

# Quantum Optics

## an Introduction (short Version)

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# CHAPTER 1

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## Introduction

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### 1.1. Quantum Optics and The Renaissance of Atomic Physics

Quantum Optics is an area of atomic, molecular and optical (AMO) physics, which is not easy to define very precisely. Its main characteristic, however, is that it deals with lasers, i. e. sources of coherent electromagnetic radiation. If we define QO in such a way, then it would be clear to everybody that Quantum Optics is one of the areas of physics that influence our every day's life particularly profoundly. Since their developments in the 60's lasers find more and more of applications: in medicine, in telecommunication, in environmental sciences, in metrology, in biology, etc. At the same time, various specific properties of the laser light can be employed for fundamental research: coherence, intensity, monochromaticity. In fact in the recent 15 years or so we observe an explosion of fundamental research with lasers that opens new perspectives for applications and technology. Some of those new perspectives seem to be directly stimulated by "Star Trek" series – nevertheless they do not belong to Sci-Fi and are firmly based scientifically! Let us mention few them (some of them will be discussed in these lectures):

- *Atom Optics and Atom Lasers*

Laser light may be employed to manipulate mechanically matter. An example of such light-matter interactions is radiation pressure. In the recent years techniques of manipulating matter at the microscopic (quantum) level have been developed. At this level atoms exhibit their wave nature, and matter waves behave very similarly to electromagnetic waves. The new area of AMO physics that deal with matter waves has been termed atom optics. Similar laser techniques allow for cooling and trapping of atoms, ions and molecules. Nobel Prizes of W. Paul, H. Dehmelt, N. Ramsey several years ago, and to C. Cohen-Tannoudji, S. Chu and W. Phillips in 1997 were related to development of those techniques. In 1995 two groups (E. Cornell- C. Wieman, and W. Ketterle) managed to condense an ensemble of cold atoms into a, so-called, Bose-Einstein condensate (BEC), which can be regarded as a trapped macroscopic coherent matter wave packet formed by statistically indistinguishable (bosonic) atoms. BE Conden-

sate can be employed then as a source of coherent, intense matter waves (atom laser). Operation of atom lasers has been already demonstrated in few laboratories. One speculate that within few years atom lasers will find fascinating applications in precision interferometry of matter waves, atomic lithography on nanometer level etc.

- *Quantum Communication and Information Processing*

We observe in the recent years an exponential growth of interest in applying quantum mechanics to information processing. In particular quantum mechanics allows for less noisy, more secure communication and information transfer. It has been already demonstrated that using single photons one can transmit confidential information over distances of kilometres in a way, which eliminates the possibility of eavesdropping. Quantum “teleportation” (i. e. copying of a unknown quantum state from one place to another) has also been demonstrated. Theorist have demonstrated the quantum computers could realize various computational tasks much faster than the classical computers used nowadays. Several laboratories in the world are starting to develop first basic elements for quantum computing and information processing. Quantum optics provides a very natural area for such implementations. Squeezed states of light, entangled state of pairs or triples of photons are being used in those experiments. Further applications involving cold atoms and ions are being considered.

The laboratories in the Physics Department are among the leading ones in the world. The research areas cover:

- atom optic and atom lasers (W. Ertmer),
- atom interferometry (W. Ertmer),
- physics of highly excited molecules (E. Tiemann),
- physics in intense laser fields (B. Wellegehausen),
- detection of gravitational waves (using interferometry on Earth and in space, K. Danzmann),
- applications in medicine and material processing.

All of the above mentioned experimental areas are equally challenging theoretically. Although the theory that one develops here is phenomenological, and is not as fundamental as quantum field theory of fundamental constituents of matter, the problems and difficulties are comparable to the most difficult problems of theoretical physics. Theoretical quantum optics is nowadays somewhat similar to astrophysics in its interdisciplinary character: it must combine knowledge of classical and quantum electrodynamics, optics and field theory, statistical physics, quantum mechanics, plasma physics, atomic and molecular physics and so forth.

## 1.2. Literature

As already stated above, quantum optics is an interdisciplinary area and requires good basis in other areas of physics. Particularly important is quantum mechanics, which is well covered in

- *Quantum Mechanics*

A good concise course is of course the Schwabl book [?]. To deepen his or hers knowledge the reader should turn to the classical texts of Landau, [?], Messiah [?] and Baym [?]. The best among the relatively new ones, the most complete, and particularly useful in the context of quantum optics is the book of Cohen-Tannoudji, Diu, and Laloë [?]. One of us (M.L) considers the book by Galindo and Pascual to be an absolute “Meisterstck” [?].

- *Quantum Optics*

There are very few good handbooks of quantum optics. We will base great parts of the lectures on the books by Walls and Milburn [?] and Vogel and Welsch [?]. The interested readers should tern to “classical” texts of Loudon [?] and Luisell [?], may consider books by Haken [?], Meystre & Sargent III [?] and Mandel & Wolf [?].

- *Laser Physics and Non-linear Optics*

A very good book for those interested in experiments is Yariv’s text [?]. The “Bible” of classical optics is still Born and Wolf [?], whereas the corresponding one of laser physics is Siegman [?]. Laser theory “freaks”, however, would enjoy more Eberly and Milonni book [?]. The book by Shen [?] can also be recommended for further reading.



# CHAPTER 2

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## Quantization of the Free EM field

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In classical electrodynamics electromagnetic fields  $\mathbf{E}$ ,  $\mathbf{D}$ , and  $\mathbf{B}$ ,  $\mathbf{H}$  are physical observables represented as time and position dependent real valued vectors. The complete microscopic theory must necessarily consider those physical observables as quantum mechanical operators. Historically, the first indication of quantum nature of light stem from Max Planck, who described spectrum of thermal radiation assuming that the energy quanta of the field correspond to excitations of harmonic oscillators of a given energy. Einstein explained the photoelectric effects assuming that electromagnetic energy is distributed in discrete quanta, later termed as photons. He has also used the concept of photons to describe the absorption and emission of light by atoms. With an advent of modern quantum theory, Dirac gave the first full description of electromagnetic fields in terms of ensembles of quantum harmonic oscillators. Further theoretical developments by Schwinger, Feynmann and other have lead to the development of the quantum electrodynamics.

Experimentally, however, it took a long time to demonstrate the quantum nature of light directly. The vast majority of physical optics experiment can be well described using classical Maxwell theory. Consider for example an analogue of Young experiment: two slit experiment with one photon incident on the slits (G. I. Taylor 1909). The results of such experiment can be equally well described in terms of interference of classical waves, or in terms of interference of probability amplitudes for the photon passing through one or other slit. Detecting photons requires more than just measuring interference patterns. The difference between classical and quantum interference become first visible on the level of higher order correlations of the EM field, for instance intensity-intensity correlations.

The first experiment of such type was Hanbury-Brown and Twiss experiment, who measured correlations in the photocurrent fluctuations at two detectors irradiated by the thermal light. The result showed enhancement of the two-time intensity correlations for short time delay, termed later as photon bunching effect. Photon bunching, however, can also be described in the classical theory assuming fluctuations of the field amplitude. The light coming from a well stabilized laser does not exhibit any correlation enhancement. When described in terms of photons, it can be regarded as

a sequence of statistically independent quanta coming to the detector with a constant mean rate. The number of photons detected in the time interval  $T$  is distributed according to Poisson distribution. Imagine, however, that an atom irradiated by a laser produces photons. Such an atom undergoes excitation (absorbs light from the laser) and emits it then in a form of fluorescence photon. The Probability of detecting two photons one right after another must in such situation cease to zero, since the physical process of absorption requires some time. The fluorescence photons emitted in this situation arrive at the detector at constant mean rate, but in a “more ordered” manner than those described by independent particle picture. This kind of correlation (termed as anti-bunching by Walls and Carmichael in 1975) cannot be described classical theory, and necessarily requires quantum mechanical description of the electromagnetic fields. Photon anti-bunching was observed by Kimble, Dagenais and Mandel in 1976, and was in fact the first direct observations of the states of electromagnetic field that have an intrinsic quantum nature. Since then the engineering of truly quantum states of light has developed enormously.

In the following we will thus postulate the quantum character of the electromagnetic fields, and quantize them using methods analogous to those used in standard quantum mechanics (canonical quantization). As we shall see, linearity of Maxwell equations implies that EM fields can indeed be regarded as ensembles of harmonic oscillators. The canonical variables for these oscillators are related to Fourier component of the electric field and electromagnetic vector potential. Since the whole quantization procedure will extensively use the concept of the harmonic oscillator, we start by reminding it to the readers.

## 2.1. Quantum harmonic oscillator revisited

In classical mechanics a harmonic oscillator in one dimension is defined as point particle of mass  $m$  moving in a quadratic potential  $V(q) = m\omega^2 q^2/2$ , or alternatively undergoing the influence of linear restoring force when displaced from the origin  $q = 0$ . Second Newton law reads

$$\ddot{q} = -\omega^2 q \tag{2.1}$$

and describes harmonic oscillations around the origin with an amplitude and frequency  $\omega$ . Here,  $\dot{q}$ , and  $\ddot{q}$  denote the first, and the second time derivatives, respectively.

The above equation can be derived using the Hamiltonian formalism. The Hamilton function is

$$\mathcal{H}(p, q) = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}, \tag{2.2}$$

where the canonical momentum  $p = m\dot{q}$ . Equation of motion (2.1) is equivalent to

Hamilton equations

$$\begin{aligned}\dot{q} &= \{q, \mathcal{H}\} = \frac{\partial \mathcal{H}}{\partial p} = p/m, \\ \dot{p} &= \{p, \mathcal{H}\} = -\frac{\partial \mathcal{H}}{\partial q} = -m\omega^2 q,\end{aligned}\tag{2.3}$$

where  $\{.,.\}$  denotes the Poisson brackets. In particular  $\{q, p\} = 1$ .

Canonical quantization consist in:

- i. replacing canonically conjugated variables by quantum mechanical self-adjoint operators  $q \rightarrow \hat{q}$ ,  $p \rightarrow \hat{p}$
- ii. replacing canonical Poisson brackets by commutators  $\{q, p\} \rightarrow [\hat{q}, \hat{p}]/i\hbar$

The quantum mechanical Hamiltonian becomes thus

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{q}^2}{2}\tag{2.4}$$

and the Heisenberg equations have an analogous form as the Newton equations

$$\frac{d\hat{q}}{dt} = -i[\hat{q}, \hat{\mathcal{H}}]/\hbar = \hat{p}/m,\tag{2.5}$$

$$\frac{d\hat{p}}{dt} = -i[\hat{p}, \hat{\mathcal{H}}]/\hbar = -m\omega^2 \hat{q}.\tag{2.6}$$

There are various ways of representing quantum mechanical position and canonical momentum operators. The most common is the position representation in which  $\hat{q} = q$ , and  $\hat{p} = -i\hbar\partial/\partial q$ . In the position representation the Hamiltonian reads:

$$\hat{\mathcal{H}} = -\frac{\hbar}{2m} \frac{\partial^2}{\partial \hat{q}^2} + \frac{m\omega^2 \hat{q}^2}{2}\tag{2.7}$$

The eigenstates of the Hamiltonian are the solutions of the stationary Schrödinger equation:

$$\hat{\mathcal{H}} |\psi\rangle = E |\psi\rangle,\tag{2.8}$$

are called usually Fock states,  $|n\rangle$  and correspond to  $E_n = \hbar\omega(n + \frac{1}{2})$ . In the position representation their wave functions  $\psi(q) = \langle q|\psi\rangle$  are

$$\psi_n(q) = \langle q|n\rangle = \left(\frac{\beta^2}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} \exp(-\beta^2 q^2/2) H_n(\beta q)\tag{2.9}$$

where  $\beta = \sqrt{m\omega/\hbar}$  has the dimension of the inverse of the length, whereas  $H_n(.)$  denote Hermite polynomials.

It is convenient to introduce dimensionless operators

$$\hat{Q} = \beta \hat{q} \quad (2.10)$$

$$\hat{P} = \hat{p}/\beta\hbar, \quad (2.11)$$

so that  $[\hat{Q}, \hat{P}] = i$ . One can then define annihilation and creations operators

$$\hat{a} = \frac{1}{\sqrt{2}} (\hat{Q} + i\hat{P}), \quad (2.12)$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}} (\hat{Q} - i\hat{P}), \quad (2.13)$$

which fulfill

$$[\hat{a}, \hat{a}^\dagger] = 1 \quad (2.14)$$

Their names follow from the fact that they annihilate, or create respectively energy quanta of the quantum oscillator

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle \quad (2.15)$$

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle \quad (2.16)$$

The operator  $\hat{n} = \hat{a}^\dagger \hat{a}$  counts the number of quanta, and  $\hat{\mathcal{H}} = \hbar\omega(\hat{n} + 1/2)$ . The term  $1/2$  represents here the energy of the *zero-point fluctuations*, or vacuum fluctuation, since it describes energy of the vacuum state  $|0\rangle$  that contains no quanta.

Note that quantum expression for the Hamiltonian differs from the classical one by the zero-point (vacuum fluctuation) energy.

## 2.2. Maxwell equations for free EM fields

In order to quantize the electromagnetic field we should proceed as above. First we have to consider evolution equation for the fields, then find the canonical variables and classical Hamilton function, and then quantize. Sounds simple, but is difficult. The reason is that it is absolutely not clear what are the proper canonical variables. Another reason is related to the gauge invariance of the Maxwell equations, and the fact that some of the Maxwell equations do not describe any dynamics, but rather constraints on the EM fields.

The Maxwell equations for the free EM fields in vacuum read

$$\nabla \cdot \mathbf{B} = 0 \quad (2.17)$$

$$\nabla \cdot \mathbf{D} = 0 \quad (2.18)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \quad (2.19)$$

$$\frac{\partial \mathbf{D}}{\partial t} - \nabla \times \mathbf{H} = 0 \quad (2.20)$$

The first two equations do not have dynamical character. Because in free space  $\mathbf{D} = \varepsilon_0 \mathbf{E}$ , and  $\mathbf{B} = \mu_0 \mathbf{H}$ , where  $\varepsilon_0$  is electric permittivity, whereas  $\mu_0$  is magnetic permeability, and  $\varepsilon_0 \mu_0 = c^{-2}$ , the second and fourth equations can be written as

$$\nabla \cdot \mathbf{E} = 0 \quad (2.21)$$

$$\frac{\partial \mathbf{E}}{c^2 \partial t} - \nabla \times \mathbf{B} = 0 \quad (2.22)$$

### 2.3. Gauge invariance

Solutions of the Maxwell can be written using EM potentials.  $\mathbf{B}$  is always sourceless ( $\nabla \cdot \mathbf{B} = 0$ ) and therefore we have  $\mathbf{B} = \nabla \times \mathbf{A}$ . Now using Maxwell equations in vacuum we get  $\mathbf{E} = -\partial \mathbf{A} / \partial t$ .

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (2.23)$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \quad (2.24)$$

Note that changing  $\mathbf{A} \rightarrow \mathbf{A} + \text{grad} \varphi$ , where  $\varphi$  is arbitrary time independent function of position  $\mathbf{r}$  does not change the fields. This is the famous *gauge invariance* in free space.

In order to identify canonical variable and then to quantize the fields we have to fix (choose) the gauge. That will automatically assure that the constraints on  $\mathbf{B}$  and  $\mathbf{E}$  will be fulfilled (divergence of both fields will be zero). The price to pay is that the definition of canonical variables will be gauge dependent, and so will be our definition of elementary field quanta, i.e. photons. The physical results concerning physical observables will, however, as they should be gauge independent, and can be expressed equivalently in any gauge. The most common choice of gauge (when we deal with non relativistic theory, and when we do not care about relativistic invariance) is the *Coulomb gauge*,

$$\boxed{\nabla \cdot \mathbf{A} = 0} \quad (2.25)$$

With this choice, the EM potential fulfills the *wave equation*

$$\boxed{\nabla^2 \mathbf{A}(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2} = 0} \quad (2.26)$$

### 2.4. Canonical quantization

We can always write the EM potential as a sum of positive and negative frequency parts,

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}^{(+)}(\mathbf{r}, t) + \mathbf{A}^{(-)}(\mathbf{r}, t), \quad (2.27)$$

where  $\mathbf{A}^{(\pm)}(\mathbf{r}, t)$  varies as  $\exp(\mp i\omega t)$  with  $\omega \geq 0$ . Obviously,  $\mathbf{A}^{(+)}(\mathbf{r}, t) = (\mathbf{A}^{(-)}(\mathbf{r}, t))^*$ , because  $\mathbf{A}(\mathbf{r}, t)$  is real.

Let us consider first the simplest situation – when the field is contained in a finite volume (a box, a resonator, etc.). The field can be expanded into a discrete sum of eigenmodes, (see next example for details)

$$\mathbf{A}^{(+)}(\mathbf{r}, t) = \sum_k \mathbf{c}_k \mathbf{u}_k(\mathbf{r}) \exp(-i\omega_k t), \quad (2.28)$$

where the eigenfunctions fulfill

$$\left( \nabla^2 + \frac{\omega_k^2}{c^2} \right) \mathbf{u}_k(\mathbf{r}) = 0 \quad (2.29)$$

In the Coulomb gauge,  $\nabla \cdot \mathbf{u}_k(\mathbf{r}) = 0$ . The eigenfunctions correspond to different eigenfrequencies  $\omega_k$ , and are enumerated by a multiindex  $k$  (see example on this page for explanation of  $k$ ). As usual, they can be constructed such that they are orthonormal

$$\int \mathbf{u}_k^*(\mathbf{r}) \mathbf{u}_{k'}(\mathbf{r}) d_3\mathbf{r} = \delta_{kk'} \quad (2.30)$$

The boundary conditions for eigenmodes depend on the physics (for instance we can have a box-like, or cylinder-like perfect resonator).

Example:

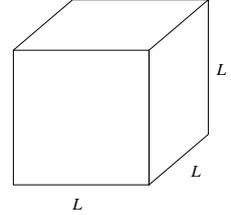
Periodic boundary conditions. They correspond to a periodic box of size  $L$ , and ‘mimic’ a ring resonator.

Periodic boundary condition imply that

$$k_x = 2\pi n_x/L, \quad k_y = 2\pi n_y/L, \quad k_z = 2\pi n_z/L,$$

with  $n_x, n_y, n_z$  integer. We assume a general form for  $\mathbf{A}(\mathbf{r}, t)$ :

$$\mathbf{A}(\mathbf{r}, t) = \sum_k \left( \frac{\hbar}{2\omega_k \varepsilon_0} \right)^{1/2} \mathbf{c}_k(t) \mathbf{u}_k(\mathbf{r}) + \text{c. c.} \quad (2.31)$$



with the *multiindex*  $k = (\mathbf{k}, \mu)$ .  $\mathbf{k}$  is here photon wave vector, whereas  $\mu = 1, 2$  enumerates two possible directions of the photon polarization.  $\mathbf{c}_k(t)$  denotes the amplitude of each mode. The eigenfunctions  $\mathbf{u}_k(\mathbf{r})$  are of the form

$$\mathbf{u}_k(\mathbf{r}) = \frac{1}{L^{3/2}} e^{i\mathbf{k}\mathbf{r}}. \quad (2.32)$$

From the gauge condition ( $\nabla \cdot \mathbf{A} = 0$ ) follows the transversality of  $\mathbf{c}_k$ :  $\mathbf{k} \cdot \mathbf{c}_k(t) = 0$ . Using the wave equation Eq.(2.26)

$$\begin{aligned} \nabla^2 \mathbf{A}(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2} &= 0 \\ \Rightarrow \left( -k^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{c}_k(t) &= 0 \end{aligned}$$

we find that the amplitudes are simple harmonic oscillators

$$\ddot{\mathbf{c}}_k(t) = -\omega_k^2 \mathbf{c}_k(t) \quad (2.33)$$

Thus

$$\mathbf{c}_k(t) = \mathbf{a}_k e^{-i\omega_k t} + \mathbf{a}_k^* e^{+i\omega_k t} \quad (2.34)$$

Although the amplitude  $\mathbf{c}_k(t)$  is complex  $\mathbf{A}(\mathbf{r}, t)$  has to be real. Therefore  $\mathbf{c}_k^* = \mathbf{c}_k$ . The  $\mathbf{a}_k$  specifies the amplitude of both polarisations

$$\mathbf{a}_k = \sum_{\mu=1}^2 a_{\mathbf{k}\mu} \boldsymbol{\varepsilon}_{\mathbf{k}\mu} \quad (2.35)$$

where  $\boldsymbol{\varepsilon}_{\mathbf{k}\mu}$  denotes the polarisation.

Properties of the polarisation  $\boldsymbol{\varepsilon}_{\mathbf{k}\mu}$ :

$$\begin{aligned} \mathbf{k} \cdot \boldsymbol{\varepsilon}_{\mathbf{k}\mu} &= 0 && \text{orthogonal to k-vector} \\ \boldsymbol{\varepsilon}_{\mathbf{k},\mu_1}^* \cdot \boldsymbol{\varepsilon}_{\mathbf{k},\mu_2} &= \delta_{\mu_1\mu_2} && \text{orthogonal} \\ \boldsymbol{\varepsilon}_{\mathbf{k},\mu_1}^* \times \boldsymbol{\varepsilon}_{\mathbf{k},\mu_2} &= \frac{\mathbf{k}}{|\mathbf{k}|} && \text{right handed} \end{aligned} \quad (2.36)$$

Insert Eq.(2.34) in Eq.(2.31):

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= \sum_{\mu} \sum_{\mathbf{k}} \left( \frac{\hbar}{2\omega_k \varepsilon_0 L^3} \right)^{1/2} \left[ a_{\mathbf{k}\mu} \boldsymbol{\varepsilon}_{\mathbf{k}\mu} e^{-i\omega_k t} e^{+i\mathbf{k}\mathbf{r}} + a_{\mathbf{k}\mu}^* \boldsymbol{\varepsilon}_{\mathbf{k}\mu}^* e^{+i\omega_k t} e^{-i\mathbf{k}\mathbf{r}} \right] \\ &= \sum_{\mu} \sum_{\mathbf{k}} \left( \frac{\hbar}{2\omega_k \varepsilon_0} \right)^{1/2} \left[ a_{\mathbf{k}\mu} \frac{\boldsymbol{\varepsilon}_{\mathbf{k}\mu}}{L^{3/2}} e^{+i\mathbf{k}\mathbf{r}} e^{-i\omega_k t} + a_{\mathbf{k}\mu} \frac{\boldsymbol{\varepsilon}_{\mathbf{k}\mu}}{L^{3/2}} e^{-i\mathbf{k}\mathbf{r}} e^{+i\omega_k t} \right] \end{aligned}$$

rewrite:  $\mathbf{u}_k(\mathbf{r}) = \frac{\boldsymbol{\varepsilon}_{\mathbf{k}\mu}}{L^{3/2}} e^{+i\mathbf{k}\mathbf{r}}$

For any general orthogonal basis expansion the vector potential reads as

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \left( \frac{\hbar}{2\omega_k \varepsilon_0} \right)^{1/2} (a_{\mathbf{k}} \mathbf{u}_{\mathbf{k}}(\mathbf{r}) e^{-i\omega_k t} + a_{\mathbf{k}}^* \mathbf{u}_{\mathbf{k}}^*(\mathbf{r}) e^{+i\omega_k t}) \quad (2.37)$$

This is the general form of the EM potential. The equivalent form of the electric field is then ( $\mathbf{E} = -\partial\mathbf{A}/\partial t$ )

$$\mathbf{E}(\mathbf{r}, t) = i \sum_{\mathbf{k}} \left( \frac{\hbar\omega_k}{2\varepsilon_0} \right)^{1/2} (a_{\mathbf{k}} \mathbf{u}_{\mathbf{k}}(\mathbf{r}) e^{-i\omega_k t} - a_{\mathbf{k}}^* \mathbf{u}_{\mathbf{k}}^*(\mathbf{r}) e^{+i\omega_k t}). \quad (2.38)$$

We have introduced here appropriate normalization of the amplitudes  $c_k$ , so that  $a_k$  are dimensionless.

