbol{A}\cdot\boldsymbol{Q}^T\cdot\boldsymbol{T}(\boldsymbol{Q}\cdot\boldsymbol{S}\cdot\boldsymbol{Q}^T,\boldsymbol{Q}\cdot\vec{v}))$$

$$= \mathrm{tr}(\boldsymbol{Q}\cdot\boldsymbol{A}\cdot\boldsymbol{Q}^T\cdot\boldsymbol{Q}\cdot\boldsymbol{T}(\boldsymbol{S},\vec{v})\cdot\boldsymbol{Q}^T) = \mathrm{tr}(\boldsymbol{Q}\cdot\boldsymbol{A}\cdot\boldsymbol{T}(\boldsymbol{S},\vec{v})\cdot\boldsymbol{Q}^T) = \mathrm{tr}(\boldsymbol{A}\cdot\boldsymbol{T}(\boldsymbol{S},\vec{v})\cdot\boldsymbol{Q}^T\cdot\boldsymbol{Q}) = F(\boldsymbol{S},\vec{v},\boldsymbol{A}) \ .$$

Under the same argument as in the previous case, function F must be represented as a linear combination of all independent invariants that are linear functions of $\boldsymbol{A}$ only, that is, in terms of

$$\mathrm{tr}\boldsymbol{A},\ \mathrm{tr}(\boldsymbol{A}\cdot\boldsymbol{S}),\ \mathrm{tr}(\boldsymbol{A}\cdot\boldsymbol{S}^2),\ \vec{v}\cdot\boldsymbol{A}\cdot\vec{v},\ \vec{v}\cdot\boldsymbol{A}\cdot\boldsymbol{S}\cdot\vec{v},\ \vec{v}\cdot\boldsymbol{A}\cdot\boldsymbol{S}^2\cdot\vec{v}. \tag{6.137}$$

Note that it is possible, but not easy to prove that there are no more independent invariants linearly dependent on $\boldsymbol{A}$. Hence, we can write

$$F = a_0 \operatorname{tr}\boldsymbol{A} + a_1 \operatorname{tr}(\boldsymbol{A}\cdot\boldsymbol{S}) + a_2 \operatorname{tr}(\boldsymbol{A}\cdot\boldsymbol{S}^2) + a_3\, \vec{v}\cdot\boldsymbol{A}\cdot\vec{v} + a_4\, \vec{v}\cdot\boldsymbol{A}\cdot\boldsymbol{S}\cdot\vec{v} + a_5\, \vec{v}\cdot\boldsymbol{A}\cdot\boldsymbol{S}^2\cdot\vec{v}\ , \quad (6.138)$$

where the coefficients a_i depend on the invariants that are independent of $\boldsymbol{A}$, that is, on the six invariants of $\boldsymbol{S}$ and $\vec{v}$. Because of the identity

$$\vec{v}\cdot\boldsymbol{A}\cdot\vec{v} = \operatorname{tr}(\boldsymbol{A}\cdot\vec{v}\otimes\vec{v})\ , \quad (6.139)$$

where $\vec{v}\otimes\vec{v}$ is the dyadic product of vector $\vec{v}$ by itself, we also have

$$F = a_0 \operatorname{tr}\boldsymbol{A} + a_1 \operatorname{tr}(\boldsymbol{A}\cdot\boldsymbol{S}) + a_2 \operatorname{tr}(\boldsymbol{A}\cdot\boldsymbol{S}^2) + a_3 \operatorname{tr}(\boldsymbol{A}\cdot\vec{v}\otimes\vec{v}) + a_4 \operatorname{tr}(\boldsymbol{A}\cdot\boldsymbol{S}\cdot\vec{v}\otimes\vec{v}) + a_5 \operatorname{tr}(\boldsymbol{A}\cdot\boldsymbol{S}^2\cdot\vec{v}\otimes\vec{v})\ . \quad (6.140)$$

The comparison of (6.140) with (6.136) results in the representation of a symmetric tensor-valued isotropic function $\boldsymbol{T}$ in the form

$$T(\boldsymbol{S},\vec{v}) = a_0\boldsymbol{I} + a_1\boldsymbol{S} + a_2\boldsymbol{S}^2 + a_3\,\vec{v}\otimes\vec{v} + a_4 \operatorname{sym}(\boldsymbol{S}\cdot\vec{v}\otimes\vec{v}) + a_5 \operatorname{sym}(\boldsymbol{S}^2\cdot\vec{v}\otimes\vec{v})\ , \quad (6.141)$$

where

$$a_i = a_i(I_S, II_S, III_S,\ \vec{v}\cdot\vec{v},\ \vec{v}\cdot\boldsymbol{S}\cdot\vec{v},\ \vec{v}\cdot\boldsymbol{S}^2\cdot\vec{v}) \quad (6.142)$$

and the symbol 'sym' stands for the symmetric part of tensor.

(b) _Symmetric tensor-valued isotropic function of two symmetric tensors._ Second, let us consider symmetric tensor-valued function $\boldsymbol{T}$ depending on two symmetric tensors $\boldsymbol{A}$ and $\boldsymbol{B}$. Tensor $\boldsymbol{T}$ is called isotropic if it satisfies the identity

$$\boldsymbol{Q}\cdot\boldsymbol{T}(\boldsymbol{A},\boldsymbol{B})\cdot\boldsymbol{Q}^T = \boldsymbol{T}(\boldsymbol{Q}\cdot\boldsymbol{A}\cdot\boldsymbol{Q}^T, \boldsymbol{Q}\cdot\boldsymbol{B}\cdot\boldsymbol{Q}^T) \quad (6.143)$$

for all orthogonal tensors $\boldsymbol{Q}$. For a tensor $\boldsymbol{C}$, let us define function F,

$$F(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}) := \operatorname{tr}(\boldsymbol{C}\cdot\boldsymbol{T}(\boldsymbol{A},\boldsymbol{B}))\ . \quad (6.144)$$

This function must be a scalar isotropic function of variables $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$. The set of all independent invariants of these three symmetric tensors that are linear functions of C is

$$\operatorname{tr}\boldsymbol{C},\ \operatorname{tr}(\boldsymbol{C}\cdot\boldsymbol{A}),\ \operatorname{tr}(\boldsymbol{C}\cdot\boldsymbol{A}^2),\ \operatorname{tr}(\boldsymbol{C}\cdot\boldsymbol{B}),\ \operatorname{tr}(\boldsymbol{C}\cdot\boldsymbol{B}^2),\ \operatorname{tr}(\boldsymbol{C}\cdot\boldsymbol{A}\cdot\boldsymbol{B}),\ \operatorname{tr}(\boldsymbol{C}\cdot\boldsymbol{A}\cdot\boldsymbol{B}^2),\ \operatorname{tr}(\boldsymbol{C}\cdot\boldsymbol{A}^2\cdot\boldsymbol{B}),\ \operatorname{tr}(\boldsymbol{C}\cdot\boldsymbol{A}^2\cdot\boldsymbol{B}^2)\ . \quad (6.145)$$

Hence, function F can be represented in the form

$$\begin{aligned} F &= a_0 \operatorname{tr}\boldsymbol{C} + a_1 \operatorname{tr}(\boldsymbol{C}\cdot\boldsymbol{A}) + a_2 \operatorname{tr}(\boldsymbol{C}\cdot\boldsymbol{A}^2) + a_3 \operatorname{tr}(\boldsymbol{C}\cdot\boldsymbol{B}) + a_4 \operatorname{tr}(\boldsymbol{C}\cdot\boldsymbol{B}^2) + a_5 \operatorname{tr}(\boldsymbol{C}\cdot\boldsymbol{A}\cdot\boldsymbol{B}) \\ &+ a_6 \operatorname{tr}(\boldsymbol{C}\cdot\boldsymbol{A}\cdot\boldsymbol{B}^2) + a_7 \operatorname{tr}(\boldsymbol{C}\cdot\boldsymbol{A}^2\cdot\boldsymbol{B}) + a_8 \operatorname{tr}(\boldsymbol{C}\cdot\boldsymbol{A}^2\cdot\boldsymbol{B}^2)\ , \end{aligned} \quad (6.146)$$

where the coefficients a_i depend on the invariants that are independent of $\boldsymbol{C}$, that is, on the ten invariants of $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$I_A, II_A, III_A, I_B, II_B, III_B, \operatorname{tr}(\boldsymbol{A}\cdot\boldsymbol{B}), \operatorname{tr}(\boldsymbol{A}\cdot\boldsymbol{B}^2), \operatorname{tr}(\boldsymbol{A}^2\cdot\boldsymbol{B}), \operatorname{tr}(\boldsymbol{A}^2\cdot\boldsymbol{B}^2)\ . \quad (6.147)$$

The comparison of (6.146) with (6.144) yields the representation of a symmetric tensor-valued isotropic function $\boldsymbol{T}$ in the form

$$\begin{aligned}\boldsymbol{T}(\boldsymbol{A},\boldsymbol{B}) &= a_0\,\boldsymbol{I} + a_1\,\boldsymbol{A} + a_2\,\boldsymbol{A}^2 + a_3\,\boldsymbol{B} + a_4\,\boldsymbol{B}^2 + a_5\,\mathrm{sym}(\boldsymbol{A}\cdot\boldsymbol{B}) + a_6\,\mathrm{sym}(\boldsymbol{A}\cdot\boldsymbol{B}^2)\\ &+ a_7\,\mathrm{sym}(\boldsymbol{A}^2\cdot\boldsymbol{B}) + a_8\,\mathrm{sym}(\boldsymbol{A}^2\cdot\boldsymbol{B}^2)\ ,\end{aligned} \tag{6.148}$$

where

$$a_i = a_i(I_A, II_A, III_A, I_B, II_B, III_B, \mathrm{tr}(\boldsymbol{A}\cdot\boldsymbol{B}), \mathrm{tr}(\boldsymbol{A}\cdot\boldsymbol{B}^2), \mathrm{tr}(\boldsymbol{A}^2\cdot\boldsymbol{B}), \mathrm{tr}(\boldsymbol{A}^2\cdot\boldsymbol{B}^2))\ . \tag{6.149}$$

6.17 Examples of isotropic materials with bounded memory

6.17.1 Elastic solid

A solid is called *elastic* if the stress tensor $\boldsymbol{t}$ depends only on the deformation gradient $\boldsymbol{F}$ at the present time, not on the temperature θ and not on the entire past thermomechanical history:

$$\boldsymbol{t} = \hat{\boldsymbol{t}}(\boldsymbol{F})\ . \tag{6.150}$$

Hence, the stress in an elastic material at each particle is uniquely determined by the present deformation from a fixed reference configuration. The material objectivity requires that the dependence on $\boldsymbol{F}$ is not arbitrary, but it has the form (6.29):

$$\boldsymbol{t} = \boldsymbol{R}\cdot\hat{\boldsymbol{t}}(\boldsymbol{C})\cdot\boldsymbol{R}^T\ . \tag{6.151}$$

For isotropic materials, $\hat{\boldsymbol{t}}$ must be an isotropic function of $\boldsymbol{C}$. Making use of the representation theorem for a tensor isotropic function, we have

$$\boldsymbol{t} = \boldsymbol{R}\cdot(a_0\boldsymbol{I} + a_1\boldsymbol{C} + a_2\boldsymbol{C}^2)\cdot\boldsymbol{R}^T\ , \tag{6.152}$$

where a_i are scalar functions of three principal invariants of $\boldsymbol{C}$,

$$a_i = a_i(I_C, II_C, III_C)\ . \tag{6.153}$$

With the help of the Finger deformation tensor $\boldsymbol{b}$ and its second power,

$$\boldsymbol{b} = \boldsymbol{F}\cdot\boldsymbol{F}^T = \boldsymbol{V}^2 = \boldsymbol{V}^T\cdot\boldsymbol{V} = \boldsymbol{R}\cdot\boldsymbol{F}^T\cdot\boldsymbol{F}\cdot\boldsymbol{R}^T = \boldsymbol{R}\cdot\boldsymbol{C}\cdot\boldsymbol{R}^T\ ,$$

$$\boldsymbol{b}^2 = \boldsymbol{R}\cdot\boldsymbol{C}\cdot\boldsymbol{R}^T\cdot\boldsymbol{R}\cdot\boldsymbol{C}\cdot\boldsymbol{R}^T = \boldsymbol{R}\cdot\boldsymbol{C}^2\cdot\boldsymbol{R}^T\ ,$$

and the fact that the tensors $\boldsymbol{C}$ and $\boldsymbol{b}$ have the same invariants since the tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ have the same eigenvectors, we obtain a general form of the constitutive equation of an isotropic elastic material:

$$\boldsymbol{t} = a_0\boldsymbol{I} + a_1\boldsymbol{b} + a_2\boldsymbol{b}^2\ , \qquad a_i = a_i(I_b, II_b, III_b)\ . \tag{6.154}$$

This does not imply that the constitutive equation can only be a quadratic in the components of $\boldsymbol{b}$, since scalar functions a_i may be non-linear in terms of the components of $\boldsymbol{b}$.

If the Lagrangean description is considered instead of Eulerian representation, the second Piola-Kirchhoff stress tensor $\boldsymbol{T}^{(2)}$ is employed. The constitutive equation (6.34) for an elastic material (even without isotropy) is reduced to the simple form:

$$\boldsymbol{T}^{(2)} = \hat{\boldsymbol{T}}^{(2)}(\boldsymbol{C})\ . \tag{6.155}$$

The function $\hat{\boldsymbol{T}}^{(2)}$ for isotropic materials can be represented in the form:

$$\boldsymbol{T}^{(2)} = a_0\boldsymbol{I} + a_1\boldsymbol{C} + a_2\boldsymbol{C}^2 \ , \qquad a_i = a_i(I_C, II_C, III_C) \ , \tag{6.156}$$

which is another general form, equivalent to (6.154), of the constitutive equation for isotropic elastic solids.

We now have a look at the case of small deformation when the displacement gradient $\boldsymbol{H}$ is sufficiently small such that the geometric linearization can be applied and the difference between the reference configuration and the present configuration is not necessary to consider. Within the limit of geometric linearization, the Finger deformation tensor is equal to the Green deformation tensor. Consequently, the linearized constitutive equation for the second Piola-Kirchhoff tensor coincides with that for the Cauchy stress tensor. Let us, for instance, linearize the constitutive equation (6.156). Because of (1.106), we have

$$\boldsymbol{C} = \boldsymbol{U}^2 = \boldsymbol{I}+2\tilde{\boldsymbol{E}} \ , \quad \boldsymbol{C}^2 = \boldsymbol{I}+4\tilde{\boldsymbol{E}} \ , \quad I_C = 3+2\,\mathrm{tr}\tilde{\boldsymbol{E}} \ , \quad II_C = 3+4\,\mathrm{tr}\tilde{\boldsymbol{E}} \ , \quad III_C = 1+2\,\mathrm{tr}\tilde{\boldsymbol{E}} \ , \tag{6.157}$$

where $\tilde{\boldsymbol{E}}$ is the infinitesimal strain tensor. Then the constitutive equations (6.156) and (6.154) may be linearized as

$$\boldsymbol{T}^{(2)} \cong \boldsymbol{t} = \lambda(\mathrm{tr}\tilde{\boldsymbol{E}})\boldsymbol{I} + 2\mu\tilde{\boldsymbol{E}} \ . \tag{6.158}$$

This is the constitutive equation of a linear, isotropic elastic material, known also as *Hooke's law.* The both parameters λ and μ are called the *Lamé elastic parameters.*

6.17.2 Thermoelastic solid

The effect of heat conduction can be incorporated in the behavior of elastic materials by including temperature and temperature gradients among the constitutive variables. From (6.117) we deduce that the classical (non-classical) thermoelastic solid is defined by $N_C = 0$, $N_\theta = 0(1)$ and $N_g = 0$. Hence, the constitutive equation for a thermoelastic solid will have the form:

$$\boldsymbol{T}^{(2)} = \hat{\boldsymbol{T}}^{(2)}(\boldsymbol{C}, \theta, \dot{\theta}, \vec{G}_\theta) \ , \qquad \vec{Q} = \hat{\vec{Q}}(\boldsymbol{C}, \theta, \dot{\theta}, \vec{G}_\theta) \ , \qquad \varepsilon = \hat{\varepsilon}(\boldsymbol{C}, \theta, \dot{\theta}, \vec{G}_\theta) \ . \tag{6.159}$$

For isotropic materials, functions $\hat{\boldsymbol{T}}^{(2)}$, $\hat{\vec{Q}}$ and $\hat{\varepsilon}$ must be isotropic functions of their arguments:

$$\begin{aligned}
\boldsymbol{T}^{(2)} &= a_0\boldsymbol{I} + a_1\boldsymbol{C} + a_2\boldsymbol{C}^2 + a_3\,\vec{G}_\theta \otimes \vec{G}_\theta + a_4\,\mathrm{sym}(\boldsymbol{C}\cdot\vec{G}_\theta \otimes \vec{G}_\theta) + a_5\,\mathrm{sym}(\boldsymbol{C}^2\cdot\vec{G}_\theta \otimes \vec{G}_\theta) \ , \\
\vec{Q} &= -\kappa\,\vec{G}_\theta + c_1\,\boldsymbol{C}\cdot\vec{G}_\theta + c_2\,\boldsymbol{C}^2\cdot\vec{G}_\theta \ , \\
\varepsilon &= \hat{\varepsilon}(\theta, \dot{\theta}, I_C, II_C, III_C, \ \vec{G}_\theta\cdot\vec{G}_\theta, \ \vec{G}_\theta\cdot\boldsymbol{C}\cdot\vec{G}_\theta, \ \vec{G}_\theta\cdot\boldsymbol{C}^2\cdot\vec{G}_\theta) \ ,
\end{aligned} \tag{6.160}$$

where the scalar function a_i, κ and c_i depend on the same set of arguments as the function $\hat{\varepsilon}$ of the internal energy. We can see that the heat flux vector $\vec{Q}$ depends linearly on the temperature gradient $\vec{G}_\theta$ but contains two non-linear terms as the Green deformation tensor arises together with $\vec{G}_\theta$. The coefficient κ is called the *heat conductivity*, the remaining coefficients c_1 and c_2 do not have separate names.

6.17.3 Kelvin-Voigt viscoelastic solid

In general, under the *viscoelastic solid* we understand the material for which the stress is dependent on the strain, strain rate and stress rate tensors and not on the temperature. From (6.117) we deduce that the simplest viscoelastic solid is defined by $N_C = 1$:

$$\boldsymbol{T}^{(2)} = \hat{\boldsymbol{T}}^{(2)}(\boldsymbol{C}, \dot{\boldsymbol{C}}\ ) \ . \tag{6.161}$$

This simple strain-rate dependent solid is called the *Kelvin-Voigt viscoelastic solid.* Since the Piola-Kirchhoff stress tensor depends on two symmetric tensors, for isotropic materials it may be represented according to (6.148):

$$\begin{aligned} \boldsymbol{T}^{(2)} &= a_0\,\boldsymbol{I} + a_1\,\boldsymbol{C} + a_2\,\boldsymbol{C}^2 + a_3\,\dot{\boldsymbol{C}} + a_4\,\dot{\boldsymbol{C}}^2 + a_5\,\mathrm{sym}(\boldsymbol{C}\cdot\dot{\boldsymbol{C}}) + a_6\,\mathrm{sym}(\boldsymbol{C}\cdot\dot{\boldsymbol{C}}^2) \\ &+ a_7\,\mathrm{sym}(\boldsymbol{C}^2\cdot\dot{\boldsymbol{C}}) + a_8\,\mathrm{sym}(\boldsymbol{C}^2\cdot\dot{\boldsymbol{C}}^2)\ , \end{aligned} \tag{6.162}$$

where

$$a_i = a_i(\mathrm{tr}\boldsymbol{C}, \mathrm{tr}\boldsymbol{C}^2, \mathrm{tr}\boldsymbol{C}^3, \mathrm{tr}\,\dot{\boldsymbol{C}}, \mathrm{tr}\,\dot{\boldsymbol{C}}^2, \mathrm{tr}\,\dot{\boldsymbol{C}}^3\ \mathrm{tr}(\boldsymbol{C}\cdot\dot{\boldsymbol{C}}), \mathrm{tr}(\boldsymbol{C}\cdot\dot{\boldsymbol{C}}^2), \mathrm{tr}(\boldsymbol{C}^2\cdot\dot{\boldsymbol{C}}), \mathrm{tr}(\boldsymbol{C}^2\cdot\dot{\boldsymbol{C}}^2))\ . \tag{6.163}$$

6.17.4 Maxwell viscoelastic solid

A special case of constitutive equation (6.6) expressed in terms of the second Piola-Kirchhoff stress tensor is

$$(\boldsymbol{T}^{(2)})^{\bullet} = \hat{\boldsymbol{T}}^{(2)}(\boldsymbol{T}^{(2)}, \dot{\boldsymbol{C}}\,)\ . \tag{6.164}$$

This simple stress-rate dependent solid is called the *Maxwell viscoelastic solid.* Since $\boldsymbol{T}^{(2)}$ is an objective scalar, see $(6.18)_2$, and the material time derivative of an objective scalar is again an objective scalar, the constitutive equation (6.164) satisfies the principle of material objectivity. If $\hat{\boldsymbol{T}}^{(2)}$ is an isotropic function, we can proceed similarly as for the Kelvin-Voigt viscoelastic solid.

6.17.5 Elastic fluid

An elastic material may be a solid or a fluid. For an *elastic fluid*, the functions in the argument of the constitutive function $\hat{\boldsymbol{t}}$ in (6.121) are reduced to the dependence on the current density ϱ:

$$\boldsymbol{t} = \hat{\boldsymbol{t}}(\varrho)\ , \qquad \vec{q} = \hat{\vec{q}}(\varrho)\ , \qquad \varepsilon = \hat{\varepsilon}(\varrho)\ . \tag{6.165}$$

According to the assumption of material objectivity, we have

$$\boldsymbol{t}^* = \hat{\boldsymbol{t}}(\varrho^*)\ , \qquad \vec{q}^* = \hat{\vec{q}}(\varrho^*)\ , \qquad \varepsilon^* = \hat{\varepsilon}(\varrho^*) \tag{6.166}$$

under a change of observer frame. Because $\varrho^* = \varrho$, $\boldsymbol{t}^* = \boldsymbol{O}\cdot\boldsymbol{t}\cdot\boldsymbol{O}^T$, $\vec{q}^* = \boldsymbol{O}\cdot\vec{q}$, and $\varepsilon^* = \varepsilon$, this implies that

$$\boldsymbol{O}\cdot\hat{\boldsymbol{t}}(\varrho)\cdot\boldsymbol{O}^T = \hat{\boldsymbol{t}}(\varrho)\ , \qquad \boldsymbol{O}\cdot\hat{\vec{q}}(\varrho) = \hat{\vec{q}}(\varrho) \tag{6.167}$$

for all orthogonal tensors $\boldsymbol{O}$. The only isotropic tensor and isotropic vector satisfying these identities are a spherical tensor and the zero vector, respectively,

$$\boldsymbol{t} = -p(\varrho)\boldsymbol{I}\ , \qquad \vec{q} = \vec{0}\ , \qquad \varepsilon = \hat{\varepsilon}(\varrho)\ . \tag{6.168}$$

Therefore the stress in an elastic fluid is always a pressure depending on the density alone. It is a matter of the second law of thermodynamics to show that there is a relationship between the internal energy and the pressure function.

6.17.6 Thermoelastic fluid

The effect of heat conduction can be incorporated in the behavior of elastic fluid by including temperature and temperature gradients among the constitutive variables. From (6.121) we deduce that the *classical (non-classical) thermoelastic (heat-conducting) fluid* is defined by $N_C = 0$, $N_\theta = 0(1)$ and $N_g = 0$. Hence, the constitutive equation for a thermoelastic fluid will have the form:

$$\boldsymbol{t} = \hat{\boldsymbol{t}}(\theta, \dot{\theta}, \vec{g}_\theta, \varrho) \ , \qquad \vec{q} = \hat{\vec{q}}(\theta, \dot{\theta}, \vec{g}_\theta, \varrho) \ , \qquad \varepsilon = \hat{\varepsilon}(\theta, \dot{\theta}, \vec{g}_\theta, \varrho) \ . \tag{6.169}$$

Making use of the representation theorems for isotropic functions, we obtain

$$\begin{aligned} \boldsymbol{t} &= \sigma \boldsymbol{I} + \tau\, \vec{g}_\theta \otimes \vec{g}_\theta \ , \\ \vec{q} &= -\kappa\, \vec{g}_\theta \ , \\ \varepsilon &= \hat{\varepsilon}(\varrho, \theta, \dot{\theta}, \vec{g}_\theta \cdot \vec{g}_\theta) \ , \end{aligned} \tag{6.170}$$

where

$$\sigma, \tau, \kappa = \sigma, \tau, \kappa(\varrho, \theta, \dot{\theta}, \vec{g}_\theta \cdot \vec{g}_\theta) \ . \tag{6.171}$$

6.17.7 Viscous fluid

Taking $N_C = 1$ in (6.121), we obtain a *viscous fluid*, an analog to a simple viscoelastic solid:

$$\boldsymbol{t} = \hat{\boldsymbol{t}}(\boldsymbol{a}_1, \varrho) = \hat{\boldsymbol{t}}(\boldsymbol{d}, \varrho) \ , \tag{6.172}$$

since $\boldsymbol{a}_1 = 2\boldsymbol{d}$. This constitutive equation can further be simplified by applying the representation theorem for a symmetric isotropic tensor:

$$\boldsymbol{t} = a_0 \boldsymbol{I} + 2\mu_v \boldsymbol{d} + a_2 \boldsymbol{d}^2 \ , \qquad a_i, \mu_v = a_i, \mu_v(\varrho, I_d, II_d, III_d) \ , \tag{6.173}$$

where μ_v is the so-called *(shear) viscosity*.

Now, we wish to show that the constitutive equation (6.173) is materially objective. Multiplying (6.173) from the left by $\boldsymbol{O}$ and from the right by $\boldsymbol{O}^T$, we obtain

$$\boldsymbol{O}\cdot\boldsymbol{t}\cdot\boldsymbol{O}^T = a_0 \boldsymbol{O}\cdot\boldsymbol{I}\cdot\boldsymbol{O}^T + 2\mu_v \boldsymbol{O}\cdot\boldsymbol{d}\cdot\boldsymbol{O}^T + a_2 \boldsymbol{O}\cdot\boldsymbol{d}\cdot\boldsymbol{O}^T\cdot\boldsymbol{O}\cdot\boldsymbol{d}\cdot\boldsymbol{O}^T \ , \qquad a_i, \mu_v = a_i, \mu_v(\varrho, I_d, II_d, III_d) \ . \tag{6.174}$$

Since $\boldsymbol{t}$ and $\boldsymbol{d}$ are objective tensors and ϱ as well as the invariants of $\boldsymbol{d}$ are objective scalars, it holds

$$\boldsymbol{t}^* = a_0 \boldsymbol{I} + 2\mu_v \boldsymbol{d}^* + a_2 (\boldsymbol{d}^*)^2 \ , \qquad a_i, \mu_v = a_i, \mu_v(\varrho^*, I_d^*, II_d^*, III_d^*) \ , \tag{6.175}$$

which is of the same form as that in the unstarred frame. This completes the proof.

Fluids characterized by the above non-linearity are called *Stokesian* or *non-Newtonian fluids*. On the other hand, linear behavior is generally referred to as *Newtonian* behavior. To get a linear relationship between stress tensor $\boldsymbol{t}$ and the strain-rate tensor $\boldsymbol{d}$, the material coefficients in (6.173) must be chosen as follows:

$$a_0 = -p(\varrho)\mathrm{tr}\,\boldsymbol{d} \ , \qquad \mu_v = \mu_v(\varrho) \ , \qquad a_2 = 0 \ . \tag{6.176}$$

The stress tensor for a *compressible Newtonian fluid* becomes

$$\boldsymbol{t} = -p(\varrho)(\mathrm{tr}\,\boldsymbol{d})\,\boldsymbol{I} + 2\mu_v(\varrho)\boldsymbol{d} \ . \tag{6.177}$$

6.17.8 Incompressible viscous fluid

For an incompressible fluid we have the kinematic constraint that the density is constant, equal to a known value: $\varrho = \varrho_0$. The mass-conservation principle then yields: $I_d = \mathrm{tr}\, \boldsymbol{d} = \mathrm{div}\, \vec{v} = 0$. Combining (6.54) and (6.121), the constitutive equation of an incompressible viscous fluid is

$$\boldsymbol{t} = -p\boldsymbol{I} + \boldsymbol{t}^D \ . \tag{6.178}$$

The pressure p is an additional unknown independent field variable and it would be determined through solutions of field equations under a given set of boundary conditions. On contrary, the deviatoric part $\boldsymbol{t}^D$ of the stress tensor $\boldsymbol{t}$ is given by a constitutive equation analogous to (6.173):

$$\boldsymbol{t}^D = 2\mu_v \boldsymbol{d} + a_2 \boldsymbol{d}^2 \ , \qquad a_i, \mu_v = a_i, \mu_v(II_d, III_d) \ . \tag{6.179}$$

The dependencies on ϱ and I_d are dropped since ϱ is a known constant and $I_d = 0$. Notice that constitutive relation (6.179) has been postulated in such a way that any spherical part of stress tensor $\boldsymbol{t}$ may, without loss of generality, be absorbed in the pressure. [22]

6.17.9 Viscous heat-conducting fluid

A *classical (non-classical) viscous heat-conducting fluid* is defined by $N_C = 1$, $N_\theta = 0(1)$ and $N_g = 0$, so that (6.121) reduces to

$$\boldsymbol{t} = \hat{\boldsymbol{t}}(\boldsymbol{d}, \theta, \dot{\theta}, \vec{g}_\theta, \varrho) \ , \qquad \vec{q} = \hat{\vec{q}}(\boldsymbol{d}, \theta, \dot{\theta}, \vec{g}_\theta, \varrho) \ , \qquad \varepsilon = \hat{\varepsilon}(\boldsymbol{d}, \theta, \dot{\theta}, \vec{g}_\theta, \varrho) \ . \tag{6.180}$$

Making use of the representation theorems for isotropic functions, we obtain

$$\begin{aligned} \boldsymbol{t} &= a_0 \boldsymbol{I} + 2\mu_v \boldsymbol{d} + a_2 \boldsymbol{d}^2 + a_3\, \vec{g}_\theta \otimes \vec{g}_\theta + a_4 \,\mathrm{sym}(\boldsymbol{d} \cdot \vec{g}_\theta \otimes \vec{g}_\theta) + a_5 \,\mathrm{sym}(\boldsymbol{d}^2 \cdot \vec{g}_\theta \otimes \vec{g}_\theta) \ , \\ \vec{q} &= -\kappa\, \vec{g}_\theta + c_1\, \boldsymbol{d} \cdot \vec{g}_\theta + c_2\, \boldsymbol{d}^2 \cdot \vec{g}_\theta \ , \\ \varepsilon &= \hat{\varepsilon}(\varrho, \theta, \dot{\theta}, I_d, II_d, III_d, \ \vec{g}_\theta \cdot \vec{g}_\theta, \ \vec{g}_\theta \cdot \boldsymbol{d} \cdot \vec{g}_\theta, \ \vec{g}_\theta \cdot \boldsymbol{d}^2 \cdot \vec{g}_\theta,) \ , \end{aligned} \tag{6.181}$$

where the scalars a_i, μ_v, κ and c_i depend on the same set of arguments as the function $\hat{\varepsilon}$ of the internal energy. The proof of the material objectivity of these constitutive equations is similar to that for the viscous fluid.

[22] The stress tensor $\boldsymbol{t}$ may, in general, be decomposed into *isotropic spherical* tensor $\sigma \boldsymbol{I}$ and the *deviatoric stress* tensor $\boldsymbol{t}^D$:

$$\boldsymbol{t} = \sigma \boldsymbol{I} + \boldsymbol{t}^D \ .$$

Under the choice of $\sigma = \mathrm{tr}\, \boldsymbol{t}/3$, the trace of the deviatoric stress vanishes, $\mathrm{tr}\, \boldsymbol{t}^D = 0$. The scalar σ is thus the mean of the normal-stress components and is called *mechanical pressure*. A characteristic feature of a fluid at rest is that it cannot support shear stresses. Consequently, the deviatoric stress identically vanishes. Choosing $p := -\sigma$, we obtain $\boldsymbol{t} = -p\boldsymbol{I}$. Therefore, the stress in a fluid at rest is the so-called *hydrostatic pressure*.

7. ENTROPY PRINCIPLE

The preceding analysis makes no use of thermodynamic considerations. The treatment of the second law of thermodynamics is a part of the material theory since the entropy principle places restrictions on material properties. Thus, the entropy principle would belong to Chapter 6. Thermodynamic requirements are, however, so important that we prefer to deal with them in a separate chapter.

7.1 The Clausius-Duhem inequality

The second law of thermodynamics states that the entropy production cannot be negative. Equation (4.35) expresses this law in spatial form,

$$\varrho\gamma := \varrho\,\dot{\eta} + \operatorname{div}\vec{s} - \varrho b \geq 0 \ , \tag{7.1}$$

where γ is the local entropy production. This law does not, however, define internal dissipation mechanisms uniquely. We must bring additional information to be able to draw conclusions on material properties. In continuum thermodynamics, there are many other dissipation postulates, some of which do not even involve entropy as a field variable. In sections 7.1–7.4, we will confine ourselves to considerations that lead to the Clausius-Duhem inequality.

- We will postulate the existence of a non-negative valued *absolute temperature* T as a measure of hotness and assume that T is a frame indifferent scalar and vanishes only at absolute zero. Moreover, based the concepts of classical thermostatics for a simple adiabatic system, it was shown that on the basis of very weak assumptions that function $T = T(\theta)$, where θ is the empirical temperature θ, exists, is independent of the material for which is defined and changes monotonically with degree of coldness, that is, with the empirical temperature. It is evident that $T = T(\theta)$ possesses some degree of universality, and was therefore called absolute temperature. It was identified with the temperature of an ideal gas which obeys equal universal properties. The absolute temperature may differ from the empirical temperature θ that can be measured in the Celsius scale and may, in contrast to T, take negative values. Later on we will present one possible way how to relate the absolute and empirical temperatures.
- We shall be dealing with *simple thermodynamic processes* for which the entropy flux $\vec{s}$ and entropy source b are taken as

  $$\vec{s} = \frac{\vec{q}}{T} \ , \qquad b = \frac{h}{T} \ . \tag{7.2}$$

  This postulate is reasonable for a one-component body. In mixtures, a more general form of the entropy flux is necessary to postulate, for instance, a further term is added in $(7.2)_1$ such that the entropy flux and heat flux are not collinear vectors.

In general, these two postulates are not provable, but they have been adopted as certain generalizations of results of special problems, such as the kinetic theory of gases. Their validity can only be proved by physical experiments.

The entropy inequality (7.1) for a simple thermodynamic process then becomes

$$\varrho\,\dot{\eta} + \operatorname{div}\left(\frac{\vec{q}}{T}\right) - \frac{\varrho h}{T} \geq 0 \ , \tag{7.3}$$

which is known as the *Clausius-Duhem inequality.* This inequality need not be satisfied for all densities ϱ, motions $\vec{x}$ and temperatures T, but for all possible thermodynamic processes, that is for all solutions of the balance laws for mass, linear momentum and energy, that is for all solutions of the field equations of thermomechanics. To satisfy these additional constraints when the Clausius-Duhem inequality is applied, we can follow two different points of view.

- The balance laws for linear momentum and energy contain two free field variables, namely, body force $\vec{f}$ and heat supply h, which may be assigned arbitrary values. It implies that arbitrary histories of density ϱ, motion $\vec{x}$ and temperature T can be chosen and still appropriate body force $\vec{f}$ and heat supply h can be found to satisfy identically the linear momentum and energy equations. Hence, the linear momentum and energy equations do not raise any restrictions when the Clausius-Duhem inequality is applied. It remains to satisfied the continuity equation which may be considered as an additional constraint to the Clausius-Duhem inequality. For instance, $\dot{\varrho}$ cannot be chosen arbitrarily, but such that $\dot{\varrho} = -\varrho \operatorname{div} \vec{v}$.
- For a given physical problem, body force $\vec{f}$ and heat supply h are specified as the input information that cannot be altered during the solution of the problem. The linear momentum and energy equations must then be considered as additional constraints to the Clausius-Duhem inequality.

Though the second criterion is more acceptable from physical point of view, the first criterion is usually applied because of its simplicity. Here, we also start with this criterion in the application of the Clausius-Duhem inequality, but later on we introduce the entropy principle in modern understanding that employs the second criterion.

With the help of the energy equation (4.34), the heat source h can be eliminated from (7.3). The entropy inequality then takes the form

$$\varrho(T\dot{\eta} - \dot{\varepsilon}) + \boldsymbol{t} : \boldsymbol{d} - \frac{\vec{q} \cdot \operatorname{grad} T}{T} \geq 0 \ , \tag{7.4}$$

which is known as the *reduced Clausius-Duhem inequality* in spatial form. The reduced Clausius-Duhem inequality referred to the reference configuration can be obtained by combining the referential form of the energy equation (4.77) and the entropy inequality (4.81):

$$\varrho_0(T\dot{\eta} - \dot{\varepsilon}) + \boldsymbol{T}^{(2)} : \dot{\boldsymbol{E}} - \frac{\vec{Q} \cdot \operatorname{Grad} T}{T} \geq 0 \ . \tag{7.5}$$

7.2 Application of the Clausius-Duhem inequality to a classical viscous heat-conducting fluid

The constitutive equations for a classical viscous heat-conducting fluid are introduced in section 6.18.9:

$$\boldsymbol{t} = \hat{\boldsymbol{t}}(\boldsymbol{d}, T, \operatorname{grad} T, \varrho) \ , \qquad \vec{q} = \hat{\vec{q}}(\boldsymbol{d}, T, \operatorname{grad} T, \varrho) \ , \qquad \varepsilon = \hat{\varepsilon}(\boldsymbol{d}, T, \operatorname{grad} T, \varrho) \ , \tag{7.6}$$

where, instead of the empirical temperature θ, the absolute temperature T is employed as a measure of hotness. The Clausius-Duhem inequality introduces a new variable, the entropy density η. Since η is not determined by the field equations, a constitutive relation must be established for it. According to the principle of equipresence, we choose the same set of independent variables:

$$\eta = \hat{\eta}(\boldsymbol{d}, T, \operatorname{grad} T, \varrho) \ . \tag{7.7}$$

Because of frame indifference and isotropy, the constitutive functions have the form (6.184). This form will be applied later.

We carry out time differentiation of ε and η according to the chain rule of differentiation and substitute the result into the reduced Clausius-Duhem inequality (7.4). This yields

$$\begin{aligned} \varrho\Bigg[& \left(T\frac{\partial\hat{\eta}}{\partial\varrho} - \frac{\partial\hat{\varepsilon}}{\partial\varrho}\right)\dot{\varrho} + \left(T\frac{\partial\hat{\eta}}{\partial T} - \frac{\partial\hat{\varepsilon}}{\partial T}\right)\dot{T} + \left(T\frac{\partial\hat{\eta}}{\partial \boldsymbol{d}} - \frac{\partial\hat{\varepsilon}}{\partial \boldsymbol{d}}\right) : \dot{\boldsymbol{d}} \\ & + \left(T\frac{\partial\hat{\eta}}{\partial(\operatorname{grad} T)} - \frac{\partial\hat{\varepsilon}}{\partial(\operatorname{grad} T)}\right)\cdot(\operatorname{grad} T)^{\bullet}\Bigg] + \boldsymbol{t} : \boldsymbol{d} - \frac{\vec{q}\cdot\operatorname{grad} T}{T} \geq 0\ . \end{aligned} \tag{7.8}$$

The inequality must be satisfied for all thermodynamic processes, that is for all solutions of the balance laws for mass, linear momentum and energy as well as the constitutive equations. As we explained before, the linear momentum and energy equations do not raise any restrictions when the Clausius-Duhem inequality is applied. On contrary, the balance law of mass imposes an additional constraint to possible thermodynamic process in such a way such that, for instance, $\dot{\varrho}$ cannot be chosen arbitrarily, but in accordance with the continuity equation:

$$\dot{\varrho} = -\varrho \operatorname{div} \vec{v} = -\varrho \operatorname{tr} \boldsymbol{d} = -\varrho \boldsymbol{I} : \boldsymbol{d}\ .$$

Substituting this value for $\dot{\varrho}$ into inequality (7.8), the first term and the term $\boldsymbol{t} : \boldsymbol{d}$ may be put together:

$$\begin{aligned} \varrho\Bigg[& \left(T\frac{\partial\hat{\eta}}{\partial T} - \frac{\partial\hat{\varepsilon}}{\partial T}\right)\dot{T} + \left(T\frac{\partial\hat{\eta}}{\partial \boldsymbol{d}} - \frac{\partial\hat{\varepsilon}}{\partial \boldsymbol{d}}\right) : \dot{\boldsymbol{d}} + \left(T\frac{\partial\hat{\eta}}{\partial(\operatorname{grad} T)} - \frac{\partial\hat{\varepsilon}}{\partial(\operatorname{grad} T)}\right)\cdot(\operatorname{grad} T)^{\bullet}\Bigg] \\ & + \left[\boldsymbol{t} - \varrho^2\left(T\frac{\partial\hat{\eta}}{\partial\varrho} - \frac{\partial\hat{\varepsilon}}{\partial\varrho}\right)\boldsymbol{I}\right] : \boldsymbol{d} - \frac{\vec{q}\cdot\operatorname{grad} T}{T} \geq 0\ . \end{aligned} \tag{7.9}$$

This inequality must hold for all fields ϱ, $\vec{x}$ and T without any additional restrictions.

