Lectures on
Topics in Algebraic K-Theory

By

Hyman Bass

Tata Institute of Fundamental Research, Bombay
1967

Lectures on
Topics in Algebraic K-Theory

By

Hyman Bass

Note by

Amit Roy

Tata Institute of Fundamental Research, Bombay
1967

Preface

These notes are based upon my lectures at the Tata Institute from December, 1965 through February, 1966. The fact that the volume of material treated was excessive for so brief a period manifests itself in the monotone increasing neglect of technical details in the last chapters. The notes are often a considerable improvement on my lectures, and I express my warm thanks to Amit Roy, who is responsible for them.

Portions of the research on which these lectures are based was carried out with NSF contract support at Columbia University.

Hyman Bass

Contents

Introduction

In order to construct a general theory of (non-singular) quadratic forms **1**
and orthogonal groups over a commutative ring k, one should first investigate the possible generalizations of the basic classical tools (when k is a field). These are

(I) Diagonalization (if char $k \neq 2$), and Witt's theorem.

(II) Construction of the classical invariants: dimension, discriminant, Hasse invariant.

This course is mostly concerned with the algebraic apparatus which is *preliminary* to a generalization of II, particularly of the Hasse invariant. Consequently, quadratic forms will receive rather little attention, and then only at the end. It will be useful, therefore, to briefly outline now the material to be covered and to indicate its ultimate relevance to quadratic forms.

We define a *quadratic module* over k to be a pair (P, q) with $P \in \underline{\underline{P}}$, the category of finitely generated projective k-modules, and with $q : P \to k$ a map satisfying $q(ax) = a^2 q(x)$ $(a\epsilon k, a\epsilon P)$ and such that $(x, y) \longmapsto q(x + y) - q(x) - q(y)$ is a bilinear form. This form then induces a homomorphism $P \to P^* = \mathrm{Hom}_k(P, k)$ (by fixing a variable), and we call (P, q) *non-singular* if $P \to P^*$ is an isomorphism.

If (P_1, q_1) and (P_2, q_2) are quadratic modules, we have the "orthogonal sum" $(P_1, q_1) \perp (P_2, q_2) = (P_1 \oplus P_2, q)$, where $q(x_1, x_2) = q_1(x_1) + q_2(x_2)$.

Given $P \in \underline{\underline{P}}$, in order to find a q so that (P, q) in non-singular we **2** must at least have $P \approx P^*$. Hence for arbitrary P, we can instead take $P \oplus P^*$, which has an obvious isomorphism, "$\begin{pmatrix} 0 & 1_{P^*} \\ 1_P & 0 \end{pmatrix}$", with its dual. Indeed this is induced by the bilinear form associated with the *hyperbolic module*

$$\mathbb{H}(P) = (P \oplus P^*, q_P),$$

where $q_p(x, f) = f(x)(x \epsilon P, f \epsilon P^*)$. The following statement is easily proved:

(P, q) is non-singular $\Leftrightarrow (P, q) \perp (P, -q) \approx \mathbb{H}(P)$.

Let $\underline{\underline{Q}}$ denote the category of non-singular quadratic modules and their isometrics. In $\underline{\underline{P}}$ we take only the *isomorphisms* as morphisms. Then we can view $\mathbb{H}$ as the *hyperbolic functor*

$$\mathbb{H} : \underline{\underline{P}} \to \underline{\underline{Q}},$$

where, for $f : P \to P'$, $\mathbb{H}(f)$ is the isometry $f \oplus f^{*^{-1}} : \mathbb{H}(P) \to \mathbb{H}(P')$. Moreover, there is a natural isomorphism