Introducing canonical Poisson brackets and using the expression of classical Hamiltonian

$$\mathcal{H} = \frac{1}{2} \int (\varepsilon_0 \mathbf{E}^2 + \mu_0 \mathbf{H}^2) d_3\mathbf{r} \quad (2.39)$$

we derive

$$\mathcal{H} = \sum_k \hbar \omega_k a_k^* a_k. \quad (2.40)$$

Thus the energy of the total EM field is the sum of the energies of different modes denoted by  $k$ .

### Canonical variables

Introduce canonical variables (with  $a_k(t) = a_k(0) e^{-i\omega_k t}$ )

$$\begin{aligned} q_k(t) &= [a_k(t) + a_k^*(t)] \\ p_k(t) &= -i\omega_k [a_k(t) - a_k^*(t)] \end{aligned} \quad (2.41)$$

These variables fulfill  $\{q, p\} = 1$ . They are the quadratures of the electromagnetic field and can be measured e. g. as squeezed light.

With the Hamilton equations Eq.(2.3)

$$\begin{aligned} \frac{\partial}{\partial t} q_k &= p_k = \frac{\partial \mathcal{H}}{\partial p} \\ \frac{\partial}{\partial t} p_k &= -\omega_k q_k = -\frac{\partial \mathcal{H}}{\partial q} \end{aligned}$$

we find the classical Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_k [p_k^2(t) + \omega_k^2 q_k^2(t)] \quad (2.42)$$

*Remarks:* As in Eq.(2.39) is this Hamiltonian a sum of independent harmonic oscillators, with one oscillator for each mode  $k = (\mathbf{k}, \mu)$  of the EM field. The EM field is specified by the canonical variables  $p_k, q_k$ .

### 2<sup>nd</sup> quantisation

This can easily be done in the following way. We replace the canonical variables by operators. These are hermitian and therefore can be observables

$$\begin{aligned} q_k(t) &\rightarrow \hat{q}_k(t) \\ p_k(t) &\rightarrow \hat{p}_k(t) \end{aligned}$$

and the Poisson brackets by commutators

$$\begin{aligned} [\hat{p}_k, \hat{p}_{k'}] &= [\hat{q}_k, \hat{q}_{k'}] = 0 \\ [\hat{p}_k, \hat{q}_{k'}] &= i\hbar\delta_{kk'} \end{aligned}$$

So the Hamiltonian changes to

$$\hat{\mathcal{H}} = \frac{1}{2} \sum_k [\hat{p}_k^2(t) + \omega_k^2 \hat{q}_k^2(t)] \quad (2.43)$$

Introduce annihilation and creation operators

$$\begin{aligned} \hat{a}_k &= \frac{1}{\sqrt{2\hbar\omega}} [\omega_k \hat{q}_k + i\hat{p}_k] \\ \hat{a}_k^\dagger &= \frac{1}{\sqrt{2\hbar\omega}} [\omega_k \hat{q}_k - i\hat{p}_k] \end{aligned} \quad (2.44)$$

These are *non-hermitian* and thus non observable!

$$[\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0 \quad (2.45)$$

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'} \quad (2.46)$$

Our observable canonical variables are represented by the annihilation and creation operators in the following way

$$\begin{aligned} \hat{q}_k &= \sqrt{\frac{\hbar}{2\omega}} [\hat{a}_k^\dagger + \hat{a}_k] \\ \hat{p}_k &= i\sqrt{\frac{\hbar\omega}{2}} [\hat{a}_k^\dagger - \hat{a}_k] \end{aligned} \quad (2.47)$$

Insert Eq.(2.47) in the Hamilton Eq.(2.43). The quantum Hamilton operator becomes then

$$\hat{\mathcal{H}} = \sum_k \hbar\omega_k (\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k^\dagger \hat{a}_k) \quad (2.48)$$

change to normal order

$$= \sum_k \hbar\omega_k (\hat{a}_k^\dagger \hat{a}_k + 1/2) \quad (2.49)$$

$$\hat{\mathcal{H}} = \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k + \text{"}\infty\text{"} \quad (2.50)$$

and contains formally infinite zero-point energy. The infinity is not physical, since we can always choose the zero of energy where we want. What is physical are variations (derivatives) of this infinity with respect to the changes of the geometry of the quantization volume. They are responsible for the *Casimir effect*. Physically it means that if I deform my resonator, I change the zero-point energy, *ergo* I execute some work. There must therefore be forces acting on my resonator walls in this process – these are Casimir forces.

The quantized EM fields (without the time dependence) are

$$\begin{aligned} \hat{\mathbf{A}}(\mathbf{r}) &= \sum_k \left( \frac{\hbar}{2\omega_k \varepsilon_0} \right)^{1/2} \left( \hat{a}_k \mathbf{u}_k(\mathbf{r}) + \hat{a}_k^\dagger \mathbf{u}_k^*(\mathbf{r}) \right) \\ \hat{\mathbf{E}}(\mathbf{r}) &= i \sum_k \left( \frac{\hbar\omega_k}{2\varepsilon_0} \right)^{1/2} \left( \hat{a}_k \mathbf{u}_k(\mathbf{r}) - \hat{a}_k^\dagger \mathbf{u}_k^*(\mathbf{r}) \right), \end{aligned} \quad (2.51)$$

and  $\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$ .

Assuming a plane wave expansion  $\mathbf{u}(\mathbf{r}) = \frac{\boldsymbol{\varepsilon}_{\mathbf{k}\mu}}{L^{3/2}} e^{+i\mathbf{k}\mathbf{r}}$

$$\hat{\mathbf{A}}(\mathbf{r}, t) = \frac{1}{L^{3/2}} \sum_k \left( \frac{\hbar}{2\omega_k \varepsilon_0} \right)^{1/2} \left[ \hat{a}_k(0) \boldsymbol{\varepsilon}_k e^{i(\mathbf{k}\mathbf{r} - \omega t)} + h. c. \right] \quad (2.52)$$

$$\hat{\mathbf{E}}(\mathbf{r}, t) = \frac{i}{L^{3/2}} \sum_k \left( \frac{\hbar\omega_k}{2\varepsilon_0} \right)^{1/2} \left[ \hat{a}_k(0) \boldsymbol{\varepsilon}_k e^{i(\mathbf{k}\mathbf{r} - \omega t)} - h. c. \right] \quad (2.53)$$

$$\hat{\mathbf{B}}(\mathbf{r}, t) = \frac{i}{L^{3/2}} \sum_k \left( \frac{\hbar}{2\omega_k \varepsilon_0} \right)^{1/2} \left[ \hat{a}_k(0) (\mathbf{r} \times \boldsymbol{\varepsilon}_k) e^{i(\mathbf{k}\mathbf{r} - \omega t)} - h. c. \right] \quad (2.54)$$

## 2.5. Number operator

Observable quantities associated with the electromagnetic field modes are represented by operators formed by taking *hermitian* combinations of  $\hat{a}$  and  $\hat{a}^\dagger$ . The most important ones are the number operator and the quadrature operator.

$$\hat{n} = \hat{a}^\dagger \hat{a} \quad (2.55)$$

This operator is hermitian (eigenvalues are real)

$$\hat{n}^\dagger = (\hat{a}^\dagger \hat{a})^\dagger = \hat{a}^\dagger \hat{a} = \hat{n} \quad (2.56)$$

Commutators with  $\hat{a}^\dagger$  and  $\hat{a}_k^\dagger$

$$[\hat{a}_k, \hat{n}_{k'}] = \hat{a}_k \delta_{kk'} \quad (2.57)$$

$$[\hat{a}_k^\dagger, \hat{n}_{k'}] = -\hat{a}_k^\dagger \delta_{kk'} \quad (2.58)$$

The number operator has real eigenvalues  $n_k$  and eigenstates  $|n_k\rangle$

$$\hat{n}_k |n_k\rangle = n_k |n_k\rangle \quad (2.59)$$

Eigenvalues of  $\hat{a}^\dagger$  and  $\hat{a}$ , ( $\hat{a}_k |0\rangle = 0$ )

$$\begin{aligned} \hat{a}_k |n_k\rangle &= \sqrt{n_k} |n_k - 1\rangle \\ \hat{a}_k^\dagger |n_k\rangle &= \sqrt{n_k + 1} |n_k + 1\rangle \end{aligned} \quad (2.60)$$

## 2.6. Quadrature operators

$$\begin{aligned} \hat{Q} &= \sqrt{\hbar/2\omega}(\hat{a} + \hat{a}^\dagger) \\ \hat{P} &= i\sqrt{2\hbar\omega}(\hat{a}^\dagger - \hat{a}) \end{aligned} \quad (2.61)$$

Then for a single mode, using a plane wave expansion Eq.(2.53)

$$\hat{\mathbf{E}}(\mathbf{r}, t) = \frac{1}{L^{3/2}} \left( \frac{\hbar\omega}{2\varepsilon_0} \right)^{1/2} \varepsilon_k \left[ \hat{Q} \sin(\mathbf{k}\mathbf{r} - \omega t) - \hat{P} \cos(\mathbf{k}\mathbf{r} - \omega t) \right] \quad (2.62)$$

The canonical variables  $\hat{P}, \hat{Q}$  are the amplitudes of the quadratures into which the oscillating EM field can be decomposed.

## 2.7. Continuous limit

Let us reconsider expansion of the EM potential in the box with periodic boundary conditions

$$\hat{\mathbf{A}}(\mathbf{r}) = \frac{1}{L^{3/2}} \sum_{\mathbf{k}, \mu} \left( \frac{\hbar}{2\omega_k \varepsilon_0} \right)^{1/2} \varepsilon_{\mathbf{k}\mu} \left( \hat{a}_{\mathbf{k}\mu} e^{+i\mathbf{k}\mathbf{r}} + \hat{a}_{\mathbf{k}\mu}^\dagger e^{-i\mathbf{k}\mathbf{r}} \right) \quad (2.63)$$

where  $\mathbf{k} = (2\pi/L)\mathbf{n}$ , and  $\mathbf{n} = (n_x, n_y, n_z)$  is a vector with integer entries.

We want to go with  $L \rightarrow \infty$  and replace the sum by an integral of  $\mathbf{k}$ , times density of states in the  $\mathbf{k}$ -space which is  $(L/2\pi)^3$ . The prescription is

$$\sum_{\mathbf{k}} \rightarrow \left( \frac{L}{2\pi} \right)^3 \int d_3\mathbf{k}$$

It is also reasonable to introduce continuous set of eigenfunctions

$$u_{\mathbf{k}\mu} = \varepsilon_{\mathbf{k}\mu} \frac{1}{(2\pi)^{3/2}} e^{+i\mathbf{k}\mathbf{r}}$$

and absorb a factor into  $\hat{a}_{\mathbf{k}\mu}$  so that

$$\hat{a}_{\mathbf{k}\mu}^{\text{new}} = \left(\frac{L}{2\pi}\right)^{3/2} \hat{a}_{\mathbf{k}\mu}$$

Such defined operators fulfil in the continuous limit the commutation relations in which Kronecker delta is replaced by Dirac's delta function,

$$[\hat{a}_{\mathbf{k}\mu}, \hat{a}_{\mathbf{k}'\mu'}^\dagger] = \left(\frac{L}{2\pi}\right)^3 \delta_{\mathbf{k}\mathbf{k}'} \delta_{\mu\mu'} \rightarrow \delta(\mathbf{k} - \mathbf{k}') \delta_{\mu\mu'} \quad (2.64)$$

The EM potential can be thus written

$$\hat{\mathbf{A}}(\mathbf{r}) = \sum_{\mu} \int d_3\mathbf{k} \left(\frac{\hbar}{2\omega_k \varepsilon_0}\right)^{1/2} \frac{\boldsymbol{\varepsilon}_{\mathbf{k}\mu}}{(2\pi)^{3/2}} \left(\hat{a}_{\mathbf{k}\mu} e^{+i\mathbf{k}\mathbf{r}} + \hat{a}_{\mathbf{k}\mu}^\dagger e^{-i\mathbf{k}\mathbf{r}}\right) \quad (2.65)$$

$$\hat{\mathbf{E}}(\mathbf{r}) = i \sum_{\mu} \int d_3\mathbf{k} \left(\frac{\hbar\omega}{2\varepsilon_0}\right)^{1/2} \frac{\boldsymbol{\varepsilon}_{\mathbf{k}\mu}}{(2\pi)^{3/2}} \left(\hat{a}_{\mathbf{k}\mu} e^{+i\mathbf{k}\mathbf{r}} - \hat{a}_{\mathbf{k}\mu}^\dagger e^{-i\mathbf{k}\mathbf{r}}\right) \quad (2.66)$$

$$\hat{\mathbf{B}}(\mathbf{r}) = \text{rot } \mathbf{A}(\mathbf{r}) \quad (2.67)$$

Hamiltonian

$$\hat{\mathcal{H}} = \sum_{\mu} \int d_3\mathbf{k} \hbar\omega_k \hat{a}_{\mathbf{k}\mu}^\dagger \hat{a}_{\mathbf{k}\mu} \quad (2.68)$$

How do we construct Hilbert space, or what states belong to it? We do it similarly as in the case of momentum representation for the free particles. Plane waves in such case are not the elements of the Hilbert space, but wave packets are. Basis can be formed using orthonormal basis of wave packets (for example wavelets). Here we construct photonic wave packets, and their creation and annihilation operators. To this aim we take square integrable functions of  $\mathbf{k}$ , and define

$$a_f = \sum_{\mu} \int d_3\mathbf{k} f(\mathbf{k}, \mu) a_{\mathbf{k},\mu}$$

and correspondingly  $\hat{a}_f^\dagger$ , so that  $[\hat{a}_f, \hat{a}_f^\dagger] = 1$ . Hilbert space consist then of vacuum state  $|\Omega\rangle$  (which is annihilated by all  $\hat{a}_f$ 's, single photon states  $\hat{a}_f^\dagger |\Omega\rangle$ , or many photon states

$$|k_{f_1}, k_{f_2}, \dots\rangle = (\hat{a}_{f_1}^\dagger)^{k_{f_1}} / \sqrt{k_{f_1}!} \dots |\Omega\rangle$$

etc.

résumé

what have we done to do the second quantisation:

- 1) Start with the Maxwell equations and fix the gauge:  $\nabla A = 0$   
(the canonical variables depend on the gauge!)
- 2) expand  $\mathbf{A}(\mathbf{r}, t)$  in a Fourier series
- 3) define the classical canonical variables  $q_k, p_k$
- 4) quantize the canonical variables and change poisson bracket to commutators
- 5) define annihilation and creation operators  $\hat{a}_k, \hat{a}_k^\dagger$
- 6) find the new Hamiltonian

Equations:

basis expansion for  $\mathbf{A}(\mathbf{r}, t)$

$$\mathbf{u}_k(\mathbf{r}) = \boldsymbol{\varepsilon}_{\mathbf{k}\mu} \frac{1}{L^{3/2}} e^{i\mathbf{k}\mathbf{r}}$$

classical electromagnetic potential

$$\mathbf{A}(\mathbf{r}, t) = \sum_k \left( \frac{\hbar}{2\omega_k \varepsilon_0} \right)^{1/2} (a_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t} + a_k^* \mathbf{u}_k^*(\mathbf{r}) e^{+i\omega_k t})$$

annihilation and creation operators

$$\hat{a}_k = \frac{1}{\sqrt{2\hbar\omega}} [\omega_k \hat{q}_k + i\hat{p}_k]$$

$$\hat{a}_k^\dagger = \frac{1}{\sqrt{2\hbar\omega}} [\omega_k \hat{q}_k - i\hat{p}_k]$$

quantum Hamiltonian

$$\mathcal{H} = \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k + \text{“}\infty\text{”}$$

quantized electromagnetic fields (without the time dependence)

$$\hat{\mathbf{A}}(\mathbf{r}) = \sum_k \left( \frac{\hbar}{2\omega_k \varepsilon_0} \right)^{1/2} (\hat{a}_k \mathbf{u}_k(\mathbf{r}) + \hat{a}_k^\dagger \mathbf{u}_k^*(\mathbf{r}))$$

$$\hat{\mathbf{E}}(\mathbf{r}) = i \sum_k \left( \frac{\hbar\omega_k}{2\varepsilon_0} \right)^{1/2} (\hat{a}_k \mathbf{u}_k(\mathbf{r}) - \hat{a}_k^\dagger \mathbf{u}_k^*(\mathbf{r})),$$

number operator

$$\hat{n} = \hat{a}^\dagger \hat{a}$$

$$\hat{n}_k |n_k\rangle = n_k |n_k\rangle$$

quadrature operators

$$\hat{Q} = \sqrt{\hbar/2\omega}(\hat{a} + \hat{a}^\dagger)$$

$$\hat{P} = \sqrt{2\hbar\omega}i(\hat{a} - \hat{a}^\dagger)$$

Continuous limit

$$\hat{\mathbf{A}}(\mathbf{r}) = \sum_{\mu} \int d_3\mathbf{k} \left( \frac{\hbar}{2\omega_k \varepsilon_0} \right)^{1/2} \frac{\boldsymbol{\varepsilon}_{\mathbf{k}\mu}}{(2\pi)^{3/2}} \left( \hat{a}_{\mathbf{k}\mu} e^{+i\mathbf{k}\mathbf{r}} + \hat{a}_{\mathbf{k}\mu}^\dagger e^{-i\mathbf{k}\mathbf{r}} \right)$$

$$\hat{\mathbf{E}}(\mathbf{r}) = i \sum_{\mu} \int d_3\mathbf{k} \left( \frac{\hbar\omega}{2\varepsilon_0} \right)^{1/2} \frac{\boldsymbol{\varepsilon}_{\mathbf{k}\mu}}{(2\pi)^{3/2}} \left( \hat{a}_{\mathbf{k}\mu} e^{+i\mathbf{k}\mathbf{r}} - \hat{a}_{\mathbf{k}\mu}^\dagger e^{-i\mathbf{k}\mathbf{r}} \right)$$

$$\hat{\mathbf{B}}(\mathbf{r}) = \text{rot } \mathbf{A}(\mathbf{r})$$

Hamiltonian

$$\hat{\mathcal{H}} = \sum_{\mu} \int d_3\mathbf{k} \hbar\omega_k \hat{a}_{\mathbf{k}\mu}^\dagger \hat{a}_{\mathbf{k}\mu}$$

## List of variables

$\mathcal{H}$	Hamiltonian
$\mathbf{E}$	electric field
$\mathbf{B}$	magnetic field
$\mathbf{A}$	electromagnetic potential field
$k$	multiindex $k = (\mathbf{k}, \mu)$
$\mathbf{k}$	wave vector
$\mu$	polarisation index
$\mathbf{c}_k$	time dependent amplitude of mode $k$
$\mathbf{a}_k(0)$	time independent amplitude of mode $k$
$\boldsymbol{\varepsilon}_{\mathbf{k},\mu}$	polarisation vector
$\mathbf{u}_k$	basis expansion for $\mathbf{A}(\mathbf{r}, t)$
$q_k, p_k$	canonical variables (classical)
$\hat{Q}_k, \hat{P}_k$	canonical variables (quantum)
$\hat{a}, \hat{a}_{\mathbf{k}\mu}$	annihilation operator
$\hat{a}^\dagger, \hat{a}_{\mathbf{k}\mu}^\dagger$	creation operator
$\hat{n}$	number operator
$\hat{Q}, \hat{P}$	quadrature operators



# CHAPTER 3

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## Quantum states of EM field

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In the following we will discuss examples of quantum states of the quantized EM field. In particular we will give examples of states that are either useful, or experimentally producible, or interesting. All of those states are quantum, but some of them are ‘more quantum than others’ (show more properties of quantum nature than others). Since we have seen that the EM field corresponds to a collection of uncoupled modes (each corresponding to a harmonic oscillator) we will discuss for simplicity a single mode. The results can be straight forwardly extended to n-modes.

### 3.1. Fock, or number states $|n\rangle$

These are defined as eigenstates of the number operator  $\hat{n}_k$ , and of the free hamiltonian  $\hat{\mathcal{H}} = \sum_k \hbar\omega_k(\hat{a}_k^\dagger\hat{a}_k + 1/2)$

$$\hat{n}_k |n_k\rangle = n_k |n_k\rangle \quad (3.1)$$

We call  $|n_k\rangle$  the *Fock state* or *number state*.

The ground state of our whole system (quantized EM field in free space) is the vacuum state, which is annihilated by all  $\hat{a}_k$

$$\hat{a}_k |\Omega\rangle = 0 \quad \text{and of course} \quad \hat{a}_k^\dagger\hat{a}_k |0_k\rangle = 0 \quad (3.2)$$

for all  $k$ . Thus,

$$|\Omega\rangle = \prod_k |0_k\rangle = |0_1\rangle \otimes \dots \otimes |0_k\rangle \otimes \dots \quad (3.3)$$

The ground state (zero-point) energy is

$$\hat{\mathcal{H}} |\Omega\rangle = \sum_k \hbar\omega_k(\hat{a}_k^\dagger\hat{a}_k + 1/2) |\Omega\rangle = \sum_k \hbar\omega_k/2 \quad (3.4)$$

In the previous chapter we have derived the expression for the EM field in a box of size  $L$ . Let us now consider this example again. The operators  $\hat{a}_k$ , and  $\hat{a}_k^\dagger$  represent annihilation and creation operators of photons of the momentum  $\mathbf{k}$  and polarization  $\epsilon_{\mathbf{k}\mu}$ . In general they annihilate and create photons in corresponding modes denoted by  $k$

$$\begin{aligned} \hat{a}_k |n_k\rangle &= \sqrt{n_k} |n_k - 1\rangle \\ \hat{a}_k^\dagger |n_k\rangle &= \sqrt{n_k + 1} |n_k + 1\rangle \end{aligned} \quad (3.5)$$

The Fock states thus have the form

$$|n_k\rangle = \frac{(\hat{a}_k^\dagger)^{n_k}}{\sqrt{n_k!}} |0_k\rangle \quad (3.6)$$

They form the orthonormal basis for each mode  $k$ ,  $\langle n_k | m_k \rangle = \delta_{nm}$ , and  $\sum_{n_k=0}^{\infty} |n_k\rangle \langle n_k| = \hat{\mathbb{I}}_k$ , where  $\hat{\mathbb{I}}_k$  is the unity operator in the  $k$ -th mode Hilbert space.

Fock states of low photon number  $|n\rangle = |0\rangle, |1\rangle, |2\rangle, \dots$  are currently created in the laboratory. One can for instance place a single atom in a high quality cavity (i. e. between two very well polished mirrors). The atom interacts with the laser, gets excited and emits a single photon which remains in the cavity after the atom has left the cavity. It is however not possible to use a similar method to create a high number of photons in the cavity.