The quantities $\dot{T}$, $\dot{\boldsymbol{d}}$ and $(\operatorname{grad} T)^{\bullet}$ are not considered as independent variables in the constitutive equations for a classical viscous heat-conducting fluid, see (7.6) and (7.7), so that they occur explicitly as linear functions only the first three constituents in (7.9). Since $\dot{T}$, $\dot{\boldsymbol{d}}$ and $(\operatorname{grad} T)^{\bullet}$ may take any arbitrary values, the inequality (7.9) would be violated unless the factors standing at $\dot{T}$, $\dot{\boldsymbol{d}}$ and $(\operatorname{grad} T)^{\bullet}$ vanish. This argument gives the following constraints:

$$\frac{\partial\hat{\eta}}{\partial T} = \frac{1}{T}\frac{\partial\hat{\varepsilon}}{\partial T}\ , \qquad \frac{\partial\hat{\eta}}{\partial \boldsymbol{d}} = \frac{1}{T}\frac{\partial\hat{\varepsilon}}{\partial \boldsymbol{d}}\ , \qquad \frac{\partial\hat{\eta}}{\partial(\operatorname{grad} T)} = \frac{1}{T}\frac{\partial\hat{\varepsilon}}{\partial(\operatorname{grad} T)}\ . \tag{7.10}$$

By this, (7.9) reduces to the so-called residual inequality

$$\Gamma(\varrho, T, \boldsymbol{d}, \operatorname{grad} T) \geq 0\ , \tag{7.11}$$

where

$$\Gamma(\varrho, T, \boldsymbol{d}, \operatorname{grad} T) := \left[\boldsymbol{t} - \varrho^2\left(T\frac{\partial\hat{\eta}}{\partial\varrho} - \frac{\partial\hat{\varepsilon}}{\partial\varrho}\right)\boldsymbol{I}\right] : \boldsymbol{d} - \frac{\vec{q}\cdot\operatorname{grad} T}{T}\ . \tag{7.12}$$

The differentiation of $(7.10)_1$ with respect to $\boldsymbol{d}$ and $(7.10)_2$ with respect to T yields

$$\frac{\partial^2\hat{\eta}}{\partial\boldsymbol{d}\,\partial T} = \frac{1}{T}\frac{\partial^2\hat{\varepsilon}}{\partial\boldsymbol{d}\,\partial T}\ , \qquad \frac{\partial^2\hat{\eta}}{\partial T\,\partial\boldsymbol{d}} = \frac{1}{T}\frac{\partial^2\hat{\varepsilon}}{\partial T\,\partial\boldsymbol{d}} - \frac{1}{T^2}\frac{\partial\hat{\varepsilon}}{\partial\boldsymbol{d}}\ .$$

By assuming the exchange of the order of partial derivatives of $\hat{\eta}$ and $\hat{\varepsilon}$ with respect to $\boldsymbol{d}$ and T, we obtain

$$\frac{1}{T}\frac{\partial^2\hat{\varepsilon}}{\partial\boldsymbol{d}\,\partial T} = \frac{1}{T}\frac{\partial^2\hat{\varepsilon}}{\partial\boldsymbol{d}\,\partial T} - \frac{1}{T^2}\frac{\partial\hat{\varepsilon}}{\partial\boldsymbol{d}} \ ,$$

which shows that $\hat{\varepsilon}$ cannot be a function of $\boldsymbol{d}$,

$$\frac{\partial\hat{\varepsilon}}{\partial\boldsymbol{d}} = 0 \ . \tag{7.13}$$

Because of $(7.10)_2$, it also holds

$$\frac{\partial\hat{\eta}}{\partial\boldsymbol{d}} = 0 \ . \tag{7.14}$$

Likewise, the cross-differentiation of $(7.10)_1$ and $(7.10)_3$ with respect to $\operatorname{grad} T$ and T, respectively, provides an analogous result:

$$\frac{\partial\hat{\varepsilon}}{\partial(\operatorname{grad} T)} = 0 \ , \qquad \frac{\partial\hat{\eta}}{\partial(\operatorname{grad} T)} = 0 \ . \tag{7.15}$$

This is the first result of the entropy principle. The constitutive functions $\hat{\varepsilon}$ and $\hat{\eta}$ cannot depend on $\boldsymbol{d}$ and $\operatorname{grad} T$, and, consequently, the constitutive equations $(7.6)_3$ and (7.7) reduces to

$$\varepsilon = \hat{\varepsilon}(\varrho, T) \ , \qquad \eta = \hat{\eta}(\varrho, T) \ . \tag{7.16}$$

Functions $\hat{\varepsilon}$ and $\hat{\eta}$ are still subject to constraints (7.10). Since $(7.10)_{2,3}$ are satisfied identically, it remains to satisfied $(7.10)_1$:

$$\frac{\partial\hat{\eta}}{\partial T} = \frac{1}{T}\frac{\partial\hat{\varepsilon}}{\partial T} \ . \tag{7.17}$$

We continue with the exploitation of the residual inequality (7.11). First, we define the *thermodynamic equilibrium* as a time-independent thermodynamic process with uniform (constant in space) and stationary (constant in time) velocity and temperature fields. Mathematically, the thermodynamic equilibrium is defined through equations

$$\vec{v}(\vec{X}, t) = \vec{v}(\vec{x}, t) = \text{const} \ , \qquad T(\vec{X}, t) = T(\vec{x}, t) = \text{const} \ , \tag{7.18}$$

which implies that

$$\boldsymbol{d} = \boldsymbol{0} \ , \qquad \operatorname{grad} T = \vec{0} \ . \tag{7.19}$$

Then (7.12) shows that

$$\Gamma(\varrho, T, \boldsymbol{0}, \vec{0}) = 0 \tag{7.20}$$

in thermodynamic equilibrium. Together with the residual inequality (7.11) we can see that Γ is minimal in thermodynamic equilibrium and the value of this minimum is zero. According to the theory extrema of functions of several variables, the necessary condition that Γ reaches a minimum is that the first derivatives of Γ with respect to $\boldsymbol{d}$ and $\operatorname{grad} T$ vanish, and the matrix of the second partial derivatives is positive semi-definite:

$$\left.\frac{\partial\Gamma}{\partial\boldsymbol{d}}\right|_{\mathrm{E}} = \boldsymbol{0} \ , \qquad \left.\frac{\partial\Gamma}{\partial(\operatorname{grad} T)}\right|_{\mathrm{E}} = \vec{0} \ , \tag{7.21}$$

and the matrix

$$\boldsymbol{A} := \begin{pmatrix} \left.\dfrac{\partial^2\Gamma}{\partial\boldsymbol{d}\,\partial\boldsymbol{d}}\right|_{\mathrm{E}} & \left.\dfrac{\partial^2\Gamma}{\partial\boldsymbol{d}\,\partial(\operatorname{grad}T)}\right|_{\mathrm{E}} \\ \left.\dfrac{\partial^2\Gamma}{\partial\boldsymbol{d}\,\partial(\operatorname{grad}T)}\right|_{\mathrm{E}} & \left.\dfrac{\partial^2\Gamma}{\partial(\operatorname{grad}T)\,\partial(\operatorname{grad}T)}\right|_{\mathrm{E}} \end{pmatrix} \tag{7.22}$$

is positive semi-definite. The subscript E denotes the state of thermodynamic equilibrium, that is, the state with $\boldsymbol{d} = \boldsymbol{0}$ and $\operatorname{grad} T = \vec{0}$.

Let us first have a look at the constraints in (7.21). Differentiating (7.12) with respect to $\boldsymbol{d}$ and $\operatorname{grad} T$, respectively, and taking the result in thermodynamic equilibrium yields

$$\left.\frac{\partial\Gamma}{\partial\boldsymbol{d}}\right|_{\mathrm{E}} = \boldsymbol{t}|_{\mathrm{E}} - \varrho^2 \left.\left(T\frac{\partial\hat{\eta}}{\partial\varrho} - \frac{\partial\hat{\varepsilon}}{\partial\varrho}\right)\right|_{\mathrm{E}} \boldsymbol{I}\ , \qquad \left.\frac{\partial\Gamma}{\partial(\operatorname{grad}T)}\right|_{\mathrm{E}} = -\frac{\vec{q}|_{\mathrm{E}}}{T}\ .$$

Using this in (7.21), we obtain

$$\boldsymbol{t}|_{\mathrm{E}} = -p|_{\mathrm{E}}\,\boldsymbol{I}\ , \qquad \vec{q}|_{\mathrm{E}} = \vec{0}\ , \tag{7.23}$$

where $p(\varrho, T)$ is the *thermodynamic pressure*,

$$p(\varrho, T) := -\varrho^2\left(T\frac{\partial\hat{\eta}}{\partial\varrho} - \frac{\partial\hat{\varepsilon}}{\partial\varrho}\right)\ . \tag{7.24}$$

The result (7.23) shows that in thermodynamic equilibrium the stress is isotropic and determined by the entropy and internal energy, and the equilibrium heat flux vanishes. The latter result does not bring any new information since the reduced constitutive equation $(6.184)_2$ involves this feature. Note that, though the thermodynamic pressure was introduced for the thermodynamic equilibrium, it can equally be defined by (7.24) for all thermodynamic processes since ε and η are, in general, that is for all thermodynamic processes only functions of ϱ and T. [23]

The constitutive equation $(6.184)_1$ in thermodynamic equilibrium reduces to

$$\boldsymbol{t}|_{\mathrm{E}} = a_0|_{\mathrm{E}}\,(\varrho, T)\boldsymbol{I}\ . \tag{7.25}$$

Comparing this with $(7.23)_1$ shows that

$$a_0|_{\mathrm{E}}\,(\varrho, T) = -p(\varrho, T)\ , \tag{7.26}$$

which motivates to decompose function a_0 in $(6.184)_1$ as

$$a_0(\varrho, T, I_d, II_d, III_d, \ldots) = -p(\varrho, T) + \nu_0(\varrho, T, I_d, II_d, III_d, \ldots)\ . \tag{7.27}$$

Consequently, the non-equilibrium part ν_0 of stress vanishes in thermodynamic equilibrium,

$$\nu_0|_{\mathrm{E}} = 0\ . \tag{7.28}$$

In view of $(7.16)_2$, the total differential $d\eta$ of η is

$$d\eta = \frac{\partial\hat{\eta}}{\partial\varrho}d\varrho + \frac{\partial\hat{\eta}}{\partial T}dT\ . \tag{7.29}$$

[23] This is typical property of the entropy principle in the Clausius-Duhem form. In other entropy principles different results may be obtained.

Let us arrange (7.24) for $\partial\hat{\eta}/\partial\varrho$,

$$\frac{\partial\hat{\eta}}{\partial\varrho} = \frac{1}{T}\left(\frac{\partial\hat{\varepsilon}}{\partial\varrho} - \frac{p}{\varrho^2}\right) ,$$

substitute it for the first term on the right-hand side of (7.29), and use (7.17) for the second term. We end up with

$$d\eta = \frac{1}{T}\left(\frac{\partial\hat{\varepsilon}}{\partial\varrho} - \frac{p}{\varrho^2}\right) d\varrho + \frac{1}{T}\frac{\partial\hat{\varepsilon}}{\partial T}dT , \tag{7.30}$$

which is known as a *Gibbs relation* for a classical viscous heat-conducting fluid. Alternatively, in view of $(7.16)_1$, we may write the total differential $d\varepsilon$ of ε:

$$d\varepsilon = \frac{\partial\hat{\varepsilon}}{\partial\varrho}d\varrho + \frac{\partial\hat{\varepsilon}}{\partial T}dT . \tag{7.31}$$

By this, the total differential $d\eta$ in (7.30) can be simplified as

$$d\eta = \frac{1}{T}d\varepsilon - \frac{p}{T}\frac{d\varrho}{\varrho^2} . \tag{7.32}$$

Introducing the *specific volume* $v := 1/\varrho$ and $dv = -d\varrho/\varrho^2$, we have

$$d\eta = \frac{1}{T}d\varepsilon + \frac{p}{T}dv , \tag{7.33}$$

which is equivalent to

$$d\varepsilon = T\,d\eta - p\,dv . \tag{7.34}$$

This is another form of the Gibbs relation, in which ε is considered as a function of η and v.

The necessary and sufficient condition that $d\eta$ in (7.30) is an exact differential is

$$\frac{\partial}{\partial\varrho}\left(\frac{1}{T}\frac{\partial\hat{\varepsilon}}{\partial T}\right) = \frac{\partial}{\partial T}\left[\frac{1}{T}\left(\frac{\partial\hat{\varepsilon}}{\partial\varrho} - \frac{p}{\varrho^2}\right)\right] ,$$

or

$$\frac{1}{T}\frac{\partial^2\hat{\varepsilon}}{\partial\varrho\partial T} = \frac{1}{T}\left(\frac{\partial^2\hat{\varepsilon}}{\partial T\partial\varrho} - \frac{1}{\varrho^2}\frac{\partial p}{\partial T}\right) - \frac{1}{T^2}\left(\frac{\partial\hat{\varepsilon}}{\partial\varrho} - \frac{p}{\varrho^2}\right) .$$

After some manipulations, we obtain

$$\frac{\partial}{\partial T}\left(\frac{p(\varrho,T)}{T}\right) = -\left(\frac{\varrho}{T}\right)^2\frac{\partial\hat{\varepsilon}(\varrho,T)}{\partial\varrho} . \tag{7.35}$$

The material equation $p = p(\varrho,T)$ is usually called the *thermal equation of state* and the equation $\varepsilon = \hat{\varepsilon}(\varrho,T)$ the *caloric equation of state.* Equation (7.35) shows that the derivative $\partial\hat{\varepsilon}/\partial\varrho$ need not be sought by caloric measurements if the thermal equation of state is known. The measurements of the latter are much simpler than the caloric measurements of $\partial\hat{\varepsilon}/\partial\varrho$.

Let us summarize the results following from the application of the entropy principle.

- The constitutive equations (6.184) for a classical viscous heat-conducting fluid are reduced to

$$\begin{aligned} \boldsymbol{t} &= (-p+\nu_0)\boldsymbol{I} + 2\mu_v \boldsymbol{d} + a_2 \boldsymbol{d}^2 + \dots \ , \\ \vec{q} &= -\kappa \operatorname{grad} T + c_1\, \boldsymbol{d}\cdot \operatorname{grad} T + c_2\, \boldsymbol{d}^2 \cdot \operatorname{grad} T \ , \\ \varepsilon &= \hat{\varepsilon}(\varrho, T) \ , \\ \eta &= \hat{\eta}(\varrho, T) \ , \end{aligned} \tag{7.36}$$

  where

$$\begin{aligned} & p = p(\varrho, T) \ , \\ & \nu_0,\ \mu_v,\ a_i,\ \kappa,\ c_i = \nu_0,\ \mu_v,\ a_i, \kappa,\ c_i(\varrho, T, I_d, II_d, III_d, \dots) \ , \\ & \nu_0|_{\mathrm{E}} = 0 \ . \end{aligned} \tag{7.37}$$

- Functions p, $\hat{\varepsilon}$ and $\hat{\eta}$ are mutually related,

$$\frac{\partial \hat{\eta}}{\partial T} = \frac{1}{T}\frac{\partial \hat{\varepsilon}}{\partial T} \ , \qquad \frac{\partial \hat{\eta}}{\partial \varrho} = \frac{1}{T}\left(\frac{\partial \hat{\varepsilon}}{\partial \varrho} - \frac{p}{\varrho^2}\right) \ , \qquad \frac{\partial}{\partial T}\left(\frac{p}{T}\right) = -\left(\frac{\varrho}{T}\right)^2 \frac{\partial \hat{\varepsilon}}{\partial \varrho} \ . \tag{7.38}$$

- Further constraints can be drawn from the positive definiteness of matrix $\boldsymbol{A}$ defined by (7.22).

We will now confine ourselves to a special type of classical viscous heat-conducting fluid which is *linear* with respect to $\boldsymbol{d}$ and $\operatorname{grad} T$. In this case, the constitutive equations (7.36) reduce to

$$\begin{aligned} \boldsymbol{t} &= (-p + \lambda_v \operatorname{tr} \boldsymbol{d})\boldsymbol{I} + 2\mu_v \boldsymbol{d} \ , \\ \vec{q} &= -\kappa \operatorname{grad} T \ , \\ \varepsilon &= \hat{\varepsilon}(\varrho, T) \ , \\ \eta &= \hat{\eta}(\varrho, T) \ , \end{aligned} \tag{7.39}$$

where

$$p,\ \lambda_v,\ \mu_v,\ \kappa = p,\ \lambda_v,\ \mu_v,\ \kappa(\varrho, T) \ . \tag{7.40}$$

The fluid characterized by the constitutive equation $(7.39)_1$ is known as the *Newton viscous fluid*, equation $(7.39)_2$ is known as *Fourier's law* of heat conduction. The parameters λ_v and μ_v are called, respectively, the *dilatational* and *shear viscosities.*

To draw conclusions from (7.22), we substitute the constitutive equations $(7.39)_1$ and $(7.39)_2$ into (7.12) and consider (7.24):

$$\begin{aligned} \Gamma(\varrho, T, \boldsymbol{d}, \operatorname{grad} T) &= (\boldsymbol{t} + p\boldsymbol{I}) : \boldsymbol{d} - \frac{\vec{q}\cdot \operatorname{grad} T}{T} \\ &= \lambda_v \operatorname{tr} \boldsymbol{d}\,(\boldsymbol{I} : \boldsymbol{d}) + 2\mu_v(\boldsymbol{d} : \boldsymbol{d}) + \kappa \frac{(\operatorname{grad} T)^2}{T} \\ &= \lambda_v (\operatorname{tr} \boldsymbol{d})^2 + 2\mu_v(\boldsymbol{d} : \boldsymbol{d}) + \kappa \frac{(\operatorname{grad} T)^2}{T} \ . \end{aligned} \tag{7.41}$$

We proceed to calculate the second derivatives of Γ occurring in (7.22). The first derivatives of Γ are

$$\frac{\partial \Gamma}{\partial d_{ii}} = 2\lambda_v \operatorname{tr} \boldsymbol{d} + 4\mu_v d_{ii} \ , \qquad \frac{\partial \Gamma}{\partial d_{ij}} = 4\mu_v d_{ij} \ , \qquad \frac{\partial \Gamma}{\partial (\operatorname{grad} T)_i} = 2\kappa \frac{(\operatorname{grad} T)_i}{T} \ , \tag{7.42}$$

where $i, j = 1, 2, 3$ and $i \neq j$ in the second term. Then the second derivatives of Γ are

$$\frac{\partial^2\Gamma}{\partial d_{ii}\partial d_{jj}} = 2\lambda_v + 4\mu_v\delta_{ij} \ , \qquad \frac{\partial^2\Gamma}{\partial d_{ij}\partial d_{k\ell}} = 4\mu_v\delta_{ik}\delta_{j\ell} \ , \qquad \frac{\partial^2\Gamma}{\partial(\mathrm{grad}\, T)_i\, \partial(\mathrm{grad}\, T)_j} = \frac{2\kappa}{T}\delta_{ij} \ , \tag{7.43}$$

where $i \neq j$ and $k \neq \ell$ in the second term. Moreover the cross-derivative of Γ with respect to $\boldsymbol{d}$ and $\mathrm{grad}\, T$ vanishes. The matrix of the second derivatives of Γ, defined by (7.22), is then block-diagonal:

$$\boldsymbol{A} = \left. \begin{pmatrix} 2\lambda_v \mathbf{1}_{3\times3} + 4\mu_v \boldsymbol{I}_{3\times3} & 0 & 0 \\ 0 & 4\mu_v \boldsymbol{I}_{6\times6} & 0 \\ 0 & 0 & \dfrac{2\kappa}{T}\boldsymbol{I}_{3\times3} \end{pmatrix} \right|_{\mathrm{E}} , \tag{7.44}$$

where $\boldsymbol{I}_{3\times3}$ and $\boldsymbol{I}_{6\times6}$ are the identity matrices of the third- and sixth-order, respectively, and the matrix $\mathbf{1}_{3\times3}$ is composed of 1 only. Consider a 12 component vector $\vec{z} = (\vec{u}, \vec{v}, \vec{w})$, where $\vec{u}$, $\vec{v}$ and $\vec{w}$ have 3, 6 and 3 components, respectively. The positive semidefiniteness of $\boldsymbol{A}$ then means that $\vec{z} \cdot \boldsymbol{A} \cdot \vec{z} \geq 0$ for all vectors $\vec{z} \neq \vec{0}$, or

$$2\lambda_v|_{\mathrm{E}}\, (\vec{u} \cdot \mathbf{1}_{3\times3} \cdot \vec{u}) + 4\mu_v|_{\mathrm{E}}\, (\vec{u} \cdot \vec{u}) + 4\mu_v|_{\mathrm{E}}\, (\vec{v} \cdot \vec{v}) + \left.\frac{2\kappa}{T}\right|_{\mathrm{E}} (\vec{w} \cdot \vec{w}) \geq 0 \quad \forall \vec{u}, \vec{v}, \vec{w} \ . \tag{7.45}$$

Since $\vec{u} \cdot \mathbf{1}_{3\times3} \cdot \vec{u} = (u_1 + u_2 + u_3)^2$, the necessary and sufficient conditions that $\boldsymbol{A}$ is positive semi-definite is

$$\lambda_v|_{\mathrm{E}} \geq 0 \ , \qquad \mu_v|_{\mathrm{E}} \geq 0 \ , \qquad \kappa|_{\mathrm{E}} \geq 0 \ . \tag{7.46}$$

We can conclude that to satisfy the Clausius-Duhem inequality for a linear classical viscous heat-conducting fluid, the bulk viscosity k_v, the shear viscosity μ_v and the heat conductivity κ must be non-negative functions of ϱ and T.

The constitutive equation $(7.39)_1$ can be written in an alternative form if the strain-rate tensor $\boldsymbol{d}$ is decomposed into spherical and deviatoric parts, $\boldsymbol{d} = (\mathrm{tr}\, \boldsymbol{d})\ \boldsymbol{I}/3 + \boldsymbol{d}^D$:

$$\boldsymbol{t} = (-p + k_v \mathrm{tr}\, \boldsymbol{d})\boldsymbol{I} + 2\mu_v \boldsymbol{d}^D \ , \tag{7.47}$$

where

$$k_v := \lambda_v + \frac{2}{3}\mu_v \tag{7.48}$$

is the *bulk viscosity*. The constraint (7.46) on the non-negativeness of λ_v and μ_v implies that

$$k_v|_{\mathrm{E}} \geq 0 \ . \tag{7.49}$$

Moreover, the functions $\hat{\varepsilon}$, $\hat{\eta}$ and p must satisfy constraints (7.38). If we introduce the *Helmholtz free energy*

$$\psi := \varepsilon - T\eta = \hat{\psi}(\varrho, T) \ , \tag{7.50}$$

we can then deduce that (7.38) are satisfied if

$$\eta = -\frac{\partial\hat{\psi}}{\partial T} \ , \qquad p = \varrho^2 \frac{\partial\hat{\psi}}{\partial\varrho} = -\frac{\partial\hat{\psi}}{\partial v} \ . \tag{7.51}$$

This demonstrates that ψ serves as a thermodynamic potential for the entropy and the pressure. The internal energy is also derivable from potential ψ by (7.50). This is one of the main results of the second law of thermodynamics: it yields the existence of the thermodynamic potential ψ which reduces the number of required constitutive relations.

7.3 Application of the Clausius-Duhem inequality to a classical viscous heat-conducting incompressible fluid

For an incompressible material, the density ϱ remains unchanged, equal to a known value, $\varrho = \varrho_0$. It implies that, compared to the preceding case, the density is excluded from a list of independent constitutive variables. The stress constitutive equation of an incompressible fluid is given by (6.181),

$$\boldsymbol{t} = -p\boldsymbol{I} + \boldsymbol{t}^D \ . \tag{7.52}$$

Note that, in contrast to the thermodynamic pressure $p(\varrho, T)$, pressure p is now an unknown field variable. The constitutive equations for $\boldsymbol{t}^D$, $\vec{q}$, ε and η have the same form as (7.6) and (7.7) except that all functions are now independent of ϱ, [24]

$$\boldsymbol{t}^D = \hat{\boldsymbol{t}}^D(\boldsymbol{d}, T, \operatorname{grad} T) \ , \qquad \vec{q} = \hat{\vec{q}}(\boldsymbol{d}, T, \operatorname{grad} T) \ , \qquad \varepsilon = \hat{\varepsilon}(\boldsymbol{d}, T, \operatorname{grad} T) \ , \qquad \eta = \hat{\eta}(\boldsymbol{d}, T, \operatorname{grad} T) \ , \tag{7.53}$$

and are subject to the additional constraint

$$\operatorname{div} \vec{v} = \operatorname{tr} \boldsymbol{d} = 0 \ . \tag{7.54}$$

Under this constraint, the term $\boldsymbol{t} : \boldsymbol{d}$ occurring in the Clausius-Duhem inequality can be arranged as follows:

$$\boldsymbol{t} : \boldsymbol{d} = (-p\boldsymbol{I} + \boldsymbol{t}^D) : \boldsymbol{d} = -p \operatorname{tr} \boldsymbol{d} + \boldsymbol{t}^D : \boldsymbol{d} = \boldsymbol{t}^D : \boldsymbol{d} \ .$$

The entropy inequality (7.8) then takes the form

$$\begin{aligned} &\varrho\Bigg[\left(T\frac{\partial\hat{\eta}}{\partial T} - \frac{\partial\hat{\varepsilon}}{\partial T}\right)\dot{T} + \left(T\frac{\partial\hat{\eta}}{\partial \boldsymbol{d}} - \frac{\partial\hat{\varepsilon}}{\partial \boldsymbol{d}}\right) : \dot{\boldsymbol{d}} \\ &+ \left(T\frac{\partial\hat{\eta}}{\partial(\operatorname{grad} T)} - \frac{\partial\hat{\varepsilon}}{\partial(\operatorname{grad} T)}\right)\cdot(\operatorname{grad} T)^{\boldsymbol{\cdot}}\Bigg] + \boldsymbol{t}^D : \boldsymbol{d} - \frac{\vec{q}\cdot\operatorname{grad} T}{T} \geq 0 \ . \end{aligned} \tag{7.55}$$

This inequality is linear in the variables $\dot{T}$, $\dot{\boldsymbol{d}}$ and $(\operatorname{grad} T)^{\boldsymbol{\cdot}}$ which may have arbitrarily assigned values. Thus the factors at these variables must vanish, which yields

$$\frac{\partial\hat{\eta}}{\partial T} = \frac{1}{T}\frac{\partial\hat{\varepsilon}}{\partial T} \ , \qquad \frac{\partial\hat{\eta}}{\partial \boldsymbol{d}} = \frac{1}{T}\frac{\partial\hat{\varepsilon}}{\partial \boldsymbol{d}} \ , \qquad \frac{\partial\hat{\eta}}{\partial(\operatorname{grad} T)} = \frac{1}{T}\frac{\partial\hat{\varepsilon}}{\partial(\operatorname{grad} T)} \ . \tag{7.56}$$

By cross-differentiation of (7.56), we again obtain that

$$\frac{\partial\hat{\varepsilon}}{\partial \boldsymbol{d}} = 0 \ , \qquad \frac{\partial\hat{\eta}}{\partial \boldsymbol{d}} = 0 \ , \qquad \frac{\partial\hat{\varepsilon}}{\partial(\operatorname{grad} T)} = 0 \ , \qquad \frac{\partial\hat{\eta}}{\partial(\operatorname{grad} T)} = 0 \ , \tag{7.57}$$

which shows that the internal energy and entropy are functions of temperature only,

$$\varepsilon = \hat{\varepsilon}(T) \ , \qquad \eta = \hat{\eta}(T) \ . \tag{7.58}$$

[24] but may, in principle, be dependent on pressure p. Does it make a sense?

Functions $\hat{\varepsilon}$ and $\hat{\eta}$ are still subject to the constraint

$$\frac{\partial \hat{\eta}}{\partial T} = \frac{1}{T} \frac{\partial \hat{\varepsilon}}{\partial T} \ . \tag{7.59}$$

We remain with the residual inequality

$$\Gamma(T, \boldsymbol{d}, \operatorname{grad} T) := \boldsymbol{t}^D : \boldsymbol{d} - \frac{\vec{q} \cdot \operatorname{grad} T}{T} \geq 0 \ . \tag{7.60}$$

In thermodynamic equilibrium, it again holds

$$\Gamma(T, \mathbf{0}, \vec{0}) = 0 \ , \tag{7.61}$$

which shows that the conditions (7.21) for the minimum of Γ are also valid here. From $(7.21)_1$ it follows that $\boldsymbol{t}^D\Big|_{\mathrm{E}} = 0$ which implies that

$$\boldsymbol{t}|_{\mathrm{E}} = -p\boldsymbol{I} \ . \tag{7.62}$$

The second constraint in (7.21) yields

$$\vec{q}|_{\mathrm{E}} = \vec{0} \ . \tag{7.63}$$

We can conclude that also in this case the stress is isotropic and the heat flux vanishes in thermodynamic equilibrium.

7.4 Application of the Clausius-Duhem inequality to a classical thermoelastic solid

The constitutive equations for a classical thermoelastic solid are introduced in section 6.18.2:

$$\begin{aligned} \boldsymbol{T}^{(2)} &= \hat{\boldsymbol{T}}^{(2)}(\boldsymbol{E}, T, \operatorname{Grad} T) \ , & \varepsilon &= \hat{\varepsilon}(\boldsymbol{E}, T, \operatorname{Grad} T) \ , \\ \vec{Q} &= \hat{\vec{Q}}(\boldsymbol{E}, T, \operatorname{Grad} T) \ , & \eta &= \hat{\eta}(\boldsymbol{E}, T, \operatorname{grad} T) \ . \end{aligned} \tag{7.64}$$

Since $\boldsymbol{C} = 2\boldsymbol{E} + \boldsymbol{I}$, we replaced the dependence on the Green deformation tensor $\boldsymbol{C}$ in (6.164) by an equivalent representation in terms of the Lagrangian strain tensor $\boldsymbol{E}$. Note that the continuity equation does not play any role here, since the balance of mass results in the statement $\varrho_0 = \varrho_0(\vec{X})$.

We carry out time differentiation of ε and η according to the chain rule of differentiation and substitute the result into the reduced Clausius-Duhem inequality (7.5). This yields

$$\begin{aligned} &\varrho_0 \Bigg[\left(T \frac{\partial \hat{\eta}}{\partial T} - \frac{\partial \hat{\varepsilon}}{\partial T} \right) \dot{T} + \left(T \frac{\partial \hat{\eta}}{\partial \boldsymbol{E}} - \frac{\partial \hat{\varepsilon}}{\partial \boldsymbol{E}} + \frac{1}{\varrho_0} \hat{\boldsymbol{T}}^{(2)} \right) : \dot{\boldsymbol{E}} \\ &+ \left(T \frac{\partial \hat{\eta}}{\partial (\operatorname{Grad} T)} - \frac{\partial \hat{\varepsilon}}{\partial (\operatorname{Grad} T)} \right) \cdot (\operatorname{Grad} T)^{\bullet} \Bigg] - \frac{\vec{Q} \cdot \operatorname{Grad} T}{T} \geq 0 \ . \end{aligned} \tag{7.65}$$

Since $\dot{T}$, $\dot{\boldsymbol{E}}$ and $(\operatorname{Grad} T)^{\bullet}$ are not considered as independent variables in the constitutive equation (7.64), the coefficients standing at $\dot{T}$, $\dot{\boldsymbol{E}}$ and $(\operatorname{Grad} T)^{\bullet}$ are independent of these quantities.

Hence, $\dot{T}$, $\dot{\boldsymbol{E}}$ and $(\operatorname{Grad} T)^{\bullet}$ occur only linearly in the inequality (7.65). This inequality cannot be maintained for all $\dot{T}$, $\dot{\boldsymbol{E}}$ and $(\operatorname{Grad} T)^{\bullet}$ unless the coefficients at these terms vanish. Hence,

$$\frac{\partial\hat{\eta}}{\partial T} = \frac{1}{T}\frac{\partial\hat{\varepsilon}}{\partial T} , \qquad \frac{1}{\varrho_0}\hat{\boldsymbol{T}}^{(2)} = \frac{\partial\hat{\varepsilon}}{\partial \boldsymbol{E}} - T\frac{\partial\hat{\eta}}{\partial \boldsymbol{E}} , \qquad \frac{\partial\hat{\eta}}{\partial(\operatorname{Grad} T)} = \frac{1}{T}\frac{\partial\hat{\varepsilon}}{\partial(\operatorname{Grad} T)} . \tag{7.66}$$

By this and since $T > 0$, inequality (7.65) reduces to

$$-\vec{Q}\cdot\operatorname{Grad} T \geq 0 . \tag{7.67}$$

This is the classical statement of the second law of thermodynamics saying that the heat flux has the orientation opposite to the temperature gradient, that is the heat energy flows from hot to cold regions.

The differentiation of $(7.66)_2$ with respect to $\operatorname{Grad} T$ and $(7.66)_3$ with respect to $\boldsymbol{E}$ and assuming the exchange of the order of differentiation of $\hat{\varepsilon}$ and $\hat{\eta}$ with respect to $\boldsymbol{E}$ and $\operatorname{Grad} T$ results in

$$\frac{\partial\hat{\boldsymbol{T}}^{(2)}}{\partial(\operatorname{Grad} T)} = 0 ,$$

which shows that the second Piola-Kirchhoff stress tensor is independent of $\operatorname{Grad} T$. By an analogous procedure, it may be shown that $\hat{\varepsilon}$ and $\hat{\eta}$ are functions of $\boldsymbol{E}$ and T only. Except the heat flux, the constitutive equations (7.64) reduces to

$$\boldsymbol{T}^{(2)} = \hat{\boldsymbol{T}}^{(2)}(\boldsymbol{E},T) , \qquad \varepsilon = \hat{\varepsilon}(\boldsymbol{E},T) , \qquad \eta = \hat{\eta}(\boldsymbol{E},T) . \tag{7.68}$$

The total differential $d\eta$ of η is

$$d\eta = \frac{\partial\hat{\eta}}{\partial\boldsymbol{E}} : d\boldsymbol{E} + \frac{\partial\hat{\eta}}{\partial T}dT . \tag{7.69}$$

Substituting from (7.66), we obtain

$$d\eta = \frac{1}{T}\left[\frac{\partial\hat{\varepsilon}}{\partial T}dT + \left(\frac{\partial\hat{\varepsilon}}{\partial\boldsymbol{E}} - \frac{1}{\varrho_0}\hat{\boldsymbol{T}}^{(2)}\right) : d\boldsymbol{E}\right] . \tag{7.70}$$

This is the *Gibbs relation* for a classical thermoelastic solid. If the constitutive equations for ε and $\boldsymbol{T}^{(2)}$ are known, it allows to determine the entropy η.

Moreover, the functions $\hat{\boldsymbol{T}}^{(2)}$, $\hat{\varepsilon}$ and $\hat{\eta}$ must satisfy constraints in (7.66). Since $(7.66)_3$ is satisfied identically, it remains to satisfy $(7.66)_{1,2}$. To accomplish it, we introduce the *Helmholtz free energy*

$$\psi := \varepsilon - T\eta = \hat{\psi}(\boldsymbol{E},T) , \tag{7.71}$$

and compute its differential

$$d\psi = \frac{\partial\hat{\psi}}{\partial\boldsymbol{E}} : d\boldsymbol{E} + \frac{\partial\hat{\psi}}{\partial T}dT . \tag{7.72}$$

Differentiating (7.71) with respect to $\boldsymbol{E}$ and T, respectively, and substituting the result to (7.72), we obtain

$$d\psi = \left(\frac{\partial\hat{\varepsilon}}{\partial\boldsymbol{E}} - T\frac{\partial\hat{\eta}}{\partial\boldsymbol{E}}\right) : d\boldsymbol{E} + \left(\frac{\partial\hat{\varepsilon}}{\partial T} - T\frac{\partial\hat{\eta}}{\partial T} - \hat{\eta}\right)dT , \tag{7.73}$$

which, in view of $(7.66)_{1,2}$, read as

$$d\psi = \frac{1}{\varrho_0}\hat{\boldsymbol{T}}^{(2)} : d\boldsymbol{E} - \hat{\eta} dT \ . \tag{7.74}$$

By comparing this with (7.72), we can deduce that

$$\boldsymbol{T}^{(2)} = \varrho_0 \frac{\partial\hat{\psi}}{\partial\boldsymbol{E}} \ , \qquad \eta = -\frac{\partial\hat{\psi}}{\partial T} \ . \tag{7.75}$$

The internal energy is also derivable from potential ψ by (7.71),

$$\varepsilon = \hat{\psi} - T\frac{\partial\hat{\psi}}{\partial T} \ . \tag{7.76}$$

The constraints $(7.66)_{1,2}$ are now satisfied identically, which demonstrates that ψ serves as a thermodynamic potential for the second Piola-Kirchhoff stress tensor, the entropy and the internal energy.

In thermodynamic equilibrium, which is defined as a process with $\text{Grad}\, T{=}0$, it is reasonable to assume that there can be no heat conduction, that is, $\vec{Q}|_{\text{E}} = 0$. This condition is satisfied provided that

$$\vec{Q} = -\boldsymbol{\kappa}\cdot \text{Grad}\, T \ , \tag{7.77}$$

where $\boldsymbol{\kappa}$ is the so-called *tensor of thermal conductivity*, which depends, in general, on $\boldsymbol{E}$, T and $\text{Grad}\, T$. The residual inequality (7.67) then becomes $\text{Grad}\, T\cdot\boldsymbol{\kappa}\cdot\text{Grad}\, T \geq 0$. Decomposing tensor $\boldsymbol{\kappa}$ into the symmetric and skew-symmetric parts, this inequality reduces to

$$\text{Grad}\, T\cdot \text{sym}(\boldsymbol{\kappa})\cdot \text{Grad}\, T \geq 0 \ , \tag{7.78}$$

which states that tensor $\text{sym}(\boldsymbol{\kappa})$ must be positive semi-definite. As far the skew-symmetric part of $\boldsymbol{\kappa}$ is concerned, nothing can be concluded from the entropy inequality. It is the statement of the so-called *Onsager reciprocity relations* that the skew-symmetric part of $\boldsymbol{\kappa}$ is equal to zero making $\boldsymbol{\kappa}$ symmetric. This assumption is extensively adopted.

We will now confine ourselves to a special type of classical thermoelastic solid for which the constitutive functions are *isotropic* and *linear* with respect to $\boldsymbol{E}$, T and $\text{Grad}\, T$. The isotropy requirement means that thermodynamic potential ψ is a function of invariants of $\boldsymbol{E}$ (and temperature T),

$$\psi = \hat{\psi}(\tilde{I}_E, \tilde{II}_E, \tilde{III}_E, T) \ , \tag{7.79}$$

where

$$\tilde{I}_E = \text{tr}\, \boldsymbol{E} \ , \qquad \tilde{II}_E = \text{tr}\, \boldsymbol{E}^2 \ , \qquad \tilde{III}_E = \text{tr}\, \boldsymbol{E}^3 \ . \tag{7.80}$$

To obtain the stress and entropy constitutive equations which are linear and uncoupled in $\boldsymbol{E}$ and T, function $\hat{\psi}$ can, at most, include polynomials of second degree in $\boldsymbol{E}$ and T, that is,

$$\varrho_0\hat{\psi} = \varrho_0\psi_0 + \pi\, \tilde{I}_E + \frac{1}{2}\lambda\left(\tilde{I}_E\right)^2 + \mu\, \tilde{II}_E - \beta T\, \tilde{I}_E - \varrho_0\eta_0 T - \frac{1}{2}\gamma T^2 \ , \tag{7.81}$$

where ψ_0, π, β, λ, μ, η_0 and γ are constants independent of $\boldsymbol{E}$ and T. The third-order terms,

$$T^3,\ T^2\tilde{I}_E,\ T\,\tilde{I}_E^2,\ T\,\tilde{II}_E,\ \tilde{III}_E,$$

and higher-order terms would generate non-linear terms in the constitutive equations and are therefore omitted from $\hat{\psi}$ in linear case. Substituting (7.81) into (7.75) and using the relations

$$\frac{\partial \tilde{I}_E}{\partial \boldsymbol{E}} = \boldsymbol{I} \,, \qquad \frac{\partial \tilde{II}_E}{\partial \boldsymbol{E}} = 2\boldsymbol{E} \,, \tag{7.82}$$

we obtain the constitutive equations for $\boldsymbol{T}^{(2)}$ and η:

$$\boldsymbol{T}^{(2)} = (\pi + \lambda \mathrm{tr}\, \boldsymbol{E} - \beta T)\boldsymbol{I} + 2\mu \boldsymbol{E} \,, \qquad \eta = \eta_0 + \frac{\beta}{\varrho_0} \mathrm{tr}\, \boldsymbol{E} + \frac{\gamma}{\varrho_0} T \,. \tag{7.83}$$

The constitutive equation for the internal energy can be derived by using (7.76):

$$\varrho_0 \varepsilon = \varrho_0 \psi_0 + \pi\, \tilde{I}_E + \frac{1}{2}\lambda \left(\tilde{I}_E\right)^2 + \mu\, \tilde{II}_E + \frac{1}{2}\gamma T^2 \,. \tag{7.84}$$

Moreover, comparing $(6.165)_2$ with (7.77) we can deduce that for a linear, isotropic thermoelastic solid the tensor of heat conduction must be a spherical tensor, that is, $\boldsymbol{\kappa} = \kappa \boldsymbol{I}$. Hence, (7.77) reduces to

$$\vec{Q} = -\kappa\, \mathrm{Grad}\, T \,, \tag{7.85}$$

where the non-negative thermal conductivity κ,

$$\kappa \geq 0 \,, \tag{7.86}$$

does not depend on $\boldsymbol{E}$, T and $\mathrm{Grad}\, T$. We can conclude that equations (7.83) and (7.84) form the constitutive equations for a linear isotropic classical thermoelastic solid. The inequality (7.86) expresses the fact that heat flows from hot to cold in such a solid.

7.5 The Müller entropy principle

Even of a broad use of the Clausius-Duhem inequality, this contains certain limitations that might be violated in some physical situations. Recall that it comes out from

- special choice (7.2) of the entropy flux $\vec{s}$ and the entropy source b,
- the necessity to postulate the existence of the absolute temperature,
- simplified treating of the balance law for mass, linear momentum and energy as additional constraints when the entropy inequality is applied (see the discussion after equation (7.3)).

We will be now dealing with a modern concept of the entropy inequality, formulated by Müller (1968), that imposes a different type of additional information to the entropy inequality. The Müller entropy principle also comes out from the second law of thermodynamics (7.1) and, in addition, assumes:

- The specific entropy η and entropy flux $\vec{s}$ are frame indifferent scalar and vector, respectively.
- η and $\vec{s}$ are material quantities for which, according to the principle of equipresence, the constitutive equations depend on the same variables as the constitutive equations for $\boldsymbol{t}$, $\vec{q}$ and ϵ.

- The source terms occurring in the balance equations do not influence the material behaviour. In particular, the entropy source b vanishes if there are no internal body forces, $\vec{f} = \vec{0}$, and no internal energy sources, $h = 0$. This requirement is, in particular, satisfied if the entropy source is linearly proportional to the body force and the heat source,

$$b = \vec{\lambda}^f \cdot \vec{f} + \lambda^h h \ , \tag{7.87}$$

where $\vec{\lambda}^f$ and λ^h are independent of $\vec{f}$ and h, respectively. Note that the entropy source in the Clausius-Duhem inequality was postulated as $b = h/T$, where T is the absolute temperature. This is obviously a special case of (7.87).
- There are impenetrable thin walls across which the empirical temperature and the normal component of the entropy flux are continuous,

$$[\theta]_-^+ = 0 \ , \qquad [\vec{n} \cdot \vec{s}]_-^+ = 0 \ , \tag{7.88}$$

meaning that there are no long-range temperature interactions between two materials separated by the wall and the entropy does not concentrate on the wall.

In view of (7.87), the entropy inequality (7.1) reads

$$\varrho \stackrel{\bullet}{\eta} + \operatorname{div} \vec{s} - \varrho(\vec{\lambda}^f \cdot \vec{f} + \lambda^h h) \geq 0 \ . \tag{7.89}$$

This inequality must be satisfied for all possible thermodynamic processes that are solutions of the balance laws for mass, linear momentum and energy. In contrast to the Clausius-Duhem inequality, the Müller entropy principle considers all these balance laws as additional restrictions to the entropy inequality (7.89). To satisfy inequality (7.89) subject to the constraints (4.37), (4.39) and (4.42), we create a new unconstrained inequality

$$\varrho \stackrel{\bullet}{\eta} + \operatorname{div} \vec{s} - \varrho(\vec{\lambda}^f \cdot \vec{f} + \lambda^h h) - \lambda^\varrho(\stackrel{\bullet}{\varrho} + \varrho \operatorname{div} \vec{v}) - \vec{\lambda}^v \cdot (\varrho \stackrel{\bullet}{\vec{v}} - \operatorname{div} \boldsymbol{t} - \varrho \vec{f}) - \lambda^\varepsilon(\varrho \stackrel{\bullet}{\varepsilon} - \boldsymbol{t} : \boldsymbol{d} + \operatorname{div} \vec{q} - \varrho h) \geq 0 \ , \tag{7.90}$$

which must be satisfied for all densities ϱ, motions $\vec{x}$ and temperatures θ, and also for all Lagrange multipliers $\vec{\lambda}^f$, λ^h, λ^ϱ, $\vec{\lambda}^v$ and λ^ε without any additional constraints.

It seems reasonable to think that all solutions of the balance laws for mass, linear momentum and energy which satisfy the original entropy inequality (7.1) also satisfy the extended entropy inequality (7.90). The inverse statement is also true, which was proved by Liu (1972). The Liu's theorem states that both statements: (i) Satisfy the extended entropy inequality (7.90) for unconstrained fields, and (ii) satisfy simultaneously both the original entropy inequality (7.1) and the field equations for mass, linear momentum and energy are equivalent. It is easy, but laborious to fulfill the extended inequality by determining the unknown Lagrange multipliers.