$$\mathbb{H}(P \oplus P') \approx \mathbb{H}(P) \perp \mathbb{H}(P').$$

With this material at hand I will now begin to describe the course. In chapter 1 we establish an exact sequence of Grothendieck groups of certain categories, in an axiomatic setting. Briefly, suppose we are
3 given a category $\mathscr{C}$ in which all morphisms are isomorphisms (i.e. a groupoid) together with a product $\perp$ which has the formal properties of $\perp$ and $\oplus$ above. We then make an abelian group out of obj $\mathscr{C}$ in which $\perp$ corresponds to $+$; it is denoted by $K_0\mathscr{C}$. A related group $K_1\mathscr{C}$, is constructed using the automorphisms of objects of $\mathscr{C}$. Its axioms resemble those for a determinant. If $H : \mathscr{C} \to \mathscr{C}'$ is a product preserving functor (i.e. $H(A \perp B) = HA \perp HB$), then it induces homomorphisms $K_iH : K_i\mathscr{C} \to K_i\mathscr{C}'$, $i = 0, 1$. We introduce a relative category ΦH, and then prove the basic theorem:

There is an exact sequence

$$K_1\mathscr{C} \to K_1\mathscr{C}' \to K_0\Phi H \to K_0\mathscr{C} \to K_0\mathscr{C}',$$

provided H is "cofinal". *Cofinal* means: given $A' \epsilon \mathscr{C}'$, there exists $B' \to \mathscr{C}'$ and $C \epsilon \mathscr{C}$ such that $A' \perp B' \approx HC$. This theorem is a special case of results of Heller [1].

The discussion above shows that the hyperbolic functor satisfies all the necessary hypotheses, so we obtain an exact sequence

$$[\mathbb{H}] \quad K_1\underline{\underline{P}} \to K_1\underline{\underline{Q}} \to K_0\Phi\mathbb{H} \to K_0\underline{\underline{P}} \to K_0\underline{\underline{Q}} \to \text{ Witt } (k) \to 0.$$

Here we define Witt (k) = coker $(k_0\mathbb{H})$. It corresponds exactly to the classical "Witt ring" of quadratic forms (see Bourbaki [2]). The $K_i\underline{\underline{P}}$, $i = 0, 1$ will be described in chapter 1. $K_1\underline{\underline{Q}}$ is related to the stable structure of the orthogonal groups over k.

The classical Hasse invariant attaches to a quadratic form over a 4
field k an element of the Brauer group Br (k). It was given an intrinsic definition by Witt [1] by means of the Clifford algebra. This necessitates a slight artifice due to the fact that the Clifford algebra of a form of odd dimension is not central simple. Moreover, this complication renders the definition unavailable over a commutative. ring in general. C.T.C. Wall [1] proposed a natural and elegant alternative. Instead of modifying the Clifford algebras he enlarged the Brauer group to accommodate them, and he calculated this "Brauer-Wall" group $BW(k)$ when k is a field. Wall's procedure generalizes naturally to any k. In order to carry this out, we present in chapters 2, 3, and 4, an exposition of the Brauer-Wall theory.

Chapter 2 contains a general theory of equivalences of categories of modules, due essentially to Morita [1] (see also Bass [2]) and Gabriel [1]. It is of general interest to algebraists, and it yields, in particular, the Wedderburn structure theory in a precise and general form. It is also a useful preliminary to chapter 2, where we deal with the Brauer group Br (k) of azumaya algebras, following the work of Auslander-Goldman [1]. In chapter 4 we study the category $\underline{\underline{Az}}_2$ of graded azumaya algebras, and extend Wall's calculation of $BW(k)$, giving only statements of results, without proofs.

Here we find a remarkable parallelism with the phenomenon wit- 5
nessed above for quadratic forms. Let $\underline{\underline{FP}}_2$ denote the category of "faith-

fully projective” k-modules P (see chapter 1 for definition), which have a grading modulo 2 : $P = P_0 \oplus P_1$. Then the full endomorphism algebra $END(P)$ (we reserve End for morphisms of degree zero) has a natural grading modulo 2, given by maps homogeneous of degree zero and one, respectively.