We can thus comprehend, that Fock states are *difficult* to create !

### 3.2. Coherent states $|\alpha\rangle$

A Coherent state is a superposition of number or Fock states  $|n\rangle$ . They are denoted by  $|\alpha\rangle$ . One could consider different possible superpositions, but the coherent state is of particular importance in practical applications. Coherent states describe to a great accuracy states of CW monochromatic lasers above threshold, and wave packets formed by coherent states describe very well states of pulsed lasers.

There are many ways of introducing coherent states: we follow here the trick introduced in classical papers by Glauber<sup>1</sup> (these states are therefore often called Glauber, or Glauber–Sudarshan states). (From now on we consider a single mode, therefore we skip the index  $k$ )

First for any complex number  $\alpha$  we define the unitary *displacement operator*.

$$\hat{D}(\alpha) \equiv e^{(\alpha \hat{a}^\dagger - \alpha^* \hat{a})} \quad (3.7)$$

<sup>1</sup>R.J.Glauber. Phys. Rev **130**, 2529 (1963)

The origin of the name becomes clear in Eq.(3.11) and Eq.(3.12). Why  $\alpha$  must be complex becomes apparend in Eq.(3.15).

Note that  $\hat{D}(\alpha)^\dagger = \hat{D}(\alpha)^{-1} = D(-\alpha)$ .

Let us remind ourselves the *Baker-Hausssdorf formula*: for two operators such that  $[A, [A, B]] = 0$  and  $[B, [A, B]] = 0$ , we have

$$e^{A+B} = e^A e^B e^{-[A,B]/2} \quad (3.8)$$

Using the above formula we get

$$\hat{D}(\alpha) = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \quad (3.9)$$

The proof of the Baker-Hausssdorf formula employs the following identity valid for arbitrary operators (without assuming vanishing of commutators):

$$e^A B e^{-A} = B + [A, B]/1! + [A, [A, B]]/2! + \dots \quad (3.10)$$

with the identity we derive

$$\hat{D}^\dagger(\alpha) \hat{a} \hat{D}(\alpha) = \hat{a} + \alpha \quad (3.11)$$

$$\hat{D}^\dagger(\alpha) \hat{a}^\dagger \hat{D}(\alpha) = \hat{a}^\dagger + \alpha^* \quad (3.12)$$

The above expressions explain why  $\hat{D}(\alpha)$  is called displacement operator.

We are now in the position to define *coherent state*  $|\alpha\rangle$ , as displaced vacuum:

$$\boxed{|\alpha\rangle \equiv \hat{D}(\alpha) |0\rangle} \quad (3.13)$$

We see that the displacement operator is equivalent to a creation operator for the coherent state

The following most important property of the coherent states is also used frequently to define them. Since

$$\begin{aligned} & \hat{D}^\dagger(\alpha) \hat{a} |\alpha\rangle \\ &= \hat{D}^\dagger(\alpha) \hat{a} \hat{D}(\alpha) |0\rangle \\ &= \alpha |0\rangle \\ &= \alpha \hat{D}^\dagger(\alpha) \hat{D}(\alpha) |0\rangle \\ &= \alpha \hat{D}^\dagger(\alpha) |\alpha\rangle \end{aligned} \quad (3.14)$$

we conclude that

$$\boxed{\hat{a} |\alpha\rangle = \alpha |\alpha\rangle} \quad (3.15)$$

and analogue

$$\boxed{\langle \alpha | \hat{a}^\dagger = \alpha^* \langle \alpha |} \quad (3.16)$$

i. e. coherent states are eigenstates of the annihilation operator. Since operator  $\hat{a}$  is not hermitian, its eigenvalues can be, and are complex.

*Note:* the state  $|\alpha\rangle$  is not an eigenstate of the creation operator.

### 3.2.1. Coherent states in Fock representation

Contrary to the Fock states that contain finite and fixed number of photons, coherent state are coherent superpositions of Fock states with arbitrary photon numbers. To prove it, we observe with  $\langle n | \hat{a} = \sqrt{n+1} \langle n+1 |$  and Eq.(3.15) that

$$\langle n | \hat{a} | \alpha \rangle = \sqrt{n+1} \langle n+1 | \alpha \rangle = \alpha \langle n | \alpha \rangle \quad \Rightarrow \quad \langle n | \alpha \rangle = \frac{\alpha}{\sqrt{n}} \langle n-1 | \alpha \rangle \quad (3.17)$$

so that

$$\langle n | \alpha \rangle = \frac{\alpha^n}{\sqrt{n!}} \langle 0 | \alpha \rangle \quad (3.18)$$

Using Eq.(3.9) we get with expansion of the exponential and  $\hat{a} | 0 \rangle = 0$

$$\langle 0 | \alpha \rangle = \langle 0 | \hat{D}(\alpha) | 0 \rangle = \exp(-|\alpha|^2/2) \quad (3.19)$$

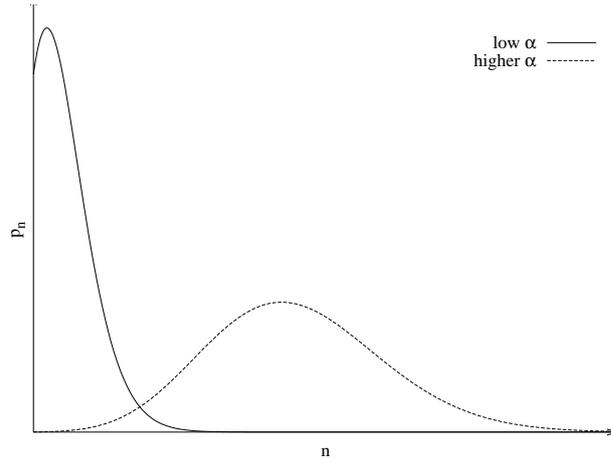
Thus with Eq.(3.18) using Eq.(3.19), the coherent states have the following Fock states representation

$$\boxed{|\alpha\rangle = \sum_{n=0}^{\infty} |n\rangle \langle n | \alpha \rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle} \quad (3.20)$$

### 3.2.2. Properties of Coherent states

The corresponding photon number distribution (probabilities of having  $n$  photons) is Poissonian:

$$p_n = |\langle n | \alpha \rangle|^2 = \frac{|\alpha|^{2n}}{n!} \exp(-|\alpha|^2) \quad (3.21)$$



**Figure 3.1.:** poisson photon number distribution for different  $\alpha$

The mean number of photons is then

$$\begin{aligned}
 \bar{n} &= \sum_n n p_n(\alpha) \\
 &= \sum_n n \cdot \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} \\
 &= e^{-|\alpha|^2} |\alpha|^2 \sum_n \frac{|\alpha|^{2(n-1)}}{(n-1)!} \\
 &= e^{-|\alpha|^2} |\alpha|^2 e^{+|\alpha|^2} = |\alpha|^2
 \end{aligned} \tag{3.22}$$

we can get the same result using the number operator  $\hat{n} = \hat{a}^\dagger \hat{a}$

$$\bar{n} = \langle \alpha | \hat{n} | \alpha \rangle = \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = \alpha^* \langle \alpha | \alpha | \alpha \rangle = \alpha^* \alpha \langle \alpha | \alpha \rangle = |\alpha|^2 \tag{3.23}$$

Another important property of the coherent states is that they are not orthogonal. Their scalar product is

$$\langle \beta | \alpha \rangle = \langle 0 | \hat{D}^\dagger(\beta) \hat{D}(\alpha) | 0 \rangle = e^{-|\alpha|^2/2 - |\beta|^2/2 + \alpha\beta^*} \tag{3.24}$$

whereas its modulus (called sometimes the *state overlap*) is

$$|\langle \beta | \alpha \rangle|^2 = e^{-|\alpha - \beta|^2}$$

and vanishes exponentially with the growth of modulus squared of the difference.

Coherent state are thus not orthogonal, but form a *complete set*! In particular,

$$\frac{1}{\pi} \int |\alpha\rangle \langle \alpha| d^2\alpha = \hat{I} \tag{3.25}$$

where  $d^2\alpha = dx dy$ , and  $\alpha = x + iy$ .

Proof

We use the Fock states representation

$$\frac{1}{\pi} \int |\alpha\rangle \langle \alpha| d_2\alpha = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|n\rangle \langle m|}{\sqrt{n!m!}} \int e^{-|\alpha|^2} (\alpha^*)^m \alpha^n d_2\alpha \quad (3.26)$$

Going to polar coordinates  $\alpha = r e^{i\varphi}$  we get

$$\begin{aligned} \frac{1}{\pi} \int |\alpha\rangle \langle \alpha| d_2\alpha &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|n\rangle \langle m|}{\sqrt{n!m!}} \int_0^{\infty} r e^{-r^2} r^{n+m} dr \int_0^{2\pi} e^{i(n-m)\varphi} d\varphi / \pi \\ &= \sum_{n=0}^{\infty} \frac{|n\rangle \langle n|}{n!} 2 \int_0^{\infty} r^{2n+1} e^{-r^2} dr \\ &= \sum_{n=0}^{\infty} |n\rangle \langle n| \end{aligned} \quad (3.27)$$

*q. e. d*

This means that coherent states are overcomplete and form a non-orthogonal basis.

*Note:* Measurements of non-orthogonal sets of states are ‘tricky’. They can be understood as measurements of orthogonal states in a bigger Hilbert space. For instance, measurement of a set of coherent states  $\{|\alpha_k\rangle\}_{k=1,2,\dots}$  can be realized as measurement of orthogonal states  $\{|\alpha_k\rangle \otimes |k\rangle\}_{k=1,2,\dots}$ , where  $|k\rangle$  are Fock states in an auxiliary Hilbert space.

Coherent states are closely related to the coherent properties of the EM field. Since they are linear superpositions of Fock states, the number of photons in each mode is badly defined, but intuitively one expects that the phase properties of the EM field will be in a coherent state well defined. Any EM field produced by a deterministic current source (*Poisson distribution*) is a coherent state. Examples are lasers, parametric oscillators and so forth.

### 3.3. Squeezed states

Let us recall the quadrature operators (which are analogues of position and momentum operators)

$$\hat{Q} = \sqrt{\hbar/2\omega}(\hat{a}^\dagger + \hat{a}) \quad (3.28)$$

$$\hat{P} = \sqrt{2\hbar\omega i}(\hat{a}^\dagger - \hat{a}) \quad (3.29)$$

Let us calculate their variance in a coherent state (the variance of an observable (hermitian operator)  $A$  is  $(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2$ ). It is easy to check that

$$(\Delta \hat{Q})_{\text{coh}}^2 = \hbar/2\omega \quad (3.30)$$

$$(\Delta \hat{P})_{\text{coh}}^2 = \hbar\omega/2 \quad (3.31)$$

so that the Heisenberg-uncertainty relation is minimized

$$(\Delta\hat{Q})_{\text{coh}}(\Delta\hat{P})_{\text{coh}} = \hbar/2 \quad (3.32)$$

Are there more states of this sort ?

The answer is yes, and has been in principle given by Schrödinger in the 20's: these are Gaussian wave-packets. Only in the 80's ago people started to look at this problem again: now from the point of view of precision measurements.

To set the question in a more general framework, and in order to free ourselves from the physical units, we consider the, so called, *generalised quadrature operators*, i.e. we represent

$$\hat{a} = (\hat{Q} + i\hat{P})/2 \quad (3.33)$$

where  $\hat{Q}$ ,  $\hat{P}$  are hermitian.

Lets write the EM field in terms of the quadrature operators (Eq.(2.62))

$$\hat{\mathbf{E}}(\mathbf{r}, t) \propto \varepsilon_k \left[ \hat{Q} \sin(\mathbf{k}\mathbf{r} - \omega t) - \hat{P} \cos(\mathbf{k}\mathbf{r} - \omega t) \right] \quad (3.34)$$

We can identify  $\hat{Q}$  as a part that oscillates as sine, and  $\hat{P}$  as a part that goes like cosine of  $\omega t$ . Since their commutator is  $[\hat{Q}, \hat{P}] = 2i$ , the Heisenberg relations says that  $\Delta\hat{Q} \cdot \Delta\hat{P} \geq 1$ . For coherent states we have

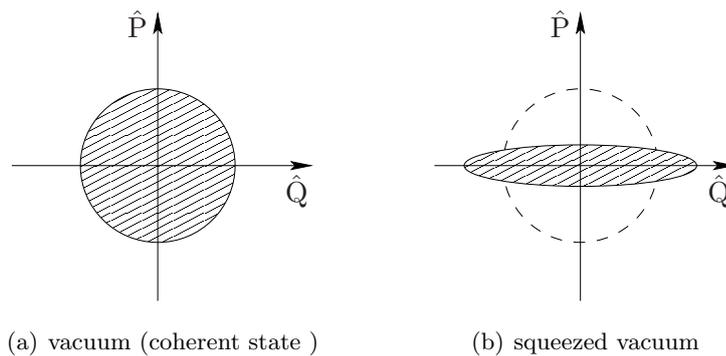
$$\Delta\hat{Q} = \Delta\hat{P} = 1.$$

For squeezed states

$$\Delta\hat{Q} < 1, \quad \Delta\hat{P} = 1/\Delta\hat{Q} > 1, \quad \text{but} \quad \Delta\hat{Q}\Delta\hat{P} \geq 1.$$

*Note:* The equality is achieved only if the state is pure.

Graphically, coherent states are represented as circles in the phase space ( $\hat{P}$ ,  $\hat{Q}$ ), whereas the squeezed states correspond to ellipses.



**Figure 3.2.:** vacuum states

Mathematically, squeezed states are constructed as follows: First we define a unitary *squeeze operator*

$$\hat{S}(\varepsilon) = \exp(\varepsilon^* \hat{a}^2 / 2 - \varepsilon (\hat{a}^\dagger)^2 / 2) \quad (3.35)$$

where  $\varepsilon = r e^{2i\varphi}$  is a complex number. Obviously,  $S^\dagger(\varepsilon) = S^{-1}(\varepsilon) = S(-\varepsilon)$ . Using the formula (3.10) we get

$$S^\dagger(\varepsilon) \hat{a} \hat{S}(\varepsilon) = \hat{a} \cosh r - \hat{a}^\dagger e^{2i\varphi} \sinh r, \quad (3.36)$$

$$S^\dagger(\varepsilon) \hat{a}^\dagger \hat{S}(\varepsilon) = \hat{a}^\dagger \cosh r - \hat{a} e^{-2i\varphi} \sinh r \quad (3.37)$$

Let us define more general hermitian quadrature operators (rotated, or phase shifted ones)

$$\hat{Q}'_1 + i\hat{P}'_2 = (\hat{Q} + i\hat{P}) e^{-i\varphi} \quad (3.38)$$

Obviously, from hermiticity we obtain

$$\hat{Q}'_1 = \hat{Q} \cos \varphi + \hat{P} \sin \varphi \quad (3.39)$$

$$\hat{P}'_2 = \hat{P} \cos \varphi - \hat{Q} \sin \varphi \quad (3.40)$$

and

$$\hat{Q}'_1 + i\hat{P}'_2 = \hat{a} e^{-i\varphi} \quad (3.41)$$

$$\hat{Q}'_1 - i\hat{P}'_2 = \hat{a}^\dagger e^{i\varphi} \quad (3.42)$$

Using above introduced properties we get

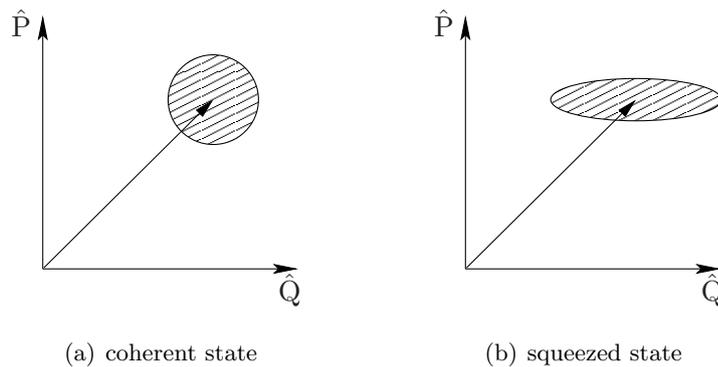
$$S^\dagger(\varepsilon) (\hat{Q}'_1 + i\hat{P}'_2) \hat{S}(\varepsilon) = e^{-i\varphi} (\hat{a} \cosh r - \hat{a}^\dagger e^{2i\varphi} \sinh r) = \hat{Q}'_1 e^{-r} + \hat{P}'_2 e^r \quad (3.43)$$

Evidently, the operator  $\hat{S}(\varepsilon)$  squeezes  $\hat{Q}'_1$ , and stretches  $\hat{P}'_2$ . The parameter  $r = |\varepsilon|$  is therefore called squeezing factor.

The squeezed states  $|\alpha, \varepsilon\rangle$  are defined

$$|\alpha, \varepsilon\rangle = \hat{D}(\alpha) \hat{S}(\varepsilon) |0\rangle \quad (3.44)$$

The state  $\hat{S}(\varepsilon) |0\rangle$  is called a squeezed vacuum.



**Figure 3.3.:** coherent and squeezed states

The most important properties of the squeezed states are

- Mean values of  $\hat{a}$  is of course  $\langle \alpha, \varepsilon | \hat{a} | \alpha, \varepsilon \rangle = \alpha$
- Variances of  $\hat{Q}'_1$  ( $\hat{P}'_2$ ) are squeezed (stretched):  $\Delta \hat{Q}'_1 = e^{-r}$ ,  $\Delta \hat{P}'_2 = e^r$
- Mean photon number  $\langle n \rangle$ , which we now denote as  $\bar{n}$  is

$$\bar{n} = |\alpha|^2 + \sinh^2 r$$

- Variance of the photon distribution is

$$(\Delta n)^2 = |\alpha \cosh r - \alpha^* e^{2i\varphi} \sinh r|^2 + 2 \cosh^2 r \sinh^2 r \quad (3.45)$$

*Note:*  $\bar{n}$  and  $(\Delta n)^2$  have not been derived, because they are difficult to calculate and beyond the scope of this introduction.

Let us make a comparison:

- For coherent states ( $r = 0$ )  $(\Delta n)^2 = \bar{n}$  and we deal with the Poissonian distribution.
- The situation is much more complex and richer for the squeezed states: For instance, for  $|\alpha| \rightarrow \infty$ ,  $(\Delta n)^2$  can be greater or smaller than  $\bar{n}$ , depending on  $\varphi$ . For example, for  $\alpha$  real we get for  $\varphi = 0$ , a *sub-Poissonian* distribution, i. e.  $(\Delta n)^2$  is smaller than  $\bar{n}$
- On the contrary, for  $\varphi = \pi/2$ ,  $(\Delta n)^2 = n e^r$ , and the distribution is *super-Poissonian*, i. e.  $(\Delta n)^2$  is greater than  $\bar{n}$

### 3.3.1. Photon number distribution

It is tempting to calculate photon number distribution for squeezed states, but that is quite a non-trivial task. The exact analytic expression is

$$p(n) = (n! \mu)^{-1} \left( \frac{\nu}{2\mu} \right)^n \left| H_n \left( \frac{\beta}{\sqrt{2\mu\nu}} \right) \right|^2 e^{-|\beta|^2 + \frac{\nu}{2\mu} \beta^2 + \frac{\nu^*}{2\mu} (\beta^*)^2}, \quad (3.46)$$

where  $\nu = e^{2i\varphi} \sinh r$ ,  $\mu = \cosh r$ , and  $\beta = \mu\alpha + \nu\alpha^*$ ;  $H_n(\cdot)$  denote here Hermite polynomials. The above formula is quite complex and not so easy to analyse. For large  $\alpha$ , and moderate  $r$ , as we know, it exhibits sub-Poissonian statistics for  $\varphi = 0$ , Poissonian statistics for  $r = 0$ , and super-Poissonian statistics for  $\varphi = \pi/2$ . For  $\alpha$  big, and bigger values of  $r$  the photon number distribution exhibits oscillations, that represent interference effects in the phase space, as has been discussed in 1987 in *Nature* by Schleich and Wheeler<sup>2</sup>.

### 3.3.2. Generation of squeezed states

Squeezed states can be generated in non-linear optical process such degenerate, or non-degenerate parametric amplifications, in which one photon of large frequency is turned in the medium into two photons of equal, or non-equal frequencies. First experiments of this sort were done by H. J. Kimbler<sup>3</sup> at Caltech, and by R.E. Slusher<sup>4</sup> at Bell Labs. Other types of non-linear optical process lead also to squeezed state generation: four wave mixing, or closely related resonance fluorescence.

## 3.4. Variance of the EM field

It easy expresses the variance of the electric field in terms of the quadrature operators. From the definition of  $\mathbf{E}$  in the previous chapter, at  $\mathbf{r} = 0$

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{L^{3/2}} \left( \frac{\hbar\omega}{2\varepsilon_0} \right)^{1/2} \left[ \hat{Q} \sin \omega t - \hat{P} \cos \omega t \right] \quad (3.47)$$

so that

$$\begin{aligned} \Delta \mathbf{E}(\mathbf{r}, t)^2 &= \langle \mathbf{E}^2 \rangle - \langle \mathbf{E} \rangle^2 \\ &= \frac{1}{L^3} \left( \frac{\hbar\omega}{2\varepsilon_0} \right)^2 \left[ (\Delta \hat{Q})^2 \sin^2 \omega t + (\Delta \hat{P})^2 \cos^2 \omega t \right]. \end{aligned} \quad (3.48)$$

Let us for simplicity assume that the mean of  $\langle \hat{Q} \rangle = 0$ , whereas the mean of  $\hat{P}$  is non-zero. Evidently, depending on the relation between  $\Delta \hat{Q}$  and  $\Delta \hat{P}$  we can have here the situation of *amplitude squeezed light* – this happens when  $\Delta \hat{P} < \Delta \hat{Q}$ , i.e. fluctuations are small, when the mean field is big (when  $\cos \omega t \simeq \pm 1$ ). Conversely, when  $\Delta \hat{P} > \Delta \hat{Q}$ , i.e. fluctuations are small, when the mean field is small (when  $\cos \omega t \simeq 0$ ), that corresponds to *phase squeezed field*.

Figures 3.4, 3.5, 3.6 are taken from the lecture: *Nonclassical light* by Dr. Roman Schnabel.