7.6 Application of the Müller entropy principle to a classical thermoelastic fluid

In section 6.18.6 we introduced a classical thermoelastic fluid as the material with the constitutive variables ϱ, θ and $\operatorname{grad} \theta$:

$$\boldsymbol{t} = \hat{\boldsymbol{t}}(\varrho, \theta, \operatorname{grad} \theta) \ , \qquad \vec{q} = \hat{\vec{q}}(\varrho, \theta, \operatorname{grad} \theta) \ , \qquad \varepsilon = \hat{\varepsilon}(\varrho, \theta, \operatorname{grad} \theta) \ . \tag{7.91}$$

According to the principle of equipresence, we choose the same set of constitutive variables for the entropy η and the entropy flux $\vec{s}$:

$$\eta = \hat{\eta}(\varrho, \theta, \operatorname{grad}\theta) \ , \qquad \vec{s} = \hat{\vec{s}}(\varrho, \theta, \operatorname{grad}\theta) \ . \tag{7.92}$$

Note that, in contrast to the Clausius-Duhem inequality, the existence of the absolute temperature T as a measure of the empirical temperature θ is not postulated in the Müller entropy principle. The reduced form of the constitutive equations (7.91) and (7.92) can be obtained from (6.175):

$$\begin{aligned} \boldsymbol{t} &= \sigma \boldsymbol{I} + \tau \operatorname{grad}\theta \otimes \operatorname{grad}\theta \ , & \eta &= \hat{\eta}(\varrho, \theta, g) \ , \\ \vec{q} &= -\kappa \operatorname{grad}\theta \ , & \vec{s} &= -\gamma \operatorname{grad}\theta \ , \\ \varepsilon &= \hat{\varepsilon}(\varrho, \theta, g) \ , & \sigma, \tau, \kappa, \gamma &= \sigma, \tau, \kappa, \gamma(\varrho, \theta, g) \ , \end{aligned} \tag{7.93}$$

where, for the sake of brevity, we introduced $g := \operatorname{grad}\theta \cdot \operatorname{grad}\theta$. The reduced constitutive equations for η and $\vec{s}$ which are not included in (6.175) are set up analogously to that of ε and $\vec{q}$ since we assume that η and $\vec{s}$ are a frame indifferent scalar and vector, respectively.

When the constitutive equations (7.91) and (7.92) are brought into (7.90) and the time differentiation is carried out according to the chain rule of differentiation, we have

$$\begin{aligned} &\varrho\left(\frac{\partial\hat{\eta}}{\partial\varrho}\,\dot{\varrho} + \frac{\partial\hat{\eta}}{\partial\theta}\,\dot{\theta} + \frac{\partial\hat{\eta}}{\partial\theta_{,i}}\,\dot{\theta}_{,i}\right) + \frac{\partial\hat{s}_i}{\partial\varrho}\varrho_{,i} + \frac{\partial\hat{s}_i}{\partial\theta}\theta_{,i} + \frac{\partial\hat{s}_i}{\partial\theta_{,j}}\theta_{,ji} - \varrho(\lambda_i^f f_i + \lambda^h h) - \lambda^\varrho(\dot{\varrho} + \varrho v_{i,i}) \\ &\qquad -\lambda_i^v \varrho\,\dot{v}_i + \lambda_i^v\left(\frac{\partial\hat{t}_{ij}}{\partial\varrho}\varrho_{,j} + \frac{\partial\hat{t}_{ij}}{\partial\theta}\theta_{,j} + \frac{\partial\hat{t}_{ij}}{\partial\theta_{,k}}\theta_{,kj}\right) + \lambda_i^v \varrho f_i \\ &-\lambda^\varepsilon\varrho\left(\frac{\partial\hat{\varepsilon}}{\partial\varrho}\,\dot{\varrho} + \frac{\partial\hat{\varepsilon}}{\partial\theta}\,\dot{\theta} + \frac{\partial\hat{\varepsilon}}{\partial\theta_{,i}}\,\dot{\theta}_{,i}\right) + \lambda^\varepsilon\hat{t}_{ij}v_{j,i} - \lambda^\varepsilon\left(\frac{\partial\hat{q}_i}{\partial\varrho}\varrho_{,i} + \frac{\partial\hat{q}_i}{\partial\theta}\theta_{,i} + \frac{\partial\hat{q}_i}{\partial\theta_{,j}}\theta_{,ji}\right) + \lambda^\varepsilon\varrho h \geq 0 \ , \end{aligned} \tag{7.94}$$

which can be rearranged to the form

$$\begin{aligned} &\left(\varrho\frac{\partial\hat{\eta}}{\partial\varrho} - \lambda^\varrho - \varrho\lambda^\varepsilon\frac{\partial\hat{\varepsilon}}{\partial\varrho}\right)\dot{\varrho} + \left(\varrho\frac{\partial\hat{\eta}}{\partial\theta} - \varrho\lambda^\varepsilon\frac{\partial\hat{\varepsilon}}{\partial\theta}\right)\dot{\theta} + \left(\varrho\frac{\partial\hat{\eta}}{\partial\theta_{,i}} - \varrho\lambda^\varepsilon\frac{\partial\hat{\varepsilon}}{\partial\theta_{,i}}\right)\dot{\theta}_{,i} - (\varrho\lambda_i^v)\,\dot{v}_i \\ &\qquad + \left(\frac{\partial\hat{s}_j}{\partial\varrho} + \lambda_i^v\frac{\partial\hat{t}_{ij}}{\partial\varrho} - \lambda^\varepsilon\frac{\partial\hat{q}_j}{\partial\varrho}\right)\varrho_{,j} + \left(\frac{\partial\hat{s}_j}{\partial\theta_{,k}} + \lambda_i^v\frac{\partial\hat{t}_{ij}}{\partial\theta_{,k}} - \lambda^\varepsilon\frac{\partial\hat{q}_j}{\partial\theta_{,k}}\right)\theta_{,kj} \\ &+ \left(-\varrho\lambda^\varrho\delta_{ij} + \lambda^\varepsilon\hat{t}_{ij}\right)v_{j,i} + \left(\lambda_i^v - \lambda_i^f\right)\varrho f_i + \left(\lambda^\varepsilon - \lambda^h\right)\varrho h + \left(\frac{\partial\hat{s}_j}{\partial\theta} + \lambda_i^v\frac{\partial\hat{t}_{ij}}{\partial\theta} - \lambda^\varepsilon\frac{\partial\hat{q}_j}{\partial\theta}\right)\theta_{,j} \geq 0 \ . \end{aligned} \tag{7.95}$$

According the third postulate of the entropy principle, that is assuming that the material properties can not be influenced by the source terms, the Lagrange multipliers λ^ϱ, λ_i^v and λ^ε are independent of b, $\vec{f}$ and h. Moreover, according to Liu's theorem, these Lagrange multipliers are only dependent on the independent constitutive variables ϱ, θ and $\operatorname{grad}\theta$. Hence, the above inequality is linear with respect to $\dot{\varrho}$, $\dot{\theta}$, $\dot{\theta}_{,i}$, $\dot{v}_i$, $\varrho_{,j}$, $\theta_{,kj}$, $v_{j,i}$, f_i and h, since these variables are not contained in the set of constitutive variables, but it is non-linear with respect to $\theta_{,j}$, since $\theta_{,j}$ is the constitutive variable and the factor at the last term in (7.95) implicitly contains $\theta_{,j}$. To maintain this inequality for all values of the variables which enter (7.95) linearly, the factors at these terms must vanish. This yields

$$\begin{aligned}
\lambda_i^v &= 0 \,, & \mathrm{sym}\left(\frac{\partial \hat{s}_j}{\partial \theta_{,k}} - \lambda^\varepsilon \frac{\partial \hat{q}_j}{\partial \theta_{,k}}\right) &= 0 \,, \\
\lambda^\varrho &= \varrho\left(\frac{\partial \hat{\eta}}{\partial \varrho} - \lambda^\varepsilon \frac{\partial \hat{\varepsilon}}{\partial \varrho}\right) \,, & \lambda^\varepsilon \hat{t}_{ij} &= \varrho \lambda^\varrho \delta_{ij} \,, \\
\frac{\partial \hat{\eta}}{\partial \theta} &= \lambda^\varepsilon \frac{\partial \hat{\varepsilon}}{\partial \theta} \,, & \lambda^h &= \lambda^\varepsilon \,, \\
\frac{\partial \hat{\eta}}{\partial \theta_{,i}} &= \lambda^\varepsilon \frac{\partial \hat{\varepsilon}}{\partial \theta_{,i}} \,, & \lambda_i^f &= \lambda_i^v = 0 \,, \\
\frac{\partial \hat{s}_j}{\partial \varrho} &= \lambda^\varepsilon \frac{\partial \hat{q}_j}{\partial \varrho} \,,
\end{aligned} \tag{7.96}$$

which must be satisfied identically. These relations constrain the constitutive equations for $\hat{\boldsymbol{t}}$, $\hat{\vec{q}}$, $\hat{\varepsilon}$, $\hat{\eta}$ and $\hat{\vec{s}}$, but they can also be viewed as determining equations for λ^ϱ, λ_i^v and λ^ε. The last interpretation implies that the Lagrange multipliers are determined by the constitutive quantities, hence they can only be dependent on the independent constitutive variables ϱ, θ, and $\mathrm{grad}\,\theta$, and cannot be dependent on $\overset{\bullet}{v}_i$. Consequently, inequality (7.95) is also linear in $\overset{\bullet}{v}_i$ which implies that the Lagrange multiplier $\lambda_i^v = 0$, which has been already considered in (7.96). By this, the inequality (7.95) reduces to

$$\left(\frac{\partial \hat{s}_j}{\partial \theta} - \lambda^\varepsilon \frac{\partial \hat{q}_j}{\partial \theta}\right) \theta_{,j} \geq 0 \,. \tag{7.97}$$

Equations $(7.96)_{8,9}$ imply that the entropy source in a classical thermoelastic fluid is of the form

$$b = \lambda^\varepsilon h \,, \tag{7.98}$$

which also means that the body force $\vec{f}$ in linear model (7.87) does not contribute to the internal entropy sources.

In the next step, we determine the Lagrange multiplier λ^ε. The relation $(7.96)_6$ together with $(7.93)_{2,5}$, represented in indicial notation, $s_j = -\gamma \theta_{,j}$ and $q_j = -\kappa \theta_{,j}$, can be arranged as follows:

$$(\gamma - \lambda^\varepsilon \kappa)\delta_{jk} + \mathrm{sym}\left(\frac{\partial \gamma}{\partial \theta_{,k}} - \lambda^\varepsilon \frac{\partial \kappa}{\partial \theta_{,k}}\right) \theta_{,j} = 0 \,.$$

This, along with

$$\frac{\partial \kappa}{\partial \theta_{,k}} = \frac{\partial \kappa}{\partial g} \frac{\partial g}{\partial \theta_{,k}} = 2 \frac{\partial \kappa}{\partial g} \theta_{,k} \,,$$

results in

$$(\gamma - \lambda^\varepsilon \kappa)\delta_{jk} + 2\left(\frac{\partial \gamma}{\partial g} - \lambda^\varepsilon \frac{\partial \kappa}{\partial g}\right) \theta_{,j}\theta_{,k} = 0 \,.$$

This equation is satisfied if both the diagonal and off-diagonal elements vanish,

$$\gamma - \lambda^\varepsilon \kappa + 2\left(\frac{\partial \gamma}{\partial g} - \lambda^\varepsilon \frac{\partial \kappa}{\partial g}\right) \theta_{,j}\theta_{,j} = 0 \,, \qquad \frac{\partial \gamma}{\partial g} - \lambda^\varepsilon \frac{\partial \kappa}{\partial g} = 0 \,,$$

which can be simplied as

$$\gamma = \lambda^\varepsilon \kappa \,, \qquad \frac{\partial \gamma}{\partial g} = \lambda^\varepsilon \frac{\partial \kappa}{\partial g} \,. \tag{7.99}$$

Upon differentiating the first equation with respect to g, we obtain

$$\frac{\partial \gamma}{\partial g} = \lambda^{\varepsilon} \frac{\partial \kappa}{\partial g} + \frac{\partial \lambda^{\varepsilon}}{\partial g} \kappa \ . \tag{7.100}$$

Since we assume that $\kappa \neq 0$ (otherwise our fluid would not conduct heat), we equate (7.100) with $(7.99)_2$ and obtain

$$\frac{\partial \lambda^{\varepsilon}}{\partial g} = 0 \ . \tag{7.101}$$

Substituting $(7.99)_1$ into the reduced constitutive equations $(7.93)_{2,5}$ for $\vec{q}$ and $\vec{s}$, we have

$$\vec{s} = \lambda^{\varepsilon} \vec{q} \ . \tag{7.102}$$

The entropy flux is thus collinear with the heat flux, whereby the proportionality factor is given by the Lagrange multiplier of the energy equation. Considering this result in $(7.96)_5$ yields

$$\frac{\partial \lambda^{\varepsilon}}{\partial \varrho} = 0 \ . \tag{7.103}$$

Equation $(7.96)_3$ can formally be understood as the determining equation for λ^{ε}. Since the factors occurring in this equation are functions of the three constitutive variables ϱ, θ and g, also λ^{ε} can only depend on these variables, $\lambda^{\varepsilon} = \lambda^{\varepsilon}(\varrho, \theta, g)$. Moreover, because of (7.101) and (7.103), λ^{ε} is only θ-dependent and we employ λ^{ε} for the definition of *absolute temperature* $T(\theta)$ as follows

$$\lambda^{\varepsilon} = \lambda^{\varepsilon}(\theta) =: \frac{1}{T(\theta)} \ . \tag{7.104}$$

This results approaches the Clausius-Duhem assumption very closely, however, λ^{ε} is a still materially dependent function of the empirical temperature.

Differentiating $(7.96)_3$ with respect to $\operatorname{grad}\theta$ and $(7.96)_4$ with respect to θ, exchanging the order of partial derivatives of $\hat{\eta}$ and $\hat{\varepsilon}$ with respect to $\operatorname{grad}\theta$ and θ, and equating the results, we obtain

$$\frac{1}{T(\theta)} \frac{\partial^2 \hat{\varepsilon}}{\partial(\operatorname{grad}\theta)\, \partial\theta} = \frac{1}{T(\theta)} \frac{\partial^2 \hat{\varepsilon}}{\partial\theta\, \partial(\operatorname{grad}\theta)} - \frac{1}{T^2(\theta)} \frac{dT(\theta)}{d\theta} \frac{\partial \hat{\varepsilon}}{\partial(\operatorname{grad}\theta)} \ , \tag{7.105}$$

which shows that

$$\frac{\partial \hat{\varepsilon}}{\partial(\operatorname{grad}\theta)} = 0 \ . \tag{7.106}$$

In view of $(7.96)_4$, we also have

$$\frac{\partial \hat{\eta}}{\partial(\operatorname{grad}\theta)} = 0 \ . \tag{7.107}$$

Consequently, the internal energy and the entropy are functions of ϱ and θ only. In addition, by $(7.96)_2$, the same holds for the Lagrange multiplier λ^{ϱ}, hence

$$\varepsilon = \hat{\varepsilon}(\varrho, \theta) \ , \qquad \eta = \hat{\eta}(\varrho, \theta) \ , \qquad \lambda^{\varrho} = \hat{\lambda}^{\varrho}(\varrho, \theta) \ . \tag{7.108}$$

Equation $(7.96)_7$ yields the representation of the stress tensor in the form

$$\boldsymbol{t} = -p(\varrho, \theta) \boldsymbol{I} \ , \tag{7.109}$$

where p is referred to as the *thermodynamic pressure*,

$$\begin{aligned} p(\varrho,\theta) &:= -\frac{\lambda^{\varrho}(\varrho,\theta)}{\lambda^{\varepsilon}(\varrho)}\,\varrho \\ &= -\varrho^2\left(T(\theta)\frac{\partial\hat{\eta}}{\partial\varrho}-\frac{\partial\hat{\varepsilon}}{\partial\varrho}\right) . \end{aligned} \tag{7.110}$$

By equating (7.109) with the constitutive equation $(7.93)_1$, we obtain

$$\sigma(\varrho,\theta,g) = -p(\varrho,\theta) \;, \qquad \tau(\varrho,\theta,g) = 0 \;. \tag{7.111}$$

Equation (7.110) can be solved for $\partial\hat{\eta}/\partial\varrho$:

$$\frac{\partial\hat{\eta}}{\partial\varrho} = \frac{1}{T(\theta)}\left(\frac{\partial\hat{\varepsilon}}{\partial\varrho}-\frac{p}{\varrho^2}\right) . \tag{7.112}$$

With the help of this and $(7.96)_3$, we can express the total differential of the entropy $\eta = \hat{\eta}(\varrho,\theta)$ in the form

$$d\eta = \frac{1}{T(\theta)}\left(\frac{\partial\hat{\varepsilon}}{\partial\theta}d\theta + \Big(\frac{\partial\hat{\varepsilon}}{\partial\varrho}-\frac{p}{\varrho^2}\Big)d\varrho\right) , \tag{7.113}$$

which is known as the *Gibbs relation*.

An integral $\int d\eta$ is assumed to be path-independent, which means that $d\eta$ must be a total differential. Hence, the integrability condition for the Gibbs relation (7.113) must hold in the form

$$\frac{1}{T(\theta)}\frac{\partial^2\hat{\varepsilon}}{\partial\varrho\,\partial\theta} = \frac{1}{T(\theta)}\left(\frac{\partial^2\hat{\varepsilon}}{\partial\theta\,\partial\varrho}-\frac{1}{\varrho^2}\frac{\partial p}{\partial\theta}\right) - \frac{1}{T^2(\theta)}\frac{dT(\theta)}{d\theta}\left(\frac{\partial\hat{\varepsilon}}{\partial\varrho}-\frac{p}{\varrho^2}\right) .$$

This equation can be solved for the derivative of $T(\theta)$,

$$\frac{d\ln T(\theta)}{d\theta} = \frac{-\dfrac{\partial p}{\partial\theta}}{\varrho^2\dfrac{\partial\hat{\varepsilon}}{\partial\varrho}-p} .$$

and integrating with respect to θ:

$$T(\theta) = T(\theta_0)\exp\left\{\int\limits_{\theta_0}^{\theta}\frac{-\dfrac{\partial p}{\partial\theta}}{\varrho^2\dfrac{\partial\hat{\varepsilon}}{\partial\varrho}-p}d\theta\right\} . \tag{7.114}$$

Choosing integration constant $T(\theta_0)$ positive, function $T(\theta)$ is a positive-valued function, as expected for an absolute temperature. To show that $T(\theta)$ is a meaningful absolute temperature, we need to prove that (i) $T(\theta)$ is a *universal* function of θ, that is, it is not different for two different materials, and (ii) $T(\theta)$ is a strict *monotonic* function of θ, that is, $T_A > T_B$ means that "A is warmer than B".

Universality

We now employ the postulate of the existence of impenetrable thin walls with the property (7.88). Let us consider two different classical thermoelastic fluids that are separated by an impenetrable thin wall. Let fluid I be placed on the positive side and fluid II on the negative side of the wall. In view of (7.102) and (7.104), the continuity condition $(7.88)_2$ of the normal component of the entropy flux across the wall becomes

$$\left[\frac{1}{T(\theta)}\vec{n}\cdot\vec{q}\right]_-^+ = 0\ . \tag{7.115}$$

By an impenetrable wall we mean that no portion of matter from one side can penetrate into the matter on the other side, that is, $\vec{n}\cdot(\,[\vec{v}]_-^+ - \vec{\nu}) = 0$. In the terminology of section 4.3, an impenetrable thin wall is a material discontinuity surface. Moreover, we assume that there is no tangential slip across the wall, so that, $\left[\vec{v}_\|\right]_-^+ = \vec{0}$, where $\vec{v}_\| := (\boldsymbol{I} - \vec{n}\otimes\vec{n})\cdot\vec{v}$. According to (4.50), the normal component of the heat flux is continuous at a welded material surface, that is $\vec{n}\cdot[\vec{q}]_-^+ = 0$, which implies that the interface condition (7.115) can be rewritten as

$$\left[\frac{1}{T(\theta)}\right]_-^+ \vec{n}\cdot\vec{q} = 0\ . \tag{7.116}$$

Since, in general, $\vec{n}\cdot\vec{q} \neq 0$, we finally obtain

$$\left[\frac{1}{T(\theta)}\right]_-^+ = 0\ , \qquad \text{or} \qquad T^{\mathrm{I}}(\theta) = T^{\mathrm{II}}(\theta)\ . \tag{7.117}$$

We can conclude that $T(\theta)$ is the same function of empirical temperature on both sides of the ideal wall. Since the fluids on both sides of the ideal wall can be arbitrary classical thermoelastic fluids, then it means that $T(\theta)$ is material independent within this class of materials. The same property of the function $T(\theta)$ can be proved for other materials. Hence, $T(\theta)$ is a material independent, universal function of the empirical temperature θ.

Monotony

First, we define the *ideal gas* as a classical thermoelastic fluid with the following thermal and caloric equations of state:

$$p(\varrho,\theta) = \frac{\varrho}{\mu}f(\theta)\ , \qquad \varepsilon(\varrho,\theta) \equiv \varepsilon(\theta) = \frac{\beta_1}{\mu}f(\theta) + \beta_2\ , \tag{7.118}$$

where a positive-valued, strict monotonic function $f(\theta)$, which is universal for all ideal gases, only depends on the choice of a measure of empirical temperature; μ is the moll-mass, and β_1 and β_2 are characteristic constants of a particular ideal gas. For instance, in the Celsius scale of temperature, we have

$$f(\theta) = R(\theta + \theta_m)\ , \tag{7.119}$$

where the *universal gas constant* R=8.314 J/(mol ^{o}C), and $\theta_m = 273.15^o$C. In this case, the equations of state are of the form

$$p = \varrho\frac{R}{\mu}(\theta + \theta_m)\ , \qquad \varepsilon = \beta_1\frac{R}{\mu}(\theta + \theta_m) + \beta_2\ . \tag{7.120}$$

With the help of (7.114), we calculate function $T(\theta)$ for the ideal gas:

$$T(\theta) = T(\theta_0)\exp\left\{\int_{\theta_0}^{\theta} \frac{-\dfrac{\varrho}{\mu}\dfrac{df(\theta)}{d\theta}}{-\dfrac{\varrho}{\mu}f(\theta)}d\theta\right\} = T(\theta_0)\exp\left\{\int_{\theta_0}^{\theta}\frac{d\ln f(\theta)}{d\theta}d\theta\right\} = T(\theta_0)\exp\left\{\ln\frac{f(\theta)}{f(\theta_0)}\right\} .$$

Hence,

$$T(\theta) = \frac{T(\theta_0)}{f(\theta_0)}f(\theta) \ . \tag{7.121}$$

Choosing integration constant $T(\theta_0)$ positive, $T(\theta)$ has the same property as function $f(\theta)$, that is, $T(\theta)$ is a positive-valued, strict monotonic function. Because of the universality of $T(\theta)$, this result holds not only for ideal gases but in general. Thus the function $T(\theta)$ satisfies both requirements of a meaningfully defined absolute temperature.

Let us have a look at the Celsius scale as a measure of empirical temperature. Putting $\theta_0 = 0^o$C and choosing the integration constant as

$$T(\theta_0) = T(0^o\mathrm{C}) = 273.15\,\mathrm{K} \ , \tag{7.122}$$

equation (7.121) along with (7.119) yields

$$T(\theta) = \frac{T(\theta_0)}{R\theta_m}R(\theta+\theta_m) = \frac{273.15\,\mathrm{K}}{273.15^o\mathrm{C}}(\theta + 273.15^o\mathrm{C}) \ ,$$

that is,

$$T(\theta) = 273.15\,\mathrm{K} + 1\,\frac{\mathrm{K}}{^o\mathrm{C}}\,\theta \ . \tag{7.123}$$

This measure of the absolute temperature is called the *Kelvin scale.* [25] Using it, the empirical temperature θ can be replaced by the absolute temperature T,

To complete the exploitation of the Müller entropy principle for a classical thermoelastic fluid, it remains to evaluate the reduced inequality (7.97). With $\vec{s} = \vec{q}/T(\theta)$, see (7.102) and (7.104), we have

$$\left(\frac{1}{T(\theta)}\frac{\partial\vec{q}}{\partial\theta} - \frac{1}{T^2(\theta)}\frac{dT(\theta)}{d\theta}\vec{q} - \frac{1}{T(\theta)}\frac{\partial\vec{q}}{\partial\theta}\right)\cdot \operatorname{grad}\theta \geq 0 \ ,$$

which reduces to

$$-\vec{q}\cdot\operatorname{grad}\theta \geq 0 \ .$$

Substituting for $\vec{q}$ from the constitutive equation $(7.93)_2$, we can see that the thermal conductivity is non-negative,

$$\kappa \geq 0 \ . \tag{7.124}$$

Let us summarize the results following from the application of the Müller entropy principle to a classical thermoelastic fluid.

- For the empirical temperature θ, it is possible to construct a positive-valued, strict monotonic and universal function $T(\theta)$,

$$T(\theta) = \frac{T(\theta_0)}{f(\theta_0)}f(\theta) \ , \tag{7.125}$$

which may serve as a measure of the absolute temperature.

[25] This relation was suggested by lord Kelvin.

- The constitutive equations (7.93) reduce to

$$\begin{aligned} \boldsymbol{t} &= -p\boldsymbol{I} , & \eta &= \hat{\eta}(\varrho,\theta) , \\ \vec{q} &= -\kappa \,\mathrm{grad}\, \theta , & \vec{s} &= \frac{\vec{q}}{T(\theta)} , \\ \varepsilon &= \hat{\varepsilon}(\varrho,\theta) , & p &= p(\varrho,\theta) , \qquad \kappa = \kappa(\varrho,\theta,g) . \end{aligned} \tag{7.126}$$

- Functions p, ε, η and T are mutually related:

$$\begin{aligned} \frac{\partial \hat{\eta}}{\partial \theta} &= \frac{1}{T(\theta)} \frac{\partial \hat{\varepsilon}}{\partial \theta} , \\ \frac{\partial \hat{\eta}}{\partial \varrho} &= \frac{1}{T(\theta)} \left(\frac{\partial \hat{\varepsilon}}{\partial \varrho} - \frac{p}{\varrho^2} \right) , \\ \frac{d \ln T(\theta)}{d\theta} &= \frac{-\dfrac{\partial p}{\partial \theta}}{\varrho^2 \dfrac{\partial \hat{\varepsilon}}{\partial \varrho} - p} . \end{aligned} \tag{7.127}$$

- The entropy source b and the heat source h are related as

$$b = \frac{h}{T(\theta)} . \tag{7.128}$$

Aside the possibility (7.125) to construct the universal absolute temperature T for a given empirical temperature θ, there is a particular choice to put

$$T = \theta . \tag{7.129}$$

Under this choice, the results of the Müller entropy principle for a classical thermoelastic fluid coincide with those of the Clausius-Duhem inequality. In particular, the relations $\vec{s} = \vec{q}/T$ and $b = h/T$, postulated by the Clausius-Duhem inequality, follow from the Müller entropy principle as the results $(7.126)_5$ and (7.128). However, these two principles do not, in general, provide equivalent restrictions on material properties. For instance, for a non-classical heat-conducting fluid (for which the constitutive equations also depend on $\dot{\theta}$), the Clausius-Duhem inequality yields that $\tau(\varrho, \theta, \dot{\theta}, g) \equiv 0$ in the stress constitutive equation, an analog to the classical case, while the Müller entropy principle allows non-vanishing τ of the stress tensor. Hence, the Clausius-Duhem inequality imposes stronger restrictions on constitutive equations than those resulting from the application of the Müller entropy principle.

8. CLASSICAL LINEAR ELASTICITY

8.1 Linear elastic solid

In section 7.4 we found that the most general constitutive equations of a classical thermoelastic solid have the forms expressed by equations (7.75)–(7.77). If thermal effects are not considered, the Helmholtz free energy and the second Piola-Kirchhoff stress tensor are functions of strain alone. According to section 6.17.1, this solid is called *elastic*. For an elastic solid, the constitutive equation $(7.75)_1$ reduces to

$$\boldsymbol{T}^{(2)} = \frac{\partial W}{\partial \boldsymbol{E}} \ , \tag{8.1}$$

where the so-called *elastic strain energy density* (strain energy per unit undeformed volume) is defined by

$$W := \varrho_0 \psi = \hat{W}(\boldsymbol{E}) \ . \tag{8.2}$$

It is worthwhile noting that elastic behavior is sometimes defined on the basis of the existence of a strain energy function from which the stress may be determined by the differentiation in (8.1). A material defined in this way is called a *hyperelastic* material. The stress is still a unique function of strain so that this "energy approach" is compatible with our earlier definition of elastic behavior in section 6.17.1.

By using the constitutive equation (8.1) of nonlinear elastic solid, we can derive various approximation theories. Expanding W about the configuration κ_0 from which the strain $\boldsymbol{E}$ is reckoned, we have

$$W(\boldsymbol{E}) = W(\boldsymbol{0}) + \frac{\partial W(\boldsymbol{0})}{\partial \boldsymbol{E}} : \boldsymbol{E} + \frac{1}{2}\boldsymbol{E} : \frac{\partial^2 W(\boldsymbol{0})}{\partial \boldsymbol{E}\,\partial \boldsymbol{E}} : \boldsymbol{E} + O(|\boldsymbol{E}|^3) \ , \tag{8.3}$$

and, from (8.1),

$$\boldsymbol{T}^{(2)} = \frac{\partial W(\boldsymbol{0})}{\partial \boldsymbol{E}} + \frac{\partial^2 W(\boldsymbol{0})}{\partial \boldsymbol{E}\,\partial \boldsymbol{E}} : \boldsymbol{E} + O(|\boldsymbol{E}|^2) \ . \tag{8.4}$$

Using different notation, we can shortly write

$$\boldsymbol{T}^{(2)} = \boldsymbol{T}_0 + \boldsymbol{C} : \boldsymbol{E} + O(|\boldsymbol{E}|^2) \ , \tag{8.5}$$

where $\boldsymbol{T}_0$ is the stress in the configuration κ_0 from which the strain $\boldsymbol{E}$ is reckoned,

$$\boldsymbol{T}_0 := \frac{\partial W(\boldsymbol{0})}{\partial \boldsymbol{E}} \ . \tag{8.6}$$

In the classical linear theory, the configuration κ_0 is used as reference and the stress $\boldsymbol{T}_0$ is considered as a tensor-valued function of the Lagrangian coordinates, $\boldsymbol{T}_0 = \boldsymbol{T}_0(\vec{X})$. The fourth-order tensor $\boldsymbol{C}$, introduced in (8.5), is defined by

$$\boldsymbol{C} := \frac{\partial^2 W(\boldsymbol{0})}{\partial \boldsymbol{E}\,\partial \boldsymbol{E}} \ , \qquad C_{KLMN} := \frac{\partial^2 W(\boldsymbol{0})}{\partial E_{KL}\,\partial E_{MN}} \ . \tag{8.7}$$

Due to the symmetry of both the stress and strain tensors, it is clear that

$$C_{KLMN} = C_{LKMN} = C_{KLNM} \ , \tag{8.8}$$

which reduces the $3^4 = 81$ components of $\boldsymbol{C}$ to 36 distinct coefficients C_{KLMN} at most. Moreover, as a consequence of the equality of the mixed partial derivatives of W, the coefficients C_{KLMN} satisfy the further symmetry relation:

$$C_{KLMN} = C_{MNKL} \ , \tag{8.9}$$

which is known as the *Maxwell relation.* Thus, the existence of a strain energy function reduces the number of distinct coefficients C_{KLMN} from 36 to 21. Further reduction for special types of elastic behavior are obtained from the material symmetry properties. Note that the same material may, in general, have different coefficients C_{KLMN} for different configurations κ_0.

In the classical linear theory, we consider only infinitesimal deformations from the reference configuration κ_0, and then the last term on the right-hand side of (8.5) is dropped. Moreover, the strain tensor $\boldsymbol{E}$ is replaced by the infinitesimal strain tensor $\tilde{\boldsymbol{E}}$, so that, the constitutive equation (8.5) reduces to

$$\boldsymbol{T}^{(2)} = \boldsymbol{T}_0 + \boldsymbol{C} : \tilde{\boldsymbol{E}} \ . \tag{8.10}$$

Within the framework of the classical infinitesimal model, we may study the effects of an infinitesimal deformation superimposed upon a reference configuration κ_0 with a finite pre-stress $\boldsymbol{T}_0$. Then (8.10) is taken as the **exact** constitutive equation defining the classical linear (infinitesimal) model of elasticity. However, within the framework of the general theory of elastic simple materials, we see that (8.10) is an approximation, for small deformations, of the exact stress relation (8.1).

In fact, we have considerable extent in the choice of tensor $\boldsymbol{C}$. The only requirement is that we satisfy the symmetries (8.8) and (8.9). It is easily verified that these relations are satisfied by a fourth-order tensor $\boldsymbol{\Xi}$ of the form

$$\begin{aligned}\Xi_{KLMN} = C_{KLMN} \ &+ a\,[T_{0,KL}\delta_{MN} + T_{0,MN}\delta_{KL}] \\ &+ b\,[T_{0,KM}\delta_{LN} + T_{0,LN}\delta_{KM}] \\ &+ c\,[T_{0,KN}\delta_{ML} + T_{0,ML}\delta_{KN}] \ ,\end{aligned} \tag{8.11}$$

or, in symbolic notation, [26]

$$\begin{aligned}\boldsymbol{\Xi} = \boldsymbol{C} \ &+ a\,[(\boldsymbol{T}_0 \otimes \boldsymbol{I})^{1234} + (\boldsymbol{I} \otimes \boldsymbol{T}_0)^{1234}] \\ &+ b\,[(\boldsymbol{T}_0 \otimes \boldsymbol{I})^{1324} + (\boldsymbol{I} \otimes \boldsymbol{T}_0)^{1324}] \\ &+ c\,[(\boldsymbol{T}_0 \otimes \boldsymbol{I})^{1432} + (\boldsymbol{I} \otimes \boldsymbol{T}_0)^{1432}] \ ,\end{aligned} \tag{8.12}$$

where a, b and c are arbitrary scalars. The expressions in parentheses multiplying a, b and c are the only three linear combinations of permutations of $\boldsymbol{T}_0 \otimes \boldsymbol{I}$ satisfying the symmetries (8.8) and (8.9). Every choice of the scalars a, b and c defines the behaviour of a linear elastic solid. We may thus replace the tensor $\boldsymbol{C}$ in the constitutive equation (8.10) by the tensor $\boldsymbol{\Xi}$:

$$\begin{aligned}\boldsymbol{T}^{(2)} &= \boldsymbol{T}_0 + \boldsymbol{\Xi} : \tilde{\boldsymbol{E}} \\ &= \boldsymbol{T}_0 + \boldsymbol{C} : \tilde{\boldsymbol{E}} + a[(\mathrm{tr}\,\tilde{\boldsymbol{E}})\boldsymbol{T}_0 + (\boldsymbol{T}_0 : \tilde{\boldsymbol{E}})\boldsymbol{I}] + (b+c)(\boldsymbol{T}_0 \cdot \tilde{\boldsymbol{E}} + \tilde{\boldsymbol{E}} \cdot \boldsymbol{T}_0) \ .\end{aligned} \tag{8.13}$$

[26] The symbols $(\ \ )^{1324}$, $(\ \ )^{1432}$, etc., denote the transpose of a quadric, e.g. $(\vec{a} \otimes \vec{b} \otimes \vec{c} \otimes \vec{d})^{1324} = \vec{a} \otimes \vec{c} \otimes \vec{b} \otimes \vec{d}$.

The linearized constitutive equations for the Cauchy stress tensor and the first Piola-Kirchhoff stress tensor can be obtained from $(8.13)_2$ using the relation $(3.25)_2$ and $(3.29)_2$:

$$\boldsymbol{T}^{(1)} = \boldsymbol{T}_0 + \boldsymbol{T}_0 \cdot \boldsymbol{H} + \boldsymbol{C} : \tilde{\boldsymbol{E}} + a[(\operatorname{tr} \tilde{\boldsymbol{E}})\boldsymbol{T}_0 + (\boldsymbol{T}_0 : \tilde{\boldsymbol{E}})\boldsymbol{I}] + (b+c)(\boldsymbol{T}_0 \cdot \tilde{\boldsymbol{E}} + \tilde{\boldsymbol{E}} \cdot \boldsymbol{T}_0) \ , \tag{8.14}$$

$$\begin{aligned} \boldsymbol{t} &= \boldsymbol{T}_0 - (\operatorname{tr} \boldsymbol{H})\, \boldsymbol{T}_0 + \boldsymbol{H}^T \cdot \boldsymbol{T}_0 + \boldsymbol{T}_0 \cdot \boldsymbol{H} + \boldsymbol{C} : \tilde{\boldsymbol{E}} + a[(\operatorname{tr} \tilde{\boldsymbol{E}})\boldsymbol{T}_0 + (\boldsymbol{T}_0 : \tilde{\boldsymbol{E}})\boldsymbol{I}] \\ &\quad + (b+c)(\boldsymbol{T}_0 \cdot \tilde{\boldsymbol{E}} + \tilde{\boldsymbol{E}} \cdot \boldsymbol{T}_0) \ . \end{aligned} \tag{8.15}$$

By the decomposition $\boldsymbol{H}^T = \tilde{\boldsymbol{E}} + \tilde{\boldsymbol{R}}$, where $\tilde{\boldsymbol{R}}$ is the infinitesimal rotation tensor, the representations (8.14) and (8.15) become

$$\begin{aligned} \boldsymbol{T}^{(1)} &= \boldsymbol{T}_0 - \boldsymbol{T}_0 \cdot \tilde{\boldsymbol{R}} + \boldsymbol{C} : \tilde{\boldsymbol{E}} + a[(\operatorname{tr} \tilde{\boldsymbol{E}})\boldsymbol{T}_0 + (\boldsymbol{T}_0 : \tilde{\boldsymbol{E}})\boldsymbol{I}] + (b+c)(\tilde{\boldsymbol{E}} \cdot \boldsymbol{T}_0) \\ &\quad + (b+c+1)(\boldsymbol{T}_0 \cdot \tilde{\boldsymbol{E}}) \ , \end{aligned} \tag{8.16}$$

$$\begin{aligned} \boldsymbol{t} &= \boldsymbol{T}_0 + \tilde{\boldsymbol{R}} \cdot \boldsymbol{T}_0 - \boldsymbol{T}_0 \cdot \tilde{\boldsymbol{R}} + \boldsymbol{C} : \tilde{\boldsymbol{E}} + (a-1)(\operatorname{tr} \tilde{\boldsymbol{E}})\boldsymbol{T}_0 + a(\boldsymbol{T}_0 : \tilde{\boldsymbol{E}})\boldsymbol{I} \\ &\quad + (b+c+1)(\boldsymbol{T}_0 \cdot \tilde{\boldsymbol{E}} + \tilde{\boldsymbol{E}} \cdot \boldsymbol{T}_0) \ . \end{aligned} \tag{8.17}$$

The first terms in (8.16) and (8.17) can be identified as the rotated initial stress, whereas the remaining terms represent the perturbation in stress due to the infinitesimal strain $\tilde{\boldsymbol{E}}$.

Upon substituting equations (8.6) and (8.7) into (8.3) and replacing the tensor $\boldsymbol{C}$ by the tensor $\boldsymbol{\Xi}$, the strain energy density can be rewritten, correct to second order in $\tilde{\boldsymbol{E}}$, in the form

$$\begin{aligned} W(\tilde{\boldsymbol{E}}) &= W(\boldsymbol{0}) + \boldsymbol{T}_0 : \tilde{\boldsymbol{E}} + \tfrac{1}{2}\, \tilde{\boldsymbol{E}} : \boldsymbol{\Xi} : \tilde{\boldsymbol{E}} \\ &= W(\boldsymbol{0}) + \boldsymbol{T}_0 : \tilde{\boldsymbol{E}} + \tfrac{1}{2}\, \tilde{\boldsymbol{E}} : \boldsymbol{C} : \tilde{\boldsymbol{E}} + a(\operatorname{tr} \tilde{\boldsymbol{E}})(\boldsymbol{T}_0 : \tilde{\boldsymbol{E}}) + (b+c)\operatorname{tr}(\tilde{\boldsymbol{E}} \cdot \boldsymbol{T}_0 \cdot \tilde{\boldsymbol{E}}) \ . \end{aligned} \tag{8.18}$$

The terms on the right-hands side of (8.18) proportional to $\boldsymbol{T}_0$ represent the work done against the initial stress, whereas the term $\frac{1}{2}\tilde{\boldsymbol{E}} : \boldsymbol{C} : \tilde{\boldsymbol{E}}$ is the classical elastic energy density in the absence of any initial stress.

8.2 The elastic tensor

A stress-free configuration is called a *natural state*. If such a configuration is used as reference, then $\boldsymbol{T}_0 = \boldsymbol{0}$, and (8.13)–(8.15) reduce to

$$\boldsymbol{\tau} = \boldsymbol{C} : \boldsymbol{\varepsilon} \ , \tag{8.19}$$

where $\boldsymbol{\tau}$ can be interpreted as either $\boldsymbol{t}$, $\boldsymbol{T}^{(1)}$ or $\boldsymbol{T}^{(2)}$ and $\boldsymbol{\varepsilon}$ either as the infinitesimal Lagrangian or Eulerian strain tensor, $\tilde{\boldsymbol{E}}$ or $\tilde{\boldsymbol{e}}$. This classical linear elastic stress–strain relation is known as *generalized Hooke's law* and the fourth-order tensor $\boldsymbol{C}$ is called the *elastic tensor*. The linear stress-strain constitutive relation for infinitesimal deformations of an anisotropic solid in its stress–free natural state are specified by 21 independent components of $\boldsymbol{C}$. In the more general case of an incremental stress superimposed upon the zeroth-order initial stress $\boldsymbol{T}_0$, the stress-strain relation is specified by 27 coefficients: the 21 components of $\boldsymbol{C}$ and 6 components of the initial stress tensor $\boldsymbol{T}_0$. Equations (8.16) and (8.17) constitute the generalization of Hooke's law to the case of a pre-stressed elastic medium.

In the particular case $\boldsymbol{T}_0 = \boldsymbol{0}$, the strain energy density reduces to

$$W(\boldsymbol{\varepsilon}) = W(\boldsymbol{0}) + \frac{1}{2}\, \boldsymbol{\tau} : \boldsymbol{\varepsilon} \ . \tag{8.20}$$

8.3 Isotropic linear elastic solid

The highest symmetry of a solid is reached if the solid possesses no preferred direction with respect to its elastic property. This also means that the linear elasticity $\boldsymbol{C}$ is invariant under any orthogonal transformation of the coordinate system. We called such a solid *isotropic*. In section 6.17.1, we showed the number of independent elastic constants reduces to 2 and the constitutive equation of an isotropic linear elastic solid is of the form

$$\boldsymbol{\tau} = \lambda\vartheta\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} \ , \tag{8.21}$$

where λ and μ are the Lamé elastic constant,

$$\vartheta := \operatorname{tr}\boldsymbol{\varepsilon} = \operatorname{div}\vec{u} \ , \tag{8.22}$$

and $\vec{u}$ is the displacement vector. Now, we will show, independently of considerations in section 6.17.1, that the generalized Hooke's law (8.19) reduces to the form (8.21) for an isotropic linear elastic solid.