Matricially, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$. These are the “trivial” algebras in $\underline{\underline{Az}}_2$; that is $BW(k)$ is the *group* of isomorphism classes of algebras in $\underline{\underline{Az}}_2$, with respect to $\otimes$, modulo those of the form $END(P)$. It is a group because of the isomorphism

$$A \otimes A^* \approx END(A),$$

where A^* is the (suitably defined) opposite algebra of A, for $A \in \underline{\underline{Az}}_2$. Morever, A is faithfully projective as a k-module. Finally we note that

$$END : \underline{\underline{FP}}_2 \to \underline{\underline{Az}}_2$$

is a functor, if in both cases we take homogeneous isomorphisms as morphisms. For, if $f : P \to P'$ and $e \in END(P)$, then $END(f)(e) = fef^{-1} \in END(P')$. Moreover, there is a natural isomorphism

$$END(P \otimes P') \approx END(P) \otimes END(P').$$

6 Consequently, we again obtain an exact sequence:

$$[END] : \; K_1\underline{\underline{FP}}_2 \to K_1\underline{\underline{Az}}_2 \to K_0\Phi END \to K_0\underline{\underline{FP}}_2 \to K_0\underline{\underline{Az}}_2 \to BW(k) \to 0.$$

Chapter 5 finally introduces the category $\underline{\underline{Q}}$ of quadratic forms. The Clifford algebra is studied, and the basic structure theorem for the Clifford algebra is proved in the following form: The diagram of (product-preserving) functors

$$\begin{tikzcd}
\underline{\underline{P}} \arrow[r, "\mathbb{H}"] \arrow[d, "\wedge"'] & \underline{\underline{Q}} \arrow[d, "\text{Clifford}"] \\
\underline{\underline{FP}}_2 \arrow[r, "\text{END}"'] & \underline{\underline{Az}}_2
\end{tikzcd}$$

commutes up to natural isomorphism. Here $\wedge$ denotes the exterior algebra, graded modulo 2 by even and odd degrees.

This result simultaneously proves that the Clifford algebras lie in $\underline{\underline{Az}}_2$, and shows that there is a natural homomorphism of exact sequences

$$
\begin{tikzcd}
{[\mathbb{H}]} \arrow[d] & : & K_1\underline{\underline{P}} \arrow[r] \arrow[d] & K_1\underline{\underline{Q}} \arrow[r] \arrow[d] & K_0\Phi\mathbb{H} \arrow[r] \arrow[d] & K_0\underline{\underline{P}} \arrow[r] \arrow[d, swap, "\text{Grassman}"] & K_0\underline{\underline{Q}} \arrow[r] \arrow[d, swap, "\text{Clifford}"] & \mathrm{Witt}(k) \arrow[r] \arrow[d, "\text{Hasse-Wall}"] & 0 \\
\mathrm{END} & : & K_1\underline{\underline{FP}}_2 \arrow[r] & K_1\underline{\underline{Az}}_2 \arrow[r] & K_0\Phi\mathrm{END} \arrow[r] & K_0\underline{\underline{FP}}_2 \arrow[r] & K_0\underline{\underline{Az}}_2 \arrow[r] & BW(k) \arrow[r] & 0
\end{tikzcd}
$$

This commutative diagram is the promised generalization of the Hasse invariant.

Chapter 1

The exact sequence of algebraic K-theory

The exact sequence of Grothendieck groups constructed in Bass [K, Chapter 3] is obtained here in an axiomatic setting. The same is done in a considerably, more general setting by A. Heller in Heller [1]. A special case of the present version was first worked out by S. Chase (unpublished). **7**

In the last sections we shall describe the Grothendieck groups of certain categories of projective modules.

1 Categories with product, and their functors

If $\mathscr{C}$ is a category, we shall denote by $\operatorname{obj}\mathscr{C}$, the class of all objects of $\mathscr{C}$, and by $\mathscr{C}(A,B)$, the set of all morphisms $A \to B$, $A, B \epsilon \operatorname{obj}\mathscr{C}$. We shall assume the isomorphism classes in our categories to form sets.