<sup>2</sup>W. Schleich, J.A. Wheeler *Nature* **326**, 574 (1987)

<sup>3</sup>L.A. Wu, H.J. Kimble, J.L. Hall *Physical Review Letters* IR2 57, 2520 (1986)

<sup>4</sup>R.E. Slusher, L.W. Hollberg, B.Yurke, J.G. Mertz, J.G. Vallery *Physical Review Letters* PR2 55, 2409 (1985)

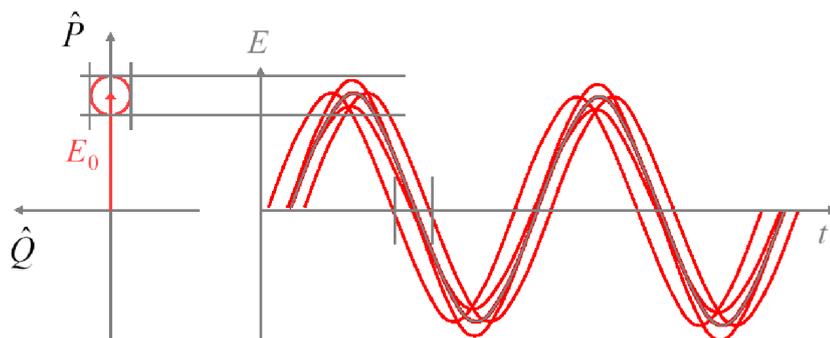


Figure 3.4.: Coherent EM field

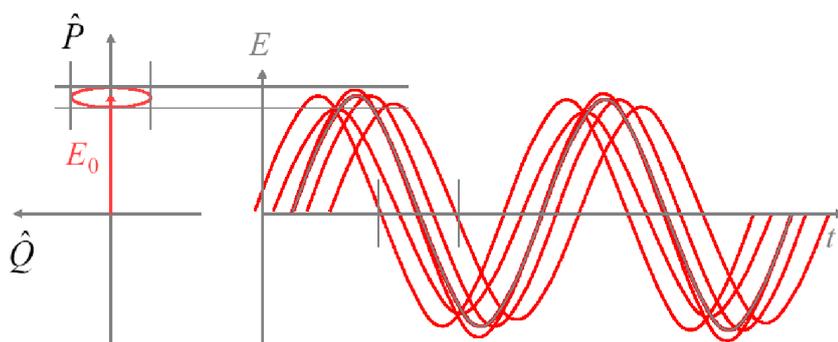


Figure 3.5.: Amplitude squeezed field

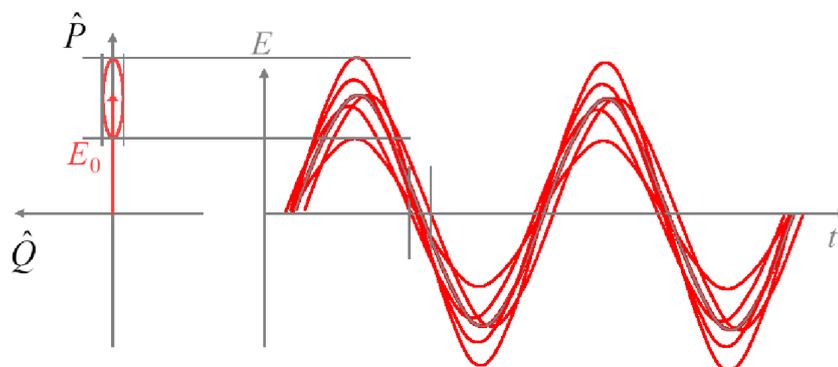


Figure 3.6.: Phase squeezed field

### 3.5. Thermal states

These are not pure states; instead they are given by a density matrix. They correspond very well to the states of radiating black body. With their help one describes thermal light, light of a bulb, etc.

The definition is ( $Z$  : partition function,  $\beta = 1/k_B T$ )

$$\rho \equiv \frac{e^{-\beta \hat{a}^\dagger \hat{a}}}{Z} = (1 - e^{-\beta \hbar \omega}) e^{-\beta \hat{a}^\dagger \hat{a}} \quad (3.49)$$

Since the mean number of photons in such state is defined as

$$\bar{n} = (1 - e^{-\beta \hbar \omega}) \sum_{n=0}^{\infty} n e^{-\beta \hbar \omega n} = e^{-\beta \hbar \omega} / (1 - e^{-\beta \hbar \omega})$$

we can rewrite

$$\rho = \frac{1}{\bar{n} + 1} \left( \frac{\bar{n}}{\bar{n} + 1} \right)^{\hat{a}^\dagger \hat{a}} \quad (3.50)$$

so the the photon number distribution is geometric

$$p_n = \frac{1}{\bar{n} + 1} \left( \frac{\bar{n}}{\bar{n} + 1} \right)^n \quad (3.51)$$

The photon number fluctuations are in such states enormous. We have

$$\overline{n^2} = \sum_{n=0}^{\infty} n^2 \bar{n}^n / (\bar{n} + 1)^{n+1} = (\bar{n} + 1)^{-1} x \frac{d}{dx} x \frac{d}{dx} \sum_{n=0}^{\infty} x^n \Big|_{x=\bar{n}/(\bar{n}+1)}$$

Elementary calculation yields to  $\overline{n^2} = 2\bar{n}^2 + \bar{n}$ .

Thus

$$(\Delta n)^2 = \overline{n^2} - \bar{n}^2 = \bar{n}^2 + \bar{n}.$$

i.e.  $(\Delta n)^2 \simeq \bar{n}$  for large  $\bar{n}$ . The variance is here of the order of the mean value, which signifies very strong fluctuations.

### 3.6. Noisy coherent states

By displacing thermal states one can construct noisy coherent states that describe for instance thermal light with coherent component, or laser with noise. They are defined as

$$\rho(\alpha, \beta) = \hat{D}(\alpha) \rho_{th}(\beta) \hat{D}^\dagger(\alpha).$$

Elementary calculations allow to determine their properties: mean photon number is just a sum of means from the coherent and thermal component,  $\bar{n} = |\alpha|^2 + \bar{n}_{th}$ . The variance has more complicated form: thermal fluctuations tend to amplify coherent state fluctuations:

$$(\Delta n)^2 = |\alpha|^2 (2\bar{n}_{th} + 1) + \bar{n}_{th}^2 + \bar{n}_{th}.$$

### 3.7. Phase of the Field

In classical optics, the phase of an EM field is not a quantity that can directly be observed, it must be inferred from an ensemble of measurements. The problem of the quantum mechanical definition of the phase as an observable, was already attacked by P. A. M. Dirac in the 20's. Dirac thought of the phase as a quantity that is a canonical conjugate of the phase operator. Such a phase is however not hermitian, i. e. not an observable. Nevertheless the question of the phase of the e.m. field remains a problem.

The question is how to define the quantum mechanical phase operator. With the recent developments in the area of precise measurements of quantum field fluctuations, the problem has gained new importance. Most important contributions have been recently done by D. Pegg and S. Barnett.<sup>5</sup>

Dirac has investigated the following definition

$$\exp(i\hat{\varphi}) = \frac{1}{\sqrt{\hat{a}\hat{a}^\dagger}} \hat{a} \quad (3.52)$$

Intuitively, one divides here the amplitude operator through the modulus operator. In order to omit the problems with dividing by zero, the modulus operator is taken as a square root of strictly positive operator  $\hat{a}\hat{a}^\dagger$ . In Fock representation

$$\exp(i\hat{\varphi}) = \sum_{n=0}^{\infty} |n\rangle \langle n+1| \quad (3.53)$$

Eigenstates of this operator do not belong to the Hilbert space (as eigenstates of the position, or momentum operators), but they make a good sense, and can be used to build well defined wave-packets. They fulfill

$$\exp(i\hat{\varphi}) |e^{i\varphi}\rangle = e^{i\varphi} |e^{i\varphi}\rangle \quad (3.54)$$

with  $\varphi$  defined modulo  $2\pi$ , for example in the interval  $[-\pi, \pi)$ . In the Fock representation

$$|e^{i\varphi}\rangle = \sum_{n=0}^{\infty} e^{in\varphi} |n\rangle \quad (3.55)$$

Indeed, the moduli squared of coefficients of this expansion are not summable. We could live with that, but the problem is the the operator  $\exp(i\hat{\varphi})$  is not unitary

$$[\exp(i\hat{\varphi}), (\exp(i\hat{\varphi}))^\dagger] = |0\rangle \langle 0| \quad (3.56)$$

and thus the operator  $\hat{\varphi}$  is not hermitian, *ergo* it is not a physical observable.

Nevertheless, the eigenstates are useful. They are not orthogonal, they scalar products can be defined as  $\langle e^{i\varphi} | e^{i\varphi'} \rangle = (1 + e^{i(\varphi' - \varphi)})^{-1}$ , but they form a complete set:

$$\int_{-\pi}^{\pi} |e^{i\varphi}\rangle \langle e^{i\varphi}| \frac{d\varphi}{2\pi} = \hat{1} \quad (3.57)$$

<sup>5</sup>D.T. Pegg, S.M. Barnett, Europhys. Letters **6**, 483 (1988)

This property allows to define in a very reasonable way the *phase probability distribution* for a given state  $|\Psi\rangle$

$$P(\varphi) = \frac{1}{2\pi} |\langle e^{i\varphi} | \Psi \rangle|^2 \quad (3.58)$$

This functions is obviously positive, and normalized,  $\int_{-\pi}^{\pi} P(\varphi) d\varphi = 1$ .

We can check what are the phase distributions for various states that we have already studied:

- Fock state  $|n_0\rangle$ .

We have  $\langle e^{i\varphi} | n_0 \rangle = e^{-in_0\varphi}$ , and thus  $P(\varphi) = 1/2\pi$ , as was easy to guess. Thus a Fock state has a completely undetermined phase, since it is independent of  $\varphi$ .

- Coherent states  $|\alpha\rangle$ , with  $\alpha = r e^{i\varphi_0}$ .

Here

$$\langle e^{i\varphi} | \alpha \rangle = e^{-r^2} \sum_{n=0}^{\infty} r^n e^{in(\varphi_0 - \varphi)} / \sqrt{n!}$$

Instead of calculating the mean of the phase and its variance, one can easily check that for any odd function of  $\varphi - \varphi_0$

$$\int_{-\pi}^{\pi} P(\varphi) f(\varphi - \varphi_0) d\varphi = 0,$$

which indicates that  $\langle \varphi \rangle = \varphi_0$ . Similarly, we can estimate for large  $r$  that

$$\langle (\varphi - \varphi_0)^2 \rangle \simeq 1 - 2 \langle \cos(\varphi - \varphi_0) \rangle \simeq 1/4r^2 = 1/4|\alpha|^2.$$

The larger the coherent state modulus square of the amplitude, the bigger the mean photon number, and the smaller the phase fluctuations. We can therefore conclude, that coherent states have a ‘well’ defined phase.

Pegg and Barnett<sup>6</sup> tried to define the hermitian phase operator, by taking a finite subspace  $\mathcal{H}_s$  of the Hilbert space, spanned by the Fock states with  $0, 1, \dots, s$  photons. They define

$$|\varphi_0\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^s e^{in\varphi_0} |n\rangle$$

Similarly they introduce

$$|\varphi_m\rangle = \exp\left(\frac{2im\pi\hat{a}^\dagger\hat{a}}{s+1}\right) |\psi_0\rangle$$

---

<sup>6</sup>Phys. Rev. A, **39**, 1665 (1989)

These new states differ in phase by  $2\pi m/(s+1)$ , i.e.  $\varphi_m = \varphi_0 + 2\pi m/(s+1)$ , and form an orthonormal basis on  $\mathcal{H}_s$ . The phase operator

$$\hat{\varphi}_{\text{PB}} = \sum_{m=0}^s \varphi_m |\varphi_m\rangle \langle \varphi_m|$$

is hermitian, and allows of course to define probability distribution

$$P_m = |\langle \varphi_m | \psi \rangle|^2$$

for any state  $|\psi\rangle$  in  $\mathcal{H}_s$ . In the limit  $s \rightarrow \infty$ , the operator  $\hat{\varphi}_{\text{PB}}$  does not converge to any Hermitian operator, but defining  $\varphi = \lim_{m,s \rightarrow \infty} \varphi_m$ , we get that

$$P(\varphi) = \lim_{s,m \rightarrow \infty} [2\pi P_m / (s+1)].$$

It is still an open question what is an exact meaning of the distribution  $P(\varphi)$ . There are strong indications, however, that it corresponds to an optimal phase measurement.

résumé

**Fock, number states**state sign  $|n_k\rangle$ definition  $\hat{n}_k |n_k\rangle = n_k |n_k\rangle$ fock state  $|n_k\rangle = \frac{(\hat{a}_k^\dagger)^{n_k}}{\sqrt{n_k!}} |0_k\rangle$ 

annihilation and creation operators

$$\begin{aligned}\hat{a}_k |n_k\rangle &= \sqrt{n_k} |n_k - 1\rangle \\ \hat{a}_k^\dagger |n_k\rangle &= \sqrt{n_k + 1} |n_k + 1\rangle\end{aligned}\tag{3.59}$$

**Coherent states**state sign  $|\alpha\rangle$ definition  $|\alpha\rangle \equiv \hat{D}(\alpha) |0\rangle$ Fock states representation  $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$ 

Displacement operator

$$\hat{D}(\alpha) \equiv e^{(\alpha \hat{a}^\dagger - \alpha^* \hat{a})}\tag{3.60}$$

annihilation and creation operators

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle\tag{3.61}$$

$$\langle \alpha | \hat{a}^\dagger = \alpha^* \langle \alpha |\tag{3.62}$$

**Squeezed states**state sign  $|\alpha, \varepsilon\rangle$ definition  $|\alpha, \varepsilon\rangle = \hat{D}(\alpha) \hat{S}(\varepsilon) |0\rangle$ 

Squeeze operator

$$\hat{S}(\varepsilon) = \exp(\varepsilon^* \hat{a}^2/2 - \varepsilon (\hat{a}^\dagger)^2/2)\tag{3.63}$$

**thermal states**definition (density matrix)  $\rho = \frac{1}{\bar{n}+1} \left( \frac{\bar{n}}{\bar{n}+1} \right)^{\hat{a}^\dagger \hat{a}}$ **Noisy coherent states**definition (density matrix)  $\rho(\alpha, \beta) = \hat{D}(\alpha) \rho_{th}(\beta) \hat{D}^\dagger(\alpha)$

**Properties**

photon number distribution (probabilities of having  $n$  photons)

fock  $p_n = \delta_{nn_0}$

coherent  $p_n = |\langle n | \alpha \rangle|^2 = \frac{|\alpha|^{2n}}{n!} \exp(-|\alpha|^2)$

squeezed  $p_n = (n!\mu)^{-1} \left(\frac{\nu}{2\mu}\right)^n \left| H_n \left( \frac{\beta}{\sqrt{2\mu\nu}} \right) \right|^2 e^{-|\beta|^2 + \frac{\nu}{2\mu}\beta^2 + \frac{\nu^*}{2\mu}(\beta^*)^2}$

thermal  $p_n = \frac{1}{\bar{n}+1} \left(\frac{\bar{n}}{\bar{n}+1}\right)^n$

noisy

mean photon number

fock  $\bar{n} = n_0$

coherent  $\bar{n} = |\alpha|^2$

squeezed  $\bar{n} = |\alpha|^2 + \sinh^2 r$

thermal  $\bar{n} = \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}$

noisy  $\bar{n} = |\alpha|^2 + \bar{n}_{th}$

photon number fluctuations

fock  $\overline{n^2} = n_0^2$

coherent  $\overline{n^2} = \bar{n}^2 + \bar{n}$

thermal  $\overline{n^2} = 2\bar{n}^2 + \bar{n}$

Variance of the photon distribution

fock  $(\Delta n)^2 = 0$

coherent  $(\Delta n)^2 = \bar{n}$

squeezed  $(\Delta n)^2 = \left| \alpha \cosh r - \alpha^* e^{2i\varphi} \sinh r \right|^2 + 2 \cosh^2 r \sinh^2 r$

thermal  $(\Delta n)^2 = \bar{n}^2 + \bar{n}$

noisy  $(\Delta n)^2 = |\alpha|^2(2\bar{n}_{th} + 1) + \bar{n}_{th}^2 + \bar{n}_{th}$

**Poisson distributions**

$(\Delta n)^2 = \bar{n}$  Poissonian

$(\Delta n)^2 < \bar{n}$  sub-Poissonian

$(\Delta n)^2 > \bar{n}$  super-Poissonian

**Phase distribution**

Fock states

$$|e^{i\Phi}\rangle = \sum_n e^{in\Phi} |n\rangle \quad (3.64)$$

$$P(\Phi) = \frac{|\langle e^{i\Phi} | \psi \rangle|^2}{2\pi}. \quad (3.65)$$

For  $|\psi\rangle = |n_0\rangle$ 

$$P(\Phi) = \frac{1}{2\pi} \quad (\text{uniform}). \quad (3.66)$$

Coherent states

$$|\alpha\rangle = |r e^{i\Phi_0}\rangle \quad (3.67)$$

$$\int_{-\pi}^{+\pi} d\Phi P(\Phi) \Phi = \Phi_0 \quad \langle \Phi \rangle = \Phi_0$$

$$\langle \Phi^2 \rangle - \langle \Phi \rangle^2 = \frac{1}{4\bar{n}} \quad \text{for large } \bar{n} \quad (3.68)$$

Thermal (chaotic states)

$$\frac{1}{2\pi} \langle e^{i\Phi} | \varrho | e^{i\Phi} \rangle = \frac{1}{2\pi} \quad (3.69)$$

## List of variables

## fock states

$\hat{n}$	number operator (photons)
$ n\rangle$	fock state
$ \Omega\rangle$	ground state (no photons)
$ 0\rangle$	vacuum state (= ground state)
$\hat{a}_k$	annihilation operator (photon)
$\hat{a}_k^\dagger$	creation operator (photon)
$\hat{I}$	unity operator

## coherent states

$\hat{D}(\alpha)$	displacement operator
$ \alpha\rangle$	coherent state
$\alpha$	eigenvalue of the annihilation operator acting on $ \alpha\rangle$
$p_n$	photon number distribution
$\bar{n}$	mean number of photons

## squeezed states

$\hat{Q}, \hat{P}$	quadrature operators
$\hat{S}(\varepsilon)$	squeeze operator
$\hat{Q}'_1, \hat{P}'_2$	general hermitian quadrature operators
$ \alpha, \varepsilon\rangle$	squeezed state
$\varepsilon = r e^{2i\varphi}$	complex number
$r =  \varepsilon $	squeezing factor

## thermal states

$\rho$	density matrix
$\overline{n^2}$	photon number fluctuations

## phase of the field

$P(\varphi)$	phase probability distribution
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# CHAPTER 4

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## Single atom – single mode interaction

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In the previous chapters we have described the properties of the free quantised EM field. We are now interested in its interaction with matter.

### 4.1. Hamiltonian with quantised EM field

The Hamiltonian describing matter-light interaction contains three different parts. The part which describes the atom, the one which describes the external EM field, and the interaction between field and atoms.

$$\mathcal{H} = \mathcal{H}_a + \mathcal{H}_F + \mathcal{H}_{\text{int}} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} \quad (4.1)$$

with

- $\mathcal{H}_a$  – the *atomic Hamiltonian*; describes an electron in an atom

$$\mathcal{H}_a = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{r}}) \quad (4.2)$$

$V(\mathbf{r})$ : is the Coulomb interaction between a bound electron and the nucleus. We make here the approximation of just one single (valence) electron in the atom, which is interacting with the field

- $\mathcal{H}_F$  – *quantised electromagnetic field Hamiltonian*

$$\begin{aligned} \mathcal{H}_F &= \frac{1}{2} \int \left( \varepsilon |\mathbf{E}(\mathbf{r}, t)|^2 + \frac{1}{\mu_0} |\mathbf{B}(\mathbf{r}, t)|^2 \right) d^3r \\ &= \sum_k \hbar\omega_k \left( \hat{\mathbf{a}}_k^\dagger \hat{\mathbf{a}}_k + \frac{1}{2} \right) \\ &= \sum_k \hbar\omega_k \left( \hat{n}_k + \frac{1}{2} \right) \end{aligned} \quad (4.3)$$

- $\mathcal{H}_{\text{int}}$  – interaction Hamiltonian.

In the dipolar approximation the interaction between the electron and the EM field simply reads  $\mathbf{r}\mathbf{E}$ . In the  $\mathbf{p}\mathbf{A}$  gauge the interaction with the EM field can be directly written as

$$\mathcal{H} = \frac{1}{2m}(\hat{\mathbf{p}} - e\hat{\mathbf{A}})^2 + \hat{V}(\mathbf{r}) + \sum_k \hbar\omega_k(\hat{a}_k^\dagger\hat{a}_k) \quad (4.4)$$

Note: neglect in the above expression the vacuum energy

$$= \underbrace{\frac{1}{2m}\hat{\mathbf{p}}^2 + \hat{V}(\mathbf{r})}_{\mathcal{H}_a} - \underbrace{\frac{e}{2m}(\hat{\mathbf{p}}\hat{\mathbf{A}} + \hat{\mathbf{A}}\hat{\mathbf{p}})}_{=\mathcal{H}_{\text{int}}} + \underbrace{\frac{e^2}{2m}\hat{\mathbf{A}}^2 + \sum_k \hbar\omega_k(\hat{a}_k^\dagger\hat{a}_k)}_{\text{EM field}} \quad (4.5)$$

Thus

$$\mathcal{H}_{\text{int}} = -\frac{e}{2m}(\hat{\mathbf{p}}\hat{\mathbf{A}} + \hat{\mathbf{A}}\hat{\mathbf{p}}) + \frac{e^2}{2m}\hat{\mathbf{A}}^2 \quad (4.6)$$

We can simplify the Hamiltonian with the following arguments

- i) value of  $[\hat{\mathbf{p}}, \hat{\mathbf{A}}] = 0$

$$\begin{aligned} [\hat{\mathbf{p}}, \hat{\mathbf{A}}]|\psi\rangle &= (\hat{\mathbf{p}}\hat{\mathbf{A}} - \hat{\mathbf{A}}\hat{\mathbf{p}})|\psi\rangle \\ &= -i\hbar\nabla(\hat{\mathbf{A}}\psi) - \hat{\mathbf{A}}(-i\hbar\nabla)\psi \quad \text{with } \hat{\mathbf{p}} = -i\hbar\nabla \\ &= -i\hbar\psi(\nabla\mathbf{A}) - i\hbar\hat{\mathbf{A}}\nabla\psi + i\hbar\hat{\mathbf{A}}\nabla\psi \\ &= 0 \end{aligned} \quad (4.7)$$

with  $\nabla\mathbf{A} = 0$  from Coulomb gauge. The Hamiltonian simplifies to

$$\mathcal{H} = \frac{1}{2m}\hat{\mathbf{p}}^2 + \hat{V}(\mathbf{r}) - \frac{e}{2m}(\hat{\mathbf{p}}\hat{\mathbf{A}}) + \frac{e^2}{2m}\hat{\mathbf{A}}^2 + \sum_k \hbar\omega_k(\hat{a}_k^\dagger\hat{a}_k) \quad (4.8)$$

- ii) We can neglect the term proportional to  $\hat{\mathbf{A}}^2$  – this term is usually very small except for very intense fields and corresponds to a two-photon process, which is in general negligible compared to a one-photon process.

We get then

$$\mathcal{H} = \frac{1}{2m}\hat{\mathbf{p}}^2 + \hat{V}(\mathbf{r}) - \frac{e}{2m}(\hat{\mathbf{p}}\hat{\mathbf{A}}) + \sum_k \hbar\omega_k(\hat{a}_k^\dagger\hat{a}_k) \quad (4.9)$$

## 4.2. Quantisation of the electron wave field (2<sup>nd</sup> quantisation)

It is sometimes useful to quantise the electron wave field in terms of atomic creation and annihilation operators  $\hat{b}^\dagger, \hat{b}$ . The quantisation is done in a similar fashion as for the electromagnetic field.