In the linear theory, it is not necessary to distinguish between the Lagrangian and Eulerian coordinate systems. We will follow this concept and use Cartesian coordinates x_k for the coinciding coordinate systems. Any orthogonal transformation of the coordinate system may be expressed by the Euclidean transformation (5.8) with $\vec{b}' = \vec{0}$, that is, by the transformation equation

$$x'_k = Q_{kl}x_l \tag{8.23}$$

subject to

$$Q_{kl}Q_{ml} = Q_{lk}Q_{lm} = \delta_{km} \ , \qquad \det Q_{kl} = \pm 1 \ . \tag{8.24}$$

The components of a second-order Cartesian tensor $\boldsymbol{\tau}$ transform under the orthogonal transformation of the coordinate system according to (5.12):

$$\tau'_{kl} = Q_{km}Q_{ln}\tau_{mn} \ , \tag{8.25}$$

which may readily be inverted with the help of the orthogonality conditions (8.24) to yield

$$\tau_{kl} = Q_{mk}Q_{nl}\tau'_{mn} \ . \tag{8.26}$$

Note carefully the location of the summed indices m and n in (8.25) and (8.26).

Proof of Hooke's law (8.21) for an isotropic linear elastic solid will be accomplished in four steps. First, we show that *for an isotropic linear elastic solid the principal axes of the stress and infinitesimal strain tensors coincide.* To proof it, we take, without loss of generality, the coordinate axes x_k in the principal directions of strain tensor $\boldsymbol{\varepsilon}$. Then $\varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0$. We shall now show that $\tau_{23} = 0$. We first have

$$\tau_{23} = A\varepsilon_{11} + B\varepsilon_{22} + C\varepsilon_{33}$$

with $A := C_{2311}$, $B := C_{2322}$, and $C := C_{2333}$. We now rotate the coordinate system through an angle of 180^o about the x_3-axis. Then $x'_1 = -x_1$, $x'_2 = -x_2$, and $x'_3 = x_3$, and the matrix of this transformation is

$$\boldsymbol{Q} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ .$$

In view of (8.25), we therefore have

$$\begin{aligned}
\tau'_{23} &= Q_{2m}Q_{3n}\tau_{mn} = -\tau_{23} \ , \\
\varepsilon'_{11} &= Q_{1m}Q_{1n}\varepsilon_{mn} = \varepsilon_{11} \ , \\
\varepsilon'_{22} &= Q_{2m}Q_{2n}\varepsilon_{mn} = \varepsilon_{22} \ , \\
\varepsilon'_{33} &= \varepsilon_{33} \ .
\end{aligned}$$

The relation

$$\tau'_{23} = A\varepsilon'_{11} + B\varepsilon'_{22} + C\varepsilon'_{33} = A\varepsilon_{11} + B\varepsilon_{22} + C\varepsilon_{33} = \tau_{23}$$

is now the consequence of isotropy since the constants A, B, and C do not depend on the reference configuration of a solid. Thus

$$-\tau_{23} = \tau'_{23} = \tau_{23} \ ,$$

which implies that $\tau_{23} = 0$. Likewise, it can be shown that $\tau_{12} = \tau_{13} = 0$. We have proved that the principal axes of stress and strain coincide. Note that the principal stresses and strains are, in general, different.

Second, consider the component τ_{11}. Taking the coordinate axes in principal directions of strain, we obtain

$$\tau_{11} = a_1\varepsilon_{11} + b_1\varepsilon_{22} + c_1\varepsilon_{33} \ ,$$

with $a_1 := C_{1111}$, $b_1 := C_{1122}$, and $c_1 := C_{1133}$. We now rotate the coordinate system through an angle 90^o about the x_1-axis in such a way that

$$x'_1 = x_1 \ , \quad x'_2 = x_3 \ , \quad x'_3 = -x_2 \ .$$

The matrix of this transformation is

$$\boldsymbol{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \ ,$$

and we have

$$\begin{aligned}
\tau'_{11} &= Q_{1m}Q_{1n}\tau_{mn} = \tau_{11} \ , \\
\varepsilon'_{11} &= \varepsilon_{11} \ , \\
\varepsilon'_{22} &= Q_{2m}Q_{2n}\varepsilon_{mn} = \varepsilon_{33} \ , \\
\varepsilon'_{33} &= \varepsilon_{22} \ .
\end{aligned}$$

In view of isotropy, the constants a_1, b_1, and c_1 do not depend on the reference configuration, so that

$$\tau'_{11} = a_1\varepsilon'_{11} + b_1\varepsilon'_{22} + c_1\varepsilon'_{33} \ ,$$

and substituting for ε'_{kk}, we get

$$\tau'_{11} = a_1\varepsilon_{11} + b_1\varepsilon_{33} + c_1\varepsilon_{22} \ .$$

This implies that $b_1 = c_1$ since $\tau_{11} = a_1\varepsilon_{11} + b_1\varepsilon_{22} + c_1\varepsilon_{33}$. We can thus write τ_{11} as

$$\tau_{11} = a_1\varepsilon_{11} + b_1\left(\varepsilon_{22} + \varepsilon_{33}\right) = \lambda_1\vartheta + 2\mu_1\varepsilon_{11} \ ,$$

where

$$\lambda_1 := b_1 = C_{1122}\ , \quad 2\mu_1 := a_1 - b_1 = C_{1111} - C_{1122}\ .$$

The same relations can be obtained for subscripts 2 and 3:

$$\begin{aligned} \tau_{22} &= \lambda_2\vartheta + 2\mu_2\varepsilon_{22}\ , \\ \tau_{33} &= \lambda_3\vartheta + 2\mu_3\varepsilon_{33}\ , \end{aligned}$$

where $\lambda_2 := C_{2222}$, $2\mu_2 := C_{2211} - C_{2222}$, $\lambda_3 := C_{3322}$, $2\mu_3 := C_{3311} - C_{3322}$.

Third, rotate the coordinate system through an angle 90^o about the x_3-axis in such a way that

$$x_1' = x_2\ , \quad x_2' = -x_1\ , \quad x_3' = x_3\ .$$

The matrix of this transformation is

$$\boldsymbol{Q} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\ ,$$

and we have

$$\begin{aligned} \tau_{11}' &= \tau_{22}\ , & \varepsilon_{11}' &= \varepsilon_{22}\ , \\ \tau_{22}' &= \tau_{11}\ , & \varepsilon_{22}' &= \varepsilon_{11}\ , \\ \tau_{33}' &= \tau_{33}\ , & \varepsilon_{33}' &= \varepsilon_{33}\ . \end{aligned}$$

Combining the results of previous steps, we can write

$$\begin{aligned} \tau_{22}' &= \lambda_2\vartheta' + 2\mu_2\varepsilon_{22}' \\ &= \lambda_2\vartheta + 2\mu_2\varepsilon_{22}' \\ &= \lambda_2\vartheta + 2\mu_2\varepsilon_{11} \\ &\overset{!}{=} \tau_{11} \\ &= \lambda_1\vartheta + 2\mu_1\varepsilon_{11}\ . \end{aligned}$$

This implies that $\lambda_2 = \lambda_1$ and $\mu_2 = \mu_1$. Likewise, it can be shown that $\lambda_3 = \lambda_1$ and $\mu_3 = \mu_1$. In summary, we have

$$\begin{aligned} \tau_{11} &= \lambda\vartheta + 2\mu\varepsilon_{11}\ , \\ \tau_{22} &= \lambda\vartheta + 2\mu\varepsilon_{22}\ , \\ \tau_{33} &= \lambda\vartheta + 2\mu\varepsilon_{33}\ , \\ \tau_{kl} &= 0 \quad \text{for} \quad k \neq j\ , \end{aligned}$$

which is the generalized Hooke's law for an isotropic body in principal directions. Shortly written,

$$\tau_{kl} = \lambda\vartheta\delta_{kl} + 2\mu\varepsilon_{kl}\ .$$

Fourth, we now rotate the coordinate system with the axes x_k coinciding with principal directions of strain to arbitrary coordinate system with the axes x_k' and show that Hooke's law for an isotropic solid also holds in a rotated coordinate system x'_k. Denoting by Q_{kl} the transformation matrix of this rotation, the stress and strain tensor transform according to the transformation law (8.25) for second-order tensors. Multiplying the above equation by $Q_{mk}Q_{nl}$, using the transformation law (8.25) and the orthogonality property (8.24) of Q_{kl}, we have

$$\tau'_{mn} = \lambda\vartheta\delta_{mn} + 2\mu\varepsilon'_{mn} \ .$$

Moreover,

$$\vartheta' = \varepsilon'_{mm} = Q_{mk}Q_{ml}\varepsilon_{kl} = \delta_{kl}\varepsilon_{kl} = \varepsilon_{kk} = \vartheta \ .$$

We finally have

$$\tau'_{mn} = \lambda\vartheta'\delta_{mn} + 2\mu\varepsilon'_{mn} \ ,$$

which proves Hooke's law (8.21) for an isotropic linear elastic solid.

We have seen that the Lamé coefficients are expressed in terms of the elastic coefficients as

$$\lambda = C_{1122} \ , \qquad \lambda + 2\mu = C_{1111} \ . \tag{8.27}$$

We note without proof that the most general form of a fourth-order isotropic tensor $\boldsymbol{A}$ is

$$A_{klmn} = \lambda\delta_{kl}\delta_{mn} + \mu\left(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}\right) + \nu\left(\delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}\right) \ , \tag{8.28}$$

where λ, μ, and ν are scalars. If $\boldsymbol{A}$ is replaced by the linear elasticity $\boldsymbol{C}$, the symmetry relations $C_{klmn} = C_{lkmn} = C_{klnm}$ imply that ν must be zero since by interchanging k and l in the expression

$$\nu\left(\delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}\right) = \nu\left(\delta_{lm}\delta_{kn} - \delta_{ln}\delta_{km}\right)$$

we see that $\nu = -\nu$ and, consequently, $\nu = 0$. Thus, the linear elasticity for an isotropic solid has the form

$$C_{klmn} = \lambda\delta_{kl}\delta_{mn} + \mu\left(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}\right) \ . \tag{8.29}$$

The Hooke's law for an isotropic solid takes a particularly simple form in spherical and deviatoric parts of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\tau}$. Let us decompose the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ into the spherical and deviatoric parts:

$$\boldsymbol{\varepsilon} = \frac{1}{3}\vartheta\boldsymbol{I} + \boldsymbol{\varepsilon}^D \ , \tag{8.30}$$

where the *mean normal strain* ϑ is defined by (8.22), and the *deviatoric strain tensor* $\boldsymbol{\varepsilon}^D$ is a trace-free, symmetric, second-order tensor:

$$\boldsymbol{\varepsilon}^D := \boldsymbol{\varepsilon} - \frac{1}{3}\vartheta\boldsymbol{I} \ . \tag{8.31}$$

The same decomposition of the stress tensor $\boldsymbol{\tau}$ reads

$$\boldsymbol{\tau} = -p\boldsymbol{I} + \boldsymbol{\tau}^D \ , \tag{8.32}$$

where the scalar p is the negative of the mean normal stress and called *mechanical pressure*,

$$p := -\frac{1}{3}\mathrm{tr}\,\boldsymbol{\tau} \ , \tag{8.33}$$

and the *deviatoric stress tensor* $\boldsymbol{\tau}^D$ is a trace-free, symmetric, second-order tensor, defined by

$$\boldsymbol{\tau}^D := \boldsymbol{\tau} + p\boldsymbol{I} \; . \tag{8.34}$$

By substituting (8.30) and (8.32) into (8.21), Hooke's law for an isotropic linear elastic solid may be written separately for the spherical and deviatoric parts of the stress and strain:

$$-p = k\vartheta \; , \qquad \boldsymbol{\tau}^D = 2\mu\boldsymbol{\varepsilon}^D \; , \tag{8.35}$$

where

$$k := \lambda + \frac{2}{3}\mu \tag{8.36}$$

is referred to as the *elastic bulk modulus.* Finally, we derive the strain energy density function for an isotropic linear elastic solid. For this purpose, we successively substitute (8.21) into (8.20), giving

$$W(\boldsymbol{\varepsilon}) = W(\boldsymbol{0}) + \frac{1}{2}\lambda\vartheta^2 + \mu(\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}) \; , \tag{8.37}$$

which, using (8.31) and (8.36), can be rewritten as

$$W(\boldsymbol{\varepsilon}) = W(\boldsymbol{0}) + \frac{1}{2}k\vartheta^2 + \mu(\boldsymbol{\varepsilon}^D : \boldsymbol{\varepsilon}^D) \; . \tag{8.38}$$

8.4 Restrictions on elastic coefficients

In this section, we analyze several hypothetical experiments and consequent restrictions that must be placed upon elastic moduli in order that they may represent a real material adequately.

We first assume that Hooke's law (8.21) for an isotropic linear elastic solid is invertible for $\boldsymbol{\varepsilon}$. Applying the trace operator on this equation gives

$$\operatorname{tr}\boldsymbol{\tau} = (3\lambda + 2\mu)\operatorname{tr}\boldsymbol{\varepsilon} \; . \tag{8.39}$$

Now, by solving (8.21) for $\boldsymbol{\varepsilon}$ and substituting from (8.39) for $\operatorname{tr}\boldsymbol{\varepsilon}$, we obtain the inverse form of Hooke's law for an isotropic solid,

$$\boldsymbol{\varepsilon} = \frac{\boldsymbol{\tau}}{2\mu} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\operatorname{tr}\boldsymbol{\tau}\,\boldsymbol{I} \; . \tag{8.40}$$

We observe that for $\boldsymbol{\varepsilon}$ to be uniquely determined by $\boldsymbol{\tau}$ we must have

$$\mu \neq 0 \; , \qquad 3\lambda + 2\mu \neq 0 \; . \tag{8.41}$$

(i) Hydrostatic pressure. Experimental observations indicate that under hydrostatic pressure the volume of an elastic solid diminishes. For hydrostatic compression, the stress tensor has the form

$$\boldsymbol{\tau} = -p\boldsymbol{I} \; , \qquad p > 0 \; . \tag{8.42}$$

From $(8.35)_1$ we see that

$$p = -k\vartheta \; , \tag{8.43}$$

where $\vartheta := \operatorname{tr}\boldsymbol{\varepsilon} = \operatorname{div}\vec{u} = J - 1 = (dv - dV)/dV$ is the cubical dilatation. In the case of hydrostatic pressure, clearly $dv < dV$ and hence $\vartheta < 0$. We must thus have $k > 0$ which implies that

$$3\lambda + 2\mu > 0 \ . \tag{8.44}$$

(ii) Simple shear. Consider a simple constant shear in which

$$\tau_{12} \neq 0 \ , \qquad \tau_{kl} = 0 \quad \text{otherwise.} \tag{8.45}$$

In this case (8.40) gives

$$\tau_{12} = 2\mu\varepsilon_{12} \ . \tag{8.46}$$

Experimental observations of small deformations of elastic solids subjected to simple shear indicate that τ_{12} and ε_{12} have the same direction. Consequently,

$$\mu > 0 \ . \tag{8.47}$$

Under conditions (8.44) and (8.47) and provided that the strain energy density function vanishes at the unstrained natural state, $W(\mathbf{0}) = 0$, the strain energy density function $W(\boldsymbol{\varepsilon})$ for an isotropic linear elastic solid is a positive definite, quadratic function of the infinitesimal strain $\boldsymbol{\varepsilon}$. This fact is basic for the proof of the uniqueness of solution of boundary-value problems in elastic equilibrium.

The conditions (8.44) and (8.47) can be shown to be sufficient but not necessary for an isotropic linear isotropic solid to get plausible results from the infinitesimal field theory. The necessary and sufficient conditions for an isotropic linear elastic solid are weaker than (8.44) and (8.47), namely,

$$\lambda + 2\mu > 0 \ , \qquad \mu > 0 \ . \tag{8.48}$$

It can be shown that these conditions give positive wave speeds in isotropic linear elastic solids. A number of special **a priori** inequalities have been proposed based on certain "reasonable" physical expectations when an isotropic elastic solid is subjected to pressure, tensions, and shears. For a discussion of these we refer the reader to Truesdell and Noll (1965).

In the classical infinitesimal theory of elasticity other material constants are often used in place of the Lamé constants λ and μ. Following are relations among some of these constants:

$$\begin{aligned}
&E := \mu(3\lambda + 2\mu)/(\lambda + \mu) \ , && \nu := \lambda/2(\lambda + \mu) \ , \\
&\lambda = E\nu/(1+\nu)(1-2\nu) = 2G\nu/(1-2\nu) \ , && \mu \equiv G = E/2(1+\nu) \ , \\
&k = \lambda + \tfrac{2}{3}\mu = E/3(1-2\nu) \ ,
\end{aligned} \tag{8.49}$$

where E is called *Young's modulus*, G is the *elastic shear modulus*, or *modulus of rigidity*, which, as noted, is identical to the Lamé constant μ, and ν is *Poisson's ratio.* The inequalities

$$G > 0 \ , \qquad -1 < \nu < \frac{1}{2} \ , \tag{8.50}$$

are equivalent to (8.44) and (8.47).

8.5 Field equations

The field equations in continuum mechanics consist of the balance laws, valid for all continua, and a particular constitutive equation. Here we shall consider isotropic linear elastic or linear thermoelastic solids. The field equations may be expressed in Lagrangian or Eulerian form but for the linearized theory of elasticity the two forms are identical.

8.5.1 Isotropic linear elastic solid

Within the frame of the infinitesimal strain theory, the continuity equation (4.60) in the referential form can be arranged as follows

$$\varrho = \frac{\varrho_0}{J} = \frac{\varrho_0}{1 + \operatorname{tr} \tilde{\boldsymbol{E}}} + O(|\tilde{\boldsymbol{E}}|^2) = \varrho_0 + O(|\tilde{\boldsymbol{E}}|) \ . \tag{8.51}$$

Making use of this approximation and provided, **in addition**, that the volume force $\vec{f}$ is also infinitesimally small, the equation of motion (4.39) in the present configuration coincides with the equation of motion (4.67) in the referential configuration:

$$\operatorname{div} \boldsymbol{\tau} + \varrho_0 \vec{f} = \varrho_0 \frac{\partial^2 \vec{u}}{\partial t^2} \ , \tag{8.52}$$

where we have substituted for $\vec{v} = \partial \vec{u}/\partial t$ since $\vec{u} = \vec{u}(\vec{X}, t)$ and the partial derivative with respect to the material coordinates held constant. We can see that in the infinitesimal strain and stress theory the distinction between the reference and present configuration of a material disappears. Moreover, the continuity equation does not play any role here, since the balance of mass results in the statement $\varrho_0 = \varrho_0(\vec{X})$.

We substitute Hooke's law (8.21) for an isotropic linear elastic solid into the equation of motion (8.52). With the help of vector differential identities introduced in Appendix A, the divergence of the stress tensor $\boldsymbol{\tau}$ can be arranged as follows

$$\begin{aligned}
\operatorname{div} \boldsymbol{\tau} &= \operatorname{div} (\lambda \vartheta \boldsymbol{I} + 2\mu \boldsymbol{\varepsilon}) \\
&= \operatorname{grad} (\lambda \vartheta) + 2\mu \operatorname{div} \boldsymbol{\varepsilon} + 2 \operatorname{grad} \mu \cdot \boldsymbol{\varepsilon} \\
&= \lambda \operatorname{grad} \operatorname{div} \vec{u} + \operatorname{div} \vec{u} \operatorname{grad} \lambda + \mu \operatorname{div}(\operatorname{grad} \vec{u} + \operatorname{grad}^T \vec{u}) + \operatorname{grad} \mu \cdot (\operatorname{grad} \vec{u} + \operatorname{grad}^T \vec{u}) \\
&= (\lambda + \mu) \operatorname{grad} \operatorname{div} \vec{u} + \mu \nabla^2 \vec{u} + \operatorname{div} \vec{u} \operatorname{grad} \lambda + \operatorname{grad} \mu \cdot (\operatorname{grad} \vec{u} + \operatorname{grad}^T \vec{u}) \\
&= (\lambda + 2\mu) \operatorname{grad} \operatorname{div} \vec{u} - \mu \operatorname{rot} \operatorname{rot} \vec{u} + \operatorname{div} \vec{u} \operatorname{grad} \lambda + \operatorname{grad} \mu \cdot (\operatorname{grad} \vec{u} + \operatorname{grad}^T \vec{u}) \ .
\end{aligned}$$

The linearized equation of motion (8.52) can now be expressed in the form

$$(\lambda + 2\mu) \operatorname{grad} \operatorname{div} \vec{u} - \mu \operatorname{rot} \operatorname{rot} \vec{u} + \operatorname{div} \vec{u} \operatorname{grad} \lambda + \operatorname{grad} \mu \cdot (\operatorname{grad} \vec{u} + \operatorname{grad}^T \vec{u}) + \varrho_0 \vec{f} = \varrho_0 \frac{\partial^2 \vec{u}}{\partial t^2} \ , \tag{8.53}$$

which is known as the *Navier–Cauchy equation* of an isotropic linear elastic solid. It consists of three second-order differential equations for the three displacement components. Note that the vector form (8.53) is independent of the coordinate system and may be expressed in curvilinear coordinates by the methods of Appendix C. In particular, for a homogeneous solid, λ and μ are constants (independent of $\vec{X}$) and the Navier-Cauchy equation reduces to

$$(\lambda + 2\mu) \operatorname{grad} \operatorname{div} \vec{u} - \mu \operatorname{rot} \operatorname{rot} \vec{u} + \varrho_0 \vec{f} = \varrho_0 \frac{\partial^2 \vec{u}}{\partial t^2} \ , \tag{8.54}$$

or, alternatively,

$$(\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\vec{u} + \mu\nabla^2\vec{u} + \varrho_0\vec{f} = \varrho_0\frac{\partial^2\vec{u}}{\partial t^2}\ . \tag{8.55}$$

Note that *elastostatics* is restricted to those situations in which inertia forces $\varrho_0\partial^2\vec{u}/\partial t^2$ may be neglected. Since (8.52) does not involve displacements, an alternative formulation of elastostatic problems in terms of stresses alone is possible.

8.5.2 Incompressible isotropic linear elastic solid

For incompressible solids we have

$$\mathrm{div}\,\vec{u} = 0\ , \tag{8.56}$$

and the density remains unchanged, equal to a known value, $\varrho = \varrho_0$. Hooke's law (8.21) need to be modified such that

$$\boldsymbol{\tau} = -p\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}\ , \tag{8.57}$$

where p is a new unknown field variable. The Navier-Cauchy equation of motion (8.53) transforms to the form

$$-\mathrm{grad}\,p + \mu\,\nabla^2\vec{u} + \mathrm{grad}\,\mu\cdot(\mathrm{grad}\,\vec{u} + \mathrm{grad}^T\vec{u}) + \varrho_0\vec{f} = \varrho_0\frac{\partial^2\vec{u}}{\partial t^2}\ . \tag{8.58}$$

Hence, (8.56) and (8.58) represents four differential equations for four unknown field variables: pressure p and three components of displacement vector $\vec{u}$.

8.5.3 Isotropic linear thermoelastic solid

Equation (7.83)–(7.86) form the constitutive equations for an isotropic linear thermoelastic solid. We now substitute these constitutive equations to the energy equation in the referential form (4.77). We obtain

$$\pi(\tilde{I}_E)^{\bullet} + \lambda(\tilde{I}_E^2)^{\bullet} + 2\mu(\tilde{II}_E)^{\bullet} + \gamma T\,\dot{T} = (\pi + \lambda\tilde{I}_E - \beta T)(\tilde{I}_E)^{\bullet} + 2\mu\boldsymbol{E} : \dot{\boldsymbol{E}} + \mathrm{Div}\,(\kappa\,\mathrm{Grad}\,T) + \varrho_0 h\ . \tag{8.59}$$

The material time derivatives of principal invariants of strain can be arranged as follows:

$$\begin{aligned}(\tilde{I}_{\tilde{E}})^{\bullet} &= \mathrm{Div}\,\dot{\vec{u}}\ ,\\ (\tilde{I}_E^2)^{\bullet} &= 2\tilde{I}_E(\tilde{I}_E)^{\bullet}\ ,\\ (\tilde{II}_E)^{\bullet} &= (\mathrm{tr}\,\boldsymbol{E}^2)^{\bullet} = \mathrm{tr}\,(\boldsymbol{E}^2)^{\bullet} = 2\mathrm{tr}\,(\boldsymbol{E}\cdot\dot{\boldsymbol{E}})\ .\end{aligned} \tag{8.60}$$

In view of these expressions, equation (8.59) is non-linear with respect to $\boldsymbol{E}$. Within the framework of linear theory, we consider only infinitesimal deformations from the reference configuration. Then the non-linear terms in (8.59) can be omitted and the strain tensor is replaced by the infinitesimal strain tensor $\tilde{\boldsymbol{E}}$. The linearization with respect to strain brings (8.59) into the form

$$\gamma T\,\dot{T} = -\beta T\,\mathrm{Div}\,\dot{\vec{u}} + \mathrm{Div}\,(\kappa\,\mathrm{Grad}\,T) + \varrho_0 h\ . \tag{8.61}$$

This equation is still non-linear with respect to temperature. Further linearization can be made by assuming that the instantaneous absolute temperature T in the present configuration differs from the temperature $T_0(\vec{X})$ (> 0) in the reference configuration by a small quantity, that is,

$$T = T_0 + T_1\ , \qquad |T_1| \ll T_0\ . \tag{8.62}$$

The linearization of (8.62) with respect to T yields

$$\gamma T_0 \overset{\bullet}{T_1} = -\beta T_0 \,\mathrm{Div}\, \overset{\bullet}{\vec{u}} + \mathrm{Div}\,(\kappa\, \mathrm{Grad}\, T_1) + \mathrm{Div}\,(\kappa\, \mathrm{Grad}\, T_0) + \varrho_0 h \ . \tag{8.63}$$

If we introduce the *specific heat* c_v at constant deformation $\mathrm{Div}\, \overset{\bullet}{\vec{u}} = 0$ and no heat supply $h = 0$ through the equation

$$\mathrm{Div}\, \vec{Q} = \varrho_0 c_v \overset{\bullet}{T_1} \ , \tag{8.64}$$

then $\gamma T_0 = \varrho_0 c_v$ and the energy equation becomes

$$\varrho_0 c_v \overset{\bullet}{T_1} = -\beta T_0 \,\mathrm{Div}\, \overset{\bullet}{\vec{u}} + \mathrm{Div}\,(\kappa\, \mathrm{Grad}\, T_1) + \mathrm{Div}\,(\kappa\, \mathrm{Grad}\, T_0) + \varrho_0 h \ . \tag{8.65}$$

This equation, known as the *coupled heat conduction equation*, gives the relationship between the rate of change of the temperature and the strain with the heat conduction.

Under the decomposition $(8.62)_1$, the linearized constitutive equation $(7.83)_1$ for the second Piola-Kirchhoff stress tensor becomes

$$\boldsymbol{T}^{(2)} = (\pi - \beta\, T_0)\boldsymbol{I} + \lambda(\mathrm{tr}\, \boldsymbol{E})\boldsymbol{I} + 2\mu \boldsymbol{E} - \beta\, T_1 \boldsymbol{I} \ . \tag{8.66}$$

Again, in the infinitesimal theory $\boldsymbol{T}^{(2)}$ and $\boldsymbol{E}$ can be replaced by the infinitesimal stress tensor $\boldsymbol{\tau}$ and the infinitesimal strain tensor $\boldsymbol{\varepsilon}$, respectively. Note that when the reference configuration is stress free then $\pi - \beta\, T_0 = 0$. The constitutive equation (8.66) is known as the *Duhamel-Neumann* form of Hooke's law. It may be inverted to give

$$\boldsymbol{E} = -\frac{1}{3\lambda + 2\mu}(\pi - \beta\, T_0)\boldsymbol{I} + \frac{1}{2\mu}\boldsymbol{T}^{(2)} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\mathrm{tr}\, \boldsymbol{T}^{(2)}\, \boldsymbol{I} + \frac{\beta}{3\lambda + 2\mu} T_1 \boldsymbol{I} \ , \tag{8.67}$$

where

$$\alpha := \beta/(3\lambda + 2\mu) \tag{8.68}$$

is the *linear coefficient of thermal expansion.*

The coupled heat conduction equation, along with the equation of motion and thermoelastic stress–strain equation constitute the basic set of field equations for coupled thermoelastic problems. There are, however, problems in which the heat conduction equation can further be simplified. For instance, if the reference configuration of the thermoelastic solid coincides with the thermostatic equilibrium, then $\mathrm{Grad}\, T_0 = \vec{0}$ and the third term on the right-hand side of (8.65) vanishes. Another simplification arises when the isotropic linear thermoelastic solid is also *incompressible.* Then $\mathrm{Div}\, \vec{u} = 0$, and also $\mathrm{Div}\, \overset{\bullet}{\vec{u}} = 0$, so that, the first term on the right-hand side of (8.65) vanishes. In this case, the heat conduction equation is decoupled from the equation of motion and the thermoelastic problem is decomposed into two separate problems, which must be solved consecutively, but independently.

8.5.4 Example: The deformation of a plate under its own weight

We consider a two-dimensional inclined plate of constant thickness H (*slab*) infinitely extended in the x- and y-directions and tightly connected to the bed (Figure 8.1). The material of slab is isotropic and linear elastic. We aim to determine the deformation of the slab by its own weight.

Solution. We assume that the inclination angle α of the slab is constant. Because of this, infinite x- and y-dimensions and constant thickness of the slab, the field variables can only depend on the vertical coordinate z; all derivatives with respect to x and y must vanish, $(\partial/\partial x)(\cdot) = (\partial/\partial y)(\cdot) = 0$. Moreover, the y-component of the displacement vector is taken as zero, and the remaining

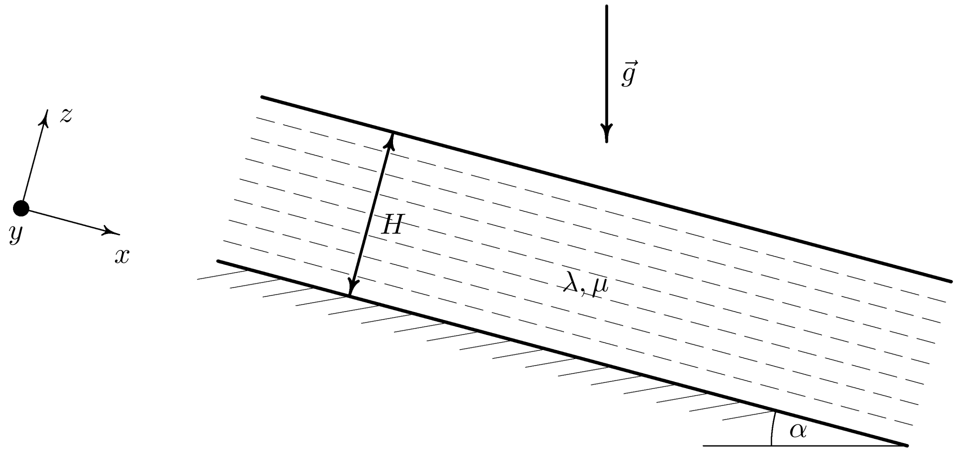

Figure 8.1. The deformation of a plate under its own weight.

components are considered as functions of z only,

$$u_x = u_x(z) \ , \qquad u_z = u_z(z) \ . \tag{8.69}$$

Consequently, the strain components $\varepsilon_{xx} = \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{yy} = 0$, while ε_{zz} and ε_{xz} are functions of z only,

$$\varepsilon_{zz} = \frac{du_z}{dz} \ , \qquad \varepsilon_{xz} = \frac{1}{2}\frac{du_x}{dz} \ . \tag{8.70}$$

Moreover, we neglect the inertia forces and treat the problem as elastostatic.

As far the boundary conditions are concerned, the displacement vector vanishes at the bed, and the surface traction vanishes at the upper free surface, that is,

$$u_x = u_z = 0 \qquad \text{for} \quad z = 0 \tag{8.71}$$

$$\tau_{xz} = \tau_{zz} = 0 \qquad \text{for} \quad z = H. \tag{8.72}$$

Since $\operatorname{div} \vec{u} = du_z/dz$, the stress components have the form

$$\begin{aligned} \tau_{xz} &= 2\mu\varepsilon_{xz} = \mu\frac{du_x}{dz} \ , \\ \tau_{zz} &= \lambda \operatorname{div} \vec{u} + 2\mu\varepsilon_{zz} = (\lambda + 2\mu)\frac{du_z}{dz} \ . \end{aligned}$$

In view of this, the boundary conditions (8.72) reduce to

$$\left.\frac{du_x}{dz}\right|_{z=H} = \left.\frac{du_z}{dz}\right|_{z=H} = 0 \ . \tag{8.73}$$

Under the above simplifications, the $x-$ and z-component of the Navier-Cauchy equation (8.55), that is,

$$\begin{aligned} (\lambda + \mu)\frac{\partial}{\partial x}(\operatorname{div} \vec{u}) + \mu\nabla^2 u_x + \varrho_0 f_x &= \varrho_0 \frac{\partial^2 u_x}{\partial t^2} \ , \\ (\lambda + \mu)\frac{\partial}{\partial z}(\operatorname{div} \vec{u}) + \mu\nabla^2 u_z + \varrho_0 f_z &= \varrho_0 \frac{\partial^2 u_z}{\partial t^2} \ , \end{aligned}$$

where f_x and f_z are components of the constant gravity acceleration g,

$$f_x = g \sin\alpha \ , \qquad f_z = -g\cos\alpha \ , \tag{8.74}$$

reduce to the form

$$\frac{d^2u_x}{dz^2} + \frac{\varrho_0 g \sin\alpha}{\mu} = 0 \ , \qquad \frac{d^2u_z}{dz^2} - \frac{\varrho_0 g\cos\alpha}{\lambda+2\mu} = 0 \ . \tag{8.75}$$

The double integration with respect to z yields a general solution of these equations:

$$u_x(z) = -\frac{\varrho_0 g\sin\alpha}{2\mu}z^2 + C_1 z + C_2 \ , \qquad u_z(z) = \frac{\varrho_0 g\cos\alpha}{2(\lambda+2\mu)}z^2 + C_3 z + C_4 \ , \tag{8.76}$$

where C_i, $i = 1,\dots,4$, are constants. The boundary condition (8.71) yields $C_2 = C_4 = 0$, while (8.73) implies that $C_1 = \varrho_0 gH\sin\alpha/\mu$ and $C_3 = -\varrho_0 gH\cos\alpha/(\lambda+2\mu)$. The solution for the displacement components can finally be written in the form

$$u_x(z) = \frac{\varrho_0 gH\sin\alpha}{\mu}\left(z - \frac{z^2}{2H}\right) \ , \qquad u_z(z) = -\frac{\varrho_0 gH\cos\alpha}{\lambda+2\mu}\left(z-\frac{z^2}{2H}\right) \ . \tag{8.77}$$

We can see that the displacement components increase monotonically from zero values at the base to their maximum values at the upper free surface. The parabolic displacement $u_x(z)$ within the slab is sketched in Figure 8.2. Note that the y-component of the Navier-Cauchy equation is identically satisfied. Hence, the problem is fully solved by (8.77).

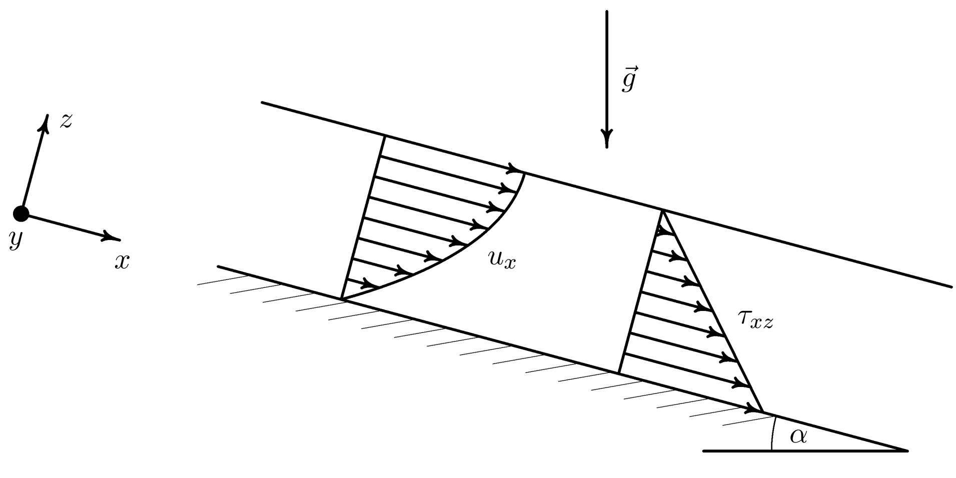

Figure 8.2. The displacement u_x and stress τ_{xz} in the elastic plate deformed by its own weight.

We further determine the components of the stress tensor by Hooke's law (8.21) for a linear elastic, isotropic body:

$$\tau_{xx} = \lambda\frac{\partial u_z}{\partial z} = -\frac{\lambda}{\lambda+2\mu}\varrho_0 gH\cos\alpha\left(1-\frac{z}{H}\right) \ ,$$
$$\tau_{yy} = \tau_{xx} \ ,$$

$$
\begin{aligned}
\tau_{zz} &= (\lambda + 2\mu)\frac{\partial u_z}{\partial z} = -\varrho_0 g H \cos\alpha\Big(1 - \frac{z}{H}\Big) \ , \\
\tau_{xy} &= \tau_{yz} = 0 \ , \\
\tau_{xz} &= \mu\frac{\partial u_x}{\partial z} = \varrho_0 g H \sin\alpha\Big(1 - \frac{z}{H}\Big) \ .
\end{aligned} \tag{8.78}
$$

The stress components are linear functions of the height of the elastic slab and vanish at the upper free surface. Figure 8.2 shows schematically the linear behaviour of stress τ_{xz} within the slab. It is interesting to note that the normal component τ_{yy} does not vanish even all the strain components vanish in y-direction.

In the particular case when the plate is not inclined ($\alpha = 0$), equation (8.77) reduces to

$$
u_x(z) = 0 \ , \qquad u_z(z) = -\frac{\varrho_0 g H}{\lambda + 2\mu}\left(z - \frac{z^2}{2H}\right) \ ; \tag{8.79}
$$

the upper free surface deforms due to the own weight of the plate by

$$
u_z(H) = -\frac{\varrho_0 g H^2}{2(\lambda + 2\mu)} \ . \tag{8.80}
$$

For the Earth's lithosphere of 100 km thickness with material parameters $\lambda = 1.5 \times 10^{11}$ N/m^2, $\mu = 0.67 \times 10^{11}$ N/m^2 and $\varrho_0 = 3300$ kg/m^3, we have

$$
u_z(H) = -\frac{3300\,\mathrm{kg/m^3} \times 9.81\,\mathrm{m/s^2} \times 10^{10}\,\mathrm{m^2}}{5.68 \times 10^{11}\,\mathrm{N/m^2}} \approx -570\,\mathrm{m} \ .
$$

9. SMALL MOTIONS IN A MEDIUM WITH A FINITE PRE-STRESS

9.1 Equations for the initial state

We consider a body $\mathcal{B}$ to be composed of a number of solid and fluid regions. We denote the volume of solid regions by V_S, and the volume of fluid regions by V_F. The entire volume of the body will be denoted by $V = V_S \cup V_F$. The solid and fluid regions are separated by non-intersecting, smooth and closed surfaces, called internal discontinuities. We denote all the internal welded solid-solid discontinuities between solid regions by Σ_{SS}, and all the internal slipping fluid-solid discontinuities between fluid and solid regions by Σ_{FS}. The union of all internal discontinuities will be denoted by $\Sigma = \Sigma_{SS} \cup \Sigma_{FS}$. The exterior surface of $\mathcal{B}$ will be denoted by ∂V.

Let the body $\mathcal{B}$ occupy at time $t = 0$ the volume V in the configuration κ_0, which we hereafter use as the reference configuration of the body. We suppose that this configuration does **not** correspond to the natural, stress-free state of the body $\mathcal{B}$ but the body in the configuration κ_0 is pre-stressed by a finite stress in such a way that $\mathcal{B}$ is in a static equilibrium. In this configuration, the Cauchy stress and the two Piola-Kirchhoff stresses coincide; we denote this initial static stress by $\boldsymbol{t}_0(\vec{X})$, where $\vec{X}$ denotes the position of a material particle in the configuration κ_0. The static equilibrium is guaranteed by the static linear momentum equation:

$$\operatorname{Div} \boldsymbol{t}_0 + \varrho_0 \vec{f}_0 = 0 \qquad \text{in } V - \Sigma \,, \tag{9.1}$$

where $\varrho_0(\vec{X})$ is the density of the body in the configuration κ_0 and $\vec{f}_0(\vec{X})$ is the body force per unit mass in κ_0. Since the body $\mathcal{B}$ contains fluid regions, it is convenient to decompose the initial static stress $\boldsymbol{t}_0$ into the isotropic and deviatoric parts,

$$\boldsymbol{t}_0 = -p_0 \boldsymbol{I} + \boldsymbol{t}_0^D \,, \tag{9.2}$$

where the pressure $p_0 = -\frac{1}{3} \operatorname{tr} \boldsymbol{t}_0$ and the trace of the deviatoric part vanishes, $\operatorname{tr} \boldsymbol{t}_0^D = 0$. Equation of static equilibrium then transforms to the form

$$-\operatorname{Grad} p_0 + \operatorname{Div} \boldsymbol{t}_0^D + \varrho_0 \vec{f}_0 = 0 \qquad \text{in } V - \Sigma \,. \tag{9.3}$$

Since a fluid is unable to support shear stresses when it is in the static equilibrium, the static stress deviatoric $\boldsymbol{t}_0^D$ vanishes in fluid regions and the static linear momentum equation reduces to:

$$-\operatorname{Grad} p_0 + \varrho_0 \vec{f}_0 = 0 \qquad \text{in } V_F - \Sigma_{FS} \,. \tag{9.4}$$

The internal discontinuities Σ within the body are assumed to be material surfaces. The interface condition at welded discontinuities between two solids and at slipping discontinuities between a solid and a fluid is given by (4.46):

$$\left[\vec{N} \cdot \boldsymbol{t}_0 \right]_-^+ = \vec{0} \qquad \text{on } \Sigma \,, \tag{9.5}$$

where $\vec{N}$ is the unit outward normal to the discontinuity Σ. In addition, at a slipping discontinuity between a solid and a fluid, there can be no shear stresses in the static equilibrium on the fluid side and the stress vector $\vec{N} \cdot \boldsymbol{t}_0$ must be in the direction of the normal $\vec{N}$; the stress vector can be expressed on the both sides of the discontinuity in the form (4.53):

$$\vec{N} \cdot \boldsymbol{t}_0 = -p_0 \vec{N} \qquad \text{on } \Sigma_{\text{FS}} , \tag{9.6}$$

where the initial pressure p_0, $p_0 = -(\vec{N} \cdot \boldsymbol{t}_0 \cdot \vec{N})$, passes through Σ_{FS} continuously,

$$[p_0]_-^+ = 0 \qquad \text{on } \Sigma_{\text{FS}} . \tag{9.7}$$

On the outer free surface of $\mathcal{B}$, the stress vector must vanish,

$$\vec{N} \cdot \boldsymbol{t}_0 = \vec{0} \qquad \text{on } \partial V . \tag{9.8}$$

9.2 Application of an infinitesimal deformation

We now consider a time and space dependent infinitesimal deformation field, characterized by the displacement $\vec{u}(\vec{X}, t)$, superimposed upon the initial configuration κ_0. The superimposed displacement field deforms the body into another time-dependent configuration κ_t, which occupies volume V_t. Because of the smallness of the superimposed displacement field, the present configuration κ_t is very closed to the reference configuration κ_0. This motivates us to use the same coordinates to describe the position of particles in both the configurations κ_0 and κ_t. We adopt the Lagrangian description of motion and write the position of a material particle in the configuration κ_t in the form

$$\vec{x} = \vec{X} + \vec{u}(\vec{X}, t) . \tag{9.9}$$

We will apply the principle of geometrical linearization and develop field equations and interface conditions correct to first order in $\|\boldsymbol{H}\|$, $\boldsymbol{H} = \text{Grad}\, \vec{u}$, and neglect all terms of higher order than $O(\|\boldsymbol{H}\|)$. From mathematical point of view, the subsequent linearization process is self-consistent, and correct to first order in $\|\boldsymbol{H}\|$, if the ratio of an incremental stress caused by the infinitesimal deformation (9.9) compared to the initial static stress is also of the order of $O(\|\boldsymbol{H}\|)$, see Section 9.5.