The atomic Hamiltonian for a single atom reads:

$$\mathcal{H}_a = \frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{r}) = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) \quad (4.10)$$

Eigenfunctions of the atomic Hamiltonian are found by solving the Schrödinger equation:

$$\mathcal{H}_a\psi_j(\mathbf{r}) = E_j\psi_j(\mathbf{r}) \quad (4.11)$$

where  $\psi_j(\mathbf{r})$  are the eigenfunctions of the hamiltonian. Any general wavefunction  $\psi(\mathbf{r})$  can be expanded in an eigenfunction basis of the hamiltonian, where  $j$  enumerates the eigenstates of  $\mathcal{H}_a$ .

$$\psi(\mathbf{r}) \equiv \sum_j b_j\psi_j(\mathbf{r}) \quad (4.12)$$

In 2<sup>nd</sup> quantisation we now replace the amplitudes  $b_j$  of the expansion by operators  $\hat{b}_j$ . Thus we get the *atomic field operators*

$$\begin{aligned} \hat{\psi}(\mathbf{r}) &= \sum_j \hat{b}_j\psi_j(\mathbf{r}) \\ \hat{\psi}^\dagger(\mathbf{r}) &= \sum_j \hat{b}_j^\dagger\psi_j^*(\mathbf{r}) \end{aligned} \quad (4.13)$$

For the atomic Hamiltonian we derive, using the condition of orthonormality for  $\hat{\psi}_j$ , i. e.  $\int \hat{\psi}_i\hat{\psi}_j^* = \delta_{ij}$

$$\begin{aligned} \hat{\mathcal{H}}_{\text{at}} &= \int \hat{\psi}^\dagger(\mathcal{H}_a)\hat{\psi} d^3r \\ &= \sum_{ij} \hat{b}_j^\dagger\hat{b}_i \int \psi_j^\dagger \underbrace{\left(\frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{r})\right) \psi_i}_{E_i\psi_i} d^3r \\ &= \sum_{ij} \hat{b}_j^\dagger\hat{b}_i E_i \int \underbrace{\psi_j^*\psi_i}_{\delta_{ij}} d^3r \end{aligned} \quad (4.14)$$

$$\hat{\mathcal{H}}_{\text{at}} = \sum_j E_j \hat{b}_j^\dagger\hat{b}_j \quad (4.15)$$

If we identify  $\hat{b}_j^\dagger \hat{b}_j$  with a number operator for the occupation number of electrons in level  $j$ , as we did for the photons, then we see that the Hamiltonian is simply the sum over all energy levels with the energy of the level times the number of electrons in this level.

#### 4.2.1. Properties of atomic operators $\hat{b}_j, \hat{b}_j^\dagger$

We identify the two atomic operators  $\hat{b}_j^\dagger$  and  $\hat{b}_j$  with the operators that create or annihilate one electron in the state  $j$ .

In the following we see that  $\hat{b}_j^\dagger |0\rangle$  describes one electron in level  $j$ :

$$\begin{aligned}\hat{b}_1^\dagger |0\rangle &= |1\rangle_1 && \text{(One electron in level 1)} \\ \hat{b}_2^\dagger |0\rangle &= |1\rangle_2 && \text{(One electron in level 2)}\end{aligned}\tag{4.16}$$

Pauli exclusion principle implies (no possibility of two electrons in the same state)

$$\hat{b}_j^\dagger \hat{b}_j^\dagger |0\rangle = 0\tag{4.17}$$

and that holds for any arbitrary state  $|\varphi\rangle$ .

Atomic operators acting on the vacuum state:

$\begin{aligned}\hat{b}_j^\dagger  0\rangle &=  1\rangle_j \\ \hat{b}_j  0\rangle &= 0\end{aligned}$	<p>With <math>\langle 0   \hat{b}_j = \langle 1  </math> follows <math>\langle 0   \hat{b}_j  0\rangle = 0</math>  <math>\Rightarrow</math> The vacuum is annihilated by <math>\hat{b}_j</math></p>	(4.18)
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and analogously on the excited state:

$$\begin{aligned}\hat{b}_j^\dagger |1\rangle_j &= 0 \\ \hat{b}_j |1\rangle_j &= |0\rangle\end{aligned}\tag{4.19}$$

The above properties follow from the assumption that the operators  $\hat{b}_j$  obey anticommutator relations:

$$\{\hat{b}_i, \hat{b}_j^\dagger\} = \hat{b}_i \hat{b}_j^\dagger + \hat{b}_j^\dagger \hat{b}_i \equiv \delta_{ij}\tag{4.20}$$

$$\{\hat{b}_i, \hat{b}_j\} = 0 \quad \text{and} \quad \{\hat{b}_i^\dagger, \hat{b}_j^\dagger\} = 0\tag{4.21}$$

So fermionic operators obey anticommutation algebra while bosonic operators obey commutation algebra.

We obtain

$$(\hat{b}_i \hat{b}_j^\dagger + \hat{b}_j^\dagger \hat{b}_i) |0\rangle = \hat{b}_i \hat{b}_j^\dagger |0\rangle = \hat{b}_i |1\rangle_j = 0 \quad \text{for } i \neq j\tag{4.22}$$

$$(\hat{b}_i \hat{b}_i^\dagger + \hat{b}_i^\dagger \hat{b}_i) |0\rangle = \hat{b}_i \hat{b}_i^\dagger |0\rangle = \hat{b}_i |1\rangle_i = |0\rangle\tag{4.23}$$

In analogy to the number operator  $\hat{n} = \hat{a}^\dagger \hat{a}$  we identify  $\hat{b}_j^\dagger \hat{b}_j$  with the operator for the occupation number of electrons in level  $j$ .

### 4.3. Full Hamiltonian in 2<sup>nd</sup> quantisation

Let us rewrite the parts of the full Hamiltonian we have found up to now:

$$\hat{\mathcal{H}}_a = \sum_j E_j \hat{b}_j^\dagger \hat{b}_j \quad (4.24)$$

$$\hat{\mathcal{H}}_F = \sum_k \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k \quad (4.25)$$

$$\hat{\mathcal{H}}_{\text{int}} = \int \hat{\psi}^\dagger(\mathbf{r}) \left( -\frac{e}{m} \hat{\mathbf{A}} \hat{\mathbf{p}} \right) \hat{\psi}(\mathbf{r}) d^3r \quad (4.26)$$

We want to derive the value of  $\mathcal{H}_{\text{int}}$ . We insert therefore these values

$$\begin{aligned} \hat{\psi}(\mathbf{r}) &= \sum_j \hat{b}_j \psi_j(\mathbf{r}) \\ \hat{\psi}^\dagger(\mathbf{r}) &= \sum_j \hat{b}_j^\dagger \psi_j^*(\mathbf{r}) \\ \hat{\mathbf{A}} &= \sum_k \sqrt{\frac{\hbar}{2\omega\varepsilon_0}} [\hat{a}_k \mathbf{u}_k(\mathbf{r}) + h.c.] \end{aligned}$$

and obtain the following expression:

$$\hat{\mathcal{H}}_{\text{int}} = -\frac{e}{m} \sum_{ijk} \hat{b}_i^\dagger \hat{b}_j \int \psi_i^*(\mathbf{r}) \sqrt{\frac{\hbar}{2\omega\varepsilon_0}} \{ [\hat{a}_k \mathbf{u}_k(\mathbf{r}) + h.c.] \hat{\mathbf{p}} \} \psi_j(\mathbf{r}) d^3r \quad (4.27)$$

which we simplify by introducing a *coupling constant*  $g_{kij}$

$$g_{kij} = -\frac{e}{m} \sqrt{\frac{1}{2\hbar\omega\varepsilon_0}} \int \psi_i^*(\mathbf{r}) (\mathbf{u}_k(\mathbf{r}) \cdot \hat{\mathbf{p}}) \psi_j(\mathbf{r}) d^3r \quad (4.28)$$

and get

$$\boxed{\hat{\mathcal{H}}_{\text{int}} = \hbar \sum_{ijk} \hat{b}_i^\dagger \hat{b}_j (g_{kij} \hat{a}_k + g_{kij}^* \hat{a}_k^\dagger)} \quad (4.29)$$

where  $\hat{a}_k$  is the annihilation operator for the EM field of mode  $k$ .

The full Hamiltonian in 2<sup>nd</sup> quantization reads

$$\hat{\mathcal{H}} = \sum_j E_j \hat{b}_j^\dagger \hat{b}_j + \sum_k \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k + \hbar \sum_{ijk} \hat{b}_i^\dagger \hat{b}_j (g_{ijk} \hat{a}_k + g_{ijk}^* \hat{a}_k^\dagger) \quad (4.30)$$

## 4.4. Approximations to treat this problem

### 4.4.1. Dipole approximation

The magnitude of the wavelength of typical light sources are in the nm range whereas the atom size is several orders of magnitude smaller. Therefore we can assume that the electron does not see the change in space of the electric field, in other words  $\mathbf{E}(\mathbf{r}, t) = \overline{\mathbf{E}}(\mathbf{r}_0, t)$  and the electron sees a constant electric field.

$$\mathbf{u}_k(\mathbf{r}) \equiv \mathbf{u}_k(\mathbf{r}_0) = e^{i\mathbf{k}\mathbf{r}_0} \quad (4.31)$$

where  $\mathbf{r}_0$  are the coordinates of the atom. We now want to solve the integral from Eq.(4.28)

$$\int \psi_i^*(\mathbf{r}) (\mathbf{u}_k(\mathbf{r}) \cdot \hat{\mathbf{p}}) \psi_j(\mathbf{r}) d^3r \rightarrow \mathbf{u}_k(\mathbf{r}_0) \int \psi_i^*(\mathbf{r}) \hat{\mathbf{p}} \psi_j(\mathbf{r}) d^3r$$

with  $\hat{\mathbf{p}}$  written in terms of the time evolution of the  $\hat{\mathbf{r}}$  operator (Heisenberg equation)

$$\hat{\mathbf{p}} = m\hat{\mathbf{v}} = m\frac{d\hat{\mathbf{r}}}{dt} = m\left(\frac{i}{\hbar}[\hat{\mathbf{r}}, \hat{\mathcal{H}}_a]\right)$$

we can solve

$$\begin{aligned} \int \psi_i^*(\mathbf{r}) \hat{\mathbf{p}} \psi_j(\mathbf{r}) d^3r &= \int \psi_i^*(\mathbf{r}) \frac{mi}{\hbar} [\hat{\mathbf{r}}, \hat{\mathcal{H}}_a] \psi_j(\mathbf{r}) d^3r \\ &= \frac{mi}{\hbar} \int \psi_i^*(\mathbf{r}) (\hat{\mathcal{H}}\hat{\mathbf{r}} - \hat{\mathbf{r}}\hat{\mathcal{H}}) \psi_j(\mathbf{r}) d^3r \end{aligned}$$

using:  $\psi_i^*(\mathbf{r})\hat{\mathcal{H}} = E_i\psi_i^*(\mathbf{r})$  and  $\hat{\mathcal{H}}\psi_j(\mathbf{r}) = E_j\psi_j(\mathbf{r})$

$$\begin{aligned} \int \psi_i^*(\mathbf{r}) \hat{\mathbf{p}} \psi_j(\mathbf{r}) d^3r &= \frac{mi}{\hbar} \int (E_i\psi_i^*(\mathbf{r})\hat{\mathbf{r}}\psi_j(\mathbf{r}) - E_j\psi_i^*(\mathbf{r})\hat{\mathbf{r}}\psi_j(\mathbf{r})) d^3r \\ &= \frac{mi}{\hbar} (E_i - E_j) \underbrace{\int \psi_i^*(\mathbf{r})\hat{\mathbf{r}}\psi_j(\mathbf{r}) d^3r}_{d_{ij} \equiv \text{dipole matrix element}} \\ &= \frac{mi}{\hbar} (E_i - E_j) d_{ij} \end{aligned}$$

We have thus derived

$$g_{kij} = -i \frac{e}{m} \sqrt{\frac{\hbar}{2\omega\varepsilon_0}} \cdot \mathbf{u}_k(\mathbf{r}_0) \frac{m}{\hbar} (E_i - E_j) d_{ij} \quad (4.32)$$

in the dipole approximation.

For simplicity we can chose the phase of  $\mathbf{u}_k(\mathbf{r}_0)$  such that  $g_{kij}$  is real. Then

$$\hat{\mathcal{H}}_{\text{int}} = \hbar \sum_{ijk} \hat{\mathbf{b}}_i^\dagger \hat{\mathbf{b}}_j g_{kij} (\hat{\mathbf{a}}_k + \hat{\mathbf{a}}_k^\dagger) \quad (4.33)$$

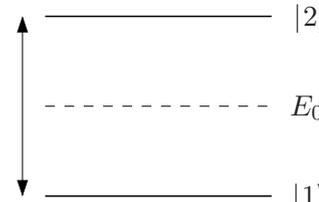
## 4.4.2. Single mode interaction with 2 level Atom

We simplify our atom which interacts with the incident electromagnetic field to have only two levels. This approximation, although it sounds only academic, has full sense, because in many problems the incident light is resonant or quasi resonant with only two internal states of the atom and does not couple to any other internal state. Another approximation we do is to neglect the electronic spin.

Since we are only dealing with a two level atom (one electron only) we can neglect the information about the number of electrons and simplify our notation in the following way:

$$\left. \begin{array}{l} |1\rangle_1 \equiv |1\rangle \\ |1\rangle_2 \equiv |2\rangle \end{array} \right\} \text{ in our notation} \quad (4.34)$$

Definitions:

upper state	$ 2\rangle$	$\hbar\omega_0$ 
lower state	$ 1\rangle$	
energy between states	$\hbar\omega_0$	
energy of upper level	$E_2 \equiv E_0 + \frac{1}{2}\hbar\omega_0 \equiv \hbar\omega_0$	
energy of lower level	$E_1 \equiv E_0 - \frac{1}{2}\hbar\omega_0 \equiv -\hbar\omega_0$	

$|1\rangle, |2\rangle$  are eigenstates of the atomic hamiltonian.

$$\hat{\mathcal{H}}_a |1, 2\rangle = E_{1,2} |1, 2\rangle \quad (4.35)$$

Properties of this system

$$|1\rangle\langle 1| + |2\rangle\langle 2| = 1 \quad (4.36)$$

$$\langle 1|2\rangle = \delta_{12} \quad (4.37)$$

with these properties a 2-level atom is formally a spin 1/2 particle.

Recall

$$g_{kij} = -i \frac{e}{m} \sqrt{\frac{1}{2\hbar\omega\varepsilon_0}} \cdot u_k(\mathbf{r}_0) \frac{m}{\hbar} (E_i - E_j) d_{ij} \quad (4.32)$$

with  $(E_2 - E_1) = \hbar\omega_0$  we can simplify this to

$$g \equiv g_{12} = -ie \sqrt{\frac{1}{2\hbar\omega\varepsilon_0}} \cdot \mathbf{u}(\mathbf{r}_0) \vec{\varepsilon}\omega_0 d_{12} \quad (4.38)$$

In this picture we have the following Hamiltonians

$$\begin{aligned}
 \hat{\mathcal{H}}_a &= E_1 \hat{b}_1^\dagger \hat{b}_1 + E_2 \hat{b}_2^\dagger \hat{b}_2 \\
 \hat{\mathcal{H}}_F &= \hbar \omega \hat{a}^\dagger \hat{a} \\
 \hat{\mathcal{H}}_{\text{int}} &= \hbar \left( \hat{b}_1^\dagger \hat{b}_2 g_{12} (\hat{a} + \hat{a}^\dagger) + \hat{b}_2^\dagger \hat{b}_1 g_{21} (\hat{a} + \hat{a}^\dagger) \right)
 \end{aligned} \tag{4.39}$$

#### 4.4.3. Rotating wave approximation (RWA)

In the Heisenberg picture we have found the time evolution of the operators  $\hat{a}$  and  $\hat{b}$

$$\hat{b}_i(t) = \hat{b}_i(0) e^{-\frac{i}{\hbar} E_i t} \tag{4.40}$$

$$\hat{b}_j^\dagger(t) = \hat{b}_j^\dagger(0) e^{+\frac{i}{\hbar} E_j t} \tag{4.41}$$

$$\hat{a}(t) = \hat{a}(0) e^{-i\omega t} \tag{4.42}$$

$$\hat{a}^\dagger(t) = \hat{a}^\dagger(0) e^{+i\omega t} \tag{4.43}$$

The product of two  $\hat{b}$  operators times the operator  $\hat{a}$  is then

$$\begin{aligned}
 \hat{b}_1^\dagger \hat{b}_2 \hat{a} &\propto e^{+\frac{i}{\hbar}(E_1 - E_2)t} e^{-i\omega t} \\
 &\simeq e^{-i\omega_0 t} e^{-i\omega t} \\
 &= e^{-i(\omega_0 + \omega)t}
 \end{aligned}$$

and analogue for the other variations. Altogether we find

$$\hat{b}_1^\dagger \hat{b}_2 \hat{a} \propto e^{-i(\omega_0 + \omega)t} \tag{4.44}$$

$$\hat{b}_1^\dagger \hat{b}_2 \hat{a}^\dagger \propto e^{-i(\omega_0 - \omega)t} \tag{4.45}$$

$$\hat{b}_2^\dagger \hat{b}_1 \hat{a} \propto e^{+i(\omega_0 - \omega)t} \tag{4.46}$$

$$\hat{b}_2^\dagger \hat{b}_1 \hat{a}^\dagger \propto e^{-i(\omega_0 + \omega)t} \tag{4.47}$$

If  $\omega_0 \simeq \omega$ , which occurs if the EM field is close to the atomic resonance, as it is the case if only these 2 atomic levels are relevant, the terms proportional to  $\omega_0 + \omega$  oscillate like  $2\omega$ . These terms can be neglected in comparison with the slow oscillating term  $\omega_0 - \omega$ . They correspond to highly non-energy conserving processes. This is known as the rotating wave approximation (RWA).

Therefore remain only  $\hat{b}_1^\dagger \hat{b}_2 \hat{a}^\dagger$  and  $\hat{b}_2^\dagger \hat{b}_1 \hat{a}$ , so that we can conclude

$$\mathcal{H}^{\text{RWA}} = E_1 \hat{b}_1^\dagger \hat{b}_1 + E_2 \hat{b}_2^\dagger \hat{b}_2 + \hbar \omega \hat{a}^\dagger \hat{a} + \hbar (g_{12} \hat{b}_1^\dagger \hat{b}_2 \hat{a}^\dagger + h.c.) \tag{4.48}$$

This Hamiltonian is exactly solvable!

## 4.4.4. Pauli Spin matrices

Instead of using fermionic operators ( $\hat{b}_i^\dagger, \hat{b}_i$ ) for the atom it is sometimes more convenient to use a corresponding combination of Pauli spin matrices  $\sigma_i$  (spin 1/2 formalism)

$$\hat{R}_i = \frac{1}{2}\hat{\sigma}_i \quad \text{with} \quad \hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.49)$$

Let us do the following association:

electron operator	→	spin operator
$\hat{b}_2^\dagger \hat{b}_1$	→	$\hat{R}_+$
$\hat{b}_1^\dagger \hat{b}_2$	→	$\hat{R}_-$
$\frac{1}{2}(\hat{b}_2^\dagger \hat{b}_2 - \hat{b}_1 \hat{b}_1^\dagger)$	→	$\hat{R}_3$

where

$\hat{R}_\pm \equiv \hat{R}_1 \pm i\hat{R}_2$

(4.50)

and  $\hat{R}_3$  in terms of  $\hat{R}_\pm$

$$\hat{R}_3 = \frac{1}{2}(\hat{R}_+ \hat{R}_- - \hat{R}_- \hat{R}_+) \quad (4.51)$$

These operators act on accessible states in the same way as the  $\hat{b}_i, \hat{b}_i^\dagger$  operators do. Sometimes the operators  $\hat{R}_\pm$  are also called raising and lowering operators, since it is easy to check that for spin 1/2 system:

$$\hat{R}_+ \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (4.52)$$

using as a basis the 2 level system  $|1\rangle$  and  $|2\rangle$  (equivalent to spin down  $\downarrow \equiv |1\rangle$  and up  $\uparrow \equiv |2\rangle$ )

$$\hat{R}_+ \equiv |2\rangle\langle 1| \quad \hat{R}_- \equiv |1\rangle\langle 2| \quad (4.53)$$

Properties of raising and lowering operators:

$$\hat{R}_+ |2\rangle = 0 \quad (4.54)$$

$$\hat{R}_+ |1\rangle = |2\rangle \quad (4.55)$$

$$\hat{R}_- |1\rangle = |0\rangle \quad (4.56)$$

$$\hat{R}_- |2\rangle = |1\rangle \quad (4.57)$$

It follows

$$\hat{R}_- \hat{R}_+ |2\rangle = 0 \quad (4.58)$$

$$\hat{R}_- \hat{R}_+ |1\rangle = \hat{R}_- |2\rangle = |1\rangle \quad (4.59)$$

$$\hat{R}_+ \hat{R}_- |1\rangle = \hat{R}_+ |0\rangle = |1\rangle \quad (4.60)$$

$$\hat{R}_+ \hat{R}_- |2\rangle = \hat{R}_+ |1\rangle = |2\rangle \quad (4.61)$$

We can see that  $\hat{R}_- \hat{R}_+$  plays the role of number of particles in level  $|1\rangle$  while  $\hat{R}_+ \hat{R}_-$  plays the role of number of particles in level  $|2\rangle$ .

We can rewrite the free hamiltonian in the rotating wave approximation with the spin operators.

$$\mathcal{H}^{\text{RWA}} = \hbar\omega_0 \hat{R}_3 + \hbar\omega \hat{a}^\dagger \hat{a} + i\hbar g (\hat{R}_+ \hat{a} - \hat{a}^\dagger \hat{R}_-) \quad (4.62)$$

Where we have used

$$\begin{aligned} \hat{\mathcal{H}}_a &= E_1 \hat{b}_1^\dagger \hat{b}_1 + E_2 \hat{b}_2^\dagger \hat{b}_2 \\ &= \frac{1}{2}(E_2 + E_1)(\hat{b}_1 \hat{b}_1^\dagger + \hat{b}_2 \hat{b}_2^\dagger) + \frac{1}{2}(E_2 - E_1) (\hat{b}_2^\dagger \hat{b}_2 - \hat{b}_1 \hat{b}_1^\dagger) \\ &= E_0(\hat{b}_1 \hat{b}_1^\dagger + \hat{b}_2 \hat{b}_2^\dagger) + \frac{1}{2}\hbar\omega_0 (\hat{b}_2^\dagger \hat{b}_2 - \hat{b}_1 \hat{b}_1^\dagger) \end{aligned} \quad (4.63)$$

and omitted the offset  $E_0$  (set to zero).

By expressing  $g = ig_{12}$  we make the notation more compact (and  $g$  real).

We observe that  $\hat{R}_3$  “counts” the number of “atomic” excitations,  $\hat{a}^\dagger \hat{a}$  counts the number of photons, while  $\hbar g (\hat{R}_+ \hat{a} - \hat{R}_- \hat{a}^\dagger)$  giving us information about the coupling in the system.

*Note:* We assume the number of photons to be conserved.

## 4.5. Jaynes Cummings model

The model we have used to derive the Hamiltonian  $\mathcal{H}^{\text{RWA}}$  is termed in the literature the Jaynes Cummings model. Our goal is now to find a solution of the Jaynes Cummings model.

We introduce the *Number operator*, which is counting the total number of excitations

$$\hat{N} \equiv \hat{a}^\dagger \hat{a} + \hat{R}_3 \quad (4.64)$$

Properties of  $\hat{R}_+$ ,  $\hat{R}_-$ ,  $\hat{R}_3$

$$\begin{aligned} [\hat{R}_+, \hat{R}_-] &= 2\hat{R}_3 \\ [\hat{R}_+, \hat{R}_3] &= \hat{R}_+ \\ [\hat{R}_-, \hat{R}_3] &= -\hat{R}_- \end{aligned} \quad (4.65)$$

$$\hat{R}_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \hat{R}_3^2 = \frac{1}{4} \mathbf{1}_2 \quad (4.66)$$

Our plan to solve the Jaynes Cummings model

- i) Analyse  $\mathcal{H}^{\text{RWA}}$
  - ii) find equation of motion for  $\hat{R}_3$
  - iii) solve this equation of motion
- i) find commutator of  $\hat{N}$  and  $\mathcal{H}^{\text{RWA}}$

To find  $[\hat{N}, \mathcal{H}^{\text{RWA}}]$  we have to calculate these commutators:

$$\begin{aligned} [\hat{N}, \hat{R}_3] &= [\hat{a}^\dagger \hat{a} + \hat{R}_3, \hat{R}_3] \\ &= [\hat{a}^\dagger \hat{a}, \hat{R}_3] + [\hat{R}_3, \hat{R}_3] \\ &= \hat{a}^\dagger [\hat{a}, \hat{R}_3] + [\hat{a}^\dagger, \hat{R}_3] \hat{a} \\ &= 0 \end{aligned}$$

$$\begin{aligned} [\hat{N}, \hat{a}^\dagger \hat{a}] &= [\hat{a}^\dagger \hat{a} + \hat{R}_3, \hat{a}^\dagger \hat{a}] \\ &= [\hat{a}^\dagger \hat{a}, \hat{a}^\dagger \hat{a}] + [\hat{R}_3, \hat{a}^\dagger \hat{a}] \\ &= \hat{a}^\dagger [\hat{R}_3, \hat{a}] + [\hat{R}_3, \hat{a}^\dagger] \hat{a} \\ &= 0 \end{aligned}$$

$$\begin{aligned} [\hat{N}, \hat{R}_+ \hat{a}] &= [\hat{a}^\dagger \hat{a} + \hat{R}_3, \hat{R}_+ \hat{a}] \\ &= [\hat{a}^\dagger \hat{a}, \hat{R}_+ \hat{a}] + [\hat{R}_3, \hat{R}_+ \hat{a}] \\ &= \hat{a}^\dagger [\hat{a}, \hat{R}_+ \hat{a}] + [\hat{a}^\dagger, \hat{R}_+ \hat{a}] \hat{a} + \hat{R}_+ [\hat{R}_3, \hat{a}] + [\hat{R}_3, \hat{R}_+] \hat{a} \\ &= \hat{a}^\dagger \hat{R}_+ [\hat{a}, \hat{a}] + \hat{a}^\dagger [\hat{a}, \hat{R}_+] \hat{a} + \hat{R}_+ [\hat{a}^\dagger, \hat{a}] \hat{a} + [\hat{a}^\dagger, \hat{R}_+] \hat{a} \hat{a} - \hat{R}_+ \hat{a} \\ &= \hat{R}_+ \hat{a} - \hat{R}_+ \hat{a} \\ &= 0 \end{aligned}$$

$$\begin{aligned} [\hat{N}, \hat{R}_- \hat{a}^\dagger] &= [\hat{a}^\dagger \hat{a} + \hat{R}_3, \hat{R}_- \hat{a}^\dagger] \\ &= [\hat{a}^\dagger \hat{a}, \hat{R}_- \hat{a}^\dagger] + [\hat{R}_3, \hat{R}_- \hat{a}^\dagger] \\ &= \hat{a}^\dagger [\hat{a}, \hat{R}_- \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{R}_- \hat{a}^\dagger] \hat{a} + \hat{R}_- [\hat{R}_3, \hat{a}^\dagger] + [\hat{R}_3, \hat{R}_-] \hat{a}^\dagger \\ &= \hat{a}^\dagger \hat{R}_- [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger [\hat{a}, \hat{R}_-] \hat{a}^\dagger + \hat{R}_- [\hat{a}^\dagger, \hat{a}^\dagger] \hat{a} + [\hat{a}^\dagger, \hat{R}_-] \hat{a}^\dagger \hat{a} + \hat{R}_- \hat{a}^\dagger \\ &= -\hat{R}_- \hat{a}^\dagger + \hat{R}_- \hat{a}^\dagger \\ &= 0 \end{aligned}$$

With all the commutators equal to zero it follows

$$\boxed{[\hat{N}, \mathcal{H}^{\text{RWA}}] = 0} \quad (4.67)$$

This symmetry implies the conservation of  $\mathcal{H}^{\text{RWA}}$  under the operator  $\hat{N}$ .

ii) find equation of motion for  $\hat{R}_3$

Use Heisenberg equation of motion:

$$i\hbar\dot{\hat{R}}_3 = [\hat{R}_3, \mathcal{H}^{\text{RWA}}] \quad (4.68)$$

To make the evaluation of the commutator easier we rewrite the Hamiltonian with

$$\boxed{\begin{aligned} \bar{\omega} &\equiv \omega_0 + \omega \\ \Delta &\equiv \omega_0 - \omega \quad \text{detuning} \end{aligned}} \quad (4.69)$$

$$\begin{aligned} \mathcal{H}^{\text{RWA}} &= \hbar\omega_0\hat{R}_3 + \hbar\omega\hat{a}^\dagger\hat{a} + i\hbar g(\hat{R}_+\hat{a} - \hat{a}^\dagger\hat{R}_-) \\ &= \dots \\ &= \frac{1}{2}\hbar\bar{\omega}\hat{N} + \frac{1}{2}\hbar\Delta(\hat{R}_3 - \hat{a}^\dagger\hat{a}) + i\hbar g(\hat{R}_+\hat{a} - \hat{a}^\dagger\hat{R}_-) \end{aligned} \quad (4.70)$$

We know  $\hat{R}_3$  commutes with  $\hat{N}$  and  $(\hat{R}_3 - \hat{a}^\dagger\hat{a})$ , so we are left with

$$\begin{aligned} [\hat{R}_3, \hat{R}_+\hat{a}] &= [\hat{R}_3, \hat{R}_+] \hat{a} + \hat{R}_+ [\hat{R}_3, \hat{a}] = \hat{R}_+\hat{a} \\ [\hat{R}_3, \hat{a}^\dagger\hat{R}_-] &= [\hat{R}_3, \hat{a}^\dagger] \hat{R}_- + \hat{a}^\dagger [\hat{R}_3, \hat{R}_-] = -\hat{a}^\dagger\hat{R}_- \\ \Rightarrow [\hat{R}_3, \mathcal{H}^{\text{RWA}}] &= i\hbar g(\hat{R}_+\hat{a} + \hat{a}^\dagger\hat{R}_-) \end{aligned} \quad (4.71)$$

This result is not in terms of  $\mathcal{H}^{\text{RWA}}$ , so we try the second derivative:

$$\begin{aligned} i\hbar\ddot{\hat{R}}_3 &= \frac{\partial}{\partial t} [\hat{R}_3, \mathcal{H}^{\text{RWA}}] \\ &= g(i\hbar\frac{\partial}{\partial t}(\hat{R}_+\hat{a}) + i\hbar\frac{\partial}{\partial t}(\hat{a}^\dagger\hat{R}_-)) \\ &= g([\hat{R}_+\hat{a}, \mathcal{H}^{\text{RWA}}] + [\hat{a}^\dagger\hat{R}_-, \mathcal{H}^{\text{RWA}}]) \end{aligned} \quad (4.72)$$

1.) Evaluate  $[\hat{R}_+\hat{a}, \mathcal{H}^{\text{RWA}}]$

$$\begin{aligned} [\hat{R}_+\hat{a}, \hat{R}_3] &= [\hat{R}_+, \hat{R}_3] \hat{a} = -\hat{R}_+\hat{a} \\ [\hat{R}_+\hat{a}, \hat{a}^\dagger\hat{a}] &= \hat{R}_+ [\hat{a}, \hat{a}^\dagger\hat{a}] + [\hat{R}_+, \hat{a}^\dagger\hat{a}] \hat{a} \\ &= \hat{R}_+ [\hat{a}, \hat{a}^\dagger] \hat{a} + \hat{R}_+\hat{a}^\dagger[\hat{a}, \hat{a}] + \hat{a}^\dagger[\hat{R}_+, \hat{a}] \hat{a} + [\hat{R}_+, \hat{a}^\dagger] \hat{a}\hat{a} \\ &= \hat{R}_+\hat{a} \end{aligned}$$

then follows:  $[\hat{R}_+, \hat{a}, \hat{N}] = 0$

$$\begin{aligned} [\hat{R}_+, \hat{a}, \hat{R}_+ \hat{a}] &= 0 \\ [\hat{R}_+, \hat{a}, \hat{a}^\dagger \hat{R}_-] &= \hat{R}_+ [\hat{a}, \hat{a}^\dagger \hat{R}_-] + [\hat{R}_+, \hat{a}^\dagger \hat{R}_-] \hat{a} \\ &= \hat{R}_+ \hat{a}^\dagger [\hat{a}, \hat{R}_-] + \hat{R}_+ [\hat{a}, \hat{a}^\dagger] \hat{R}_- + \hat{a}^\dagger [\hat{R}_+, \hat{R}_-] \hat{a} + [\hat{R}_+, \hat{a}^\dagger] \hat{a} \hat{R}_- \\ &= \hat{R}_+ \hat{R}_- + \hat{a}^\dagger [\hat{R}_+, \hat{R}_-] \hat{a} \end{aligned}$$

We know from atom physics

$$\mathbf{R}^2 = (S+1)S \rightarrow \left(\frac{1}{2} + 1\right) \frac{1}{2} = \frac{3}{4} \mathbf{1}$$

together with  $\hat{R}_3 = \frac{1}{2} \mathbf{1}$  we rewrite

$$\begin{aligned} \hat{R}_+ \hat{R}_- &= \frac{3}{4} \mathbf{1} - \frac{1}{4} \mathbf{1} + \hat{R}_3 = \frac{1}{2} \mathbf{1} + \hat{R}_3 \\ &= 2\hat{R}_3^2 + \hat{R}_3 = (2\hat{R}_3 + 1)\hat{R}_3 \end{aligned}$$

with these results we can continue

$$\begin{aligned} [\hat{R}_+, \hat{a}, \hat{a}^\dagger \hat{R}_-] &= \hat{R}_+ \hat{R}_- + \hat{a}^\dagger [\hat{R}_+, \hat{R}_-] \hat{a} \\ &= \hat{R}_+ \hat{R}_- + 2\hat{R}_3 \hat{a}^\dagger \hat{a} \\ &= (2\hat{R}_3 + 1)\hat{R}_3 + 2\hat{R}_3 \hat{a}^\dagger \hat{a} \\ &= \left[ (2\hat{R}_3 + 1) + 2\hat{a}^\dagger \hat{a} \right] \hat{R}_3 \\ &= \left[ 2(\hat{R}_3 + \hat{a}^\dagger \hat{a}) + 1 \right] \hat{R}_3 \\ &= (2\hat{N} + 1)\hat{R}_3 \end{aligned}$$

2.) Evaluate  $[\hat{a}^\dagger \hat{R}_-, \mathcal{H}^{\text{RWA}}]$

$$\begin{aligned} [\hat{a}^\dagger \hat{R}_-, \hat{R}_3] &= \hat{a}^\dagger [\hat{R}_-, \hat{R}_3] \hat{a} = \hat{a}^\dagger \hat{R}_- \\ [\hat{a}^\dagger \hat{R}_-, \hat{a}^\dagger \hat{a}] &= \hat{a}^\dagger [\hat{R}_-, \hat{a}^\dagger \hat{a}] + [\hat{a}^\dagger, \hat{a}^\dagger \hat{a}] \hat{R}_- \\ &= \hat{a}^\dagger \hat{a}^\dagger [\hat{R}_-, \hat{a}] + \hat{a}^\dagger [\hat{R}_-, \hat{a}^\dagger] \hat{a} + \hat{a}^\dagger [\hat{a}^\dagger, \hat{a}] \hat{R}_- + [\hat{a}^\dagger, \hat{a}^\dagger \hat{a}] \hat{R}_- \\ &= -\hat{a}^\dagger \hat{R}_- \end{aligned}$$

then follows:  $[\hat{R}_+, \hat{N}] = 0$

$$\begin{aligned}
[\hat{a}^\dagger \hat{R}_-, \hat{a}^\dagger \hat{R}_-] &= 0 \\
[\hat{a}^\dagger \hat{R}_-, \hat{R}_+ \hat{a}] &= \hat{a}^\dagger [\hat{R}_-, \hat{R}_+ \hat{a}] + [\hat{a}^\dagger, \hat{R}_+ \hat{a}] \hat{R}_- \\
&= \hat{a}^\dagger \hat{R}_+ [\hat{R}_-, \hat{a}] + \hat{a}^\dagger [\hat{R}_-, \hat{R}_+] \hat{a} + \hat{R}_+ [\hat{a}^\dagger, \hat{a}] \hat{R}_- + [\hat{a}^\dagger, \hat{R}_+] \hat{a} \hat{R}_- \\
&= -\hat{R}_+ \hat{R}_- + \hat{a}^\dagger [\hat{R}_-, \hat{R}_+] \hat{a} \\
&= -(2\hat{R}_3 + 1)\hat{R}_3 - 2\hat{R}_3 \hat{a}^\dagger \hat{a} \\
&= -\left[(2\hat{R}_3 + 1) + 2\hat{a}^\dagger \hat{a}\right] \hat{R}_3 \\
&= -\left[2(\hat{R}_3 + \hat{a}^\dagger \hat{a}) + 1\right] \hat{R}_3 \\
&= -(2\hat{N} + 1)\hat{R}_3
\end{aligned}$$

Recall

$$\mathcal{H}^{\text{RWA}} = \hbar\omega_0 \hat{R}_3 + \hbar\omega \hat{a}^\dagger \hat{a} + i\hbar g(\hat{R}_+ \hat{a} - \hat{a}^\dagger \hat{R}_-) \quad (4.62)$$

$$i\hbar \ddot{\hat{R}}_3 = g\left([\hat{R}_+ \hat{a}, \mathcal{H}^{\text{RWA}}] + [\hat{a}^\dagger \hat{R}_-, \mathcal{H}^{\text{RWA}}]\right) \quad (4.72)$$

We can now insert all values we have calculated before

$$\begin{aligned}
i\hbar \ddot{\hat{R}}_3 &= g\left(\frac{1}{2}\hbar\Delta(-2\hat{R}_+ \hat{a}) - i\hbar g(2\hat{N} + 1)\hat{R}_3\right. \\
&\quad \left. + \frac{1}{2}\hbar\Delta(2\hat{a}^\dagger \hat{R}_-) - i\hbar g(2\hat{N} + 1)\hat{R}_3\right) \\
i\hbar \ddot{\hat{R}}_3 &= \hbar g\Delta(-\hat{R}_+ \hat{a} + \hat{a}^\dagger \hat{R}_-) - 2i\hbar g^2(2\hat{N} + 1)\hat{R}_3
\end{aligned}$$

The second derivative is now in terms of the Hamiltonian

$$\begin{aligned}
\ddot{\hat{R}}_3 &= ig\Delta(\hat{R}_+ \hat{a} - \hat{a}^\dagger \hat{R}_-) - 2g^2(2\hat{N} + 1)\hat{R}_3 \\
&= \Delta \left\{ \frac{\mathcal{H}^{\text{RWA}}}{\hbar} - \frac{1}{2}(\bar{\omega} - \Delta)\hat{N} \right\} - \underbrace{\left\{ \Delta^2 + 2g^2(2\hat{N} + 1) \right\}}_{\hat{\Omega}^2} \hat{R}_3
\end{aligned}$$

with  $\hbar\omega^{\text{RWA}} = \mathcal{H}^{\text{RWA}}$  we have finally found the *equation of motion* for  $\hat{R}_3$

$$\boxed{\left[ \frac{d^2}{dt^2} + \hat{\Omega}^2 \right] \hat{R}_3(t) = \Delta \left[ \omega^{\text{RWA}} - \frac{1}{2}(\bar{\omega} - \Delta)\hat{N} \right]} \quad (4.73)$$

with the *quantum Rabi frequency*

$$\boxed{\hat{\Omega}^2 = \Delta^2 + 2g^2(2\hat{N} + 1)} \quad (4.74)$$

## 4.6. Spontaneous emission

We want to study the decay of an excited state in a 2-level atom in the presence of vacuum. We shall use the following notation: Our initial state  $|\psi(t=0)\rangle \equiv |2,0\rangle$  meaning an atom whose electron is in the excited state and zero photons (vacuum).

We use the conservation of the number operator  $\hat{N} = \hat{R}_3 + \sum_k \hat{a}_k^\dagger \hat{a}_k$  and  $\mathcal{H}^{\text{RWA}}$ , and the Schrödinger equation

$$i\hbar \frac{d|\psi\rangle}{dt} = \mathcal{H}^{\text{RWA}} |\psi\rangle \quad \Rightarrow \quad i\hbar \frac{d|\psi\rangle}{dt} = \mathcal{H}^{\text{RWA}} |\psi\rangle \quad (4.75)$$

In general

$$|\psi\rangle = \alpha(t) |2,0\rangle + \sum_{\mu} \int \beta(k, \mu, t) |1, 1_{k,\mu}\rangle d^3k \quad (4.76)$$

where  $\alpha(t)$  and  $\beta(k, \mu, t)$  are the amplitudes of having the atom in the excited state and zero photons  $|2,0\rangle$  and having the atom in the ground state and 1 photon of momentum  $k$  and polarisation  $\mu$  emitted  $|1, 1_{k,\mu}\rangle$ .

Our initial conditions are

$$\alpha(t=0) = 1$$

$$\beta(k, \mu, t=0) = 0$$

The Schrödinger equation becomes thus

$$i\hbar \frac{\partial}{\partial t} \left\{ \alpha(t) |2,0\rangle + \sum_{\mu} \int \beta(k, \mu, t) |1, 1_{k,\mu}\rangle d^3k \right\} = \mathcal{H}^{\text{RWA}} |\psi\rangle \quad (4.77)$$

By inserting the initial values into the Schrödinger equation Eq.(4.77) we find the differential equation:

$$i\hbar \dot{\alpha}(t) = \frac{\hbar\omega_0}{2} \alpha + \sum_{\mu} \int g(k, \mu) \beta(k, \mu, t) d^3k \quad (4.78)$$

and

$$i\hbar \dot{\beta}(k, \mu, t) = \left( -\frac{\hbar\omega_0}{2} + \omega_k \right) \beta(k, \mu, t) + g^*(k, \mu) \alpha(t) \quad (4.79)$$

From these, one finds the solution

$$\dot{\alpha} = \alpha(t) [-i\tilde{\omega}_0 - \gamma] \quad (4.80)$$

here we used  $\tilde{\omega}_0 = \omega_o + \delta\omega_0$ , where  $\delta\omega_0$  denotes the Lamb shift. This is a very small  $\delta\omega_0 \lll \omega_0$  quantum correction to the energy of the atomic transition. Second we have used the *natural linewidth*  $\gamma$  of an atomic transition

$$\boxed{\gamma = \frac{d^2\omega_0^2}{6\pi\epsilon_0 c^2}} \quad (4.81)$$

With *dipole moment*  $d = \langle 1 | e\vec{r}\vec{E} | 2 \rangle$   
 From Eq.(4.80) follows

$$\alpha(t) = \alpha(0) e^{(-i\tilde{\omega}_0 - \gamma)t} \quad (4.82)$$

Since  $\alpha(t)$  is nothing else than the population of the excited state, we can rewrite Eq.(4.82) as

$$\frac{d}{dt} P_2(t) = -\gamma P_2(t) \quad (4.83)$$

This gives us the decay of population, called *Weisskopf Formula* (1930)

$$P_2(t) = e^{-\gamma t} P_2(0) \quad (4.84)$$

Typical values for  $\gamma$  are  $10^{-9}s$  and for metastable states  $10^{-3}s$

résumé

**Hamiltonian with quantised em-fields**

Hamiltonian describing matter-light interaction

$$\mathcal{H} = \mathcal{H}_a + \mathcal{H}_F + \mathcal{H}_{\text{int}} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}$$

atomic Hamiltonian

$$\mathcal{H}_a = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{r}})$$

quantised electromagnetic field Hamiltonian

$$\mathcal{H}_F = \sum_k \hbar\omega_k \left( \hat{n}_k + \frac{1}{2} \right)$$

interaction Hamiltonian

$$\mathcal{H}_{\text{int}} = -\frac{e}{2m} (\hat{\mathbf{p}}\hat{\mathbf{A}} + \hat{\mathbf{A}}\hat{\mathbf{p}}) + \frac{e^2}{2m} \hat{\mathbf{A}}^2$$

**Quantisation of the electron wave field (2<sup>nd</sup> quantisation)**

quantized EM field Hamiltonian

$$\hat{\mathcal{H}}_F = \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k$$

quantized atomic Hamiltonian

$$\hat{\mathcal{H}}_{\text{at}} = \sum_j E_j \hat{b}_j^\dagger \hat{b}_j$$

quantized interaction Hamiltonian

$$\hat{\mathcal{H}}_{\text{int}} = \hbar \sum_{ijk} \hat{b}_i^\dagger \hat{b}_j \left( g_{kij} \hat{a}_k + g_{kij}^* \hat{a}_k^\dagger \right)$$

$$g_{kij} = -\frac{e}{m} \sqrt{\frac{1}{2\hbar\omega\varepsilon_0}} \int \psi_i^*(\mathbf{r}) (\mathbf{u}_k(\mathbf{r}) \cdot \hat{\mathbf{p}}) \psi_j(\mathbf{r}) d^3r$$

**Approximations**

dipole approximation: quantized interaction Hamiltonian

$$\hat{\mathcal{H}}_{\text{int}} = \hbar \sum_{ijk} \hat{b}_i^\dagger \hat{b}_j g_{kij} (\hat{a}_k + \hat{a}_k^\dagger)$$

$$g_{kij} = -i \frac{e}{m} \sqrt{\frac{\hbar}{2\omega\varepsilon_0}} \cdot \mathbf{u}_k(\mathbf{r}_0) \frac{m}{\hbar} (E_i - E_j) d_{ij}$$

single mode interaction with 2 level atom

$$\hat{\mathcal{H}}_a = E_1 \hat{b}_1^\dagger \hat{b}_1 + E_2 \hat{b}_2^\dagger \hat{b}_2$$

$$\hat{\mathcal{H}}_F = \hbar\omega \hat{a}^\dagger \hat{a}$$

$$\hat{\mathcal{H}}_{\text{int}} = \hbar \left( \hat{b}_1^\dagger \hat{b}_2 (g_{12} \hat{a} + g_{12}^* \hat{a}^\dagger) + \hat{b}_2^\dagger \hat{b}_1 (g_{21} \hat{a} + g_{21}^* \hat{a}^\dagger) \right)$$