9.3 Lagrangian and Eulerian increments

Time changes of any physical quantity $\mathcal{Q}$ that has a non-zero initial static value can be described by the *Eulerian (local)* or *Lagrangian (material)* increments q^E and q^L, defined by

$$\begin{aligned} q^E(\vec{x}, t) &:= q(\vec{x}, t) - q(\vec{x}, 0) , \\ q^L(\vec{X}, t) &:= Q(\vec{X}, t) - q(\vec{X}, 0) , \end{aligned} \tag{9.10}$$

where $q(\vec{x}, t)$ and $Q(\vec{X}, t)$ are the Eulerian and Lagrangian descriptions of quantity $\mathcal{Q}$ at the present configuration κ_t, respectively, and $q(\vec{x}, 0)$ and $q(\vec{X}, 0)$ are the initial static values of $\mathcal{Q}$ at the configurations κ_t and κ_0, respectively. The former are related by (1.22) and the latter by

$$\begin{aligned} q(\vec{x}, 0) &= q(\vec{X} + \vec{u}, 0) \\ &= q(\vec{X}, 0) + \vec{u} \cdot \text{Grad}\, q(\vec{X}, 0) + O(|\vec{u}|^2) . \end{aligned} \tag{9.11}$$

Note that the initial static values of $\mathcal{Q}$ at the configuration κ_0 can be denoted by either $q(\vec{X},0)$ or $Q(\vec{X},0)$, that is $q(\vec{X},0) \equiv Q(\vec{X},0)$. Inserting (9.9)–(9.11) into $(1.22)_1$ yields

$$\begin{aligned} Q(\vec{X},t) &= q(\vec{x}(\vec{X},t),t) \\ &= q(\vec{X}+\vec{u},0) + q^E(\vec{X}+\vec{u},t) \\ &= q(\vec{X},0) + \vec{u}\cdot \operatorname{Grad} q(\vec{X},0) + q^E(\vec{X}+\vec{u},t) + O(|\vec{u}|^2) \\ &= q(\vec{X},0) + \vec{u}\cdot \operatorname{Grad} q(\vec{X},0) + q^E(\vec{X},t) + O(|\vec{u}|^2) \\ &\overset{!}{=} q(\vec{X},0) + q^L(\vec{X},t)\ . \end{aligned}$$

Hence, correct to first order in $|\vec{u}|$, the Lagrangian and Eulerian increments are related by

$$q^L = q^E + \vec{u}\cdot \operatorname{Grad} q_0(\vec{X})\ , \tag{9.12}$$

where $q_0(\vec{X}) \equiv q(\vec{X},0)$. Note that we have dropped the dependence of q^L and q^E on the positions $\vec{X}$ and $\vec{x}$, respectively, since, in first-order theory, it is immaterial whether the increments q^L and q^E are regarded as functions of $\vec{X}$ or $\vec{x}$.

9.4 Linearized continuity equation

Adopting the above concept, the Eulerian and Lagrangian increments in density are defined by

$$\begin{aligned} \varrho^E &:= \varrho(\vec{x},t) - \varrho_0(\vec{x})\ , \\ \varrho^L &:= \varrho\!\!\!\varrho(\vec{X},t) - \varrho_0(\vec{X})\ , \end{aligned} \tag{9.13}$$

where $\varrho(\vec{x},t)$ and $\varrho\!\!\!\varrho(\vec{X},t)$ is the Eulerian and Lagrangian description of the density, respectively; they can be converted to each other by (1.22). The increments satisfy the first-order relation

$$\varrho^L = \varrho^E + \vec{u}\cdot \operatorname{Grad} \varrho_0(\vec{X})\ . \tag{9.14}$$

These two increments can be expressed in terms of displacement $\vec{u}$ by linearizing the Eulerian and Lagrangian conservation of mass laws. Inserting the decomposition $(9.13)_1$ into the Eulerian continuity equation (4.16) and integrating it with respect to time, we find that

$$\varrho^E = -\operatorname{Div}\left[\varrho_0(\vec{X})\vec{u}\right]\ , \tag{9.15}$$

correct to first order in $\|\boldsymbol{H}\|$. Inserting (1.106) together with $(9.13)_2$ into the Lagrangian conservation of mass equation (4.60) yields

$$\varrho^L = -\varrho_0(\vec{X})\operatorname{Div}\vec{u}\ , \tag{9.16}$$

correct to the same order. It is easy to validate that the last two expressions for the Eulerian and Lagrangian increments in density are consistent with the general expression (9.14).

9.5 Increments in stress

We now consider the state of stress in the present configuration κ_t. The stress in κ_t referred to the area element in the reference configuration κ_0 is characterized by the first Piola-Kirchhoff stress tensor $\boldsymbol{T}^{(1)}$ (see section 3.5). Since the initial static stress for $\boldsymbol{T}^{(1)}$ (and also for the Cauchy stress tensor $\boldsymbol{t}$) is $\boldsymbol{t}_0(\vec{X})$, we express $\boldsymbol{T}^{(1)}$ in the form analogous to $(9.10)_2$

$$\boldsymbol{T}^{(1)}(\vec{X},t) = \boldsymbol{t}_0(\vec{X}) + \boldsymbol{T}^{(1),L} \ , \tag{9.17}$$

where $\boldsymbol{T}^{(1),L}$ is the Lagrangian increment of the first Piola-Kirchhoff stress tensor caused by the infinitesimal displacement $\vec{u}(\vec{X},t)$. We assume that the increment in stresses caused by the infinitesimal deformation is also small, of the order of $O(\|\boldsymbol{H}\|$:

$$\frac{\|\boldsymbol{T}^{(1),L}\|}{\|\boldsymbol{t}_0\|} \approx O(\|\boldsymbol{H}\|) \ . \tag{9.18}$$

The Lagrangian Cauchy stress tensor $\boldsymbol{t}$ is related to $\boldsymbol{T}^{(1)}$ by $(3.21)_2$:

$$\boldsymbol{t}(\vec{X},t) = J^{-1}\boldsymbol{F}\cdot\boldsymbol{T}^{(1)}(\vec{X},t) \ , \tag{9.19}$$

where J and $\boldsymbol{F}$ is the Jacobian and the deformation gradient of the infinitesimal deformation (9.9), respectively. Correct to the first order in $\|\boldsymbol{H}\|$, relation (9.19) can be expressed by $(3.29)_1$:

$$\boldsymbol{t} = (1 - \operatorname{tr}\boldsymbol{H})\,\boldsymbol{T}^{(1)} + \boldsymbol{H}^T\cdot\boldsymbol{T}^{(1)} + O(\|\boldsymbol{H}\|^2) \ . \tag{9.20}$$

Substituting from (9.17) into (9.20) yields the decomposition:

$$\boldsymbol{t}(\vec{X},t) = \boldsymbol{t}_0(\vec{X}) + \boldsymbol{t}^L \ , \tag{9.21}$$

where $\boldsymbol{t}^L$ is the Lagrangian increment of the Lagrangian Cauchy stress tensor $\boldsymbol{t}(\vec{X},t)$, which is related to the Lagrangian increment of the first Piola-Kirchhoff stress tensor by the relation

$$\boldsymbol{t}^L = \boldsymbol{T}^{(1),L} - (\operatorname{tr}\boldsymbol{H})\boldsymbol{t}_0(\vec{X}) + \boldsymbol{H}^T\cdot\boldsymbol{t}_0(\vec{X}) + O(\|\boldsymbol{H}\|^2) \ . \tag{9.22}$$

Equations (9.17) and (9.22) can be combined to give

$$\boldsymbol{T}^{(1)}(\vec{X},t) = \boldsymbol{t}_0(\vec{X}) + (\operatorname{tr}\boldsymbol{H})\boldsymbol{t}_0(\vec{X}) - \boldsymbol{H}^T\cdot\boldsymbol{t}_0(\vec{X}) + \boldsymbol{t}^L + O(\|\boldsymbol{H}\|^2) \ . \tag{9.23}$$

Therefore, the knowledge of displacement gradients, the initial stress distribution and the increment in the Cauchy stress tensor determine the first Piola-Kirchhoff stress tensor.

Beside the Lagrangian increment $\boldsymbol{t}^L$ of the Cauchy stress tensor we can introduce the Eulerian increment $\boldsymbol{t}^E$ of the Cauchy stress tensor by definition

$$\boldsymbol{t}^E := \boldsymbol{t}(\vec{x},t) - \boldsymbol{t}_0(\vec{x}) \ , \tag{9.24}$$

where $\boldsymbol{t}(\vec{x},t)$ is the Eulerian description of the Cauchy stress tensor. At a fixed point in space, the Lagrangian and Eulerian increments are related by the first-order relation (9.12):

$$\boldsymbol{t}^L = \boldsymbol{t}^E + \vec{u}\cdot\operatorname{Grad}\boldsymbol{t}_0(\vec{X}) \ . \tag{9.25}$$

9.6 Linearized equation of motion

The exact form of the equation of motion in the Lagrangian description is given by (4.67):

$$\operatorname{Div} \boldsymbol{T}^{(1)} + \varrho_0 \vec{F} = \varrho_0 \frac{\partial^2 \vec{u}}{\partial t^2} \qquad \text{in } V - \Sigma \ , \tag{9.26}$$

where $\vec{F}(\vec{X}, t)$ is the Lagrangian description of the body force per unit mass. Time changes of the body force can be described by either the Eulerian increment $\vec{f}^E$ or the Lagrangian increments $\vec{f}^L$, defined by

$$\begin{aligned} \vec{f}^E &:= \vec{f}(\vec{x}, t) - \vec{f}_0(\vec{x}) \ , \\ \vec{f}^L &:= \vec{F}(\vec{X}, t) - \vec{f}_0(\vec{X}) \ , \end{aligned} \tag{9.27}$$

where $\vec{f}(\vec{x}, t)$ is the Eulerian description of the body force, $\vec{f}(\vec{x}, t) = \vec{F}(\vec{X}(\vec{x}, t), t)$. The increments satisfy the usual first-order relation:

$$\vec{f}^L = \vec{f}^E + \vec{u} \cdot \operatorname{Grad} \vec{f}_0(\vec{X}) \ . \tag{9.28}$$

To obtain the equation of motion in the Lagrangian increments, we substitute the representation (9.17) and $(9.27)_2$ into the exact relation (9.26). Subtracting the static equilibrium equation (9.1), we obtain

$$\operatorname{Div} \boldsymbol{T}^{(1),L} + \varrho_0 \vec{f}^L = \varrho_0 \frac{\partial^2 \vec{u}}{\partial t^2} \qquad \text{in } V - \Sigma \ . \tag{9.29}$$

This equation of motion is exactly valid in the initial configuration κ_0 of the body, that is, everywhere in the volume $V - \Sigma$.

We shall now use this form for linearization in order to express the equation of motion in terms of the Lagrangian increment of the Cauchy stress tensor, which is more convenient to be defined by the constitutive equation than the Lagrangian increment of the first Piola-Kirchhoff stress tensor. We substitute (9.22) into (9.29), neglect the terms of second order in $\|\boldsymbol{H}\|$, and obtain

$$\operatorname{Div} \boldsymbol{t}^L + \operatorname{Div} [(\operatorname{tr} \boldsymbol{H}) \boldsymbol{t}_0] - \operatorname{Div} (\boldsymbol{H}^T \cdot \boldsymbol{t}_0) + \varrho_0 \vec{f}^L = \varrho_0 \frac{\partial^2 \vec{u}}{\partial t^2} \qquad \text{in } V - \Sigma \ . \tag{9.30}$$

By making use of the differential identities (A.14), (A.22) and (A.24), and recalling that $\operatorname{tr} \boldsymbol{H} = \operatorname{Div} \vec{u}$, the second and the third term on the left-hand side of (9.30) can be simplified as

$$\operatorname{Div} [(\operatorname{tr} \boldsymbol{H}) \boldsymbol{t}_0] - \operatorname{Div} (\boldsymbol{H}^T \cdot \boldsymbol{t}_0) = (\operatorname{tr} \boldsymbol{H}) \operatorname{Div} \boldsymbol{t}_0 - \boldsymbol{H} : \operatorname{Grad} \boldsymbol{t}_0 \ .$$

In view of this and the static linear momentum equation (9.1), the equation of motion (9.30) becomes

$$\operatorname{Div} \boldsymbol{t}^L + \varrho_0 \vec{f}^L + \varrho^L \vec{f}_0 - \boldsymbol{H} : \operatorname{Grad} \boldsymbol{t}_0 = \varrho_0 \frac{\partial^2 \vec{u}}{\partial t^2} \qquad \text{in } V - \Sigma \ . \tag{9.31}$$

Decomposing the initial static stress $\boldsymbol{t}_0$ into the isotropic and deviatoric parts according to (9.2), the second term on the left-hand side of (9.31) can be arranged as

$$\boldsymbol{H} : \operatorname{Grad} \boldsymbol{t}_0 = -\boldsymbol{H} \cdot \operatorname{Grad} p_0 + \boldsymbol{H} : \operatorname{Grad} \boldsymbol{t}_0^D \ .$$

Eliminating $\operatorname{Grad} p_0$ by means of (9.3), the linearized form of the Lagrangian equation of motion can finally be written in the form

$$\operatorname{Div} \boldsymbol{t}^L + \varrho_0 (\vec{f}^L + \boldsymbol{H} \cdot \vec{f}_0) + \varrho^L \vec{f}_0 - \boldsymbol{H} : \operatorname{Grad} \boldsymbol{t}_0^D + \boldsymbol{H} \cdot \operatorname{Div} \boldsymbol{t}_0^D = \varrho_0 \frac{\partial^2 \vec{u}}{\partial t^2} \qquad \text{in } V - \Sigma \ . \tag{9.32}$$

In the fluid regions $\boldsymbol{t}_0^D = \boldsymbol{0}$, and the equation of motion reduces to

$$\operatorname{Div} \boldsymbol{t}^L + \varrho_0(\vec{f}^L + \boldsymbol{H} \cdot \vec{f}_0) + \varrho^L \vec{f}_0 = \varrho_0 \frac{\partial^2 \vec{u}}{\partial t^2} \qquad \text{in } V_\text{F} - \Sigma_\text{FS} \ . \tag{9.33}$$

The equation of motion (9.31) can be rewritten explicitly in terms of the Eulerian increment in the Cauchy stress. Applying operator Div on (9.25), using the differential identity (A.23) and substituting for $\operatorname{Div} \boldsymbol{t}_0$ from the static momentum equation (9.1), we obtain

$$\operatorname{Div} \boldsymbol{t}^L = \operatorname{Div} \boldsymbol{t}^E + \boldsymbol{H} : \operatorname{Grad} \boldsymbol{t}_0 - \vec{u} \cdot \operatorname{Grad} (\varrho_0 \vec{f}_0) \ .$$

In view of this, the differential identity (A.2) and the relation (9.28) between the Lagrangian and Eulerian increments in body force, the equation of motion (9.31) transforms to

$$\operatorname{Div} \boldsymbol{t}^E + \varrho_0 \vec{f}^E + \varrho^E \vec{f}_0 = \varrho_0 \frac{\partial^2 \vec{u}}{\partial t^2} \ , \tag{9.34}$$

where ϱ^E is the Eulerian increment in density. Strictly speaking, this equation is valid at a point $\vec{x}$ within the volume $v(t) - \sigma(t)$ of the present configuration κ_t of the body rather than at a point $\vec{X}$ within the volume $V - \Sigma$ of the initial configuration κ_0. However, correct to first order in $\|\boldsymbol{H}\|$, this distinction is immaterial.

The final interesting form of the equation of motion in this context is

$$\operatorname{Div} \boldsymbol{t}^L + \operatorname{Grad} (\varrho_0 \vec{u} \cdot \vec{f}_0) + \varrho_0 \vec{f}^E + \varrho^E \vec{f}_0 + \operatorname{Grad} (\vec{u} \cdot \operatorname{Div} \boldsymbol{t}_0^D) - \operatorname{Div} (\vec{u} \cdot \operatorname{Grad} \boldsymbol{t}_0^D) = \varrho_0 \frac{\partial^2 \vec{u}}{\partial t^2} \ , \tag{9.35}$$

which can be derived from (9.32) by making use of the identity

$$-\boldsymbol{H} : \operatorname{Grad} \boldsymbol{t}_0^D + \boldsymbol{H} \cdot \operatorname{Div} \boldsymbol{t}_0^D = \operatorname{Grad} (\vec{u} \cdot \operatorname{Div} \boldsymbol{t}_0^D) - \operatorname{Div} (\vec{u} \cdot \operatorname{Grad} \boldsymbol{t}_0^D) + \operatorname{Grad} (\varrho_0 \vec{f}_0) \cdot \vec{u} - \vec{u} \cdot \operatorname{Grad} (\varrho_0 \vec{f}_0). \tag{9.36}$$

The most suitable forms of the equation of motion valid within the volume $V - \Sigma$ are summarized for convenience in Table 9.1.

Field variables	Linearized equation of motion
$\boldsymbol{T}^{(1),L}$, $\vec{f}^L$, $\vec{u}$	$\operatorname{Div} \boldsymbol{T}^{(1),L} + \varrho_0 \vec{f}^L = \varrho_0 \frac{\partial^2 \vec{u}}{\partial t^2}$ (exact)
$\boldsymbol{t}^L$, $\vec{f}^L$, $\vec{u}$	$\operatorname{Div} \boldsymbol{t}^L + \varrho_0 \vec{f}^L + \varrho^L \vec{f}_0 - \operatorname{Grad} \vec{u} : \operatorname{Grad} \boldsymbol{t}_0 = \varrho_0 \frac{\partial^2 \vec{u}}{\partial t^2}$
	$\operatorname{Div} \boldsymbol{t}^L + \varrho_0(\vec{f}^L + \operatorname{Grad} \vec{u} \cdot \vec{f}_0) + \varrho^L \vec{f}_0$ $-\operatorname{Grad} \vec{u} : \operatorname{Grad} \boldsymbol{t}_0^D + \operatorname{Grad} \vec{u} \cdot \operatorname{Div} \boldsymbol{t}_0^D = \varrho_0 \frac{\partial^2 \vec{u}}{\partial t^2}$
$\boldsymbol{t}^L$, $\vec{f}^E$, $\vec{u}$	$\operatorname{Div} \boldsymbol{t}^L + \varrho_0(\vec{f}^E + \vec{u} \cdot \operatorname{Grad} \vec{f}_0) + \varrho^L \vec{f}_0 - \operatorname{Grad} \vec{u} : \operatorname{Grad} \boldsymbol{t}_0 = \varrho_0 \frac{\partial^2 \vec{u}}{\partial t^2}$
	$\operatorname{Div} \boldsymbol{t}^L + \operatorname{Grad} (\varrho_0 \vec{u} \cdot \vec{f}_0) + \varrho_0 \vec{f}^E + \varrho^E \vec{f}_0$ $+\operatorname{Grad} (\vec{u} \cdot \operatorname{Div} \boldsymbol{t}_0^D) - \operatorname{Div} (\vec{u} \cdot \operatorname{Grad} \boldsymbol{t}_0^D) = \varrho_0 \frac{\partial^2 \vec{u}}{\partial t^2}$
$\boldsymbol{t}^E$, $\vec{f}^E$, $\vec{u}$	$\operatorname{Div} \boldsymbol{t}^E + \varrho_0 \vec{f}^E + \varrho^E \vec{f}_0 = \varrho_0 \frac{\partial^2 \vec{u}}{\partial t^2}$

Table 9.1. Summary of various forms of the equation of motion.

9.7 Linearized interface conditions

The linearized equation of motion must be supplemented by the kinematic and dynamic interface conditions at the internal discontinuities.

9.7.1 Kinematic interface conditions

The kinematic interface condition on the welded solid-solid discontinuities is given by (4.49). We transform the Eulerian form (4.49) to the Lagrangian form and obtain

$$[\vec{V}]_-^+ = \vec{0} \qquad \text{on } \Sigma_{\text{SS}} \ , \tag{9.37}$$

where $\vec{V}(\vec{X},t) := \vec{v}(\vec{x}(\vec{X},t),t)$ is the Lagrangian description of the velocity. We substitute for $\vec{V}(\vec{X},t)$ from $(2.5)_1$ and integrate the result with respect to time. Choosing $\vec{u}(\vec{X},0) = \vec{0}$, the interface condition (9.37) transforms to

$$[\vec{u}]_-^+ = \vec{0} \qquad \text{on } \Sigma_{\text{SS}} \ . \tag{9.38}$$

Note that the interface condition (9.38) is exact, like the incremental Lagrangian linear momentum equation (9.29).

On the fluid-solid discontinuities, tangential slip is allowed and, hence, the interface condition (4.49) must be replaced by (4.51). Considering the transformation (1.123) between the unit normal $\vec{n}$ to the deformed discontinuity σ and the unit normal $\vec{N}$ to the undeformed discontinuity Σ and integrating (4.51) with respect to time, we obtain

$$[\vec{N}\cdot\vec{u}]_-^+ = 0 \qquad \text{on } \Sigma_{\text{FS}} \ , \tag{9.39}$$

correct to first order in $\|\boldsymbol{H}\|$. This equation is the first-order condition that guarantees that there is no separation or interpenetration of the material on either side of the fluid-solid discontinuity Σ_{FS}.

9.7.2 Dynamic interface conditions

The dynamic interface condition on the welded boundaries Σ_{SS} can readily be obtained from the interface condition (4.69). Realizing that Σ_{SS} are material non-slipping discontinuities across which the initial surface element dA changes continuously, scalar factor dA can be dropped from (4.69) and we have

$$[\vec{N}\cdot\boldsymbol{T}^{(1)}]_-^+ = \vec{0} \qquad \text{on } \Sigma_{\text{SS}} \ . \tag{9.40}$$

Substituting for $\boldsymbol{T}^{(1)}$ from (9.17) and subtracting the static interface condition (9.5), we obtain

$$[\vec{N}\cdot\boldsymbol{T}^{(1),L}]_-^+ = \vec{0} \qquad \text{on } \Sigma_{\text{SS}} \ . \tag{9.41}$$

The corresponding condition on the outer free surface ∂V is simply

$$\vec{N}\cdot\boldsymbol{T}^{(1),L} = \vec{0} \qquad \text{on } \partial V \ . \tag{9.42}$$

Both (9.41) and (9.42) are exact, like the kinematic interface condition (9.38).

The dynamic interface condition on the slipping fluid-solid discontinuities Σ_{FS} requires more consideration. Figure 9.1 shows a small portion of a slipping boundary; $\vec{N}^+dA^+$ and $\vec{N}^-dA^-$ are two initial elements of surface area centered on particles $\vec{X}^+$ and $\vec{X}^-$ lying on the upper and lower side of an undeformed discontinuity $\Sigma_{\rm FS}$. Upon deformation, both particles $\vec{X}^+$ and $\vec{X}^-$ move to the same point $\vec{x}$ and the surface elements $\vec{N}^+dA^+$ and $\vec{N}^-dA^-$ merge to form the continuous element $\vec{n}da$ on the deformed discontinuity σ. Since $\Sigma_{\rm FS}$ are material discontinuities, the interface condition (4.69) reduces to

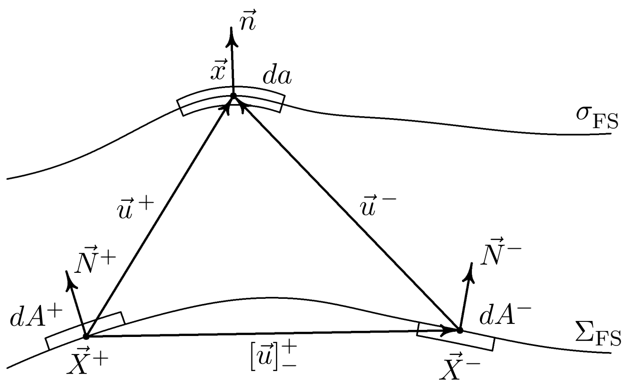

Figure 9.1. A small portion of a fluid-solid discontinuity before ($\Sigma_{\rm FS}$) and after ($\sigma_{\rm FS}$) deformation.

$$\left[\vec{N}dA\cdot\boldsymbol{T}^{(1)}\right]_-^+ = \vec{0} \qquad \text{on } \Sigma_{\rm FS}\,. \tag{9.43}$$

Substituting for $\boldsymbol{T}^{(1)}$ from (9.17) and for dA from (1.117), this exact condition can be written, correct to first order in $\|\boldsymbol{H}\|$, in the form

$$\left[\vec{N}\cdot(\boldsymbol{t}_0+\boldsymbol{T}^{(1),L})(1-\mathrm{Div}_\Sigma\vec{u})da\right]_-^+ = \vec{0} \qquad \text{on } \Sigma_{\rm FS}\,. \tag{9.44}$$

where $da = da(\vec{X},t)$ is the Lagrangian description of the deformed surface element of the discontinuity σ. The continuity of the Eulerian surface element $da(\vec{x},t)$ on σ, that is, the condition $[da(\vec{x},t)]_-^+ = 0$ on σ, implies the continuity of the Lagrangian surface element $da(\vec{X},t)$ on Σ, that is, the condition $\left[da(\vec{X},t)\right]_-^+ = 0$ on Σ. Consequently, the factor da can be dropped from the condition (9.44). In addition, since the initial static stress vector $\vec{N}\cdot\boldsymbol{t}_0$ is continuous, the offset quantities $\vec{N}^+\cdot\boldsymbol{t}_0^+$ and $\vec{N}^-\cdot\boldsymbol{t}_0^-$ are related by

$$\vec{N}^+\cdot\boldsymbol{t}_0^+ = \vec{N}^-\cdot\boldsymbol{t}_0^- - [\vec{u}]_-^+\cdot\mathrm{Grad}_\Sigma(\vec{N}\cdot\boldsymbol{t}_0) \qquad \text{on } \Sigma_{\rm FS}\,, \tag{9.45}$$

where the superscripts $\pm$ denote evaluation at $\vec{X}^\pm$. Correct to first order in $|\vec{u}|$, it is immaterial whether the tensor $\mathrm{Grad}_\Sigma(\vec{N}\cdot\boldsymbol{t}_0)$ is taken at $\vec{X}^\pm$. Using (9.45) and the differential identity $(B.18)_5$, equation (9.44) can be put into the form

$$\left[\vec{N}\cdot\boldsymbol{T}^{(1),L} - \mathrm{Div}_\Sigma[\vec{u}\otimes(\vec{N}\cdot\boldsymbol{t}_0)]\right]_-^+ = \vec{0} \qquad \text{on } \Sigma_{\rm FS}\,. \tag{9.46}$$

This linearized interface condition guarantees the continuity of stress vector across a slipping boundary. At a welded discontinuity, which experiences zero slip, the displacement $\vec{u}$ is continuous and condition (9.46) reduces to (9.41).

In addition, on the fluid-solid boundaries, the initial stress vector is of the form (9.6). Using this, (9.46) reduces to

$$\left[\vec{N}\cdot\boldsymbol{T}^{(1),L} + \mathrm{Div}_\Sigma(p_0\vec{u}\otimes\vec{N})\right]_-^+ = \vec{0} \qquad \text{on } \Sigma_{\rm FS}\,, \tag{9.47}$$

which, by making use of the differential identity $(B.18)_5$, can also be written as

$$\left[\vec{N}\cdot\boldsymbol{T}^{(1),L} + \vec{N}\,\mathrm{Div}_\Sigma(p_0\vec{u}) + p_0\vec{u}\cdot\mathrm{Grad}_\Sigma\vec{N}\right]_-^+ = \vec{0} \qquad \text{on } \Sigma_{\rm FS}\,. \tag{9.48}$$

The kinematic condition (9.39) requires the normal component of displacement $\vec{u}$ to be continuous across $\Sigma_{\rm FS}$. This means that the changes in $\vec{N}\cdot\vec{u}$ must be equally large on both sides of the discontinuity $\Sigma_{\rm FS}$, that is,

$$\Big[\text{Grad}_\Sigma(\vec{N}\cdot\vec{u})\Big]_-^+ = \vec{0} \qquad \text{on } \Sigma_{\rm FS}\ . \tag{9.49}$$

Employing the identity (B.18)$_2$ and the symmetry of surface curvature tensor, $(\text{Grad}_\Sigma\vec{N})^T = \text{Grad}_\Sigma\vec{N}$, equation (9.49) can be considered in the form

$$\Big[\text{Grad}_\Sigma\vec{u}\cdot\vec{N} + \vec{u}\cdot\text{Grad}_\Sigma\vec{N}\Big]_-^+ = \vec{0} \qquad \text{on } \Sigma_{\rm FS}\ . \tag{9.50}$$

If we substitute this form for the last term in the condition (9.48) and use the continuity of the pressure p_0, the condition (9.48) can be written in the alternative form

$$\Big[\vec{N}\cdot\boldsymbol{T}^{(1),L} + \vec{N}\,\text{Div}_\Sigma(p_0\vec{u}) - p_0\text{Grad}_\Sigma\vec{u}\cdot\vec{N}\Big]_-^+ = \vec{0} \qquad \text{on } \Sigma_{\rm FS}\ . \tag{9.51}$$

This is the most useful form of the linearized dynamical continuity condition on $\Sigma_{\rm FS}$.

If the fluid-solid boundary is, in addition, frictionless, the Cauchy stress vector $\vec{n}\cdot\boldsymbol{t}$ must be in the direction of normal $\vec{n}$ to the deformed fluid-solid boundary $\sigma_{\rm FS}$, that is, its projection on $\sigma_{\rm FS}$ must vanish:

$$\vec{n}\cdot\boldsymbol{t}\cdot(\boldsymbol{I}-\vec{n}\otimes\vec{n}) = \vec{0} \qquad \text{on } \sigma_{\rm FS}\ . \tag{9.52}$$

Making use of (1.118), (9.21) and (9.22), the Eulerian Cauchy stress vector $\vec{n}\cdot\boldsymbol{t}$ at the deformed discontinuity $\sigma_{\rm FS}$ can be expressed in terms of the first Piola-Kirchhoff stress tensor on the undeformed discontinuity $\Sigma_{\rm FS}$. Correct to first order in $\|\boldsymbol{H}\|$, it holds

$$\begin{aligned}
\vec{n}\cdot\boldsymbol{t} &= [\vec{N} + (\vec{N}\cdot\boldsymbol{H}\cdot\vec{N})\vec{N} - \boldsymbol{H}\cdot\vec{N}]\cdot[\boldsymbol{t}_0 + \boldsymbol{T}^{(1),L} - (\text{tr}\,\boldsymbol{H})\boldsymbol{t}_0 + \boldsymbol{H}^T\cdot\boldsymbol{t}_0]\\
&= \vec{N}\cdot\boldsymbol{t}_0 + \vec{N}\cdot\boldsymbol{T}^{(1),L} - (\text{tr}\,\boldsymbol{H})(\vec{N}\cdot\boldsymbol{t}_0) + \vec{N}\cdot(\boldsymbol{H}^T\cdot\boldsymbol{t}_0) + (\vec{N}\cdot\boldsymbol{H}\cdot\vec{N})(\vec{N}\cdot\boldsymbol{t}_0) - (\boldsymbol{H}\cdot\vec{N})\cdot\boldsymbol{t}_0\\
&= \vec{N}\cdot\boldsymbol{t}_0 + \vec{N}\cdot\boldsymbol{T}^{(1),L} - (\text{tr}\,\boldsymbol{H} - \vec{N}\cdot\boldsymbol{H}\cdot\vec{N})(\vec{N}\cdot\boldsymbol{t}_0)\ ,
\end{aligned}$$

where we have abbreviated $\boldsymbol{t}_0 \equiv \boldsymbol{t}_0(\vec{X})$ and used the identity $\vec{N}\cdot(\boldsymbol{H}^T\cdot\boldsymbol{t}_0) = (\boldsymbol{H}\cdot\vec{N})\cdot\boldsymbol{t}_0$. The last term in the above equation can further be expressed in terms of the surface divergence of the displacement $\vec{u}$, see (1.120):

$$\vec{n}\cdot\boldsymbol{t} = \vec{N}\cdot\boldsymbol{t}_0 + \vec{N}\cdot\boldsymbol{T}^{(1),L} - (\text{Div}_\Sigma\vec{u})(\vec{N}\cdot\boldsymbol{t}_0) \qquad \text{on } \Sigma_{\rm FS}\ . \tag{9.53}$$

The dyadic product $\vec{n}\otimes\vec{n}$ occurring in the interface condition (9.52) can be written, correct to first order in $\|\boldsymbol{H}\|$, in the form

$$\vec{n}\otimes\vec{n} = \vec{N}\otimes\vec{N} - (\text{Grad}_\Sigma\vec{u}\cdot\vec{N})\otimes\vec{N} - \vec{N}\otimes(\text{Grad}_\Sigma\vec{u}\cdot\vec{N})\ . \tag{9.54}$$

Substituting (9.53) and (9.54) into (9.52) yields

$$[\vec{N}\cdot\boldsymbol{t}_0 + \vec{N}\cdot\boldsymbol{T}^{(1),L} - (\text{Div}_\Sigma\vec{u})(\vec{N}\cdot\boldsymbol{t}_0)]\cdot(\boldsymbol{I}-\vec{N}\otimes\vec{N}) + (\vec{N}\cdot\boldsymbol{t}_0)\cdot[(\text{Grad}_\Sigma\vec{u}\cdot\vec{N})\otimes\vec{N} + \vec{N}\otimes(\text{Grad}_\Sigma\vec{u}\cdot\vec{N})] = \vec{0}\ ,$$

which is valid on $\Sigma_{\rm FS}$. At this discontinuity, the initial stress vector $\vec{N}\cdot\boldsymbol{t}_0$ has the form (9.6), which helps to reduce the last equation:

$$(\vec{N}\cdot\boldsymbol{T}^{(1),L})\cdot(\boldsymbol{I}-\vec{N}\otimes\vec{N}) - p_0(\text{Grad}_\Sigma\vec{u}\cdot\vec{N}) = \vec{0}\ .$$

Using the identity (B.33), we can also write

$$[\vec{N} \cdot \boldsymbol{T}^{(1),L} - p_0(\mathrm{Grad}_\Sigma \vec{u} \cdot \vec{N})] \cdot (\boldsymbol{I} - \vec{N} \otimes \vec{N}) = \vec{0} \ .$$

Finally, we can add the term $\vec{N}\,\mathrm{Div}_\Sigma(p_0\vec{u}) \cdot (\boldsymbol{I} - \vec{N} \otimes \vec{N}) = \vec{0}$ for convenience:

$$[\vec{N} \cdot \boldsymbol{T}^{(1),L} + \vec{N}\,\mathrm{Div}_\Sigma(p_0\vec{u}) - p_0(\mathrm{Grad}_\Sigma \vec{u} \cdot \vec{N})] \cdot (\boldsymbol{I} - \vec{N} \otimes \vec{N}) = \vec{0} \ . \tag{9.55}$$

Correct to first order in $\|\boldsymbol{H}\|$, this equation guarantees that there is no shear stress vector on the fluid-solid boundary Σ_{FS}.

By comparing condition (9.51) and (9.55), we see that the quantity

$$\vec{\tau}^{(1),L} := \vec{N} \cdot \boldsymbol{T}^{(1),L} + \vec{N}\,\mathrm{Div}_\Sigma(p_0\vec{u}) - p_0(\mathrm{Grad}_\Sigma \vec{u} \cdot \vec{N}) \tag{9.56}$$

is a continuous, normal vector on Σ_{FS}:

$$\left[\vec{\tau}^{(1),L}\right]_-^+ = \vec{0} \ , \tag{9.57}$$

$$\vec{\tau}^{(1),L} = (\vec{N} \cdot \vec{\tau}^{(1),L})\vec{N} \ . \tag{9.58}$$

At a solid-solid discontinuity Σ_{SS}, the last two term in the definition (9.56) of $\vec{\tau}^{(1),L}$ are continuous and the condition (9.54) coincides with the condition (9.41). Therefore, frictionless fluid-solid and welded solid-solid discontinuities are distinguished by the normality condition (9.58) only.

All above derived condition can be expressed in terms of the incremental Lagrangian Cauchy stress vector $\vec{N} \cdot \boldsymbol{t}^L$ rather than $\vec{N} \cdot \boldsymbol{T}^{(1),L}$ using the relation (9.22). The left scalar product of (9.22) with the normal $\vec{N}$ yields

$$\vec{N} \cdot \boldsymbol{T}^{(1),L} = \vec{N} \cdot \boldsymbol{t}^L + (\mathrm{tr}\,\boldsymbol{H})(\vec{N} \cdot \boldsymbol{t}_0) - (\boldsymbol{t}_0 \cdot \boldsymbol{H}) \cdot \vec{N} \ ,$$

where we have used the identity $\vec{N} \cdot (\boldsymbol{H}^T \cdot \boldsymbol{t}_0) = (\boldsymbol{t}_0 \cdot \boldsymbol{H}) \cdot \vec{N}$ which follows from the symmetry of $\boldsymbol{t}_0$. Decomposing the initial stress into the isotropic and deviatoric parts, see (9.2), and using (B.30) and (B.31) results in

$$\vec{N} \cdot \boldsymbol{T}^{(1),L} = \vec{N} \cdot \boldsymbol{t}^L + p_0(\mathrm{Grad}_\Sigma \vec{u} \cdot \vec{N} - \vec{N}\,\mathrm{Div}_\Sigma \vec{u}) + (\mathrm{tr}\,\boldsymbol{H})(\vec{N} \cdot \boldsymbol{t}_0^D) - (\boldsymbol{t}_0^D \cdot \boldsymbol{H}) \cdot \vec{N} \ , \tag{9.59}$$

which is valid for both the solid-solid interface Σ_{SS} and the fluid-solid interface Σ_{FS}. Moreover, the auxiliary vector $\vec{\tau}^{(1),L}$, which is defined at the fluid-solid interface Σ_{FS}, can be expressed in the form

$$\vec{\tau}^{(1),L} = \vec{N} \cdot \boldsymbol{t}^L + (\mathrm{Grad}_\Sigma p_0 \cdot \vec{u})\vec{N} \ . \tag{9.60}$$

The complete set of the interface conditions on Σ is summarized for convenience in Table 9.2.

Discontinuity type	Exact and linearized interface condition
∂V: free surface	$\vec{N}\cdot\boldsymbol{T}^{(1),L}=\vec{0}$ (exact)
Σ_{SS}: solid-solid	$[\vec{u}]_-^+=\vec{0}$ (exact)
	$[\vec{N}\cdot\boldsymbol{T}^{(1),L}]_-^+=\vec{0}$ (exact)
Σ_{FS}: fluid-solid	$[\vec{N}\cdot\vec{u}]_-^+=0$
	$[\vec{\tau}^{(1),L}]_-^+=[\vec{N}\cdot\vec{\tau}^{(1),L}]_-^+\vec{N}=\vec{0}$
$\vec{\tau}^{(1),L}:=\vec{N}\cdot\boldsymbol{T}^{(1),L}+\vec{N}\,\mathrm{Div}_\Sigma(p_0\vec{u})-p_0(\mathrm{Grad}_\Sigma\vec{u}\cdot\vec{N})$	

Table 9.2. Summary of exact and linearized interface conditions.

9.8 Linearized elastic constitutive equation

All of the equations we have derived so far can be regarded as linearized balance laws of either geometry or physics. They are valid regardless of the constitution of the material. To complete these equations, we specify the constitutive relation between the incremental stresses and displacement gradient. Here we will assume that the material behavior for the infinitesimal superimposed deformation is linear and elastic, as discussed in section 8.1.

The linearized constitutive equations for the incremental first Piola-Kirchhoff stress $\boldsymbol{T}^{(1),L}$ the incremental Lagrangian Cauchy stress $\boldsymbol{t}^L$ can be obtained from (8.16) and (8.17) using the decompositions (9.17) and (9.21):

$$\begin{aligned}\boldsymbol{T}^{(1),L} &= \boldsymbol{C}:\tilde{\boldsymbol{E}}-\boldsymbol{t}_0\cdot\tilde{\boldsymbol{R}}+a[(\mathrm{tr}\,\tilde{\boldsymbol{E}})\boldsymbol{t}_0+(\boldsymbol{t}_0:\tilde{\boldsymbol{E}})\boldsymbol{I}]+(b+c)(\tilde{\boldsymbol{E}}\cdot\boldsymbol{t}_0)\\ &\quad+(b+c+1)(\boldsymbol{t}_0\cdot\tilde{\boldsymbol{E}})\ , \qquad (9.61)\\ \boldsymbol{t}^L &= \boldsymbol{C}:\tilde{\boldsymbol{E}}+\tilde{\boldsymbol{R}}\cdot\boldsymbol{t}_0-\boldsymbol{t}_0\cdot\tilde{\boldsymbol{R}}+(a-1)(\mathrm{tr}\,\tilde{\boldsymbol{E}})\boldsymbol{t}_0+a(\boldsymbol{t}_0:\tilde{\boldsymbol{E}})\boldsymbol{I}\\ &\quad+(b+c+1)(\boldsymbol{t}_0\cdot\tilde{\boldsymbol{E}}+\tilde{\boldsymbol{E}}\cdot\boldsymbol{t}_0)\ . \qquad (9.62)\end{aligned}$$

Every choice of the scalars a, b and c defines the behavior of a linear elastic solid. The most convenient alternative, adopted by Dahlen and Tromp (1998), is

$$a=-b=-c=\frac{1}{2}\ . \qquad (9.63)$$

Under this choice, the equation (9.61) and (9.62) reduce to

$$\boldsymbol{T}^{(1),L}=\boldsymbol{C}:\tilde{\boldsymbol{E}}-\boldsymbol{t}_0\cdot\tilde{\boldsymbol{R}}-\tilde{\boldsymbol{E}}\cdot\boldsymbol{t}_0+\frac{1}{2}(\mathrm{tr}\,\tilde{\boldsymbol{E}})\boldsymbol{t}_0+\frac{1}{2}(\boldsymbol{t}_0:\tilde{\boldsymbol{E}})\boldsymbol{I}\ , \qquad (9.64)$$

$$\boldsymbol{t}^L=\boldsymbol{C}:\tilde{\boldsymbol{E}}+\tilde{\boldsymbol{R}}\cdot\boldsymbol{t}_0-\boldsymbol{t}_0\cdot\tilde{\boldsymbol{R}}-\frac{1}{2}(\mathrm{tr}\,\tilde{\boldsymbol{E}})\boldsymbol{t}_0+\frac{1}{2}(\boldsymbol{t}_0:\tilde{\boldsymbol{E}})\boldsymbol{I}\ . \qquad (9.65)$$

By the decomposition (9.2) of the initial static stress into the isotropic and deviatoric parts, we also have

$$\boldsymbol{T}^{(1),L}=\boldsymbol{C}:\tilde{\boldsymbol{E}}+p_0[\tilde{\boldsymbol{E}}+\tilde{\boldsymbol{R}}-(\mathrm{tr}\,\tilde{\boldsymbol{E}})\boldsymbol{I}]-\boldsymbol{t}_0^D\cdot\tilde{\boldsymbol{R}}-\tilde{\boldsymbol{E}}\cdot\boldsymbol{t}_0^D+\frac{1}{2}(\mathrm{tr}\,\tilde{\boldsymbol{E}})\boldsymbol{t}_0^D+\frac{1}{2}(\boldsymbol{t}_0^D:\tilde{\boldsymbol{E}})\boldsymbol{I}, \qquad (9.66)$$

$$\boldsymbol{t}^L=\boldsymbol{C}:\tilde{\boldsymbol{E}}+\tilde{\boldsymbol{R}}\cdot\boldsymbol{t}_0^D-\boldsymbol{t}_0^D\cdot\tilde{\boldsymbol{R}}-\frac{1}{2}(\mathrm{tr}\,\tilde{\boldsymbol{E}})\boldsymbol{t}_0^D+\frac{1}{2}(\boldsymbol{t}_0^D:\tilde{\boldsymbol{E}})\boldsymbol{I}\ . \qquad (9.67)$$

It deserves to note that the incremental Lagrangian Cauchy stress $\boldsymbol{t}^L$ does not depend explicitly on the initial pressure p_0, but only on the elastic tensor $\boldsymbol{C}$ and the initial deviatoric stress $\boldsymbol{t}_0^D$. In fact, this is what motivated us to set $a = -b = -c = 1/2$ in equations (9.61) and (9.62). This is the only choice that completely eliminates the explicit dependence of $\boldsymbol{t}^L$ on p_0.

9.9 Gravitational potential theory

So far we have not specified a body force acting in body $\mathcal{B}$. We will now assume that the Newton *gravitational attraction* is the body force acting in the body $\mathcal{B}$; we call $\mathcal{B}$ a *self-gravitating body.* Since the Newton gravitation is a *conservative force*, the Eulerian gravitational force per unit mass $\vec{g}(\vec{x}, t)$ is deriveable from an *Eulerian gravitational potential* $\phi(\vec{x}, t)$:

$$\vec{g} = -\operatorname{grad} \phi \ . \tag{9.68}$$

Given an instantaneous Eulerian density distribution $\varrho(\vec{x}, t)$, we can find $\phi(\vec{x}, t)$ by the Newton integral:

$$\phi(\vec{x}, t) = -G \int_{v(t)} \frac{\varrho(\vec{x}', t)}{\|\vec{x} - \vec{x}'\|} \, dv' \ , \tag{9.69}$$

where G is Newton's gravitational constant and the primes denote the dummy point of integration. The integration in (9.69) is carried out over the volume $v(t)$ which is filled in by a material body at time t, that is over all point $\vec{x}'$ where $\varrho(\vec{x}', t) > 0$. By combining (9.68) and (9.69), we obtain the gravitational attraction $\vec{g}(\vec{x}, t)$:

$$\vec{g}(\vec{x}, t) = -G \int_{v(t)} \frac{\varrho(\vec{x}', t)(\vec{x} - \vec{x}')}{\|\vec{x} - \vec{x}'\|^3} \, dv' \ . \tag{9.70}$$

It can be verified that the gravitational potential satisfies Poisson's equation:

$$\operatorname{div} \operatorname{grad} \phi = 4\pi G \varrho \qquad \text{in} \ \ v(t) - \sigma(t) \ . \tag{9.71}$$

At the internal discontinuities across which the density ϱ has a jump, the Poisson equation (9.71) must be supplemented by the interface conditions

$$[\phi]_-^+ = 0 \qquad \text{on} \ \ \sigma(t) \ , \tag{9.72}$$

$$[\vec{n} \cdot \operatorname{grad} \phi]_-^+ = 0 \qquad \text{on} \ \ \sigma(t) \ . \tag{9.73}$$

Note that the continuity of the scalars ϕ and $(\vec{n} \cdot \operatorname{grad} \phi)$ implies the continuity of the gravitational attraction vector $\vec{g} = -\operatorname{grad} \phi$.

Classical gravitational potential theory, as summarized above, is an inherently Eulerian theory. The quantities ϕ and $\vec{g}$ are the gravitational potential and gravitational attraction at a fixed point $\vec{x}$ in space. The corresponding Lagrangian variables can, however, readily be found from the relation

$$\Phi(\vec{X}, t) := \phi(\vec{x}(\vec{X}, t), t) \qquad \vec{G}(\vec{X}, t) := \vec{g}(\vec{x}(\vec{X}, t), t) \ . \tag{9.74}$$

Using (1.79) and (4.60), we transform (9.69) and (9.70) into integrals over the corresponding reference volume V,

$$\Phi(\vec{X}, t) = -G \int_V \frac{\varrho_0(\vec{X}')}{\|\vec{x}(\vec{X}, t) - \vec{x}(\vec{X}', t)\|} \, dV' \ , \tag{9.75}$$

$$\vec{G}(\vec{X}, t) = -G \int_V \frac{\varrho_0(\vec{X}')[\vec{x}(\vec{X}, t) - \vec{x}(\vec{X}', t)]}{\|\vec{x}(\vec{X}, t) - \vec{x}(\vec{X}', t)\|^3} \, dV' \ . \tag{9.76}$$

Making use of $(1.46)_3$, it is straightforward to rewrite the Poisson equation (9.71) in the Lagrangian form:

$$\boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-T} : \text{Grad}\,\text{Grad}\,\Phi + \text{div}\,\boldsymbol{F}^{-T} \cdot \text{Grad}\,\Phi = 4\pi G \varrho \qquad \text{in } V - \Sigma \;, \tag{9.77}$$

where $\varrho(\vec{X}, t)$ is the Lagrangian description of the density $\varrho(\vec{x}, t)$. To transform the interface conditions (9.72) and (9.73) to the undeformed discontinuity Σ, we apply $(1.46)_1$ and (1.77) and obtain:

$$[\Phi]_-^+ = 0 \qquad \text{on } \Sigma \;, \tag{9.78}$$

$$\left[\frac{\vec{N} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-T}}{\sqrt{\vec{N} \cdot \boldsymbol{B} \cdot \vec{N}}} \cdot \text{Grad}\,\Phi \right]_-^+ = 0 \qquad \text{on } \Sigma \;. \tag{9.79}$$

9.9.1 Equations for the initial state

Let the density $\varrho_0(\vec{X})$ in the initial configuration κ_0 generates the initial gravitational potential $\phi_0(\vec{X})$, and let

$$\vec{g}_0 = -\text{Grad}\,\phi_0 \tag{9.80}$$

denote the corresponding initial gravitational attraction. The two fields ϕ_0 and $\vec{g}_0$ are given in terms of ϱ_0 by

$$\phi_0(\vec{X}) = -G \int_V \frac{\varrho_0(\vec{X}')}{\|\vec{X} - \vec{X}'\|}\, dV' \;, \tag{9.81}$$

and

$$\vec{g}_0(\vec{X}) = -G \int_V \frac{\varrho_0(\vec{X}')(\vec{X} - \vec{X}')}{\|\vec{X} - \vec{X}'\|^3}\, dV' \;. \tag{9.82}$$

The density ϱ_0 is assumed to vanish outside the body, but ϕ_0 and $\vec{g}_0$ are both non-zero everywhere outside and inside the body. The potential ϕ_0 satisfies Poisson's equation [27]

$$\nabla^2 \phi_0 = 4\pi G \varrho_0 \qquad \text{in } V - \Sigma \;, \tag{9.83}$$

together with the interface conditions at the internal discontinuities across which the density ϱ_0 has a jump:

$$[\phi_0]_-^+ = 0 \qquad \text{on } \Sigma \;, \tag{9.84}$$

$$\left[\vec{N} \cdot \text{Grad}\,\phi_0 \right]_-^+ = 0 \qquad \text{on } \Sigma \;. \tag{9.85}$$

In the region outside the body, the potential is harmonic,

$$\nabla^2 \phi_0 = 0 \qquad \text{outside } V \;. \tag{9.86}$$

9.9.2 Increments in gravitation

[27] $\nabla^2 \equiv \text{Div}\,\text{Grad}$; we do not introduce a nabla operator for the Eulerian operator div grad .

The Eulerian and Lagrangian increments in gravitational potential are defined by

$$\begin{aligned} \phi(\vec{x},t) &=: \phi_0(\vec{x}) + \phi^E \ , \\ \Phi(\vec{X},t) &=: \phi_0(\vec{X}) + \phi^L \ , \end{aligned} \tag{9.87}$$

where $\phi(\vec{x},t)$ and $\Phi(\vec{X},t)$ is the Eulerian and Lagrangian description of the gravitational potential, respectively; they are related to each other by $(9.74)_1$. The increments satisfy the usual first-order relation

$$\phi^L = \phi^E + \vec{u} \cdot \mathrm{Grad}\,\phi_0(\vec{X}) \ . \tag{9.88}$$

Applying the operator $-\mathrm{grad}$ on $(9.87)_1$, the Eulerian gravitation attraction can be arranged as follows:

$$\vec{g}(\vec{x},t) = -\mathrm{grad}\,\phi(\vec{x},t) = -\mathrm{grad}\,\phi_0(\vec{x}) - \mathrm{grad}\,\phi^E = \vec{g}_0(\vec{x}) - \mathrm{grad}\,\phi^E = \vec{g}_0(\vec{x}) - \mathrm{Grad}\,\phi^E \ ,$$

where, in the last step, we have neglected the term $\boldsymbol{H} \cdot \mathrm{Grad}\,\phi^E$, as can be seen from $(1.108)_1$. Defining the Eulerian increment in gravitation in accordance with $(9.10)_1$,

$$\vec{g}^E := \vec{g}(\vec{x},t) - \vec{g}_0(\vec{x}) \ , \tag{9.89}$$

the above result shows that $\vec{g}^E$ is expressible as the gradient of the corresponding potential increment:

$$\vec{g}^E = -\mathrm{Grad}\,\phi^E \ . \tag{9.90}$$

The corresponding result for $\vec{g}^L$ is more complicated. Writing the Lagrangian description of the gravitational attraction in the form

$$\begin{aligned} \vec{G}(\vec{X},t) &= \vec{g}(\vec{x}(\vec{X},t),t) = \vec{g}(\vec{X}+\vec{u},t) = \vec{g}_0(\vec{X}+\vec{u},t) + \vec{g}^E = \vec{g}_0(\vec{X}) + \vec{u} \cdot \mathrm{Grad}\,\vec{g}_0(\vec{X}) + \vec{g}^E \ , \\ &\stackrel{!}{=} \vec{g}_0(\vec{X}) + \vec{g}^L \ , \end{aligned}$$

it shows that the two increments in gravitation are related by

$$\vec{g}^L = \vec{g}^E + \vec{u} \cdot \mathrm{Grad}\,\vec{g}_0(\vec{X}) \tag{9.91}$$

in agreement with the general relation (9.12). Inserting for $\vec{g}^E$ from (9.90) and for $\vec{g}_0(\vec{X})$ from (9.80), we obtain

$$\vec{g}^L = -\mathrm{Grad}\,\phi^E - \vec{u} \cdot \mathrm{Grad}\,\mathrm{Grad}\phi_0(\vec{X}) \ . \tag{9.92}$$

On the other hand, applying the operator Grad on (9.88) and using the identity (A.3), we obtain

$$\mathrm{Grad}\,\phi^L = \mathrm{Grad}\,\phi^E + \mathrm{Grad}\,\vec{u} \cdot \mathrm{Grad}\,\phi_0(\vec{X}) + \mathrm{Grad}\,\mathrm{Grad}\,\phi_0(\vec{X}) \cdot \vec{u} \ .$$

Combining the last two relations and using the symmetry of tensor $\mathrm{Grad}\,\mathrm{Grad}\,\phi_0$, the Lagrangian increment in gravitation can be expressed in an alternative form

$$\vec{g}^L = -\mathrm{Grad}\,\phi^L + \mathrm{Grad}\,\vec{u} \cdot \mathrm{Grad}\,\phi_0(\vec{X}) \ . \tag{9.93}$$

The difference between the two relations (9.90) and (9.93) reflects the inherently Eulerian nature of classical gravitational field theory.

9.9.3 Linearized Poisson's equation

To obtain the linearized form of Poisson equation, we apply geometrical linearization $(1.108)_3$ of the Laplace operator to the exact relation (9.71) and find that

$$\nabla^2\Phi - 2\boldsymbol{H} : \mathrm{Grad}\,\mathrm{Grad}\,\Phi - \mathrm{Div}\,\boldsymbol{H}\cdot\mathrm{Grad}\,\Phi = 4\pi G\varrho \qquad \text{in } V-\Sigma\,, \tag{9.94}$$

correct to the first order in $\|\boldsymbol{H}\|$. Substituting for Φ from $(9.87)_2$ and for ϱ from $(9.13)_1$, subtracting Poisson's equation (9.83) for the initial configuration, we obtain

$$\nabla^2\phi^L - 2\boldsymbol{H} : \mathrm{Grad}\,\mathrm{Grad}\,\phi_0 - \mathrm{Div}\,\boldsymbol{H}\cdot\mathrm{Grad}\,\phi_0 = 4\pi G\varrho^L \qquad \text{in } V-\Sigma\,, \tag{9.95}$$

where ϱ^L is the Lagrangian increment in density. Making use of the differential identities (A.37) and (A.38), and the symmetry of tensor Grad Grad ϕ_0, we can derive the identity

$$\nabla^2(\vec{u}\cdot\mathrm{Grad}\,\phi_0) = 2\boldsymbol{H} : \mathrm{Grad}\,\mathrm{Grad}\,\phi_0 + \mathrm{Div}\,\boldsymbol{H}\cdot\mathrm{Grad}\,\phi_0 + \vec{u}\cdot\mathrm{Grad}\,(\nabla^2\phi_0)\,. \tag{9.96}$$

Substituting for the expression $2\boldsymbol{H} : \mathrm{Grad}\,\mathrm{Grad}\,\phi_0 + \mathrm{Div}\,\boldsymbol{H}\cdot\mathrm{Grad}\,\phi_0$ from (9.96) into (9.95), using Poisson's equation (9.83) for the initial potential, and considering (9.15) for the Lagrangian increment in density, Poisson's equation for ϕ^L can be put into an alternative form

$$\nabla^2\phi^L + \nabla^2(\vec{u}\cdot\vec{g}_0) = -4\pi G\,\mathrm{Div}\,(\varrho_0\vec{u}) \qquad \text{in } V-\Sigma\,. \tag{9.97}$$

The linearized Poisson equation can also be written in terms of the Eulerian increments in potential and density. By inspection of (9.14), the right-hand side of (9.97) is proportional to the Eulerian increment in density ϱ^E. Moreover, taking into account (9.89), $\phi^L = \phi^E - \vec{u}\cdot\vec{g}_0$, and we obtain

$$\nabla^2\phi^E = 4\pi G\,\varrho^E \qquad \text{in } V-\Sigma\,. \tag{9.98}$$

9.9.4 Linearized interface condition for potential increments

The exact continuity condition (9.78) for the potential on the undeformed discontinuity Σ can be expressed in terms of the Lagrangian increment in potential as

$$\left[\phi_0(\vec{X}) + \phi^L\right]_-^+ = 0 \qquad \text{on } \Sigma\,. \tag{9.99}$$

At a welded discontinuity Σ_{SS}, which experiences zero slip, the initial potential ϕ_0 is continuous and the condition (9.99) reduces to

$$\left[\phi^L\right]_-^+ = 0 \qquad \text{on } \Sigma_{\mathrm{SS}}\,. \tag{9.100}$$

At a fluid-solid discontinuity Σ_{FS}, which may exhibit a tangential slip, the offset quantities ϕ_0^+ and ϕ_0^- are related by an expansion analogous to (9.45). Correct to first order in $\|\vec{u}\|$, it holds

$$\phi_0^+ = \phi_0^- - [\vec{u}]_-^+\cdot\mathrm{Grad}_\Sigma\phi_0 \qquad \text{on } \Sigma_{\mathrm{FS}}\,, \tag{9.101}$$

where the superscripts $\pm$ denote evaluation at $\vec{X}^\pm$. Within the same accuracy, it is immaterial whether $\mathrm{Grad}_\Sigma\phi_0$ is taken at $\vec{X}^+$ or $\vec{X}^-$. Substituting (9.101) into (9.99) yields

$$\left[\phi^L - \vec{u}\cdot\mathrm{Grad}_\Sigma\phi_0\right]_-^+ = 0 \qquad \text{on } \Sigma_{\mathrm{FS}}\,. \tag{9.102}$$

Since the normal component of displacement and the normal component of the initial gravitation are continuous across $\Sigma_{\rm FS}$, we can subtract the term $(\vec{N}\cdot\vec{u})(\vec{N}\cdot{\rm Grad}\phi_0)$ from the left-hand side of (9.102) for convenience and obtain

$$\Big[\phi^L - \vec{u}\cdot{\rm Grad}\,\phi_0\Big]_-^+ = 0 \qquad \text{on } \Sigma_{\rm FS}\ . \tag{9.103}$$

Note that equation (9.100) is exact whereas (9.103) is correct to first order in $\|\vec{u}\|$.

The corresponding results for ϕ^E can be obtained from the relation (9.88) between the increments ϕ^E and ϕ^L. At a welded discontinuity $\Sigma_{\rm SS}$, we have

$$\Big[\phi^E + \vec{u}\cdot{\rm Grad}\phi_0\Big]_-^+ = 0\ .$$

Since the displacement and the initial gravitation are continuous on $\Sigma_{\rm SS}$, the interface condition simplifies to

$$\Big[\phi^E\Big]_-^+ = 0 \qquad \text{on } \Sigma_{\rm SS}\ . \tag{9.104}$$

At a fluid-solid discontinuity $\Sigma_{\rm FS}$, the interface condition (9.103) transforms to a condition of the same form as (9.104). Hence, for the both types of discontinuities, the Eulerian increment in potential satisfies the interface condition of the form

$$\Big[\phi^E\Big]_-^+ = 0 \qquad \text{on } \Sigma\ , \tag{9.105}$$

where $\Sigma = \Sigma_{\rm SS}\cup\Sigma_{\rm FS}$.

9.9.5 Linearized interface condition for increments in gravitation

To obtain the linearized form of the interface condition for gravitation, we apply the geometrical linearization $(1.107)_1$ of $\boldsymbol{F}^{-1}$ and the linearized equation (1.116) in the exact interface condition (9.79) and find that

$$\Big[\vec{N}\cdot{\rm Grad}\,\Phi - \vec{N}\cdot\boldsymbol{H}\cdot{\rm Grad}\,\Phi - {\rm Grad}\,\Phi\cdot\boldsymbol{H}\cdot\vec{N} + (\vec{N}\cdot\boldsymbol{H}\cdot\vec{N})(\vec{N}\cdot{\rm Grad}\,\Phi)\Big]_-^+ = 0 \qquad \text{on } \Sigma \tag{9.106}$$

correct to the first order in $\|\boldsymbol{H}\|$. Upon inserting the decomposition $(9.87)_2$ of the Lagrangian gravitational potential into (9.106), we find, correct to first order in $\|\vec{u}\|$ that

$$\Big[-\vec{N}\cdot\vec{g}_0 + \vec{N}\cdot{\rm Grad}\,\phi^L + \vec{N}\cdot\boldsymbol{H}\cdot\vec{g}_0 + \vec{g}_0\cdot\boldsymbol{H}\cdot\vec{N} - (\vec{N}\cdot\boldsymbol{H}\cdot\vec{N})(\vec{N}\cdot\vec{g}_0)\Big]_-^+ = 0\ , \tag{9.107}$$

where the initial gravitation $\vec{g}_0$ is introduced for convenience. On the other hand, equations (B.5) and (B.28) can be combined to give

$$\vec{g}_0\cdot\boldsymbol{H}\cdot\vec{N} - (\vec{N}\cdot\boldsymbol{H}\cdot\vec{N})(\vec{N}\cdot\vec{g}_0) = {\rm Grad}_\Sigma(\vec{N}\cdot\vec{u})\cdot(\vec{g}_0)_\Sigma\ , \tag{9.108}$$

where $(\vec{g}_0)_\Sigma$ is the tangential part of $\vec{g}_0$. Since both the vectors on the right-hand side of (9.108) are continuous on Σ, we obtain

$$\Big[\vec{g}_0\cdot\boldsymbol{H}\cdot\vec{N} - (\vec{N}\cdot\boldsymbol{H}\cdot\vec{N})(\vec{N}\cdot\vec{g}_0)\Big]_-^+ = 0 \qquad \text{on } \Sigma\ . \tag{9.109}$$

In view of this, the interface condition (9.107) reduces to

$$\left[-\vec{N}\cdot\vec{g}_0+\vec{N}\cdot\mathrm{Grad}\,\phi^L+\vec{N}\cdot\boldsymbol{H}\cdot\vec{g}_0\right]_-^+=0 \qquad \text{on } \Sigma\ . \tag{9.110}$$

Note that the interface condition (9.110) is valid at all discontinuities Σ. At a welded discontinuity $\Sigma_{\rm SS}$, the normal component of the initial gravitation is continuous and the condition (9.110) further reduces to

$$\left[\vec{N}\cdot\mathrm{Grad}\,\phi^L+\vec{N}\cdot\boldsymbol{H}\cdot\vec{g}_0\right]_-^+=0 \qquad \text{on } \Sigma_{\rm SS}\ . \tag{9.111}$$

At a fluid-solid discontinuity $\Sigma_{\rm FS}$ which admits a tangential slip, the quantities $\vec{N}^\pm\cdot\vec{g}_0^\pm$ are related by an expansion analogous to (9.101):

$$\vec{N}^+\cdot\vec{g}_0^+=\vec{N}^-\cdot\vec{g}_0^- -[\vec{u}]_-^+\cdot\mathrm{Grad}_\Sigma(\vec{N}\cdot\vec{g}_0) \qquad \text{on } \Sigma_{\rm FS}\ , \tag{9.112}$$

correct to first order in $\|\vec{u}\|$. Substituting (9.112) into (9.110) yields

$$\left[\vec{N}\cdot\mathrm{Grad}\,\phi^L+\vec{N}\cdot\boldsymbol{H}\cdot\vec{g}_0+\vec{u}\cdot\mathrm{Grad}_\Sigma(\vec{N}\cdot\vec{g}_0)\right]_-^+=0 \qquad \text{on } \Sigma_{\rm FS}\ . \tag{9.113}$$

The last term in the brackets can be developed by means of the differential identity $(\mathrm{B}.18)_2$ resulting in:

$$\left[\vec{N}\cdot\mathrm{Grad}\,\phi^L+\vec{N}\cdot\boldsymbol{H}\cdot\vec{g}_0+\vec{u}\cdot\mathrm{Grad}_\Sigma\vec{g}_0\cdot\vec{N}+\vec{u}\cdot\mathrm{Grad}_\Sigma\vec{N}\cdot\vec{g}_0\right]_-^+=0 \qquad \text{on } \Sigma_{\rm FS}\ . \tag{9.114}$$

In view of (9.50) and the continuity of the initial gravitation, the last term in the brackets can be rewritten in a slightly different form

$$\left[\vec{N}\cdot\mathrm{Grad}\,\phi^L+\vec{N}\cdot\boldsymbol{H}\cdot\vec{g}_0+\vec{u}\cdot\mathrm{Grad}_\Sigma\vec{g}_0\cdot\vec{N}-\vec{g}_0\cdot\mathrm{Grad}_\Sigma\vec{u}\cdot\vec{N}\right]_-^+=0 \qquad \text{on } \Sigma_{\rm FS}\ . \tag{9.115}$$

By combining (B.32) and the interface condition (9.111), it is readily to show that the last term in the brackets is continuous on Σ and can, thus, be dropped from the brackets, leaving

$$\left[\vec{N}\cdot\mathrm{Grad}\,\phi^L+\vec{N}\cdot\boldsymbol{H}\cdot\vec{g}_0+\vec{u}\cdot\mathrm{Grad}_\Sigma\vec{g}_0\cdot\vec{N}\right]_-^+=0 \qquad \text{on } \Sigma_{\rm FS}\ . \tag{9.116}$$

The corresponding results for ϕ^E can be obtained from the relation (9.88) between the increments ϕ^E and ϕ^L. Substituting (9.88) into (9.111) and employing the differential identity (A.3), we obtain

$$\left[\vec{N}\cdot\mathrm{Grad}\,\phi^E-\vec{N}\cdot\mathrm{Grad}\,\vec{g}_0\cdot\vec{u}\right]_-^+=0 \qquad \text{on } \Sigma_{\rm SS}\ . \tag{9.117}$$

Making use of the identity (B.37), the continuity of $\vec{u}$, ϕ_0 and the normal component of $\vec{g}_0$ on $\Sigma_{\rm SS}$, and the Poisson's equation (9.83), we find that

$$\left[\vec{N}\cdot\mathrm{Grad}\,\vec{g}_0\cdot\vec{u}\right]_-^+=\left[(\vec{N}\cdot\vec{u})\mathrm{Div}\,\vec{g}_0\right]_-^+=-4\pi G[\varrho_0]_-^+(\vec{N}\cdot\vec{u}) \qquad \text{on } \Sigma_{\rm SS}\ . \tag{9.118}$$

The interface condition (9.118) can then be put into the form

$$\left[\vec{N}\cdot\mathrm{Grad}\,\phi^E+4\pi G\varrho_0(\vec{N}\cdot\vec{u})\right]_-^+=0 \qquad \text{on } \Sigma_{\rm SS}\ . \tag{9.119}$$

At a fluid-solid discontinuity $\Sigma_{\rm FS}$, condition (9.119), expressed in terms of ϕ^E, becomes

$$\Big[\vec{N}\cdot{\rm Grad}\,\phi^E - \vec{N}\cdot{\rm Grad}\,\vec{g}_0\cdot\vec{u} + \vec{u}\cdot{\rm Grad}_\Sigma\vec{g}_0\cdot\vec{N}\Big]_-^+ = 0 \qquad \text{on} \;\; \Sigma_{\rm FS}\;. \tag{9.120}$$

Because of the symmetry of the second-order tensor ${\rm Grad}\,\vec{g}_0 = -{\rm Grad}\,{\rm Grad}\phi_0$, we can write $\vec{N}\cdot{\rm Grad}\,\vec{g}_0\cdot\vec{u} = \vec{u}\cdot{\rm Grad}\,\vec{g}_0\cdot\vec{N}$. Moreover, the identity (B.32), applied to the initial gravitation vector $\vec{g}_0$, yields

$$ {\rm Grad}\,\vec{g}_0\cdot\vec{N} - {\rm Grad}_\Sigma\vec{g}_0\cdot\vec{N} = \vec{N}(\vec{N}\cdot{\rm Grad}\,\vec{g}_0\cdot\vec{N})\;. \tag{9.121}$$

The interface condition for the expression $\vec{N}\cdot{\rm Grad}\,\vec{g}_0\cdot\vec{N}$ follows from the identity (B.37), the continuity of the initial potential and normal component of initial gravitation, and Poisson's equation (9.83):

$$\Big[\vec{N}\cdot{\rm Grad}\,\vec{g}_0\cdot\vec{N}\Big]_-^+ = [{\rm Div}\,\vec{g}_0]_-^+ = -4\pi G[\varrho_0]_-^+ \qquad \text{on} \;\; \Sigma_{\rm FS}\;. \tag{9.122}$$

A combination of (9.123) and (9.124) results in

$$\Big[\vec{N}\cdot{\rm Grad}\,\vec{g}_0\cdot\vec{u} - \vec{u}\cdot{\rm Grad}_\Sigma\vec{g}_0\cdot\vec{N}\Big]_-^+ = -4\pi G[\varrho_0]_-^+(\vec{N}\cdot\vec{u}) \qquad \text{on} \;\; \Sigma_{\rm FS}\;. \tag{9.123}$$

In view of this, the interface condition (9.122) for gravitation at a discontinuity $\Sigma_{\rm FS}$ takes the form (9.121), that is valid at a discontinuity $\Sigma_{\rm SS}$. In summary, for the both types of discontinuities, the linearized form of the interface condition for gravitation is

$$\Big[\vec{N}\cdot{\rm Grad}\,\phi^E + 4\pi G\varrho_0(\vec{N}\cdot\vec{u})\Big]_-^+ = 0 \qquad \text{on} \;\; \Sigma\;. \tag{9.124}$$

The complete set of linearized interface conditions at a density discontinuity is summarized for convenience in Table 9.3.

Discontinuity type	Linearized interface condition
∂V: free surface	$\left[\phi^L\right]_-^+ = 0 \qquad \text{(exact)}$
	$\left[\phi^E\right]_-^+ = 0$
	$\left[\vec{N}\cdot\mathrm{Grad}\,\phi^L\right]_-^+ - \vec{N}\cdot\mathrm{Grad}\,\vec{u}^-\cdot\vec{g}_0^- = 0$
	$\left[\vec{N}\cdot\mathrm{Grad}\,\phi^E\right]_-^+ - 4\pi G\varrho_0^-(\vec{N}\cdot\vec{u}^-) = 0$
Σ_{SS}: solid-solid	$\left[\phi^L\right]_-^+ = 0 \qquad \text{(exact)}$
	$\left[\phi^E\right]_-^+ = 0$
	$\left[\vec{N}\cdot\mathrm{Grad}\,\phi^L + \vec{N}\cdot\mathrm{Grad}\,\vec{u}\cdot\vec{g}_0\right]_-^+ = 0$
	$\left[\vec{N}\cdot\mathrm{Grad}\,\phi^E + 4\pi G\varrho_0(\vec{N}\cdot\vec{u})\right]_-^+ = 0$
Σ_{FS}: fluid-solid	$\left[\phi^L + \vec{u}\cdot\vec{g}_0\right]_-^+ = 0$
	$\left[\phi^E\right]_-^+ = 0$
	$\left[\vec{N}\cdot\mathrm{Grad}\,\phi^L + \vec{N}\cdot\mathrm{Grad}\,\vec{u}\cdot\vec{g}_0 + \vec{u}\cdot\mathrm{Grad}_\Sigma\vec{g}_0\cdot\vec{N}\right]_-^+ = 0$
	$\left[\vec{N}\cdot\mathrm{Grad}\,\phi^E + 4\pi G\varrho_0(\vec{N}\cdot\vec{u})\right]_-^+ = 0$

Table 9.3. Summary of linearized gravitational interface conditions.

9.9.6 Boundary-value problem for the Eulerian potential increment

The gravitational increments ϕ^E and ϕ^L can be obtained as a function of particle displacement $\vec{u}$ in two ways. Either by solving the boundary-value problem for linearized Poisson's equations or by the linearization of the exact Newton integrals. We consider both approaches here and demonstrate their equivalence.

The Eulerian potential increment ϕ^E is the solution to the linearized Poisson's equation

$$\nabla^2\phi^E = 4\pi G\,\varrho^E \qquad \text{in } V-\Sigma\,, \tag{9.125}$$

subject to the interface conditions derived above:

$$\left[\phi^E\right]_-^+ = 0 \qquad \text{on } \Sigma\,, \tag{9.126}$$

$$\left[\vec{N}\cdot\mathrm{Grad}\,\phi^E\right]_-^+ = -4\pi G[\varrho_0]_-^+(\vec{N}\cdot\vec{u}) \qquad \text{on } \Sigma\,. \tag{9.127}$$

Outside the body $\mathcal{B}$, where $\varrho^E = 0$, the incremental potential is harmonic, $\nabla^2\phi^E = 0$. The solution to the boundary-value problem (9.127)–(9.130) is

$$\phi^E = -G\int_V \frac{\varrho^{E\,\prime}}{L}\,dV' + G\int_\Sigma \frac{[\varrho_0']_-^+(\vec{N}'\cdot\vec{u}\,')}{L}\,d\Sigma'\,, \tag{9.128}$$

where L is the distance between the computation point $\vec{X}$ and a dummy point of integration $\vec{X}'$,

$$L := \|\vec{X} - \vec{X}'\| \ . \tag{9.129}$$

The first term in (9.130) accounts for the volumetric density distribution ϱ^E in V and the second term accounts for the surface mass distribution due to the normal displacement of the boundary Σ. Substituting $\varrho^{E\,\prime} = -\mathrm{Div}\,'(\varrho_0'\vec{u}\,')$ in (9.130) and applying Green's theorem,

$$\int_V \frac{\mathrm{Div}\,'\vec{v}\,'}{L}\, dV' = -\int_\Sigma \frac{[\vec{N}' \cdot \vec{v}\,']_-^+}{L}\, d\Sigma' - \int_V \vec{v}\,' \cdot \mathrm{Grad}\,'\Big(\frac{1}{L}\Big)\, dV' \ , \tag{9.130}$$

which is valid for a differentiable vector function $\vec{v}$, we find that the surface integrals cancel, leaving simply

$$\phi^E = -G \int_V \varrho_0'\vec{u}\,' \cdot \mathrm{Grad}\,'\Big(\frac{1}{L}\Big)\, dV' \ . \tag{9.131}$$

By direct differentiation we find that

$$\mathrm{Grad}\,'\Big(\frac{1}{L}\Big) = \frac{\vec{X} - \vec{X}'}{\|\vec{X} - \vec{X}'\|^3} \ . \tag{9.132}$$

Equation (9.133) is the most convenient explicit analytical representation of the Eulerian potential increment ϕ^E as a linear function of the particle displacement $\vec{u}$. The corresponding representation of the incremental Eulerian gravitation $\vec{g}^E = -\mathrm{Grad}\,\phi^E$ can be written in the form

$$\vec{g}^E = G \int_V \varrho_0'(\vec{u}\,' \cdot \mathbf{\Pi}) dV' \ , \tag{9.133}$$

where

$$\mathbf{\Pi} := \frac{\boldsymbol{I}}{\|\vec{X} - \vec{X}'\|^3} - \frac{3(\vec{X} - \vec{X}') \otimes (\vec{X} - \vec{X}')}{\|\vec{X} - \vec{X}'\|^5} \ . \tag{9.134}$$

9.9.7 Boundary-value problem for the Lagrangian potential increment

The Lagrangian potential increment ϕ^L satisfies the linearized Poisson's equation

$$\nabla^2\phi^L = -\nabla^2(\vec{u} \cdot \vec{g}_0) - 4\pi G\,\mathrm{Div}\,(\varrho_0\vec{u}) \qquad \text{in} \ \ V - \Sigma \ , \tag{9.135}$$

subject to the interface conditions derived above:

$$\Big[\phi^L\Big]_-^+ = 0 \qquad \text{on} \ \ \Sigma_{\mathrm{SS}} \ , \tag{9.136}$$

$$\Big[\vec{N} \cdot \mathrm{Grad}\,\phi^L\Big]_-^+ = -\vec{N} \cdot [\boldsymbol{H}]_-^+ \cdot \vec{g}_0 \qquad \text{on} \ \ \Sigma_{\mathrm{SS}} \ , \tag{9.137}$$

$$\Big[\phi^L\Big]_-^+ = -[\vec{u}]_-^+ \cdot \vec{g}_0 \qquad \text{on} \ \ \Sigma_{\mathrm{FS}} \ , \tag{9.138}$$

$$\Big[\vec{N} \cdot \mathrm{Grad}\,\phi^L\Big]_-^+ = -\vec{N} \cdot [\boldsymbol{H}]_-^+ \cdot \vec{g}_0 - [\vec{u}]_-^+ \cdot \mathrm{Grad}_\Sigma\vec{g}_0 \cdot \vec{N} \qquad \text{on} \ \ \Sigma_{\mathrm{FS}} \ . \tag{9.139}$$

Outside the body $\mathcal{B}$, where $\varrho_0 = 0$ and $\vec{u} = \vec{0}$, the incremental potential is harmonic, $\nabla^2\phi^L = 0$. The solution to the boundary-value problem (9.137)–(9.141) is

$$\phi^L = \frac{1}{4\pi} \int_V \frac{\nabla^{2\,\prime}(\vec{u}\,' \cdot \vec{g}_0\,') + 4\pi G\,\mathrm{Div}\,'(\varrho_0'\vec{u}\,')}{L}\, dV' + \frac{1}{4\pi} \int_{\Sigma_{\mathrm{SS}}} \frac{\vec{N}' \cdot [\boldsymbol{H}']_-^+ \cdot \vec{g}_0\,'}{L}\, d\Sigma'$$

$$+\frac{1}{4\pi}\int_{\Sigma_{\rm FS}} \frac{\vec{N}'\cdot[\boldsymbol{H}']_-^+\cdot\vec{g}_0{}'+[\vec{u}\,']_-^+\cdot{\rm Grad}'_\Sigma\vec{g}_0{}'\cdot\vec{N}'}{L}\,d\Sigma' - \frac{1}{4\pi}\int_{\Sigma_{\rm FS}} ([\vec{u}\,']_-^+\cdot\vec{g}_0{}')\Big(\vec{N}'\cdot{\rm Grad}\,'\Big(\frac{1}{L}\Big)\Big)\,d\Sigma' \,. \tag{9.140}$$

The first two surface integrals on the right-hand side of (9.142) can be regarded as the gravitational potentials of single layers placed on $\Sigma_{\rm SS}$ and $\Sigma_{\rm FS}$ with surface densities $\frac{1}{4\pi G}(\vec{N}\cdot[\boldsymbol{H}]_-^+\cdot\vec{g}_0)$ and $\frac{1}{4\pi G}(\vec{N}\cdot[\boldsymbol{H}]_-^+\cdot\vec{g}_0+[\vec{u}]_-^+\cdot{\rm Grad}_\Sigma\vec{g}_0\cdot\vec{N})$, respectively. The last surface integral in (9.142) represents the potential of a double layer placed on $\Sigma_{\rm FS}$ with the density $-\frac{1}{4\pi G}[\vec{u}]_-^+\cdot\vec{g}_0$.

Applying Green's theorem (9.132) on the volume integral, we obtain

$$\begin{aligned}
\phi^L \;=\; & -\frac{1}{4\pi}\int_{\Sigma_{\rm SS}} \frac{\vec{N}'\cdot[{\rm Grad}\,'(\vec{u}\,'\cdot\vec{g}_0{}')+4\pi G\varrho_0'\vec{u}\,']_-^+}{L}\,d\Sigma' \\
& +\frac{1}{4\pi}\int_{\Sigma_{\rm SS}} \frac{\vec{N}'\cdot[\boldsymbol{H}']_-^+\cdot\vec{g}_0{}'}{L}\,d\Sigma' \\
& -\frac{1}{4\pi}\int_{\Sigma_{\rm FS}} \frac{\vec{N}'\cdot[{\rm Grad}\,'(\vec{u}\,'\cdot\vec{g}_0{}')+4\pi G\varrho_0'\vec{u}\,']_-^+}{L}\,d\Sigma' \\
& +\frac{1}{4\pi}\int_{\Sigma_{\rm FS}} \frac{\vec{N}'\cdot[\boldsymbol{H}']_-^+\cdot\vec{g}_0{}'+[\vec{u}\,']_-^+\cdot{\rm Grad}'_\Sigma\vec{g}_0{}'\cdot\vec{N}'}{L}\,d\Sigma' \\
& -\frac{1}{4\pi}\int_{\Sigma_{\rm FS}} ([\vec{u}\,']_-^+\cdot\vec{g}_0{}')\Big(\vec{N}'\cdot{\rm Grad}\,'\Big(\frac{1}{L}\Big)\Big)\,d\Sigma' \\
& -\frac{1}{4\pi}\int_V [{\rm Grad}\,'(\vec{u}\,'\cdot\vec{g}_0{}')+4\pi G\varrho_0'\vec{u}\,']\cdot{\rm Grad}\,'\Big(\frac{1}{L}\Big)\,dV' \,.
\end{aligned}$$

It is necessary to apply Green's theorem to each sub-volumes of the volume V separately, and add the results, since the expression $\nabla^2(\vec{u}\cdot\vec{g}_0)+4\pi G\,{\rm Div}\,(\varrho_0\vec{u})$ in the integrand may possess different jumps at the discontinuities $\Sigma_{\rm SS}$ and $\Sigma_{\rm FS}$. Making use of the identity

$${\rm Grad}\,'(\vec{u}\,'\cdot\vec{g}_0{}') = \boldsymbol{H}'\cdot\vec{g}_0{}' + {\rm Grad}\,'\vec{g}_0{}'\cdot\vec{u}\,' \,,$$

and the continuity of the initial gravitation, the surface integrals containing the displacement gradient $\boldsymbol{H}'$ cancel. Moreover, the last contribution in the volume integral is exactly the Eulerian potential increment ϕ^E referred to the particle position $\vec{X}$. Hence, we have

$$\begin{aligned}
\phi^L = \phi^E & -\frac{1}{4\pi}\int_{\Sigma_{\rm SS}} \frac{\vec{N}'\cdot[{\rm Grad}\,'\vec{g}_0{}'\cdot\vec{u}\,'+4\pi G\varrho_0'\vec{u}\,']_-^+}{L}\,d\Sigma' \\
& -\frac{1}{4\pi}\int_{\Sigma_{\rm FS}} \frac{\vec{N}'\cdot[{\rm Grad}\,'\vec{g}_0{}'\cdot\vec{u}\,'+4\pi G\varrho_0'\vec{u}\,']_-^+}{L}\,d\Sigma' \\
& +\frac{1}{4\pi}\int_{\Sigma_{\rm FS}} \frac{[\vec{u}\,']_-^+\cdot{\rm Grad}'_\Sigma\vec{g}_0{}'\cdot\vec{N}'}{L}\,d\Sigma' \\
& -\frac{1}{4\pi}\int_{\Sigma_{\rm FS}} ([\vec{u}\,']_-^+\cdot\vec{g}_0{}')\Big(\vec{N}'\cdot{\rm Grad}\,'\Big(\frac{1}{L}\Big)\Big)\,d\Sigma' \\
& -\frac{1}{4\pi}\int_V {\rm Grad}\,'(\vec{u}\,'\cdot\vec{g}_0{}')\cdot{\rm Grad}\,'\Big(\frac{1}{L}\Big)\,dV' \,.
\end{aligned}$$

The surface integral over Σ_{SS} and the first two surface integrals over Σ_{FS} vanish because of the interface condition (9.120) and (9.125), respectively. As a result, we have

$$\phi^L = \phi^E - \frac{1}{4\pi}\int_{\Sigma_{FS}} ([\vec{u}\,']_-^+ \cdot \vec{g}_0\,')\Big(\vec{N}' \cdot \mathrm{Grad}\,'\Big(\frac{1}{L}\Big)\Big)\, d\Sigma' - \frac{1}{4\pi}\int_V \mathrm{Grad}\,'(\vec{u}\,' \cdot \vec{g}_0\,') \cdot \mathrm{Grad}\,'\Big(\frac{1}{L}\Big)\, dV' \ . \tag{9.141}$$

We now intend to apply the following Green's theorem to the volume integral on the right-hand side of (9.144). For a differentiable scalar function ϕ, it holds

$$\int_V \mathrm{Grad}\,'\phi' \cdot \mathrm{Grad}\,'\Big(\frac{1}{L}\Big) dV' = -\int_\Sigma [\phi']_-^+ \Big(\vec{N}' \cdot \mathrm{Grad}\,'\Big(\frac{1}{L}\Big)\Big) d\Sigma' - \int_V \phi' \nabla^{2\,\prime}\Big(\frac{1}{L}\Big) dV' \ . \tag{9.142}$$

Since

$$\nabla^{2\,\prime}\Big(\frac{1}{L}\Big) = -4\pi\delta(\vec{X} - \vec{X}') \ , \tag{9.143}$$

where $\delta(\vec{X} - \vec{X}')$ is the three-dimensional delta function with the property

$$\int_V \phi(\vec{X}')\delta(\vec{X} - \vec{X}')dV' = \phi(\vec{X}) \ , \tag{9.144}$$

the Green's theorem (9.144) is simplified to

$$\int_V \mathrm{Grad}\,'\phi' \cdot \mathrm{Grad}\,'\Big(\frac{1}{L}\Big) dV' = 4\pi\phi - \int_\Sigma [\phi']_-^+ \Big(\vec{N}' \cdot \mathrm{Grad}\,'\Big(\frac{1}{L}\Big)\Big) d\Sigma' \ . \tag{9.145}$$

Applying (9.147) to the volume integral in (9.143), we find that the surface integrals over Σ_{FS} cancel, leaving

$$\phi^L = \phi^E - \vec{u} \cdot \vec{g}_0 + \frac{1}{4\pi}\int_{\Sigma_{SS}} [\vec{u}\,' \cdot \vec{g}_0\,']_-^+ \Big(\vec{N}' \cdot \mathrm{Grad}\,'\Big(\frac{1}{L}\Big)\Big) d\Sigma' \ .$$

The surface integral over Σ_{SS} vanishes because of the continuity of $\vec{u}$ and $\vec{g}_0$ on the discontinuity Σ_{SS}. This equation becomes precisely $\phi^L = \phi^E - \vec{u} \cdot \vec{g}_0$ in agreement with the general relation (9.88). Substituting for ϕ^E from (9.133) and for $\vec{g}_0$ from (9.82), we finally obtain

$$\phi^L = -G\int_V \varrho_0'(\vec{u}\,' - \vec{u}) \cdot \mathrm{Grad}\,'\Big(\frac{1}{L}\Big)\, dV' \ . \tag{9.146}$$

9.9.8 Linearized integral relations

Alternatively, we can obtain gravitational increments ϕ^L, $\vec{g}^L$, ϕ^E and $\vec{g}^E$ by linearizing the exact Newton integrals (9.75) and (9.76). Considering (9.75), for example, we rewrite it in the form

$$\phi_0(\vec{X}) + \phi^L = -G\int_V \frac{\varrho_0'}{\|\vec{X} + \vec{u} - \vec{X}' - \vec{u}\,'\|}\, dV' \ , \tag{9.147}$$

and expand the right-hand side in powers of $\vec{u} - \vec{u}\,'$. The zeroth-order term is simply ϕ_0, whereas the first-order term is

$$\phi^L = -G\int_V \frac{\varrho_0'(\vec{u}\,' - \vec{u}) \cdot (\vec{X} - \vec{X}')}{\|\vec{X} - \vec{X}'\|^3}\, dV' \ . \tag{9.148}$$

which is identical to (9.148). The expansion of

$$\vec{g}_0(\vec{X}) + \vec{g}^L = -G \int_V \frac{\varrho_0'(\vec{X} + \vec{u} - \vec{X}' - \vec{u}\,')}{\|\vec{X} + \vec{u} - \vec{X}' - \vec{u}\,'\|^3}\, dV' \tag{9.149}$$

likewise leads to

$$\vec{g}^L = G \int_V \varrho_0'(\vec{u} - \vec{u}\,') \cdot \mathbf{\Pi}\, dV' \ , \tag{9.150}$$

which is identical to $\vec{g}^L = \vec{g}^E + \vec{u} \cdot \operatorname{Grad} \vec{g}_0 = -\operatorname{Grad} \phi^E - \vec{u} \cdot \operatorname{Grad} \operatorname{Grad} \phi_0$.