Hamiltonian in rotating wave approximation

$$\mathcal{H}^{\text{RWA}} = E_1 \hat{b}_1^\dagger \hat{b}_1 + E_2 \hat{b}_2^\dagger \hat{b}_2 + \hbar\omega \hat{a}^\dagger \hat{a} + \hbar(g_{12} \hat{b}_1^\dagger \hat{b}_2 \hat{a}^\dagger + h.c.)$$

### Pauli Spin matrices

atom raise and lower operators

$$\hat{R}_+ \equiv \hat{R}_1 + i\hat{R}_2$$

$$\hat{R}_- \equiv \hat{R}_1 - i\hat{R}_2$$

number of excitations

$$\hat{R}_3 = \frac{1}{2}(\hat{R}_+ \hat{R}_- - \hat{R}_- \hat{R}_+)$$

### Jaynes Cummings model

$$\mathcal{H}^{\text{RWA}} = \hbar\omega_0 \hat{R}_3 + \hbar\omega \hat{a}^\dagger \hat{a} + i\hbar g(\hat{R}_+ \hat{a} - \hat{a}^\dagger \hat{R}_-)$$

Number operator (total number of excitations)

$$\hat{N} \equiv \hat{a}^\dagger \hat{a} + \hat{R}_3$$

$$\bar{\omega} \equiv \omega_0 + \omega$$

$$\Delta \equiv \omega_0 - \omega \quad \text{detuning}$$

equation\* of motion for  $\hat{R}_3$

$$\left[ \frac{d^2}{dt^2} + \hat{\Omega}^2 \right] \hat{R}_3(t) = \Delta \left[ \omega^{\text{RWA}} - \frac{1}{2}(\bar{\omega} - \Delta)\hat{N} \right]$$

quantum Rabi frequency

$$\hat{\Omega}^2 = \Delta^2 + 2g^2(2\hat{N} + 1)$$

### Spontaneous emission

natural linewidth

$$\gamma = \frac{d^2 \omega_0^2}{6\pi \epsilon_0 c^2}$$

decay of population – Weisskopf Formula

$$P_2(t) = e^{-\gamma t} P_2(0)$$

## List of variables

$\mathcal{H}_a$	atomic Hamiltonian
$\mathcal{H}_F$	electromagnetic Hamiltonian
$\mathcal{H}_{\text{int}}$	interaction Hamiltonian
$\hat{\mathbf{p}}$	quantum impulse operator
$\hat{\mathbf{A}}$	electromagnetic potential field operator
$\hat{a}$	annihilation operator (field)
$\hat{a}^\dagger$	creation operator (field)
$\hat{\psi}, \hat{\psi}^\dagger$	atomic field operators
$\hat{b}_j$	annihilation operator (electron)
$\hat{b}_j^\dagger$	creation operator (electron)
$g_{kij}$	coupling constant: atom-field
$d_{ij}$	dipole matrix element
$ 2\rangle$	upper state (atoms)
$ 1\rangle$	lower state (atoms)
$\hat{R}_1, \hat{R}_2, \hat{R}_3$	pseudo spin 1/2 operators
$\hat{R}_3$	counting number of excited electrons
$\mathcal{H}^{\text{RWA}}$	Hamiltonian in rotating wave approximation
$\hat{R}_+$	raise atom operator
$\hat{R}_-$	lower atom operator
$\hat{N}$	Number operator (counting the total number of excitations)
$\Delta$	detuning
$\hat{\Omega}$	quantum Rabi frequency
$n$	number of photons
$m$	energy number (denotes upper or lower state)
$\alpha$	amplitude of the excited atomic state with zero photons
$\beta$	probability of decay
$\gamma$	natural linewidth
$d$	dipole moment
$P_2$	Population of upper state
$P_1$	Population of lower state



# CHAPTER 5

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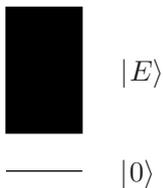
## Photo detection and photo-counting

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In order to characterize the states of quantum EM field one has to perform measurements. The simplest of such measurements conceptually is counting the number of photons, i. e. measuring the operator  $\hat{n} = \hat{a}^\dagger \hat{a}$ . In reality, however, photons are coming from the source and arriving at a detector at various times. What one measures in effect is the number of photons that have arrived at the detector within a given time interval. Repeated measurements of that number provide its probability distribution. The probability distribution of photon counts provides a lot of information about the state of the EM field, and about the correlation functions of the field that characterizes the state.

We first need to understand what do photo detectors (photo multipliers) really measure, in order to develop the theory of photo-detection and photo-counting, and to study the properties of various states of the quantum EM field from that side. To this aim we will develop a caricature model of a photo-detector – a model of a single electron atom that undergoes ionization under the influence of the field to be measured. Photocurrent due to ionization is then amplified in the photo-multiplier, but that will not be of interest of us. We will try to determine which characteristics of the EM field determine the photo-detector response.

### 5.1. Simple model of a photo-detector



We consider a one electron atom with ground state of energy  $-E_0$ , and states in the continuum of energies  $E$ . The electron interacts with incident electric field.

This is the simplest possible model that may serve our purposes. It has one bound state, and a single continuum. Coupling of  $|0\rangle$  to continuum will cause ‘ionization’, i. e. will bring the electron from groundstate  $|0\rangle$  to one of the states  $|E\rangle$ .

The Hamiltonian in the interaction picture with respect to the

field is

$$\begin{aligned} \mathcal{H} = & -E_0 |0\rangle \langle 0| + \int_0^\infty dE |E\rangle \langle E| \\ & + \hbar\kappa \int_0^\infty dE (|0\rangle \langle E| + |E\rangle \langle 0|) \left( \mathcal{E}^{(-)}(\mathbf{r}, t) + \mathcal{E}^{(+)}(\mathbf{r}, t) \right) \end{aligned} \quad (5.1)$$

The interaction part describes transitions  $|0\rangle \leftrightarrow |E\rangle$  caused by interaction with the electric field (ionization!) at  $\mathbf{r}$ .  $\kappa$  is a coupling constant. The sum  $\mathcal{E}^{(-)}(\mathbf{r}, t) + \mathcal{E}^{(+)}(\mathbf{r}, t)$  is the total electric field. We have divided it into a positive and a negative frequency parts, oscillating as  $e^{-i\omega t}$  and  $e^{+i\omega t}$ , respectively. The frequencies relevant for the process of ionization will be close to the laser frequency,  $\omega \approx \omega_L$

We solve the Schrödinger Equation

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle \quad (5.2)$$

The solution has the form

$$|\psi(t)\rangle = \alpha(t) |0\rangle + \int dE \beta(E, t) |E\rangle \quad (5.3)$$

The Schrödinger equation for the amplitudes reads

$$\dot{\alpha}(t) = \frac{i}{\hbar} E_0 \alpha(t) - i\kappa \left( \mathcal{E}^{(-)}(\mathbf{r}, t) + \mathcal{E}^{(+)}(\mathbf{r}, t) \right) \int_0^\infty \beta(E, t) dE, \quad (5.4)$$

$$\dot{\beta}(E, t) = \frac{i}{\hbar} E \beta(E, t) - i\kappa \left( \mathcal{E}^{(-)}(\mathbf{r}, t) + \mathcal{E}^{(+)}(\mathbf{r}, t) \right) \alpha(t) \quad (5.5)$$

Formal solution:

first we go to the interaction picture

$$\alpha = e^{i\frac{E_0}{\hbar}t} \tilde{\alpha}, \quad (5.6)$$

$$\beta = e^{i\frac{E}{\hbar}t} \tilde{\beta}. \quad (5.7)$$

Thus

$$\dot{\tilde{\alpha}} = -i\kappa \int_0^\infty e^{-\frac{i}{\hbar}(E+E_0)t} \left( \mathcal{E}^{(-)}(\mathbf{r}, t) + \mathcal{E}^{(+)}(\mathbf{r}, t) \right) \tilde{\beta}(E, t) dE \quad (5.8)$$

$$\dot{\tilde{\beta}} = -i\kappa e^{\frac{i}{\hbar}(E+E_0)t} \left( \mathcal{E}^{(-)}(\mathbf{r}, t) + \mathcal{E}^{(+)}(\mathbf{r}, t) \right) \tilde{\alpha}(E, t) \quad (5.9)$$

We know that

$$\mathcal{E}^{(-)}(\mathbf{r}, t) \sim e^{+i\omega_L t} \quad (5.10)$$

$$\mathcal{E}^{(+)}(\mathbf{r}, t) \sim e^{-i\omega_L t} \quad (5.11)$$

We have  $\omega_L \sim 10^{15}$  Hz,  $(E + E_0)/\hbar \sim 10^{15}$  Hz. We can safely forget rapidly oscillating terms in the equation above! So

$$\tilde{\beta}(t) = -i\kappa \int_0^t e^{\frac{i}{\hbar}(E+E_0)t'} \mathcal{E}^{(+)}(\mathbf{r}, t') \tilde{\alpha}(t') dt' \quad (5.12)$$

$$\dot{\tilde{\alpha}}(t) = -i\kappa \int_0^\infty dE e^{-\frac{i}{\hbar}(E+E_0)t} \mathcal{E}^{(-)}(\mathbf{r}, t) \tilde{\beta}(E, t) \quad (5.13)$$

Putting the equations together, we get

$$\dot{\tilde{\alpha}}(t) = -\kappa^2 \int_0^\infty dE \int_0^t dt' e^{-\frac{i}{\hbar}(E+E_0)(t-t')} \mathcal{E}^{(-)}(\mathbf{r}, t) \mathcal{E}^{(+)}(\mathbf{r}, t') \tilde{\alpha}(t) \quad (5.14)$$

But we can approximate

$$\int_0^\infty dE e^{-\frac{i}{\hbar}E(t-t')} \approx \int_{-\infty}^{+\infty} dE e^{-\frac{i}{\hbar}E(t-t')} \approx 2\pi\hbar\delta(t-t') \quad (5.15)$$

so that with the delta function we find

$$\dot{\tilde{\alpha}}(t) = -\pi\hbar\kappa^2 \mathcal{E}^{(-)}(\mathbf{r}, t) \mathcal{E}^{(+)}(\mathbf{r}, t) \tilde{\alpha}(t) = -\Gamma(t) \tilde{\alpha}(t) \quad (5.16)$$

where  $\Gamma(t)$  is the instantaneous ionization rate. We take here

$$\int_0^t \delta(t-t') dt' = \frac{1}{2} \quad \text{“Fermi’s Golden Rule”} \quad (5.17)$$

The ionization rate (instantaneous) is proportional to the intensity

$$I_i(\mathbf{r}, t) = \mathcal{E}^{(-)}(\mathbf{r}, t) \mathcal{E}^{(+)}(\mathbf{r}, t) \quad (5.18)$$

Mean ionization rate is then related to the mean intensity

$$\boxed{I(\mathbf{r}, t) = \langle \mathcal{E}^{(-)}(\mathbf{r}, t) \mathcal{E}^{(+)}(\mathbf{r}, t) \rangle} \quad (5.19)$$

where  $\langle \dots \rangle$  is the quantum average in quantum electrodynamics (i. e. when fields are quantized), or average over stochastic character of classical field, if we use classical model of  $\mathcal{E}(\mathbf{r}, t)$ .

Note that the ionization rate determines counting rate of the detector!

$$I(\mathbf{r}, t) \propto \left\langle \mathcal{E}^{(-)}(\mathbf{r}, t) \mathcal{E}^{(+)}(\mathbf{r}, t) \right\rangle \quad (5.20)$$

Since Eq.(2.53)

$$\mathcal{E}^{(-)} \propto \hat{a}^\dagger \quad (\text{photon creation operator}) \quad (5.21)$$

$$\mathcal{E}^{(+)} \propto \hat{a} \quad (\text{photon annihilation operator}) \quad (5.22)$$

$I(\mathbf{r}, t) \propto$  normally ordered correlation function of  $\mathcal{E}^{(-)}$ ,  $\mathcal{E}^{(+)}$ . Normal ordering means  $\hat{a}^\dagger$ 's to the left of  $\hat{a}$ 's.

Normal ordering is the consequence of energy conservation (neglecting of the rapidly oscillating terms in the model). Photo-detectors measure normally ordered correlation functions!

## 5.2. Correlation functions

Correlation functions play a very important role in physics, since they characterize spatially and temporally physical processes. Typically, if we deal with some physical (fluctuating) field  $\varphi(x, t)$ , we consider various averages of the type  $\langle \varphi(x, t) \rangle$ ,  $\langle \varphi(x_1, t_1) \varphi(x_2, t_2) \rangle$ . When the field  $\varphi(x_1, t_1)$  is quantum one has to be careful with orders of terms in averages since  $\varphi(x_1, t_1)$  and  $\varphi(x_2, t_2)$  do not necessarily commute. Appropriately ordered averages are then called correlation functions of certain kinds.

We define two point correlation function

$$G^{(1)}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2) \equiv \left\langle \mathcal{E}^{(-)}(\mathbf{r}_1, t_1) \mathcal{E}^{(+)}(\mathbf{r}_2, t_2) \right\rangle \quad (5.23)$$

Counting rate thus is

$$I(\mathbf{r}, t) \propto G^{(1)}(\mathbf{r}, t, \mathbf{r}, t) \quad (5.24)$$

In general, standard photo-detection allows to measure more general, higher order correlation functions

$$G^{(n)}(\mathbf{r}_1, t_1, \dots, \mathbf{r}_n, t_n, \mathbf{r}_{n+1}, t_{n+1}, \dots, \mathbf{r}_{2n}, t_{2n}) \propto \left\langle \mathcal{E}^{(-)}(\mathbf{r}_1, t_1) \dots \mathcal{E}^{(-)}(\mathbf{r}_n, t_n) \mathcal{E}^{(+)}(\mathbf{r}_{n+1}, t_{n+1}) \dots \mathcal{E}^{(+)}(\mathbf{r}_{2n}, t_{2n}) \right\rangle \quad (5.25)$$

where

$$t_1 \leq t_2 \leq \dots \leq t_n \quad (5.26)$$

$$t_{2n} \leq t_{2n-1} \leq \dots \leq t_{n+1} \quad (5.27)$$

These correlation functions are mean values of normally ordered and apex-time ordered products of  $\mathcal{E}^{(-)}$ ,  $\mathcal{E}^{(+)}$ . Apex-time ordering means that creation operators are ordered from earliest to the latest times from left to right, annihilation operators conversely, from the latest to the earliest. The correlation functions Eq.(5.25) are related to counting rates of  $n$  photons at various times.

Coincidence rate of detecting a photon at  $(\mathbf{r}_1, t_1)$ , then another at  $(\mathbf{r}_2, t_2)$ , etc. (for  $t_1 \leq t_2 \leq \dots \leq t_n$ ) is given by

$$\begin{aligned} W^{(n)}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2, \dots, \mathbf{r}_n, t_n) &\propto G^{(n)}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2, \dots, \mathbf{r}_n, t_n) \\ &= \left\langle \mathcal{E}^{(-)}(\mathbf{r}_1, t_1) \cdots \mathcal{E}^{(-)}(\mathbf{r}_n, t_n) \mathcal{E}^{(+)}(\mathbf{r}_n, t_n) \cdots \mathcal{E}^{(+)}(\mathbf{r}_1, t_1) \right\rangle \end{aligned} \quad (5.28)$$

This is a direct generalization of the result of a photo-detector to two photo-detectors, or in general to  $n$  photo-detectors .

Physically:

The simple photo-counting rate is proportional to normally ordered mean intensity. The coincidence rate is proportional to the normally and apex ordered product of intensities!

### 5.2.1. Properties of the correlation functions

We are going to use in this section the fact that  $\langle \hat{A}^\dagger \hat{A} \rangle \geq 0$  for any operator  $\hat{A}$ , where the mean value is either

$$\langle \psi | \hat{A}^\dagger \hat{A} | \psi \rangle, \quad \text{or} \quad \text{Tr}(\hat{A}^\dagger \hat{A} \rho),$$

where  $\rho$  is the density matrix.

Obviously we have

$$G^{(n)}(x_1, \dots, x_n; x_n, \dots, x_1) \geq 0, \quad (5.29)$$

where

$$x_k = (\mathbf{r}_k, t_k).$$

Similarly, taking

$$\hat{A} = \sum_i \lambda_i \mathcal{E}^{(+)}(x_i), \quad (5.30)$$

$$\hat{A}^\dagger = \sum_j \lambda_j^* \mathcal{E}^{(-)}(x_j), \quad (5.31)$$

we get

$$\sum_{i,j} \lambda_j^* \lambda_i \langle \mathcal{E}^{(-)}(x_j) \mathcal{E}^{(+)}(x_i) \rangle \geq 0, \quad (5.32)$$

so that

$$\sum_{i,j} \lambda_j^* \lambda_i G^{(1)}(x_j, x_i) \geq 0. \quad (5.33)$$

$G^{(1)}(x_j, x_i)$  regarded as a matrix must be positively defined. So have to be all of its minors. In particular, Cauchy-Schwartz inequality holds

$$\det \left( G^{(1)}(x_i, x_j) \right) \geq 0, \quad (5.34)$$

so for  $i, j = 1, 2$  we have

$$G^{(1)}(x_1, x_1)G^{(1)}(x_2, x_2) \geq \left| G^{(1)}(x_1, x_2) \right|^2. \quad (5.35)$$

Here we have also used

$$G^{(1)}(x_1, x_2) = \left( G^{(1)}(x_2, x_1) \right)^* \quad (5.36)$$

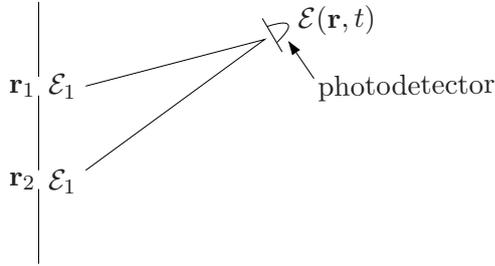
More generally

$$\begin{aligned} & G^{(n)}(x_1, \dots, x_n; x_n, \dots, x_1) G^{(n)}(x_{n+1}, \dots, x_{2n}; x_{2n}, \dots, x_{n+1}) \\ & \geq \left| G^{(n)}(x_1, \dots, x_n; x_{n+1}, \dots, x_{2n}) \right|^2. \end{aligned} \quad (5.37)$$

Apart from the mathematical sense mean have all these properties an important physical sense. For instance, the inequality Eq.(5.35) means that the interference term  $|G^{(1)}(x_1, x_2)|$  cannot be larger than the geometric mean of the intensities.

### 5.3. Correlations and optical coherence

We consider a double slit interference



We have obviously

$$\mathcal{E}^{(+)}(\mathbf{r}, t) = \mathcal{E}_1^{(+)}(\mathbf{r}, t) + \mathcal{E}_2^{(+)}(\mathbf{r}, t) \quad (5.38)$$

The fields propagate from the slits to  $\mathbf{r}$  as spherical waves, so that

$$\mathcal{E}_i^{(+)}(\mathbf{r}, t) \propto \underbrace{\mathcal{E}_i^{(+)}(\mathbf{r}_i, t - \frac{s_i}{c})}_{1} \frac{1}{s_i} e^{i(k-\omega/c)s_i}, \quad (5.39)$$

where optical path is

$$s_i = |\mathbf{r}_i - \mathbf{r}| \approx R. \quad (5.40)$$

(We assume that slits are close one to another!)

Since  $k = \omega/c$ , we get

$$\mathcal{E}^{(+)}(\mathbf{r}, t) \propto \frac{1}{R} \left( \underbrace{\mathcal{E}_1^{(+)}(\mathbf{r}_1, t - \frac{s_1}{c})}_2 + \underbrace{\mathcal{E}_2^{(+)}(\mathbf{r}_2, t - \frac{s_2}{c})}_2 \right). \quad (5.41)$$

Counting rate is  $(x_i = (\mathbf{r}_i, t - \frac{s_i}{c}))$

$$I(\mathbf{r}, t) \propto \left[ G^{(1)}(x_1, x_1) + G^{(1)}(x_2, x_2) + 2\text{Re} G^{(1)}(x_1, x_2) \right] \quad (5.42)$$

*Note:*  $G^{(1)}(x_1, x_2)$  determines interference pattern!

<sup>1</sup>retarded time

<sup>2</sup>here we keep  $s_1, s_2$  to account for phase difference!

## 5.4. Photon correlations measurements

### 5.4.1. Classical measurements

*One should note:*  $G^{(1)}(x_1, x_2)$  may be the same for coherent and Fock states (for instance for 1-photon states). To get real insight into quantum statistical character of em-fields we need more!

What one does is two-time coincidence measurement. We measure coincidence that a detector at  $\mathbf{r}$  clicks at time  $t$  and  $t + \tau$ . Since we have only one detector at  $\mathbf{r}$ , we skip  $\mathbf{r}$  in the following.

We define

$$G^{(2)}(\tau) \equiv \left\langle \mathcal{E}^{(-)}(t) \mathcal{E}^{(-)}(t + \tau) \mathcal{E}^{(+)}(t + \tau) \mathcal{E}^{(+)}(t) \right\rangle \quad (5.43)$$

By means of intensities

$$G^{(2)}(\tau) = \underbrace{\langle : I(t) I(t + \tau) : \rangle}_3 \quad (5.44)$$

Often one uses the normalized correlation

$$g^{(2)}(\tau) = \frac{G^{(2)}(\tau)}{|G^{(1)}(0)|^2}. \quad (5.45)$$

Let us consider various examples of classical fields:

#### Stationary Fields

We assume here that fields are stationary, i. e.

$$G^{(1)}(t, t) = G^{(1)}(t', t') = G^{(1)}(0, 0). \quad (5.46)$$

In this case  $G^{(2)}$  defined above depends on  $\tau$  only and not on  $t$ !

We call the field to possess a second order coherence, if

$$G^{(2)}(\tau) = \mathcal{E}^{(-)}(t) \mathcal{E}^{(-)}(t + \tau) \mathcal{E}^{(+)}(t + \tau) \mathcal{E}^{(+)}(t), \quad (5.47)$$

i. e.  $G^{(2)}(\tau)$  is a product of complex numbers. In the stationary case, we then must have

$$\mathcal{E}^{(+)}(t) \propto e^{-i\omega t}, \quad (5.48)$$

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<sup>3</sup>normally and apex ordered correlation of intensities

so that

$$G^{(2)}(\tau) = (G^{(1)}(0))^2. \quad (5.49)$$

In that situation

$$\boxed{g^{(2)}(\tau) = 1}. \quad (5.50)$$

For classical fields, which do not fluctuate, and fields with  $\mathcal{E}^{(+)}(t) = \mathcal{E} e^{-i\omega t}$  this is the case.

### Fluctuating Fields

However, for classical fields, which do fluctuate, for instance fields having fluctuating amplitude  $\mathcal{E}$ , we have

$$G^{(2)}(\tau) = \int P(\mathcal{E}) |\mathcal{E}|^4 d\mathcal{E}, \quad (5.51)$$

where  $P(\mathcal{E})$  is the probability distribution of the amplitude  $\mathcal{E}$ .