9.10 Equation of motion for a self-gravitating body

We now deduce the linearized equation of motion describing infinitesimal deformation of a body initially pre-stressed by its own gravitation. For this purpose, the general theory presented in Section 9.6 will be applied to a particular case that the gravitational force is the only force acting in the body. In particular, we substitute the initial gravitational attraction $\vec{g}_0$ for the initial body force $\vec{f}_0$ in static linear momentum equation (9.1), that is

$$\vec{f}_0 = -\operatorname{Grad} \phi_0 \ . \tag{9.151}$$

The Lagrangian increments in body force ϕ^L, defined by $(9.27)_2$, that occurs in the equation of motion will be replaced by the Lagrangian increments in gravitation $\vec{g}^L$ which can be expressed either in terms of the Eulerian increment in gravitational potential ϕ^E, see (9.92), or in terms of the Lagrangian increment in gravitational potential ϕ^L, see (9.93):

$$\begin{aligned} \vec{f}^L &= -\operatorname{Grad} \phi^E - \vec{u} \cdot \operatorname{Grad} \operatorname{Grad} \phi_0 \\ &= -\operatorname{Grad} \phi^L + \operatorname{Grad} \vec{u} \cdot \operatorname{Grad} \phi_0 \ . \end{aligned} \tag{9.152}$$

Likewise, the Eulerian increments in body force ϕ^E, defined by $(9.27)_1$, is replaced by the Eulerian increments in gravitation $\vec{g}^E$ which can be expressed either in terms of the Eulerian increment in gravitational potential ϕ^E, see (9.90), or in terms of the Lagrangian increment in gravitational potential ϕ^L, see (9.88):

$$\begin{aligned} \vec{f}^E &= -\operatorname{Grad} \phi^E \\ &= -\operatorname{Grad} \phi^L + \vec{u} \cdot \operatorname{Grad} \phi_0 \ . \end{aligned} \tag{9.153}$$

All forms of the equation of motion derived in Section 9.6 can be expressed in terms of the Lagrangian or Eulerian incremental gravitational potential using relations (9.153)–(9.155). Since this substitution is straightforward, we shall not write explicitly the resulting relations.

Appendix A. Vector and tensor differential identities

Let ϕ and ψ be differentiable scalar fields, $\vec{u}$ and $\vec{v}$ differentiable vector fields, $\boldsymbol{A}$ and $\boldsymbol{B}$ differentiable second-order tensor fields and $\boldsymbol{I}$ the second-order unit tensor. The following vector differential identities can be verified:

$$\operatorname{grad}(\phi\psi) = \phi \operatorname{grad}\psi + \psi \operatorname{grad}\phi \tag{A.1}$$

$$\operatorname{grad}(\phi\vec{u}) = \phi \operatorname{grad}\vec{u} + \operatorname{grad}\phi \otimes \vec{u} \tag{A.2}$$

$$\operatorname{grad}(\vec{u}\cdot\vec{v}) = \operatorname{grad}\vec{u}\cdot\vec{v} + \operatorname{grad}\vec{v}\cdot\vec{u} \tag{A.3}$$

$$\operatorname{grad}(\vec{u}\times\vec{v}) = \operatorname{grad}\vec{u}\times\vec{v} - \operatorname{grad}\vec{v}\times\vec{u} \tag{A.4}$$

$$\begin{aligned} \operatorname{grad}(\vec{u}\otimes\vec{v}) &= \operatorname{grad}\vec{u}\otimes\vec{v} + (\vec{u}\otimes\operatorname{grad}\vec{v})^{213} \\ &= \operatorname{grad}\vec{u}\otimes\vec{v} + (\operatorname{grad}\vec{v}\otimes\vec{u})^{132} \end{aligned} \tag{A.5}$$

$$\vec{w}\cdot\operatorname{grad}(\vec{u}\otimes\vec{v}) = (\vec{w}\cdot\operatorname{grad}\vec{u})\otimes\vec{v} + \vec{u}\otimes(\vec{w}\cdot\operatorname{grad}\vec{v}) \tag{A.6}$$

$$\operatorname{grad}(\phi\boldsymbol{A}) = \phi\operatorname{grad}\boldsymbol{A} + \operatorname{grad}\phi\otimes\boldsymbol{A} \tag{A.7}$$

$$\operatorname{grad}(\boldsymbol{A}\cdot\vec{u}) = \operatorname{grad}\boldsymbol{A}\cdot\vec{u} + \operatorname{grad}\vec{u}\cdot\boldsymbol{A}^T \tag{A.8}$$

$$\operatorname{grad}(\vec{u}\cdot\boldsymbol{A}) = \operatorname{grad}\vec{u}\cdot\boldsymbol{A} + \vec{u}\cdot(\operatorname{grad}\boldsymbol{A})^{213} \tag{A.9}$$

$$\operatorname{grad}\vec{v}\cdot\vec{u} - \vec{u}\cdot\operatorname{grad}\vec{v} = \vec{u}\times\operatorname{rot}\vec{v} \tag{A.10}$$

$$\operatorname{grad}\vec{u} - (\operatorname{grad}\vec{u})^T = -\boldsymbol{I}\times\operatorname{rot}\vec{u} \tag{A.11}$$

$$\operatorname{div}(\phi\vec{u}) = \phi\operatorname{div}\vec{u} + \operatorname{grad}\phi\cdot\vec{u} \tag{A.12}$$

$$\operatorname{div}(\vec{u}\times\vec{v}) = \operatorname{rot}\vec{u}\cdot\vec{v} - \vec{u}\cdot\operatorname{rot}\vec{v} \tag{A.13}$$

$$\operatorname{div}(\vec{u}\otimes\vec{v}) = (\operatorname{div}\vec{u})\vec{v} + \vec{u}\cdot\operatorname{grad}\vec{v} \tag{A.14}$$

$$\operatorname{div}(\phi\boldsymbol{A}) = \phi\operatorname{div}\boldsymbol{A} + \operatorname{grad}\phi\cdot\boldsymbol{A} \tag{A.15}$$

$$\operatorname{div}(\phi\boldsymbol{I}) = \operatorname{grad}\phi \tag{A.16}$$

$$\operatorname{div}(\boldsymbol{A}\cdot\vec{u}) = \operatorname{div}\boldsymbol{A}\cdot\vec{u} + \boldsymbol{A}^T : \operatorname{grad}\vec{u} \tag{A.17}$$

$$\operatorname{div}(\vec{u}\cdot\boldsymbol{A}) = \operatorname{grad}\vec{u} : \boldsymbol{A} + \vec{u}\cdot\operatorname{div}\boldsymbol{A}^T \tag{A.18}$$

$$\operatorname{div}(\boldsymbol{A}\times\vec{u}) = \operatorname{div}\boldsymbol{A}\times\vec{u} + \boldsymbol{A}^T \overset{\cdot}{\times} \operatorname{grad}\vec{u} \tag{A.19}$$

$$\operatorname{div}(\vec{u}\times\boldsymbol{A}) = \operatorname{rot}\vec{u}\cdot\boldsymbol{A} - \vec{u}\cdot\operatorname{rot}\boldsymbol{A} \tag{A.20}$$

$$\operatorname{div}(\boldsymbol{I}\times\vec{u}) = \operatorname{rot}\vec{u} \tag{A.21}$$

$$\operatorname{div}\operatorname{rot}\vec{u} = 0 \tag{A.22}$$

$$\operatorname{div}(\boldsymbol{A}\cdot\boldsymbol{B}) = \operatorname{div}\boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{A}^T : \operatorname{grad}\boldsymbol{B} \tag{A.23}$$

$$\operatorname{div}(\vec{u}\cdot\operatorname{grad}\boldsymbol{A}) = \operatorname{grad}\vec{u} : \operatorname{grad}\boldsymbol{A} + \vec{u}\cdot\operatorname{grad}\operatorname{div}\boldsymbol{A} \tag{A.24}$$

$$\operatorname{div}[(\operatorname{grad}\vec{u})^T] = \operatorname{grad}\operatorname{div}\vec{u} \tag{A.25}$$

$$\operatorname{rot}(\phi\vec{u}) = \phi\operatorname{rot}\vec{u} + \operatorname{grad}\phi\times\vec{u} \tag{A.26}$$

$$\operatorname{rot}(\vec{u}\times\vec{v}) = \vec{v}\cdot\operatorname{grad}\vec{u} - \vec{u}\cdot\operatorname{grad}\vec{v} + \vec{u}\operatorname{div}\vec{v} - \vec{v}\operatorname{div}\vec{u} \tag{A.27}$$

$$\operatorname{rot}(\vec{u}\otimes\vec{v}) = \operatorname{rot}\vec{u}\otimes\vec{v} - \vec{u}\times\operatorname{grad}\vec{v} \tag{A.28}$$

$$\operatorname{rot}(\phi\boldsymbol{A}) = \phi\operatorname{rot}\boldsymbol{A} + \operatorname{grad}\phi\times\boldsymbol{A} \tag{A.29}$$

$$\operatorname{rot}(\phi\boldsymbol{I}) = \operatorname{grad}\phi\times\boldsymbol{I} \tag{A.30}$$

$$\operatorname{rot}(\vec{u}\times\boldsymbol{I}) = \operatorname{rot}(\boldsymbol{I}\times\vec{u}) = (\operatorname{grad}\vec{u})^T - (\operatorname{div}\vec{u})\boldsymbol{I} \tag{A.31}$$

$$\operatorname{rot}\operatorname{rot}\vec{u} = \operatorname{grad}\operatorname{div}\vec{u} - \operatorname{div}\operatorname{grad}\vec{u} \tag{A.32}$$

$$\operatorname{rot}\operatorname{grad}\phi = \vec{0} \tag{A.33}$$

$$\operatorname{rot}(\operatorname{grad}\vec{u})^T = (\operatorname{grad}\operatorname{rot}\vec{u})^T \tag{A.34}$$

$$\boldsymbol{I} : \operatorname{grad}\vec{u} = \operatorname{div}\vec{u} \tag{A.35}$$

$$\boldsymbol{I} \dot{\times} \operatorname{grad}\vec{u} = \operatorname{rot}\vec{u} \tag{A.36}$$

$$\nabla^2(\phi\psi) = \psi\,\nabla^2\phi + \phi\,\nabla^2\psi + 2(\operatorname{grad}\phi)\cdot(\operatorname{grad}\psi) \tag{A.37}$$

$$\nabla^2(\vec{u}\cdot\vec{v}) = (\nabla^2\vec{u})\cdot\vec{v} + \vec{u}\cdot\nabla^2\vec{v} + 2(\operatorname{grad}\vec{u})^T : \operatorname{grad}\vec{v} \tag{A.38}$$

$$\nabla^2(\operatorname{grad}\phi) = \operatorname{grad}(\nabla^2\phi) \tag{A.39}$$

$$\nabla^2(\operatorname{rot}\vec{u}) = \operatorname{rot}(\nabla^2\vec{u}) \tag{A.40}$$

$$\nabla^2(\operatorname{rot}\operatorname{rot}\vec{u}) = \operatorname{rot}\operatorname{rot}(\nabla^2\vec{u})\ . \tag{A.41}$$

The symbols $:$ and $\dot{\times}$ denote, respectively, the double-dot product of vectors and the dot-cross product of vectors. The symbols $(\ \ )^T$, $(\ \ )^{213}$ and $(\ \ )^{132}$ denote, respectively, the transpose of a dyadic, the left transpose of a triadic and the right transpose of a triadic, e.g. $(\vec{u}\otimes\vec{v})^T = \vec{v}\otimes\vec{u}$, $(\vec{u}\otimes\vec{v}\otimes\vec{w})^{213} = (\vec{v}\otimes\vec{u}\otimes\vec{w})$, $(\vec{u}\otimes\vec{v}\otimes\vec{w})^{132} = (\vec{u}\otimes\vec{w}\otimes\vec{v})$.

Appendix B. Fundamental formulae for surfaces

In a three-dimensional space, let Σ be an oriented surface with the unit normal $\vec{n}$. Let $(\vartheta_1, \vartheta_2)$ be the orthogonal curvilinear coordinates on Σ with the coordinate lines tangent to Σ in the directions of the principal curvatures of Σ with principal curvature radii ϱ_1 and ϱ_2. If ϑ_1 and ϑ_2 are chosen as angles, the elements of length on the surface in the principal curvature directions are given by

$$ds_\alpha = \varrho_\alpha d\vartheta_\alpha \ , \qquad \alpha = 1, 2 \ . \tag{B.1}$$

The vectors defined by

$$\vec{e}_\alpha := \frac{\partial \vec{n}}{\partial \vartheta_\alpha} \ , \qquad \alpha = 1, 2 \ . \tag{B.2}$$

are tangent to the curvilinear coordinate lines on the surface. Due to the definition of $(\vartheta_1, \vartheta_2)$, the coordinate lines on the surface coincide with two lines of principal directions. Hence, $\vec{e}_1$ and $\vec{e}_2$ are mutually perpendicular. In addition, they are perpendicular to $\vec{n}$,

$$\vec{n} \cdot \vec{e}_\alpha = \vec{e}_1 \cdot \vec{e}_2 = 0 \ . \tag{B.3}$$

We assume that the surface Σ is sufficiently smooth such that, for instance, the curvature of Σ is bounded at all points of Σ. Because of this regularity assumption, we can imagine an auxiliary surface $\Sigma + d\Sigma$ that is parallel with Σ at a differential distance dn reckoned along the normal $\vec{n}$. A local three-dimensional orthogonal curvilinear coordinates $(n, \vartheta_1, \vartheta_2)$ with the coordinate n measured in the direction of the normal $\vec{n}$ can be employed to describe the position of a point in the layer bounded by surfaces Σ and $\Sigma + d\Sigma$. Consequently,

$$\frac{\partial \vec{n}}{\partial n} = \vec{0} \tag{B.4}$$

at all points in the layer.

B.1 Tangent vectors and tensors

An arbitrary vector defined on an oriented surface Σ with the unit normal $\vec{n}$ can be decomposed into a normal and a tangential part:

$$\vec{u} = \vec{n} u_n + \vec{u}_\Sigma \ , \tag{B.5}$$

where $u_n = \vec{n} \cdot \vec{u}$ and $\vec{u}_\Sigma = \vec{u} \cdot (\boldsymbol{I} - \vec{n} \otimes \vec{n})$. Obviously, $\vec{n} \cdot \vec{u}_\Sigma = 0$. The quantity u_n is the normal component of $\vec{u}$. Any vector $\vec{u}_\Sigma$ with the property $\vec{n} \cdot \vec{u}_\Sigma = 0$ is referred to as a *tangent vector* to Σ. The component form of a tangent vector is

$$\vec{u}_\Sigma = \sum_\alpha u_\alpha \vec{e}_\alpha \ , \tag{B.6}$$

where the summation index α takes values 1 and 2. An arbitrary second-order tensor $\boldsymbol{T}$ can be decomposed in an analogous manner:

$$\boldsymbol{T} = (\vec{n} \otimes \vec{n}) T_{nn} + \vec{n} \otimes \vec{T}_{n\Sigma} + \vec{T}_{\Sigma n} \otimes \vec{n} + \boldsymbol{T}_{\Sigma\Sigma} \ , \tag{B.7}$$

where $T_{nn} = \vec{n} \cdot \boldsymbol{T} \cdot \vec{n}$. The quantities $\vec{T}_{n\Sigma}$ and $\vec{T}_{\Sigma n}$ are tangent vectors satisfying $\vec{n} \cdot \vec{T}_{n\Sigma} = \vec{n} \cdot \vec{T}_{\Sigma n} = 0$, whereas $\boldsymbol{T}_{\Sigma\Sigma}$ is a so-called *tangent tensor*, defined by $\vec{n} \cdot \boldsymbol{T}_{\Sigma\Sigma} = \boldsymbol{T}_{\Sigma\Sigma} \cdot \vec{n} = \vec{0}$. The transpose of $\boldsymbol{T}$ is given by

$$\boldsymbol{T}^T = (\vec{n} \otimes \vec{n}) T_{nn} + \vec{T}_{n\Sigma} \otimes \vec{n} + \vec{n} \otimes \vec{T}_{\Sigma n} + (\boldsymbol{T}_{\Sigma\Sigma})^T \ . \tag{B.8}$$

If $\boldsymbol{T}$ is a symmetric tensor, then $\vec{T}_{\Sigma n} = \vec{T}_{n\Sigma}$ and $(\boldsymbol{T}_{\Sigma\Sigma})^T = \boldsymbol{T}_{\Sigma\Sigma}$. The component form of tangent vectors $\vec{T}_{n\Sigma}$, $\vec{T}_{\Sigma n}$, and tangent tensor $\boldsymbol{T}_{\Sigma\Sigma}$ are

$$\vec{T}_{n\Sigma} = \sum_\alpha T_{n\alpha}\vec{e}_\alpha \ , \qquad \vec{T}_{\Sigma n} = \sum_\alpha T_{\alpha n}\vec{e}_\alpha \ , \qquad \boldsymbol{T}_{\Sigma\Sigma} = \sum_{\alpha\beta} T_{\alpha\beta}(\vec{e}_\alpha \otimes \vec{e}_\beta) \ . \tag{B.9}$$

The scalar product of two vectors is

$$\vec{u}\cdot\vec{v} = u_n v_n + \vec{u}_\Sigma \cdot \vec{v}_\Sigma \ , \tag{B.10}$$

and the left or right scalar product of a tensor and a vector is

$$\begin{aligned} \boldsymbol{T}\cdot\vec{u} &= \vec{n}(T_{nn}u_n + \vec{T}_{n\Sigma}\cdot\vec{u}_\Sigma) + \vec{T}_{\Sigma n}u_n + \boldsymbol{T}_{\Sigma\Sigma}\cdot\vec{u}_\Sigma \ , \\ \vec{u}\cdot\boldsymbol{T} &= \vec{n}(T_{nn}u_n + \vec{T}_{\Sigma n}\cdot\vec{u}_\Sigma) + \vec{T}_{n\Sigma}u_n + \vec{u}_\Sigma\cdot\boldsymbol{T}_{\Sigma\Sigma} \ . \end{aligned} \tag{B.11}$$

Obviously, it holds

$$\begin{aligned} \boldsymbol{T}\cdot\vec{n} &= \vec{n}T_{nn} + \vec{T}_{\Sigma n} \ , \\ \vec{n}\cdot\boldsymbol{T} &= \vec{n}T_{nn} + \vec{T}_{n\Sigma} \ , \end{aligned} \tag{B.12}$$

and

$$\begin{aligned} \vec{n}\cdot\boldsymbol{T}\cdot\vec{u} - (\vec{n}\cdot\boldsymbol{T}\cdot\vec{n})(\vec{n}\cdot\vec{u}) &= \vec{T}_{n\Sigma}\cdot\vec{u}_\Sigma \ , \\ \vec{u}\cdot\boldsymbol{T}\cdot\vec{n} - (\vec{n}\cdot\boldsymbol{T}\cdot\vec{n})(\vec{n}\cdot\vec{u}) &= \vec{T}_{\Sigma n}\cdot\vec{u}_\Sigma \ . \end{aligned} \tag{B.13}$$

B.2 Surface gradient

The three-dimensional gradient operator can also be decomposed into a normal and a tangential part:

$$\mathrm{grad} = \vec{n}\frac{\partial}{\partial n} + \mathrm{grad}_\Sigma \ , \tag{B.14}$$

where $\dfrac{\partial}{\partial n} := \vec{n}\cdot\mathrm{grad}$ and $\vec{n}\cdot\mathrm{grad}_\Sigma = 0$. The tangential part grad_Σ is called the *surface gradient* operator; its component form is

$$\mathrm{grad}_\Sigma = \sum_\alpha \frac{\vec{e}_\alpha}{\varrho_\alpha}\frac{\partial}{\partial\vartheta_\alpha} \ . \tag{B.15}$$

Since grad_Σ involves only differentiation in directions tangent to the surface Σ, it can be applied to any scalar, vector or tensor field defined on Σ, whether that field is defined elsewhere or not. Likewise, the three-dimensional divergence operator can be decomposed into a normal and a tangential part:

$$\mathrm{div} = \vec{n}\cdot\frac{\partial}{\partial n} + \mathrm{div}_\Sigma \ . \tag{B.16}$$

The component form of the *surface divergence* operator is

$$\mathrm{div}_\Sigma = \sum_\alpha \frac{\vec{e}_\alpha}{\varrho_\alpha}\cdot\frac{\partial}{\partial\vartheta_\alpha} \ . \tag{B.17}$$

B.3 Identities

Let ϕ and ψ be differentiable scalar fields, $\vec{u}$ and $\vec{v}$ differentiable vector fields and $\boldsymbol{T}$ a differentiable second-order tensor field. The following vector differential identities can be verified:

$$\begin{aligned}
\mathrm{grad}_\Sigma(\phi\psi) &= \phi\,\mathrm{grad}_\Sigma\psi + \psi\,\mathrm{grad}_\Sigma\phi \ , \\
\mathrm{grad}_\Sigma(\vec{u}\cdot\vec{v}) &= \mathrm{grad}_\Sigma\vec{u}\cdot\vec{v} + \mathrm{grad}_\Sigma\vec{v}\cdot\vec{u} \ , \\
\mathrm{grad}_\Sigma(\phi\vec{u}) &= \phi\,\mathrm{grad}_\Sigma\vec{u} + (\mathrm{grad}_\Sigma\phi)\otimes\vec{u} \ , \\
\mathrm{div}_\Sigma(\phi\vec{u}) &= \phi\,\mathrm{div}_\Sigma\vec{u} + \mathrm{grad}_\Sigma\phi\cdot\vec{u} \ , \\
\mathrm{div}_\Sigma(\vec{u}\otimes\vec{v}) &= (\mathrm{div}_\Sigma\vec{u})\vec{v} + \vec{u}\cdot\mathrm{grad}_\Sigma\vec{v} \ , \\
\mathrm{div}_\Sigma(\phi\boldsymbol{T}) &= \phi\,\mathrm{div}_\Sigma\boldsymbol{T} + \mathrm{grad}_\Sigma\phi\cdot\boldsymbol{T} \ .
\end{aligned} \tag{B.18}$$

B.4 Curvature tensor

The surface gradient of the unit normal $\mathrm{grad}_\Sigma\vec{n}$, the so-called *surface curvature tensor*, is often used to classify the curvature of the surface Σ. Combining (B.2) and (B.15), we obtain

$$\mathrm{grad}_\Sigma\vec{n} = \sum_\alpha \frac{1}{\varrho_\alpha}(\vec{e}_\alpha\otimes\vec{e}_\alpha) \ . \tag{B.19}$$

Obviously, $\mathrm{grad}_\Sigma\vec{n}$ is a symmetric tensor,

$$(\mathrm{grad}_\Sigma\vec{n})^T = \mathrm{grad}_\Sigma\vec{n} \ . \tag{B.20}$$

The surface divergence of the unit normal can be obtained from (B.2) and (B.17):

$$\mathrm{div}_\Sigma\vec{n} = \frac{1}{\varrho_1} + \frac{1}{\varrho_2} =: 2\bar{c} \ , \tag{B.21}$$

where $\bar{c}$ is the *mean curvature* of Σ. Another relation of interest in this context is

$$\mathrm{div}_\Sigma(\vec{n}\otimes\vec{n}) = (\mathrm{div}_\Sigma\vec{n})\vec{n} \ , \tag{B.22}$$

where the identity $(\mathrm{B.18})_5$ has been used.

B.5 Divergence of vector, Laplacian of scalar

In applying the operators (B.15) and (B.17) to decomposed vectors and second-order tensors of the form (B.5) and (B.7), it must be remembered that the surface gradient acts on the unit normal $\vec{n}$. For instance, the surface divergence of a vector field is given by

$$\mathrm{div}_\Sigma\vec{u} = \mathrm{div}_\Sigma(\vec{n}u_n + \vec{u}_\Sigma) = (\mathrm{div}_\Sigma\vec{n})u_n + \mathrm{grad}_\Sigma u_n\cdot\vec{n} + \mathrm{div}_\Sigma\vec{u}_\Sigma \ ,$$

where the identity $(\mathrm{B.18})_4$ has been used. In view of the orthogonality property $(\mathrm{B.3})_1$, the second term on the right-hand side vanishes, and we obtain

$$\mathrm{div}_\Sigma\vec{u} = (\mathrm{div}_\Sigma\vec{n})u_n + \mathrm{div}_\Sigma\vec{u}_\Sigma \ . \tag{B.23}$$

This enables us to express the three-dimensional divergence of a vector field as

$$\mathrm{div}\,\vec{u} = \vec{n}\cdot\frac{\partial\vec{u}}{\partial n} + \mathrm{div}_\Sigma\vec{u} = \vec{n}\cdot\Big(\frac{\partial\vec{n}}{\partial n}u_n + \vec{n}\frac{\partial u_n}{\partial n}\Big) + \mathrm{div}_\Sigma\vec{u} \ .$$

In view of (B.4), this can be written in the form

$$\operatorname{div}\vec{u} = \frac{\partial u_n}{\partial n} + \operatorname{div}_\Sigma \vec{u} \ . \tag{B.24}$$

The equations derived above will now be employed to decompose the three-dimensional Laplacian operator of a three-dimensional scalar field ϕ. Since the normal and tangential part of gradient of ϕ is $\dfrac{\partial \phi}{\partial n}$ and $\operatorname{grad}_\Sigma \phi$, respectively, (B.23 and (B.24) yield

$$\operatorname{div}\operatorname{grad}\phi = \frac{\partial^2 \phi}{\partial n^2} + (\operatorname{div}_\Sigma \vec{n})\frac{\partial \phi}{\partial n} + \operatorname{div}_\Sigma \operatorname{grad}_\Sigma \phi \ , \tag{B.25}$$

where $\operatorname{div}_\Sigma \operatorname{grad}_\Sigma$ is usually called the *Beltrami surface operator*.

B.6 Gradient of vector

The surface gradient of a decomposed vector field of the form (B.5) is given by

$$\operatorname{grad}_\Sigma \vec{u} = (\operatorname{grad}_\Sigma u_n) \otimes \vec{n} + u_n \operatorname{grad}_\Sigma \vec{n} + \operatorname{grad}_\Sigma \vec{u}_\Sigma \ , \tag{B.26}$$

where the identity $(B.18)_3$ has been used. This enables us to express the three-dimensional gradient of a vector field in the form:

$$\operatorname{grad}\vec{u} = \vec{n} \otimes \frac{\partial \vec{u}}{\partial n} + \operatorname{grad}_\Sigma \vec{u} = \vec{n} \otimes \Big(\frac{\partial \vec{n}}{\partial n} u_n + \vec{n}\frac{\partial u_n}{\partial n} + \frac{\partial \vec{u}_\Sigma}{\partial n} \Big) + \operatorname{grad}_\Sigma \vec{u} \ ,$$

or, in view of (B.4),

$$\operatorname{grad}\vec{u} = (\vec{n} \otimes \vec{n})\frac{\partial u_n}{\partial n} + \vec{n} \otimes \frac{\partial \vec{u}_\Sigma}{\partial n} + \operatorname{grad}_\Sigma \vec{u} \ . \tag{B.27}$$

Combining (B.26) and (B.27), $\operatorname{grad}\vec{u}$ can be put to a general form (B.7):

$$\begin{aligned} \boldsymbol{H} &:= \operatorname{grad}\vec{u} \\ &= (\vec{n} \otimes \vec{n})H_{nn} + \vec{n} \otimes \vec{H}_{n\Sigma} + \vec{H}_{\Sigma n} \otimes \vec{n} + \boldsymbol{H}_{\Sigma\Sigma} \ , \end{aligned} \tag{B.28}$$

where

$$\begin{aligned} H_{nn} &:= \frac{\partial u_n}{\partial n} \ , & \vec{H}_{\Sigma n} &:= \operatorname{grad}_\Sigma u_n \ , \\ \vec{H}_{n\Sigma} &:= \frac{\partial \vec{u}_\Sigma}{\partial n} \ , & \boldsymbol{H}_{\Sigma\Sigma} &:= u_n \operatorname{grad}_\Sigma \vec{n} + \operatorname{grad}_\Sigma \vec{u}_\Sigma \ . \end{aligned} \tag{B.29}$$

Note that $\vec{n} \cdot \boldsymbol{H}_{\Sigma\Sigma} = \vec{0}$, but, in general, $\boldsymbol{H}_{\Sigma\Sigma} \cdot \vec{n} \neq \vec{0}$ since the vector $\partial \vec{e}_\alpha / \partial \vartheta_\beta$ may have a component in the direction normal to Σ. Equations (B.24) and (B.27) can alternatively be written in the form

$$\operatorname{div}_\Sigma \vec{u} = \operatorname{div}\vec{u} - (\vec{n} \cdot \operatorname{grad}\vec{u} \cdot \vec{n}) \ , \tag{B.30}$$

$$\operatorname{grad}_\Sigma \vec{u} = \operatorname{grad}\vec{u} - \vec{n} \otimes (\vec{n} \cdot \operatorname{grad}\vec{u}) \ . \tag{B.31}$$

The right scalar product of the last equation with the unit normal $\vec{n}$ results in an important identity

$$\operatorname{grad}\vec{u} \cdot \vec{n} - \operatorname{grad}_\Sigma \vec{u} \cdot \vec{n} = \vec{n}(\vec{n} \cdot \operatorname{grad}\vec{u} \cdot \vec{n}) \ . \tag{B.32}$$

Obviously, $\vec{n} \cdot (\mathrm{grad}_\Sigma \vec{u} \cdot \vec{n}) = 0$, and vector $\mathrm{grad}_\Sigma \vec{u} \cdot \vec{n}$ is tangent to Σ. This implies that

$$\mathrm{grad}_\Sigma \vec{u} \cdot \vec{n} = (\mathrm{grad}_\Sigma \vec{u} \cdot \vec{n}) \cdot (\boldsymbol{I} - \vec{n} \otimes \vec{n}) \ . \tag{B.33}$$

B.7 Divergence of tensor

The surface divergence of a decomposed second-order tensor field of the form (B.7) is given by

$$\begin{aligned} \mathrm{div}_\Sigma \boldsymbol{T} &= \mathrm{div}_\Sigma [(\vec{n} \otimes \vec{n}) T_{nn} + \vec{n} \otimes \vec{T}_{n\Sigma} + \vec{T}_{\Sigma n} \otimes \vec{n} + \boldsymbol{T}_{\Sigma\Sigma}] \\ &= T_{nn} \mathrm{div}_\Sigma (\vec{n} \otimes \vec{n}) + (\mathrm{grad}_\Sigma T_{nn}) \cdot (\vec{n} \otimes \vec{n}) + (\mathrm{div}_\Sigma \vec{n}) \vec{T}_{n\Sigma} + \vec{n} \cdot \mathrm{grad}_\Sigma \vec{T}_{n\Sigma} \\ &\quad + (\mathrm{div}_\Sigma \vec{T}_{\Sigma n}) \vec{n} + \vec{T}_{\Sigma n} \cdot \mathrm{grad}_\Sigma \vec{n} + \mathrm{div}_\Sigma \boldsymbol{T}_{\Sigma\Sigma} \ , \end{aligned}$$

where the identity $(\mathrm{B.19})_{5,6}$ have been used. The second and the fourth terms vanish because of the orthogonality property $(\mathrm{B.3})_1$. Simplifying the first term according to (B.22), we obtain

$$\mathrm{div}_\Sigma \boldsymbol{T} = T_{nn} (\mathrm{div}_\Sigma \vec{n}) \vec{n} + (\mathrm{div}_\Sigma \vec{T}_{\Sigma n}) \vec{n} + (\mathrm{div}_\Sigma \vec{n}) \vec{T}_{n\Sigma} + \vec{T}_{\Sigma n} \cdot \mathrm{grad}_\Sigma \vec{n} + \mathrm{div}_\Sigma \boldsymbol{T}_{\Sigma\Sigma} \ . \tag{B.34}$$

B.8 Vector $\vec{n} \cdot \mathrm{grad}\,\mathrm{grad}\,\phi$

We are now interested in changing the order of differentiation of a scalar field ϕ with respect to n and with respect to surface curvilinear coordinates ϑ_α. Taking the derivative with respect to n of (B.15) applied to ϕ, we obtain

$$\frac{\partial}{\partial n}\Big(\mathrm{grad}_\Sigma \phi\Big) = \sum_\alpha \frac{\partial}{\partial n}\Big(\frac{\vec{e}_\alpha}{\varrho_\alpha}\frac{\partial \phi}{\partial \vartheta_\alpha}\Big) = \sum_\alpha \Big[\frac{\partial}{\partial n}\Big(\frac{\vec{e}_\alpha}{\varrho_\alpha}\Big)\frac{\partial \phi}{\partial \vartheta_\alpha} + \frac{\vec{e}_\alpha}{\varrho_\alpha}\frac{\partial^2 \phi}{\partial n \partial \vartheta_\alpha}\Big] \ .$$

Introducing tensor $\boldsymbol{C}_\Sigma$,

$$\boldsymbol{C}_\Sigma := \sum_\alpha \varrho_\alpha \frac{\partial}{\partial n}\Big(\frac{\vec{e}_\alpha}{\varrho_\alpha}\Big) \otimes \vec{e}_\alpha \ , \tag{B.35}$$

that classifies the curvature of the surface Σ, we obtain

$$\frac{\partial}{\partial n}\Big(\mathrm{grad}_\Sigma \phi\Big) = \mathrm{grad}_\Sigma \Big(\frac{\partial \phi}{\partial n}\Big) + \boldsymbol{C}_\Sigma \cdot \mathrm{grad}_\Sigma \phi \ , \tag{B.36}$$

which shows that the operators $\frac{\partial}{\partial n}$ and grad_Σ do not, in general, commute.

We are now ready to decompose vector $(\vec{n} \cdot \mathrm{grad}\,\mathrm{grad}\phi)$ into a normal and a tangential part. We have

$$\vec{n} \cdot \mathrm{grad}\,\mathrm{grad}\phi = \frac{\partial}{\partial n}(\mathrm{grad}\,\phi) = \frac{\partial}{\partial n}\Big(\vec{n}\frac{\partial \phi}{\partial n} + \mathrm{grad}_\Sigma \phi\Big) = \frac{\partial \vec{n}}{\partial n}\frac{\partial \phi}{\partial n} + \vec{n}\frac{\partial^2 \phi}{\partial n^2} + \frac{\partial}{\partial n}\Big(\mathrm{grad}_\Sigma \phi\Big) \ .$$

Making use (B.4) and (B.35), the last equation can be put to the form

$$\vec{n} \cdot \mathrm{grad}\,\mathrm{grad}\phi = \vec{n}\frac{\partial^2 \phi}{\partial n^2} + \mathrm{grad}_\Sigma \Big(\frac{\partial \phi}{\partial n}\Big) + \boldsymbol{C}_\Sigma \cdot \mathrm{grad}_\Sigma \phi \ . \tag{B.37}$$

The first term on the right-hand side of (B.37) is a normal part and the sum of the second and the third terms is a tangential part of vector $(\vec{n} \cdot \mathrm{grad}\,\mathrm{grad}\phi)$, respectively. The final relation of interest in this context arises when the second derivative of ϕ with respect to n is eliminated from (B.37) by means of (B.25):

$$\vec{n} \cdot \mathrm{grad}\,\mathrm{grad}\phi = \vec{n}\,\mathrm{div}\,\mathrm{grad}\,\phi - (\mathrm{div}_\Sigma \vec{n})\frac{\partial \phi}{\partial n}\,\vec{n} - \vec{n}\,\mathrm{div}_\Sigma \mathrm{grad}_\Sigma \phi + \mathrm{grad}_\Sigma \Big(\frac{\partial \phi}{\partial n}\Big) + \boldsymbol{C}_\Sigma \cdot \mathrm{grad}_\Sigma \phi \ . \tag{B.38}$$

Appendix C. Orthogonal curvilinear coordinates

The concepts of strain, deformation, stress, general principles applicable to all continuous media and the constitutive equations were developed in Chapters 1 through 7 using almost exclusively rectangular Cartesian coordinates. This coordinate system is the simplest system in which the three Cartesian unit base vectors have a fixed orientation in space. These coordinates, however, are suitable only for the solution of boundary-value problems involving bodies of rectangular shape. We often need to treat finite configurations with spherical, spheroidal, ellipsoidal, or another type of symmetric boundaries. We must therefore learn how to write the components of vectors and tensors in a wider class of coordinates. Here, we make a section on orthogonal curvilinear coordinates.

C.1 Coordinate transformation

In a three-dimensional space of elementary geometry, we can define a system of curvilinear coordinates by specifying three *coordinate transformation* functions x_k of the reference system of the rectangular Cartesian coordinates y_l: [28]

$$x_k = x_k(y_1, y_2, y_3), \qquad k = 1, 2, 3. \tag{C.1}$$

We assume that the three functions x_k of y_l have continuous first partial derivatives and that the jacobian determinant does not vanish, that is,

$$j := \det\left(\frac{\partial x_k}{\partial y_l}\right) \neq 0; \tag{C.2}$$

exceptions may occur at singular points or curves but never throughout any volume. Then the correspondence between x_k and y_l is one-to-one and there exists a unique inverse of (C.1) in the form

$$y_k = y_k(x_1, x_2, x_3), \qquad k = 1, 2, 3. \tag{C.3}$$

If x_1 is held constant, the three equations (C.3) define parametrically a surface, giving its rectangular Cartesian coordinates as a function of the two parameters x_2 and x_3. The first equation in (C.1), $x_1 = x_1(y_1, y_2, y_3)$, defines the same surface implicitly. Similarly, for fixed values of x_2 and x_3 we obtain other two surfaces. The three surfaces so obtained are the *curvilinear coordinate surfaces*. Each pair of coordinate surfaces intersect at a *curvilinear coordinate line*, along which only one of the three parameters x_k varies. In contrast to the rectangular Cartesian coordinates, the curvilinear coordinate lines are space curves. The three coordinate surfaces (and also all three coordinate lines) intersect each other at a single point P marked with specific values of x_1, x_2 and x_3. We may take the values of x_1, x_2 and x_3 as the *curvilinear coordinates* of point P (Figure C.1). If at each point P the coordinate lines through point P are mutually orthogonal, we have a system of *orthogonal curvilinear coordinates.*

Example. The *spherical coordinates* r, ϑ, λ, $(r \equiv x_1, \vartheta \equiv x_2, \lambda \equiv x_3)$, are defined by their relations to the rectangular Cartesian coordinates y_k by equations

$$y_1 = r \sin\vartheta \cos\lambda, \qquad y_2 = r \sin\vartheta \sin\lambda, \qquad y_3 = r\cos\vartheta, \tag{C.4}$$

[28] More generally, the coordinate transformation functions can be introduced between any two curvilinear systems.

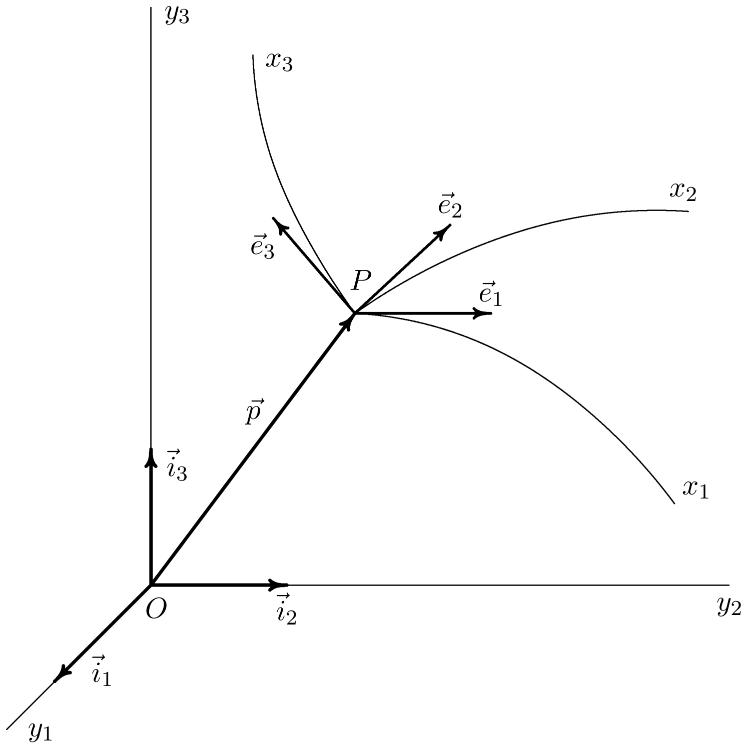

Figure C.1. Curvilinear coordinates.

or, inversely,

$$r = \sqrt{y_1^2 + y_2^2 + y_3^2}, \qquad \vartheta = \arctan\left(\frac{\sqrt{y_1^2 + y_2^2}}{y_3}\right), \qquad \lambda = \arctan\left(\frac{y_2}{y_1}\right). \tag{C.5}$$

The ranges of values are $0 \leq r < \infty$, $0 \leq \vartheta \leq \pi$, and $0 \leq \lambda \leq 2\pi$. The inverse of the jacobian j is given by

$$j^{-1} = \begin{vmatrix} \sin\vartheta\cos\lambda & \sin\vartheta\sin\lambda & \cos\vartheta \\ r\cos\vartheta\cos\lambda & r\cos\vartheta\sin\lambda & -r\sin\vartheta \\ -r\sin\vartheta\sin\lambda & r\sin\vartheta\cos\lambda & 0 \end{vmatrix} = r^2\sin\vartheta. \tag{C.6}$$

Hence, the unique inverse of (C.4) exists everywhere except at $r = 0$, $\vartheta = 0$ and $\vartheta = \pi$. The coordinate surfaces are the concentric sphere r=const. centered at the origin, the circular cones ϑ=const. centered on the y_3-axis, and the half planes λ=const. passing through the y_3-axis. The r-coordinate lines are half straights, the ϑ-coordinate lines are circles, the *meridians*, and the λ-coordinate lines are again circles, the *parallels*.