Since

$$G^{(1)}(0) = \int P(\mathcal{E}) |\mathcal{E}|^2 d\mathcal{E}, \quad (5.52)$$

we obtain

$$\begin{aligned} g^{(2)}(\tau) &= \frac{\int P(\mathcal{E}) |\mathcal{E}|^4 d\mathcal{E}}{(\int P(\mathcal{E}) |\mathcal{E}|^2 d\mathcal{E})^2} \\ &= \frac{\int P(\mathcal{E}) (|\mathcal{E}|^2 - \langle |\mathcal{E}|^2 \rangle)^2 d\mathcal{E}}{(\int P(\mathcal{E}) |\mathcal{E}|^2 d\mathcal{E})^2} + 1 \geq 1 \end{aligned} \quad (5.53)$$

Thus, for classical fluctuations

$$\boxed{g^{(2)}(\tau) \geq 1}. \quad (5.54)$$

### Gaussian fields

Let us consider another example of classical fluctuating fields, Gaussian fields. These are fields, whose probability distributions are Gaussian. Technically, if we have  $\mathcal{E}^{(-)}(t)$ ,  $\mathcal{E}^{(-)}(t + \tau)$ ,  $\mathcal{E}^{(+)}(t)$  and  $\mathcal{E}^{(+)}(t + \tau)$ , they should be regarded as complex random variables with probability distribution  $P$ ,

$$P \propto \exp\left(\text{quadratic form of } \mathcal{E}^{(-)}(t), \mathcal{E}^{(-)}(t + \tau), \mathcal{E}^{(+)}(t), \mathcal{E}^{(+)}(t + \tau)\right). \quad (5.55)$$

Practically, that means that higher order correlations factorize into a sum of products of two-point correlations for all possible pairings (this is called Wick's theorem in Quantum Field Theory). We have then

$$\begin{aligned}
& \langle \mathcal{E}^{(-)}(t)\mathcal{E}^{(-)}(t+\tau)\mathcal{E}^{(+)}(t+\tau)\mathcal{E}^{(+)}(t) \rangle \\
&= \langle \mathcal{E}^{(-)}(t)\mathcal{E}^{(-)}(t+\tau) \rangle \langle \mathcal{E}^{(+)}(t+\tau)\mathcal{E}^{(+)}(t) \rangle \\
&+ \langle \mathcal{E}^{(-)}(t)\mathcal{E}^{(+)}(t+\tau) \rangle \langle \mathcal{E}^{(-)}(t+\tau)\mathcal{E}^{(+)}(t) \rangle \\
&+ \langle \mathcal{E}^{(-)}(t)\mathcal{E}^{(+)}(t) \rangle \langle \mathcal{E}^{(-)}(t+\tau)\mathcal{E}^{(+)}(t+\tau) \rangle \\
&= G^{(1)}(0)^2 + |G^{(1)}(\tau)|^2.
\end{aligned} \tag{5.56}$$

We obtain thus

$$\boxed{g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2}, \tag{5.57}$$

where

$$g^{(1)}(\tau) = \frac{G^{(1)}(\tau)}{G^{(1)}(0)} \tag{5.58}$$

When the Gaussian field fluctuates, clearly  $\langle \mathcal{E}^{(\pm)}(t) \rangle = 0$ . This is the result of the Gaussian distribution, which is invariant with respect to the change of sign of  $\mathcal{E}^{(\pm)}(t)$ ,  $\mathcal{E}^{(\pm)}(t+\tau)$ .

In this case we always have that

$$G^{(1)}(\tau) = \langle \mathcal{E}^{(-)}\mathcal{E}^{(+)}(t+\tau) \rangle \rightarrow 0, \tag{5.59}$$

as  $\tau \rightarrow \infty$ , or in other words, the field becomes uncorrelated for large  $\tau$ 's.

Typically

$$g^{(1)}(\tau) = e^{-\gamma\tau} \quad (\text{for fields with Lorentzian power spectrum}), \tag{5.60}$$

$$g^{(1)}(\tau) = e^{-\gamma^2\tau^2} \quad (\text{for fields with Gaussian power spectrum}). \tag{5.61}$$

In any case, for such fields we have that  $g^{(1)}(\tau) \leq g^{(1)}(0)$ , and in fact that  $g^{(1)}(\tau)$  is a monotonically decreasing function of  $\tau$ .

That implies that for classical fluctuating fields we have always

$$\boxed{1 \leq g^{(2)}(0) \geq g^{(2)}(\tau)}. \tag{5.62}$$

The relation above, measured in the famous Hanbury-Brown and Twiss experiment<sup>4</sup>, expresses the fact that classical fields exhibit *photon bunching*. In other words, photons in classical fields, or better to say in classically interpretable or classically described states, like to bunch, i. e. come together. Of course, in practice, what we observe is, that the detector is more likely to click twice for short time lag than for larger ones.

<sup>4</sup>R. Hanbury-Brown and R.Q. Twiss, Nature **127** 27 (1956)

## 5.4.2. The quantum case

Classical stochastic models might work very well sometimes, but in reality the EM field is quantized. For single mode field

$$\hat{\mathcal{E}}^{(-)}(t) \propto \hat{a}^\dagger \quad (5.63)$$

$$\hat{\mathcal{E}}^{(+)}(t) \propto \hat{a} \quad (5.64)$$

where  $\hat{a}$ ,  $\hat{a}^\dagger$  are the photon annihilation and creation operators.

In this case of single mode field

$$g^{(2)}(0) = \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2} = 1 + \frac{(\Delta n)^2 - \bar{n}}{\bar{n}^2}, \quad (5.65)$$

where mean value  $\bar{n} = \langle \hat{a}^\dagger \hat{a} \rangle$ , variance  $(\Delta n)^2 = \langle (n - \bar{n})^2 \rangle = \langle (\hat{a}^\dagger \hat{a} - \bar{n})^2 \rangle$  we have:

For the Fock states  $|n\rangle \langle n|$

$$g^{(2)}(0) = 1 - \frac{1}{n},$$

$$(\Delta n)^2 = 0,$$

$$\bar{n} = n.$$

For the coherent states

$$g^{(2)}(0) = 1,$$

$$(\Delta n)^2 = \bar{n}.$$

Since typically  $g^{(2)}(\tau) \rightarrow 1$  as  $\tau \rightarrow \infty$ , then we have for the Fock states:

$$g^{(2)}(\tau) \geq g^{(2)}(0) \quad (\textit{photon antibunching}). \quad (5.66)$$

The situation when

$$g^{(2)}(\tau) \leq g^{(2)}(0) \quad (5.67)$$

is termed photon bunching.

## 5.5. Photon counting

### 5.5.1. Classical case

There are two causes of randomness of clicks:

1. detectors are quantized and work according to quantum probability laws,
2. the fields are quantized and fluctuate.

We first consider the probabilistic character of detectors plus classical fields (in general being also random). We assume, that the probability of a count (click) in the interval  $[t, t + \tau]$  is

$$\Delta p(t) = \alpha I(t) dt \quad (5.68)$$

(with:  $\alpha$  = detector efficiency,  $I(t)$  = intensity). Then the probability of zero clicks in  $[0, T]$  is

$$\begin{aligned} P_0(t, +T, t) &= \prod_{t' \in [t, t+T]} \underbrace{(1 - \Delta p(t'))}_5 \approx \prod_{t'} \exp(-\Delta p(t')) \\ &= \exp\left(-\alpha \int_t^{t+T} I(t') dt'\right). \end{aligned} \quad (5.69)$$

Analogously, the probability of one click is

$$\begin{aligned} P_1(t, +T, t) &= \sum_{t''} \Delta p(t'') \prod_{t' \neq t''} (1 - \Delta p(t')) \\ &\approx \alpha \int_t^{t+T} I(t') dt' \cdot \exp\left(-\alpha \int_t^{t+T} I(t') dt'\right) \end{aligned} \quad (5.70)$$

Generally, the probability of  $n$  clicks is

$$P_n(t, +T, t) = \frac{1}{n!} [\alpha T \bar{I}(t, T)]^n \exp[-\alpha T \bar{I}(t, T)], \quad (5.71)$$

where

$$\bar{I}(t, T) = \frac{1}{T} \int_t^{t+T} I(t') dt'. \quad (5.72)$$

Let us again consider some examples:

---

<sup>5</sup>probability of no clicks

## Stationary fluctuating fields

For stationary fluctuating fields we get

$$P_n(T) = \langle P_n(t, T) \rangle \stackrel{6}{=} \left\langle \frac{(\alpha \bar{I}(T)T)^n}{n!} \exp(-\alpha \bar{I}(T)T) \right\rangle \quad (5.73)$$

Take non-fluctuating field of constant intensity,

$$\bar{I}(t) = I. \quad (5.74)$$

we obtain Poissonian photon counting statistics,

$$P_n(T) = \frac{\bar{n}^n}{n!} \exp(-\bar{n}), \quad (5.75)$$

where

$$\boxed{\bar{n} = \alpha IT} \quad (5.76)$$

means the number of photons detected in interval  $T$ .

We have

$$\overline{n^2} - \bar{n}^2 = \bar{n} \quad (5.77)$$

This is the so called *shot noise*. It shows that the number of count's in classical probabilistic theory fluctuates according to Poisson distribution, even if the EM field is deterministic. Shot noise is the consequence of probabilistic character of detectors.

## Random intensity

$\bar{I}(T) \sim$  random variable, distributed according to  $P(\bar{I}(T))$ ;

$$\bar{n} = \langle \alpha \bar{I}(T)T \rangle = \alpha T \int \bar{I} P(\bar{I}) d\bar{I}, \quad (5.78)$$

$$\begin{aligned} \bar{n}^2 &= \langle \alpha^2 \bar{I}(T)^2 T^2 + \alpha \bar{I}(T)T \rangle \\ &= \alpha^2 T^2 \langle \bar{I}^2 \rangle + \alpha T \langle \bar{I} \rangle. \end{aligned} \quad (5.79)$$

One gets

$$\overline{n^2} - \bar{n}^2 = \bar{n} + \alpha^2 T^2 \left[ \langle \bar{I}^2 \rangle - \langle \bar{I} \rangle^2 \right] > \bar{n}. \quad (5.80)$$

For classical fluctuating fields

$$\overline{n^2} - \bar{n}^2 \geq \bar{n} \quad (5.81)$$

i. e. one obtains super Poissonian photon counting statistics.

---

<sup>6</sup>does not depend on  $t$

Thermal fields

$$P(\bar{I}) = \frac{1}{I_0} \exp\left(-\frac{I}{I_0}\right) \quad (5.82)$$

In that case

$$\langle \bar{I}^n \rangle = n! I_0^n, \quad (5.83)$$

$$\bar{n} = \alpha T I_0, \quad (5.84)$$

$$\overline{n^2} - \bar{n}^2 = \bar{n}^2 + \bar{n}, \quad (5.85)$$

and

$$\begin{aligned} P_n(T) &= \frac{(\alpha T)^n}{I_0 n!} \int_0^\infty \bar{I}^n \exp\left(-\bar{I} \left(\alpha T + \frac{1}{I_0}\right)\right) d\bar{I} \\ &= \frac{1}{1 + \bar{n}} \left(\frac{\bar{n}}{1 + \bar{n}}\right)^n. \end{aligned} \quad (5.86)$$

We get a geometric, exponential photon counting distribution.

### 5.5.2. Quantum case

The photon counting formula in the full quantum case is the same as shown by Glauber, but we have to account for normal and apex ordering of operators

$$P_n(T) = \left\langle : \frac{[\alpha T \bar{I}]^n}{n!} \exp(-\alpha T \bar{I}(T)) : \right\rangle, \quad (5.87)$$

with

$$\langle : X : \rangle = \text{Tr}(\rho : X :), \quad (5.88)$$

and  $::$  denote normal and apex ordering.

In general

$$\bar{I}(T) = \frac{1}{T} \int_t^{t+T} \mathcal{E}^{(-)}(\mathbf{r}, t') \mathcal{E}^{(+)}(\mathbf{r}, t') dt'. \quad (5.89)$$

For a single mode  $\mathcal{E}^{(+)} \propto \hat{a}$ ,  $\mathcal{E}^{(-)} \propto \hat{a}^\dagger$ , so that

$$P_n(T) = \left\langle : \frac{(\mu(T) \hat{a}^\dagger \hat{a})^n}{n!} \exp(-\mu(T) \hat{a}^\dagger \hat{a}) : \right\rangle, \quad (5.90)$$

where  $\mu(T)$  is the probability of detecting one photon in interval  $[t, t + T]$ , equal to the quantum efficiency of the detector.

For

Open systems (typically):  $\mu(T) = \lambda T$ ,

Closed systems (typically):  $\mu(t) = (1 - e^{-\lambda T})$ .

Let us define

$P_n = \langle n | \varrho | n \rangle$  – photon number distribution (no  $T$  dependence),

$P_m(T)$  – photon counting distribution.

There is a relation between the two:

$$\begin{aligned} P_m(T) &= \sum_n P_n \frac{[\mu(T)]^m}{m!} \langle n | \sum_{l=0}^{\infty} (-1)^l \frac{\mu(T)^l}{l!} (\hat{a}^\dagger)^{m+l} \hat{a}^{m+l} | n \rangle \\ &= \sum_{n=m}^{\infty} P_n \sum_{l=0}^{n-m} (-1)^l \frac{\mu(T)^l}{l!} \frac{n!}{(n-m-l)!}, \end{aligned} \quad (5.91)$$

$$\Rightarrow \boxed{P_m(T) = \sum_{n=m}^{\infty} P_n \underbrace{\binom{n}{m} \mu(T)^m (1 - \mu(T))^{n-m}}_{\text{Bernoulli distr.}}} \quad (5.92)$$

(with:  $P_n$  = probability that the state ‘has’  $n$  photons)

*Note:*  $P_m(T) = P_m$  if  $\mu(T) = 1$ , that is if quantum efficiency  $\mu(T) = 1$ . Normally, however,  $\mu(t) \ll 1$  in experiments.

Let  $\bar{n}$  denote the mean number of photons in the state, while  $\bar{m}$  is the number of detected photons in the interval  $T$ .

Examples:

- Coherent states

$$P_n = \frac{\bar{n}^n}{n!} \exp(-\bar{n}), \quad (5.93)$$

$$P_m(T) = \frac{(\mu(T)\bar{n})^m}{m!} \exp(-\mu(T)\bar{n}). \quad (5.94)$$

Both distributions are Poissonian!  $P_m(T)$  has the same form as in the classical case of constant  $I$ !

- Chaotic states (thermal light)

$$P_n = \frac{1}{1 + \bar{n}} \left( \frac{\bar{n}}{1 + \bar{n}} \right)^n, \quad (5.95)$$

$$P_m(T) = \frac{1}{1 + \bar{m}} \left( \frac{\bar{m}}{1 + \bar{m}} \right)^m, \quad (5.96)$$

where

$$\boxed{\bar{m} = \mu(t)\bar{n}}. \quad (5.97)$$

Again,  $P_m(T)$  is the same as in classical model of thermal light.

- But, for Fock states  $|n_0\rangle \langle n_0|$

$$P_n = \delta_{nn_0}, \quad (5.98)$$

$$P_m(T) = \begin{cases} 0 & \text{for } m \geq n_0 \\ \binom{n_0}{m} \mu(T)^m (1 - \mu(T))^{n_0 - m} & \text{otherwise} \end{cases} \quad (5.99)$$

We have:

$$\begin{aligned} \bar{n} &= n_0, \\ \bar{m} &= \mu(T)n_0 = \mu(T)\bar{n}, \\ \overline{m^2} &= \mu(T)n_0 + \mu(T)^2 n_0 (n_0 - 1), \\ \overline{m^2} - \bar{m}^2 &= n_0 \mu(T) (1 - \mu(T)), \end{aligned} \quad (5.100)$$

$$\begin{aligned} g^{(2)}(0) &= \sum \frac{n(n-1)P_n}{\bar{n}^2} = 1 - \frac{1}{\bar{n}}, \\ &= \sum \frac{m(m-1)P_m(T)}{\bar{m}^2}. \end{aligned} \quad (5.101)$$

The last equality,  $g^{(2)}(0)$  can be calculated from  $P_n$  or  $P_m(T)$ , the result is the same.

résumé

**Model of a photodetector**

Mean intensity

$$I(\mathbf{r}, t) = \langle \mathcal{E}^{(-)}(\mathbf{r}, t) \mathcal{E}^{(+)}(\mathbf{r}, t) \rangle$$

**Correlation functions**

Two points correlation function

$$G^{(1)}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2) \equiv \langle \mathcal{E}^{(-)}(\mathbf{r}_1, t_1) \mathcal{E}^{(+)}(\mathbf{r}_2, t_2) \rangle$$

Four points correlation function

$$G^{(2)}(\tau) \equiv \langle \mathcal{E}^{(-)}(t) \mathcal{E}^{(-)}(t + \tau) \mathcal{E}^{(+)}(t + \tau) \mathcal{E}^{(+)}(t) \rangle = \langle : I(t) I(t + \tau) : \rangle$$

normalized correlation

$$g^{(1)}(\tau) = \frac{G^{(1)}(\tau)}{G^{(1)}(0)}$$

$$g^{(2)}(\tau) = \frac{G^{(2)}(\tau)}{|G^{(1)}(0)|^2}$$

**Optical Coherence**

Double slit interference

$$I(\mathbf{r}, t) \propto \left[ G^{(1)}(x_1, x_1) + G^{(1)}(x_2, x_2) + 2\text{Re} G^{(1)}(x_1, x_2) \right]$$

**Photon correlation measurements**

classical case

Stationary fields  $g^{(2)}(\tau) = 1$

Gaussian Fields  $g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2$

Fluctuating fields  $1 \leq g^{(2)}(0) \geq g^{(2)}(\tau)$

quantum case

mean value  $\bar{n} = \langle \hat{a}^\dagger \hat{a} \rangle$

variance  $(\Delta n)^2 = \langle (n - \bar{n})^2 \rangle = \langle (\hat{a}^\dagger \hat{a} - \bar{n})^2 \rangle$

normalized correlation  $g^{(2)}(0) = \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2} = 1 + \frac{(\Delta n)^2 - \bar{n}}{\bar{n}^2}$

normalized correlation for different states

fock  $g^{(2)}(0) = 1 - \frac{1}{n}$

coherent  $g^{(2)}(0) = 1$

thermal  $g^{(2)}(0) = 2$

**Photon counting**

classical case

Probability of  $n$  clicks

$$P_n(t, +T, t) = \frac{1}{n!} [\alpha T \bar{I}(t, T)]^n \exp[-\alpha T \bar{I}(t, T)],$$

quantum case

Photon number distribution

$$P_n(T) = \left\langle : \frac{[\alpha T \bar{I}]^n}{n!} \exp(-\alpha T \bar{I}(T)) : \right\rangle$$

Photon counting distribution

$$P_m(T) = \sum_{n=m}^{\infty} P_n \binom{n}{m} \mu(T)^m (1 - \mu(T))^{n-m}$$

mean number of counted (detected) photons

$$\bar{m} = \mu(t) \bar{n}$$

**Properties for different states**

Photon counting distribution

$$\text{fock} \quad P_m(T) = \binom{n_0}{m} \mu(T)^m (1 - \mu(T))^{n_0 - m}$$

$$\text{coherent} \quad P_m(T) = \frac{(\mu(t) \bar{n})^m}{m!} e^{-\mu(t) \bar{n}}$$

$$\text{thermal} \quad P_m(T) = \left( \frac{\mu(T) \bar{n}}{1 + \mu(T) \bar{n}} \right)^m \frac{1}{1 + \mu(T) \bar{n}}$$

mean number of detected photons

$$\text{fock} \quad \bar{m}^2 = \bar{n} \mu(t) + \bar{n} (\bar{n} - 1) \mu(T)^2$$

$$\text{coherent} \quad \bar{m}^2 = \bar{m}^2 + \bar{m}$$

$$\text{thermal} \quad \bar{m}^2 = 2\bar{m}^2 + \bar{m}$$

variance of detected photons

$$\text{fock} \quad (\Delta m)^2 = \bar{n} \mu(T) (1 - \mu(T))$$

$$\text{coherent} \quad (\Delta m)^2 = \bar{m}$$

$$\text{thermal} \quad (\Delta m)^2 = \bar{m}^2 + \bar{m}$$

## List of variables

$ 0\rangle$	ground state
$ E\rangle$	excited state with energy $E$
$\kappa$	coupling constant in the photo-detector
$\mathcal{E}^{(+)}$	positive frequency part of the EM field
$\mathcal{E}^{(-)}$	negative frequency part of the EM field
$\alpha(t)$	amplitude of the ground state
$\beta(E, t)$	amplitude of the excited state
$I$	Ionization rate / counting rate
$G^{(1)}$	correlation function of 1-st order (two points)
$G^{(2)}$	correlation function of 2-nd order (four points)
$g^{(2)}$	normalized correlation
$(\Delta n)^2$	variance of the photon number
$I$	Intensity
$\alpha$	detector efficiency
$P_n$	probability of $n$ clicks
$\mu(T)$	probability of detecting one photon in $[t, t + T]$ – quantum efficiency
$P_n$	photon number distribution
$P_m$	photon counting distribution
$m$	number of detected photons
$\bar{m}$	mean number of detected photons
$(\Delta m)^2$	variance of the detected photons



# APPENDIX A

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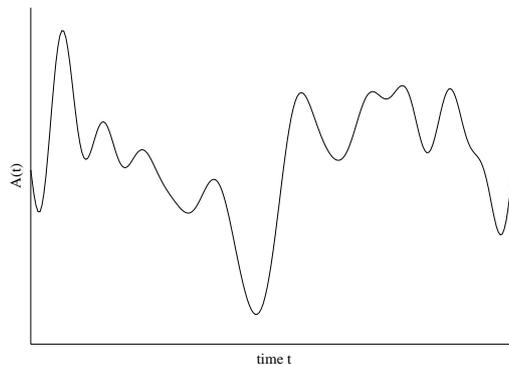
## Revision of other topics

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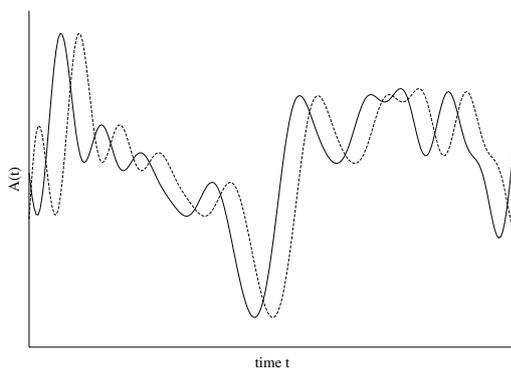
### A.1. Correlation functions

What is a correlation function ? If we consider a series of measurements with the value of the measurement  $A(t)$  changing randomly but continuously.

Then at times  $t$  and  $t'$  that are close together the values  $A(t)$  and  $A(t')$  are *correlated* if they have similar values. Whereas for the measurements at times  $t$  and  $t'$  that are far apart we could consider no relationship between the values  $A(t)$  and  $A(t')$ , so they are *uncorrelated*. The *correlation* plotted against time would then start at some value and decay with time.



If we shift the data by a time  $\tau$  and multiply the values of the new plot to the original one we get a big value if the curves have both high and low values at the same place.



The operation of multiplying two curves together and integrate them over the x-axis is called an *overlap integral*

The overlap integral is also called the *Correlation function*

$$G^{(2)}(\tau) = \langle A(t)A(t + \tau) \rangle \quad (\text{A.1})$$

The correlation is not a function of time, it is a function of the shift in time or the *correlation time*  $\tau$ .



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