C.2 Base vectors

We now intend to introduce a set of base vectors $\vec{e}_k$ in the curvilinear system. We observe that if we move along a curvilinear coordinate line, only one of the three curvilinear coordinates varies. For instance, along the x_1-coordinate line only x_1 varies while x_2 and x_3 are kept fixed. In analogy to the definition of the base vectors in the rectangular Cartesian coordinates, the unit base vector $\vec{e}_1$ in curvilinear coordinates will be defined as the tangent vector to the x_1-coordinate line. Similarly, $\vec{e}_2$ and $\vec{e}_3$ are the unit tangent vectors to the curvilinear coordinate lines of varying x_2 and x_3, respectively. The direction cosines of the base vector $\vec{e}_1$ are given by (see Figure C.2)

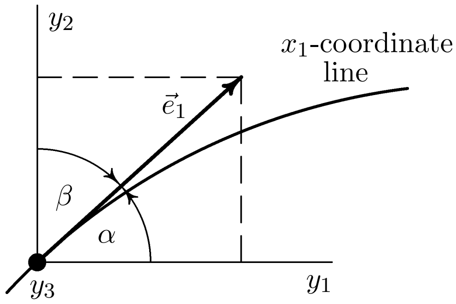

Figure C.2. The unit base vector $\vec{e}_1$.

$$\cos\alpha = \frac{1}{h_1}\frac{\partial y_1}{\partial x_1}, \ \cos\beta = \frac{1}{h_1}\frac{\partial y_2}{\partial x_1}, \ \cos\gamma = \frac{1}{h_1}\frac{\partial y_3}{\partial x_1},$$

where, for instance, α is the angle between the positive x_1-direction and the positive y_1-direction. A scale factor h_1 is introduced to satisfy the orthonormality condition for the direction cosines,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

In view of this, the scale factor is

$$h_1 = \sqrt{\sum_{l=1}^{3}\left(\frac{\partial y_l}{\partial x_1}\right)^2}.$$

Here and throughout this appendix, we will suspend the Einstein summation convention and explicitly indicate all summations over the range (1,2,3) when it is necessary. The unit base vector $\vec{e}_1$ can now be written in the form

$$\vec{e}_1 = \frac{1}{h_1}\left(\frac{\partial y_1}{\partial x_1}\vec{i}_1 + \frac{\partial y_2}{\partial x_1}\vec{i}_2 + \frac{\partial y_3}{\partial x_1}\vec{i}_3\right),$$

where $\vec{i}_1$, $\vec{i}_2$ and $\vec{i}_3$ are the rectangular Cartesian unit base vectors. If we express the position vector $\vec{p}$ of point P in terms of the rectangular Cartesian coordinates by

$$\vec{p} = \sum_{k=1}^{3} y_k \vec{i}_k, \tag{C.7}$$

the curvilinear coordinate unit base vector $\vec{e}_1$ can be expressed in the form

$$\vec{e}_1 = \frac{1}{h_1}\frac{\partial \vec{p}}{\partial x_k}.$$

Similar consideration can be carried out for the unit base vectors $\vec{e}_2$ and $\vec{e}_3$. Hence, for $k = 1, 2, 3$, we have

$$\boxed{\vec{e}_k = \frac{1}{h_k}\frac{\partial \vec{p}}{\partial x_k}.} \qquad \text{(no summation over } k\,!) \tag{C.8}$$

The function h_k are called the *scale factors*, or the *Lamé coefficients*. They are defined by relation

$$\boxed{h_k := \sqrt{\frac{\partial \vec{p}}{\partial x_k} \cdot \frac{\partial \vec{p}}{\partial x_k}}.} \tag{C.9}$$

They are non-negative functions of positions for a given curvilinear coordinate system.

In what follows, we shall consider only orthogonal curvilinear coordinates, and we will order the three curvilinear coordinates so that when each base vector $\vec{e}_k$ points in the direction of increasing x_k, the three base vectors form a right-hand system:

$$\vec{e}_k \cdot \vec{e}_l = \delta_{kl}, \qquad \vec{e}_k \times \vec{e}_l = \varepsilon_{klm} \vec{e}_m, \tag{C.10}$$

where δ_{kl} and ε_{klm} is the Kronecker delta and the Levi-Civita permutation symbol, respectively.

An infinitesimal vector at any point P can be expressed as

$$d\vec{p} = \sum_{k=1}^{3} \frac{\partial \vec{p}}{\partial x_k} dx_k = \sum_{k=1}^{3} h_k \vec{e}_k dx_k, \tag{C.11}$$

where we have substituted for $\partial \vec{p}/\partial x_k$ from (C.8). The scalar product of this with itself gives the square of the distance between two neighboring points, that is, the infinitesimal element of arc length on an arbitrary curve through point P:

$$(ds)^2 = (h_1 dx_1)^2 + (h_2 dx_2)^2 + (h_3 dx_3)^2. \tag{C.12}$$

In particular, along the x_k-coordinate line the only increment dx_k differs from zero, while the others are equal to zero. The elementary distance along the x_k-coordinate line is then

$$ds_k = h_k dx_k. \qquad \text{(no summation over } k\text{)}. \tag{C.13}$$

Note that three curvilinear coordinates x_k need not be lengths and the scale factors h_k may have dimensions. The products $h_k dx_k$ must, however, have dimension of length.

From (C.13) we may immediately develop the area element of coordinate surface and the volume element,

$$da_{kl} = ds_k ds_l = h_k h_l dx_k dx_l, \tag{C.14}$$

and

$$dV = ds_1 ds_2 ds_3 = h_1 h_2 h_3 dx_1 dx_2 dx_3. \tag{C.15}$$

Example. The scale factors in the spherical coordinates r, ϑ, λ are

$$h_r = 1, \qquad h_\vartheta = r, \qquad h_\lambda = r \sin\vartheta. \tag{C.16}$$

The square of the arc length is given by

$$(ds)^2 = (dr)^2 + r^2 (d\vartheta)^2 + r^2 \sin^2\vartheta (d\lambda)^2. \tag{C.17}$$

The unit vectors $\vec{e}_r$, $\vec{e}_\vartheta$ and $\vec{e}_\lambda$ in the direction of increasing r, ϑ, λ form a local right-handed orthonormal basis related to the rectangular Cartesian unit base vectors $\vec{i}_1$, $\vec{i}_2$ and $\vec{i}_3$ by

$$\begin{aligned}
\vec{e}_r &= \sin\vartheta \cos\lambda\, \vec{i}_1 + \sin\vartheta \sin\lambda\, \vec{i}_2 + \cos\vartheta\, \vec{i}_3, \\
\vec{e}_\vartheta &= \cos\vartheta \cos\lambda\, \vec{i}_1 + \cos\vartheta \sin\lambda\, \vec{i}_2 - \sin\vartheta\, \vec{i}_3, \\
\vec{e}_\lambda &= -\sin\lambda\, \vec{i}_1 + \cos\lambda\, \vec{i}_2.
\end{aligned} \tag{C.18}$$

Conversely,

$$\begin{aligned}
\vec{i}_1 &= \sin\vartheta\cos\lambda\,\vec{e}_r + \cos\vartheta\cos\lambda\,\vec{e}_\vartheta - \sin\lambda\,\vec{e}_\lambda,\\
\vec{i}_2 &= \sin\vartheta\sin\lambda\,\vec{e}_r + \cos\vartheta\sin\lambda\,\vec{e}_\vartheta + \cos\lambda\,\vec{e}_\lambda,\\
\vec{i}_3 &= \cos\vartheta\,\vec{e}_r - \sin\vartheta\,\vec{e}_\vartheta.
\end{aligned} \tag{C.19}$$

C.3 Derivatives of unit base vectors

Since the unit base vectors $\vec{e}_k$ are functions of position, they vary in direction as the curvilinear coordinates vary. Hence they cannot be treated as constants in differentiation. We shall next evaluate the derivatives $\partial\vec{e}_k/\partial x_l$. Because this is a vector quantity, it can be represented as a linear combination of the base vectors $\vec{e}_m$:

$$\frac{\partial \vec{e}_k}{\partial x_l} = \sum_{m=1}^{3} \binom{m}{k\,l} \vec{e}_m, \tag{C.20}$$

where the expansion coefficients $\binom{m}{k\,l}$ are known as the *Christoffel symbols*. Their meaning is simply the mth components of the derivative of the kth unit base vector along the lth coordinate. Another common notation for the Christoffel symbols is

$$\Gamma^m_{kl} \equiv \binom{m}{k\,l},$$

but we will not use it in this text because the Γ^m_{kl} look like mixed components of a third-order tensor, which they are not. [29]

Each Christoffel symbol is a function of the scale factors only. To establish the explicit form of this functional dependence, we take the scalar product of (C.20) with the unit base vector $\vec{e}_n$, use the orthonormality relation $(\text{C.10})_1$, and denote the index n again by m. We obtain the explicit expression for the Christoffel symbol:

$$\binom{m}{k\,l} = \frac{\partial \vec{e}_k}{\partial x_l} \cdot \vec{e}_m. \tag{C.21}$$

By substituting for the unit base vectors from (C.8), we can successively write

$$\begin{aligned}
\binom{m}{k\,l} &= \frac{\partial}{\partial x_l}\left(\frac{1}{h_k}\frac{\partial \vec{p}}{\partial x_k}\right) \cdot \frac{1}{h_m}\frac{\partial \vec{p}}{\partial x_m}\\
&= -\frac{1}{h_k^2 h_m}\frac{\partial h_k}{\partial x_l}\left(\frac{\partial \vec{p}}{\partial x_k}\cdot\frac{\partial \vec{p}}{\partial x_m}\right) + \frac{1}{h_k h_m}\left(\frac{\partial^2 \vec{p}}{\partial x_l \partial x_k}\cdot\frac{\partial \vec{p}}{\partial x_m}\right)\\
&= -\frac{1}{h_k}\frac{\partial h_k}{\partial x_l}\delta_{km} + \frac{1}{h_k h_m}\left(\frac{\partial^2 \vec{p}}{\partial x_l \partial x_k}\cdot\frac{\partial \vec{p}}{\partial x_m}\right),
\end{aligned}$$

[29] In general curvilinear coordinates, the Christoffel symbols of the first kind, $[kl, m]$, and the Christoffel symbols of the second kind, $\{kl, m\}$, are usually introduced. In orthogonal curvilinear coordinates, we can avoid the need of the two kinds of Christoffel's symbols, and can operate only with the Christoffel symbol $\binom{m}{k\,l}$. It should be, however, emphasized that neither the symbol $[kl, m]$ nor the symbol $\{kl, m\}$ does not reduce to the symbol $\binom{m}{k\,l}$ when general curvilinear coordinates are orthogonal.

since the base vectors are orthogonal. Let us find an explicit form for the second term on the right-hand side. Differentiating the orthonormality relation

$$\frac{\partial \vec{p}}{\partial x_k} \cdot \frac{\partial \vec{p}}{\partial x_m} = h_k h_m \delta_{km}$$

with respect to x_l, we have

$$\frac{\partial^2 \vec{p}}{\partial x_l \partial x_k} \cdot \frac{\partial \vec{p}}{\partial x_m} + \frac{\partial \vec{p}}{\partial x_k} \cdot \frac{\partial^2 \vec{p}}{\partial x_l \partial x_m} = \frac{\partial (h_k h_m)}{\partial x_l} \delta_{km}.$$

By permuting indices in the last equation, we obtain other two equations,

$$\frac{\partial^2 \vec{p}}{\partial x_m \partial x_l} \cdot \frac{\partial \vec{p}}{\partial x_k} + \frac{\partial \vec{p}}{\partial x_l} \cdot \frac{\partial^2 \vec{p}}{\partial x_m \partial x_k} = \frac{\partial (h_l h_k)}{\partial x_m} \delta_{lk}.$$

$$\frac{\partial^2 \vec{p}}{\partial x_k \partial x_m} \cdot \frac{\partial \vec{p}}{\partial x_l} + \frac{\partial \vec{p}}{\partial x_m} \cdot \frac{\partial^2 \vec{p}}{\partial x_k \partial x_l} = \frac{\partial (h_m h_l)}{\partial x_k} \delta_{ml}.$$

The last three equations can be combined to yield

$$\frac{\partial^2 \vec{p}}{\partial x_k \partial x_l} \cdot \frac{\partial \vec{p}}{\partial x_m} = \frac{1}{2} \left[\frac{\partial (h_k h_m)}{\partial x_l} \delta_{km} + \frac{\partial (h_m h_l)}{\partial x_k} \delta_{ml} - \frac{\partial (h_l h_k)}{\partial x_m} \delta_{lk} \right].$$

The Christoffel symbols can now be written in the form

$$\left({m \atop k\,l} \right) = -\frac{1}{h_k} \frac{\partial h_k}{\partial x_l} \delta_{km} + \frac{1}{2 h_k h_m} \left[\frac{\partial (h_k h_m)}{\partial x_l} \delta_{km} + \frac{\partial (h_m h_l)}{\partial x_k} \delta_{ml} - \frac{\partial (h_l h_k)}{\partial x_m} \delta_{lk} \right].$$

Simple manipulation with this expression finally results in

$$\boxed{\left({m \atop k\,l} \right) = \frac{1}{h_k} \frac{\partial h_l}{\partial x_k} \delta_{lm} - \frac{1}{h_m} \frac{\partial h_k}{\partial x_m} \delta_{kl}.} \tag{C.22}$$

If k, l and m are all different, $k \neq l \neq m$, then

$$\left({m \atop k\,l} \right) = \left({k \atop k\,k} \right) = \left({k \atop k\,l} \right) = 0. \tag{C.23}$$

The last equality is the consequence of the fact that the vector $\partial \vec{e}_k / \partial x_l$ is orthogonal to the x_k-coordinate line and, thus, has no component in the direction of $\vec{e}_k$ (but may have components in both directions orthogonal to $\vec{e}_l$). Because of (C.23), at most 12 of the 27 Christoffel symbols are non-zero:

$$\boxed{\left({l \atop k\,l} \right) = \frac{1}{h_k} \frac{\partial h_l}{\partial x_k}, \qquad \left({l \atop k\,k} \right) = -\frac{1}{h_l} \frac{\partial h_k}{\partial x_l} \qquad \text{if} \quad k \neq l.} \tag{C.24}$$

Of these, only six can be independent since it holds:

$$\left({l \atop k\,l} \right) = -\left({k \atop l\,l} \right). \tag{C.25}$$

We have shown that the Christoffel symbols are fully defined in terms of the scale factors. Since the scale factors are constant in a Cartesian coordinate system (rectangular or skew), (C.24) shows that all the Christoffel symbols vanish in a Cartesian system.

Example. Only six Christoffel symbols are non-zero in the spherical coordinates r, ϑ, λ, namely:

$$\begin{aligned} &\binom{\vartheta}{r\,\vartheta} = 1, \quad \binom{\lambda}{r\,\lambda} = \sin\vartheta, \quad \binom{\lambda}{\vartheta\,\lambda} = \cos\vartheta, \\ &\binom{r}{\vartheta\,\vartheta} = -1, \quad \binom{r}{\lambda\,\lambda} = -\sin\vartheta, \quad \binom{\vartheta}{\lambda\,\lambda} = -\cos\vartheta. \end{aligned} \tag{C.26}$$

The partial derivatives of the unit base vectors $\vec{e}_r$, $\vec{e}_\vartheta$ and $\vec{e}_\lambda$ are given by

$$\begin{aligned} &\frac{\partial \vec{e}_r}{\partial r} = 0, && \frac{\partial \vec{e}_\vartheta}{\partial r} = 0, && \frac{\partial \vec{e}_\lambda}{\partial r} = 0, \\ &\frac{\partial \vec{e}_r}{\partial \vartheta} = \vec{e}_\vartheta, && \frac{\partial \vec{e}_\vartheta}{\partial \vartheta} = -\vec{e}_r, && \frac{\partial \vec{e}_\lambda}{\partial \vartheta} = 0, \\ &\frac{\partial \vec{e}_r}{\partial \lambda} = \vec{e}_\lambda \sin\vartheta, && \frac{\partial \vec{e}_\vartheta}{\partial \lambda} = \vec{e}_\lambda \cos\vartheta, && \frac{\partial \vec{e}_\lambda}{\partial \lambda} = -\vec{e}_r \sin\vartheta - \vec{e}_\vartheta \cos\vartheta. \end{aligned} \tag{C.27}$$

C.4 Derivatives of vectors and tensors

With the formulae available for differentiating the unit base vectors we can write the partial derivatives of any vectors or tensors as follows.

Let $\vec{v}$ be an arbitrary vector represented in the form $\vec{v} = \sum_k v_k \vec{e}_k$. Then the partial derivative of $\vec{v}$ can be calculated by

$$\frac{\partial \vec{v}}{\partial x_l} = \frac{\partial}{\partial x_l}\Big(\sum_k v_k \vec{e}_k\Big) = \sum_k \Big(\frac{\partial v_k}{\partial x_l}\vec{e}_k + v_k \frac{\partial \vec{e}_k}{\partial x_l}\Big) = \sum_k \Big[\frac{\partial v_k}{\partial x_l}\vec{e}_k + v_k \sum_m \binom{m}{k\,l}\vec{e}_m\Big].$$

By interchanging the summation indices k and m in the last term we may write the vector $\partial\vec{v}/\partial x_l$ in a compact form,

$$\frac{\partial \vec{v}}{\partial x_l} = \sum_k v_{k;l}\vec{e}_k, \tag{C.28}$$

where

$$v_{k;l} := \frac{\partial v_k}{\partial x_l} + \sum_m \binom{k}{m\,l} v_m \tag{C.29}$$

is the *balanced* or *(neutral) derivative* of v_k with respect to x_l.

The partial derivatives of higher-order tensors are defined in a similar fashion. For example, if we represent a second-order tensor $\boldsymbol{T}$ as a dyadic, that is, as the linear combination of the nine dyads formed from three curvilinear coordinate unit base vectors [30] ,

$$\boldsymbol{T} = \sum_{kl} T_{kl}(\vec{e}_k \otimes \vec{e}_l), \tag{C.30}$$

[30] This is the Gibbs dyadic notation applied to orthogonal curvilinear coordinates.

the partial derivatives of $\boldsymbol{T}$ can then be calculated by

$$\begin{aligned} \frac{\partial \boldsymbol{T}}{\partial x_m} &= \sum_{kl} \left[\frac{\partial T_{kl}}{\partial x_m} (\vec{e}_k \otimes \vec{e}_l) + T_{kl} \frac{\partial \vec{e}_k}{\partial x_m} \otimes \vec{e}_l + T_{kl} \vec{e}_k \otimes \frac{\partial \vec{e}_l}{\partial x_m} \right] \\ &= \sum_{kl} \left[\frac{\partial T_{kl}}{\partial x_m} (\vec{e}_k \otimes \vec{e}_l) + T_{kl} \sum_n \binom{n}{k\,m} (\vec{e}_n \otimes \vec{e}_l) + T_{kl} \sum_n \binom{n}{l\,m} (\vec{e}_k \otimes \vec{e}_n) \right], \end{aligned}$$

where we have substituted for the derivatives of the unit base vectors from (C.20). We interchange the summation indices n and k in the second term and n and l in the last term to obtain

$$\frac{\partial \boldsymbol{T}}{\partial x_m} = \sum_{kl} T_{kl;m} (\vec{e}_k \otimes \vec{e}_l), \tag{C.31}$$

where

$$T_{kl;m} = \frac{\partial T_{kl}}{\partial x_m} + \sum_n \binom{k}{n\,m} T_{nl} + \sum_n \binom{l}{n\,m} T_{kn} \tag{C.32}$$

is the *balanced* or *(neutral) derivative* of T_{kl} with respect to x_m. Higher-order tensors can be treated in the same way by writing them as polyadics.

C.5 Invariant differential operators

The results of previous section on derivatives of unit base vectors, arbitrary vectors and tensors will now be used to formulate expressions for the gradient, divergence and curl in orthogonal curvilinear coordinates.

C.5.1 Gradient of a scalar

We start to derive the basic differential expression for the gradient operator. Let $\phi(x_1, x_2, x_3)$ be a scalar function. Then the gradient of a scalar ϕ, denoted by $\operatorname{grad}\phi$, is defined as a product of the nabla operator with ϕ,

$$\operatorname{grad}\phi := \nabla\phi. \tag{C.33}$$

Note that this definition makes no reference to any coordinate system. The gradient of scalar is thus a vector invariantly independent of coordinate system. To find its components in any orthogonal curvilinear system, we first express the nabla operator in the rectangular Cartesian coordinates,

$$\nabla = \sum_k \vec{i}_k \frac{\partial}{\partial y_k}. \tag{C.34}$$

The gradient of scalar function can then be calculated by the chain rule of calculus,

$$\operatorname{grad}\phi = \sum_l \vec{i}_l \frac{\partial \phi}{\partial y_l} = \sum_{kl} \vec{i}_l \frac{\partial \phi}{\partial x_k} \frac{\partial x_k}{\partial y_l},$$

where $\phi = \phi(x_k(y_l))$ in the second term and $\phi = \phi(x_k)$ in the third term.

Let us now find the explicit expression for $\partial x_k/\partial y_l$. In view of (C.7) and (C.8), the orthogonality relation $(C.10)_1$ has the form

$$\sum_m \frac{\partial y_m}{\partial x_k} \frac{\partial y_m}{\partial x_l} = h_k h_l \delta_{kl}.$$

Multiplying this by $\partial x_l/\partial y_n$ and summing the result over l, we obtain

$$\sum_m \frac{\partial y_m}{\partial x_k} \sum_l \frac{\partial y_m}{\partial x_l}\frac{\partial x_l}{\partial y_n} = \sum_l h_k h_l \delta_{kl} \frac{\partial x_l}{\partial y_n}.$$

Because of the one-to-one mappings (C.1) and (C.3), it holds

$$\sum_l \frac{\partial y_m}{\partial x_l}\frac{\partial x_l}{\partial y_n} = \delta_{mn},$$

so that, we have

$$\frac{\partial y_n}{\partial x_k} = h_k^2 \frac{\partial x_k}{\partial y_n}. \tag{C.35}$$

The gradient of a scalar ϕ can now be arranged as follows

$$\operatorname{grad}\phi = \sum_{kl} \frac{1}{h_k^2}\frac{\partial \phi}{\partial x_k}\frac{\partial y_l}{\partial x_k}\vec{i}_l = \sum_k \frac{1}{h_k^2}\frac{\partial \phi}{\partial x_k}\frac{\partial \vec{p}}{\partial x_k}.$$

Substituting for the orthogonal curvilinear unit base vector $\vec{e}_k$, we finally obtain

$$\boxed{\operatorname{grad}\phi = \sum_k \frac{1}{h_k}\frac{\partial \phi}{\partial x_k}\vec{e}_k.} \tag{C.36}$$

Thus, the nabla operator can be expressed in an orthogonal curvilinear coordinate system in the form

$$\nabla = \sum_k \frac{\vec{e}_k}{h_k}\frac{\partial}{\partial x_k}. \tag{C.37}$$

Example. The gradient of scalar ϕ in the spherical coordinates r, ϑ, λ is

$$\operatorname{grad}\phi = \frac{\partial \phi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \vartheta}\vec{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial \phi}{\partial \lambda}\vec{e}_\lambda. \tag{C.38}$$

C.5.2 Divergence of a vector

The divergence of vector $\vec{v}$ is a scalar function, defined in any orthogonal curvilinear coordinate system as the scalar product of the nabla operator and vector $\vec{v}$:

$$\operatorname{div}\vec{v} := \nabla\cdot\vec{v}. \tag{C.39}$$

This definition can successively be arranged as follows.

$$\begin{aligned}
\operatorname{div}\vec{v} &= \sum_k \frac{\vec{e}_k}{h_k}\cdot\frac{\partial \vec{v}}{\partial x_k} = \sum_k \frac{\vec{e}_k}{h_k}\cdot\sum_l v_{l;k}\vec{e}_l = \sum_{kl}\frac{v_{l;k}}{h_k}\delta_{kl} = \sum_k \frac{v_{k;k}}{h_k} = \sum_k \frac{1}{h_k}\left[\frac{\partial v_k}{\partial x_k} + \sum_m \left\{ {k \atop m\,k} \right\} v_m\right] \\
&= \sum_k \frac{1}{h_k}\left(\frac{\partial v_k}{\partial x_k} + \sum_{\substack{m \\ m\neq k}} \frac{1}{h_m}\frac{\partial h_k}{\partial x_m}v_m\right) = \frac{1}{h_1}\frac{\partial v_1}{\partial x_1} + \frac{1}{h_1 h_2}\frac{\partial h_1}{\partial x_2}v_2 + \frac{1}{h_1 h_3}\frac{\partial h_1}{\partial x_3}v_3 \\
&+ \frac{1}{h_2}\frac{\partial v_2}{\partial x_2} + \frac{1}{h_2 h_3}\frac{\partial h_2}{\partial x_3}v_3 + \frac{1}{h_2 h_1}\frac{\partial h_2}{\partial x_1}v_1 + \frac{1}{h_3}\frac{\partial v_3}{\partial x_3} + \frac{1}{h_3 h_1}\frac{\partial h_3}{\partial x_1}v_1 + \frac{1}{h_3 h_2}\frac{\partial h_3}{\partial x_2}v_2,
\end{aligned}$$

or, in a more compact form,

$$\boxed{\operatorname{div}\vec{v} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3v_1) + \frac{\partial}{\partial x_2}(h_3h_1v_2) + \frac{\partial}{\partial x_3}(h_1h_2v_3)\right].} \tag{C.40}$$

Example. The divergence of vector $\vec{v}$ in the spherical coordinates r, ϑ, λ is

$$\operatorname{div}\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\, v_\vartheta) + \frac{1}{r\sin\vartheta}\frac{\partial v_\lambda}{\partial\lambda}. \tag{C.41}$$

The divergence of the spherical unit base vectors is presented in the form

$$\operatorname{div}\vec{e}_r = \frac{2}{r}, \qquad \operatorname{div}\vec{e}_\vartheta = \frac{1}{r}\cot\vartheta, \qquad \operatorname{div}\vec{e}_\lambda = 0. \tag{C.42}$$

C.5.3 Curl of a vector

The curl of vector $\vec{v}$ is a vector, defined by the cross-product of the ∇ and $\vec{v}$:

$$\operatorname{rot}\vec{v} := \nabla \times \vec{v}. \tag{C.43}$$

We can successively write

$$\begin{aligned}
\operatorname{rot}\vec{v} &= \sum_k \frac{\vec{e}_k}{h_k} \times \frac{\partial\vec{v}}{\partial x_k} = \sum_k \frac{\vec{e}_k}{h_k} \times \sum_l v_{l;k}\vec{e}_l = \sum_{klm} \frac{v_{l;k}}{h_k}\varepsilon_{klm}\vec{e}_m = \sum_{klm}\frac{\varepsilon_{klm}}{h_k}\left[\frac{\partial v_l}{\partial x_k} + \sum_n \left\{ {l \atop n\,k} \right\} v_n\right]\vec{e}_m \\
&= \sum_{klm}\frac{\varepsilon_{klm}}{h_k}\left[\frac{\partial v_l}{\partial x_k} + \left\{ {l \atop k\,k} \right\} v_k\right]\vec{e}_m + \sum_{klm}\frac{\varepsilon_{klm}}{h_k}\sum_{\substack{n \\ n\neq k}}\left\{ {l \atop n\,k} \right\} v_n\vec{e}_m = \sum_{klm}\frac{\varepsilon_{klm}}{h_k}\left(\frac{\partial v_l}{\partial x_k} - \frac{1}{h_l}\frac{\partial h_k}{\partial x_l}v_k\right)\vec{e}_m \\
&= \sum_{klm}\left(\frac{\varepsilon_{klm}}{h_k}\frac{\partial v_l}{\partial x_k} + \frac{\varepsilon_{lkm}}{h_kh_l}\frac{\partial h_k}{\partial x_l}v_k\right)\vec{e}_m = \sum_{klm}\left(\frac{\varepsilon_{klm}}{h_k}\frac{\partial v_l}{\partial x_k} + \frac{\varepsilon_{klm}}{h_kh_l}\frac{\partial h_l}{\partial x_k}v_l\right)\vec{e}_m,
\end{aligned}$$

or, in a more compact form,

$$\boxed{\operatorname{rot}\vec{v} = \sum_m\left[\sum_{kl}\frac{\varepsilon_{klm}}{h_kh_l}\frac{\partial(h_lv_l)}{\partial x_k}\right]\vec{e}_m.} \tag{C.44}$$

Thus, for example, the coefficient of $\vec{e}_1$ is

$$(\operatorname{rot}\vec{v})_1 = \frac{1}{h_2h_3}\left[\frac{\partial(h_3v_3)}{\partial x_2} - \frac{\partial(h_2v_2)}{\partial x_3}\right].$$

The other components can be written down by the cyclic permutation of indices. It is often convenient to write the curl of a vector in determinant form:

$$\operatorname{rot}\vec{v} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\vec{e}_1 & h_2\vec{e}_2 & h_3\vec{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ h_1v_1 & h_2v_2 & h_3v_3 \end{vmatrix}. \tag{C.45}$$

Example. The curl of vector $\vec{v}$ in the spherical coordinates r, ϑ, λ is

$$\operatorname{rot}\vec{v} = \left[\frac{1}{r\sin\vartheta}\frac{\partial(\sin\vartheta\, v_\lambda)}{\partial\vartheta} - \frac{1}{r\sin\vartheta}\frac{\partial v_\vartheta}{\partial\lambda}\right]\vec{e}_r + \left[\frac{1}{r\sin\vartheta}\frac{\partial v_r}{\partial\lambda} - \frac{1}{r}\frac{\partial(r\, v_\lambda)}{\partial r}\right]\vec{e}_\vartheta + \left[\frac{1}{r}\frac{\partial(r\, v_\vartheta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\vartheta}\right]\vec{e}_\lambda, \tag{C.46}$$

or, in determinant form,

$$\operatorname{rot}\vec{v} = \frac{1}{r^2\sin\vartheta}\begin{vmatrix} \vec{e}_r & r\,\vec{e}_\vartheta & r\sin\vartheta\,\vec{e}_\lambda \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\vartheta} & \dfrac{\partial}{\partial\lambda} \\ v_r & r\,v_\vartheta & r\sin\vartheta\, v_\lambda \end{vmatrix}. \tag{C.47}$$

The curl of the spherical unit base vectors is presented in the form

$$\operatorname{curl}\vec{e}_r = 0, \qquad \operatorname{curl}\vec{e}_\vartheta = \frac{1}{r}\vec{e}_\lambda, \qquad \operatorname{curl}\vec{e}_\lambda = \frac{1}{r}\cot\vartheta\,\vec{e}_r - \frac{1}{r}\vec{e}_\vartheta. \tag{C.48}$$

C.5.4 Gradient of a vector

The gradient of vector $\vec{v}$ is a non-symmetric, second-order tensor, defined by the left-dyadic product of the nabla operator with a vector,

$$\operatorname{grad}\vec{v} := \nabla\otimes\vec{v}. \tag{C.49}$$

In any orthogonal curvilinear coordinates, we can successively write,

$$\begin{aligned}
\operatorname{grad}\vec{v} &= \sum_k \frac{\vec{e}_k}{h_k}\otimes\frac{\partial\vec{v}}{\partial x_k} = \sum_k \frac{\vec{e}_k}{h_k}\otimes\sum_l v_{l;k}\vec{e}_l = \sum_{kl}\frac{1}{h_k}\left[\frac{\partial v_l}{\partial x_k} + \sum_m \left(\begin{matrix} l \\ m\,k\end{matrix}\right) v_m\right](\vec{e}_k\otimes\vec{e}_l) \\
&= \sum_k \frac{1}{h_k}\left[\frac{\partial v_k}{\partial x_k} + \sum_m \left(\begin{matrix} k \\ m\,k\end{matrix}\right) v_m\right](\vec{e}_k\otimes\vec{e}_k) + \sum_k\sum_{\substack{l \\ l\neq k}}\frac{1}{h_k}\left[\frac{\partial v_l}{\partial x_k} + \sum_m \left(\begin{matrix} l \\ m\,k\end{matrix}\right) v_m\right](\vec{e}_k\otimes\vec{e}_l) \\
&= \sum_k \frac{1}{h_k}\left(\frac{\partial v_k}{\partial x_k} + \sum_{\substack{m \\ m\neq k}}\frac{1}{h_m}\frac{\partial h_k}{\partial x_m}v_m\right)(\vec{e}_k\otimes\vec{e}_k) + \sum_k\sum_{\substack{l \\ l\neq k}}\frac{1}{h_k}\left(\frac{\partial v_l}{\partial x_k} - \frac{1}{h_l}\frac{\partial h_k}{\partial x_l}v_k\right)(\vec{e}_k\otimes\vec{e}_l).
\end{aligned}$$

Hence, the orthogonal curvilinear components of the second-order tensor $\operatorname{grad}\vec{v}$ are

$$\boxed{(\operatorname{grad}\vec{v})_{kl} = \begin{cases} \dfrac{1}{h_k}\left(\dfrac{\partial v_k}{\partial x_k} + \displaystyle\sum_{\substack{m \\ m\neq k}}\dfrac{1}{h_m}\dfrac{\partial h_k}{\partial x_m}v_m\right) & \text{if} \quad l = k, \\ \dfrac{1}{h_k}\left(\dfrac{\partial v_l}{\partial x_k} - \dfrac{1}{h_l}\dfrac{\partial h_k}{\partial x_l}v_k\right) & \text{if} \quad l \neq k. \end{cases}} \tag{C.50}$$

The symmetric part of $\operatorname{grad}\vec{v}$ is the tensor $\frac{1}{2}[\operatorname{grad}\vec{v} + (\operatorname{grad}\vec{v})^T]$; its components are given by

$$\frac{1}{2}[\operatorname{grad}\vec{v} + (\operatorname{grad}\vec{v})^T]_{kl} = \begin{cases} \dfrac{1}{h_k}\left(\dfrac{\partial v_k}{\partial x_k} + \displaystyle\sum_{\substack{m \\ m\neq k}}\dfrac{1}{h_m}\dfrac{\partial h_k}{\partial x_m}v_m\right) & \text{if} \quad l = k, \\ \dfrac{1}{2}\left(\dfrac{1}{h_k}\dfrac{\partial v_l}{\partial x_k} + \dfrac{1}{h_l}\dfrac{\partial v_k}{\partial x_l} - \dfrac{1}{h_k h_l}\dfrac{\partial h_k}{\partial x_l}v_k - \dfrac{1}{h_k h_l}\dfrac{\partial h_l}{\partial x_k}v_l\right) & \text{if} \quad l \neq k. \end{cases} \tag{C.51}$$

Example. The symmetric part of the gradient of vector $\vec{v}$ in the spherical coordinates r, ϑ, λ is

$$\begin{aligned}
\frac{1}{2}[\operatorname{grad}\vec{v} + (\operatorname{grad}\vec{v})^T] = {} & \frac{\partial v_r}{\partial r}(\vec{e}_r \otimes \vec{e}_r) + \frac{1}{r}\left(\frac{\partial v_\vartheta}{\partial \vartheta} + v_r\right)(\vec{e}_\vartheta \otimes \vec{e}_\vartheta) \\
& + \frac{1}{r}\left(\frac{1}{\sin\vartheta}\frac{\partial v_\lambda}{\partial \lambda} + v_r + \cot\vartheta\, v_\vartheta\right)(\vec{e}_\lambda \otimes \vec{e}_\lambda) \\
& + \frac{1}{2}\left(\frac{\partial v_\vartheta}{\partial r} + \frac{1}{r}\frac{\partial v_r}{\partial \vartheta} - \frac{v_\vartheta}{r}\right)(\vec{e}_r \otimes \vec{e}_\vartheta + \vec{e}_\vartheta \otimes \vec{e}_r) \\
& + \frac{1}{2}\left(\frac{\partial v_\lambda}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial v_r}{\partial \lambda} - \frac{v_\lambda}{r}\right)(\vec{e}_r \otimes \vec{e}_\lambda + \vec{e}_\lambda \otimes \vec{e}_r) \\
& + \frac{1}{2r}\left(\frac{\partial v_\lambda}{\partial \vartheta} + \frac{1}{\sin\vartheta}\frac{\partial v_\vartheta}{\partial \lambda} - \cot\vartheta\, v_\lambda\right)(\vec{e}_\vartheta \otimes \vec{e}_\lambda + \vec{e}_\lambda \otimes \vec{e}_\vartheta).
\end{aligned} \tag{C.52}$$

C.5.5 Divergence of a tensor

The divergence of second-order tensor $\boldsymbol{T}$ is a vector, given in orthogonal curvilinear coordinates as follows.

$$\begin{aligned}
\operatorname{div}\boldsymbol{T} = \nabla\cdot\boldsymbol{T} = {} & \sum_m \frac{\vec{e}_m}{h_m}\cdot\frac{\partial \boldsymbol{T}}{\partial x_m} = \sum_m \frac{\vec{e}_m}{h_m}\cdot\sum_{kl} T_{kl;m}(\vec{e}_k \otimes \vec{e}_l) = \sum_{klm}\frac{T_{kl;m}}{h_m}(\vec{e}_m\cdot\vec{e}_k)\vec{e}_l = \sum_{kl}\frac{T_{kl;k}}{h_k}\vec{e}_l \\
= {} & \sum_{kl}\frac{1}{h_k}\left[\frac{\partial T_{kl}}{\partial x_k} + \sum_m \binom{k}{m\,k} T_{ml} + \sum_m \binom{l}{m\,k} T_{km}\right]\vec{e}_l \\
= {} & \sum_{kl}\frac{1}{h_k}\left[\frac{\partial T_{kl}}{\partial x_k} + \sum_{\substack{m \\ m\neq k}} \binom{k}{m\,k} T_{ml} + \binom{l}{k\,k} T_{kk} + \sum_{\substack{m \\ m\neq k}} \binom{l}{m\,k} T_{km}\right]\vec{e}_l \\
= {} & \sum_{kl}\frac{1}{h_k}\left[\frac{\partial T_{kl}}{\partial x_k} + \sum_{\substack{m \\ m\neq k}} \binom{k}{m\,k} T_{ml}\right]\vec{e}_l + \sum_l \sum_{\substack{k \\ k\neq l}} \frac{1}{h_k}\binom{l}{k\,k} T_{kk}\vec{e}_l + \sum_k \sum_{\substack{m \\ m\neq k}} \frac{1}{h_k}\binom{k}{m\,k} T_{km}\vec{e}_k \\
= {} & \sum_{kl}\frac{1}{h_k}\left[\frac{\partial T_{kl}}{\partial x_k} + \sum_{\substack{m \\ m\neq k}} \binom{k}{m\,k} T_{ml}\right]\vec{e}_l + \sum_l \sum_{\substack{k \\ k\neq l}} \left[\frac{1}{h_k}\binom{l}{k\,k} T_{kk} + \frac{1}{h_l}\binom{l}{k\,l} T_{lk}\right]\vec{e}_l \\
= {} & \sum_{kl}\frac{1}{h_k}\left(\frac{\partial T_{kl}}{\partial x_k} + \sum_{\substack{m \\ m\neq k}} \frac{1}{h_m}\frac{\partial h_k}{\partial x_m}\right)\vec{e}_l + \sum_l \sum_{\substack{k \\ k\neq l}} \frac{1}{h_k h_l}\left(\frac{\partial h_l}{\partial x_k} T_{lk} - \frac{\partial h_k}{\partial x_l} T_{kk}\right)\vec{e}_l.
\end{aligned}$$

The sum of the first two terms can be arranged in the same way as the divergence of a vector, see Section C.5.2. The lth component of the result is given by

$$\boxed{(\operatorname{div}\boldsymbol{T})_l = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 T_{1l}) + \frac{\partial}{\partial x_2}(h_3 h_1 T_{2l}) + \frac{\partial}{\partial x_3}(h_1 h_2 T_{3l})\right] + \sum_k \frac{1}{h_k h_l}\left(\frac{\partial h_l}{\partial x_k} T_{lk} - \frac{\partial h_k}{\partial x_l} T_{kk}\right).} \tag{C.53}$$

We should emphasize that tensor $\boldsymbol{T}$ has not been assumed symmetric.

Example. The divergence of a second-order tensor $\boldsymbol{T}$ in the spherical coordinates r, ϑ, λ is

$$
\begin{aligned}
\operatorname{div} \boldsymbol{T} = & \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rr}) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\, T_{\vartheta r}) + \frac{1}{r\sin\vartheta}\frac{\partial T_{\lambda r}}{\partial\lambda} - \frac{1}{r}(T_{\vartheta\vartheta} + T_{\lambda\lambda})\right]\vec{e}_r \\
& + \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\vartheta}) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\, T_{\vartheta\vartheta}) + \frac{1}{r\sin\vartheta}\frac{\partial T_{\lambda\vartheta}}{\partial\lambda} + \frac{1}{r}(T_{\vartheta r} - \cot\vartheta\, T_{\lambda\lambda})\right]\vec{e}_\vartheta \quad \text{(C.54)} \\
& + \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\lambda}) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\, T_{\vartheta\lambda}) + \frac{1}{r\sin\vartheta}\frac{\partial T_{\lambda\lambda}}{\partial\lambda} + \frac{1}{r}(T_{\lambda r} + \cot\vartheta\, T_{\lambda\vartheta})\right]\vec{e}_\lambda .
\end{aligned}
$$

C.5.6 Laplacian of a scalar and a vector

We may obtain the Laplacian of scalar ϕ by combining (C.36) and (C.40), using $\vec{v} = \operatorname{grad}\phi$. This leads to

$$
\nabla^2\phi \equiv \operatorname{div}\operatorname{grad}\,\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial x_3}\right)\right]. \quad \text{(C.55)}
$$

Example. The Laplacian of scalar ϕ in the spherical coordinates r, ϑ, λ is

$$
\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\phi}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\phi}{\partial\lambda^2}. \quad \text{(C.56)}
$$

In curvilinear coordinates, the Laplacian of a vector is more complicated than the Laplacian of a scalar due to the spatial dependence of the unit base vectors. Occasionally, the Laplacian of a vector is needed in spherical coordinates. It is best obtained by using the vector differential identity (A.32). Without detailed derivation, we introduce

$$
\begin{aligned}
(\nabla^2\vec{v})_r &= \nabla^2 v_r - \frac{2}{r^2}\,v_r - \frac{2}{r^2}\frac{\partial v_\vartheta}{\partial\vartheta} - \frac{2}{r^2}\frac{\cos\vartheta}{\sin\vartheta}\,v_\vartheta - \frac{2}{r^2\sin\vartheta}\frac{\partial v_\lambda}{\partial\lambda}, \\
(\nabla^2\vec{v})_\vartheta &= \nabla^2 v_\vartheta - \frac{1}{r^2\sin^2\vartheta}\,v_\vartheta + \frac{2}{r^2}\frac{\partial v_r}{\partial\vartheta} - \frac{2}{r^2}\frac{\cos\vartheta}{\sin^2\vartheta}\frac{\partial v_\lambda}{\partial\lambda}, \qquad \text{(C.57)} \\
(\nabla^2\vec{v})_\lambda &= \nabla^2 v_\lambda - \frac{1}{r^2\sin^2\vartheta}\,v_\lambda + \frac{2}{r^2\sin\vartheta}\frac{\partial v_r}{\partial\lambda} + \frac{2}{r^2}\frac{\cos\vartheta}{\sin^2\vartheta}\frac{\partial v_\vartheta}{\partial\lambda}.
\end{aligned}
$$

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