A *groupoid* is a category in which all morphisms are isomorphisms.

Definition. *A category with product is a groupoid $\mathscr{C}$, together with a "product" functor*

$$\perp : \mathscr{C} \times \mathscr{C} \to \mathscr{C},$$

which is assumed to be "coherently" associative and commutative in the sense of MacLane [1].

That is, we are given isomorphisms of functors

$$\perp \circ (1_{\mathscr{C}} \times \perp) \approx \perp \circ (\perp \times 1_{\mathscr{C}}) : \mathscr{C} \times \mathscr{C} \times \mathscr{C} \to \mathscr{C}$$

8 and

$$\perp \circ T \approx \perp : \mathscr{C} \times \mathscr{C} \to \mathscr{C},$$

where T is the transposition on $\mathscr{C} \times \mathscr{C}$. Moreover, these isomorphisms are compatible in the sense that isomorphisms of products of several factors obtained from these by a succession of three-fold reassociations, and two-fold permutations, are all the same. This permits us to write, unambiguously, expressions like $A_1 \perp \cdots \perp A_n = \mathop{\perp}\limits_{i=1}^{n} A_i$.

A functor $F : (\mathscr{C}, \perp) \to (\mathscr{C}', \perp')$ of categories with product is a functor $F : \mathscr{C} \to \mathscr{C}'$ which "preserves the product". More precisely, there should be an isomorphism of functors $F \circ \perp \approx \perp' \circ (F \times F) : \mathscr{C} \times \mathscr{C} \to \mathscr{C}'$, which is compatible, in an obvious sense, with the associativity and commutativity isomorphisms in the two categories.

Hereafter all products will be denoted by the same symbol $\perp$ (except for special cases where there is a standard notation) and we will usually write $\mathscr{C}$ instead of $\mathscr{C}$, $\perp$).

Examples. 1) Let k be a commutative ring and let $\underline{\underline{P}}$ denote the category of finitely generated projective modules over k with isomorphisms as morphisms. It is a category with product if we set $\perp = \oplus$.

2) The full subcategory $\underline{\underline{FP}}$ of $\underline{\underline{P}}$ with finitely generated faithful projective modules as objects. Here we set $\perp = \otimes_k$.

9 3) The full subcategory $\underline{\underline{Pic}}$ of $\underline{\underline{FP}}$ whose objects are finitely generated projective modules of rank 1. We set $\perp = \otimes_k$.

4) The category $\underline{\underline{Q}}$ of quadratic modules over k with isometries as morphisms. We take $\perp$ to be the orthogonal sum of two quadratic modules.

5) The category $\underline{\underline{Az}}$ of Azumaya algebras over k (see Chapter 3). Here take $\perp = \otimes_k$.

Let $\mathscr{C}(k)$ denote one of the categories mentioned above, and let $k \to k'$ be a homomorphism of rings. Then $k' \otimes_k$ induces a functor $\mathscr{C}(k) \to (k')$ preserving product.

If we neglect naturality conditions, then a category with product is one whose (isomorphism classes of) objects are a commutative semi-group. The *Grothendieck group* is got by formally introducing inverses and making this semi-group into a group.

Definition. *Let $\mathscr{C}$ be a category with product. The Grothendieck group of $\mathscr{C}$ is defined to be an abelian group $K_0\mathscr{C}$, together with a map*

$$()_{\mathscr{C}} : \operatorname{obj}\mathscr{C} \to K_0\mathscr{C},$$

which is universal for maps into abelian groups satisfying

$$K_0.\ \textit{if } A \approx B, \textit{ then } (A)_{\mathscr{C}} = (B)_{\mathscr{C}},$$
$$K_1.\ (A \perp B)_{\mathscr{C}} = (A)_{\mathscr{C}} + (B)_{\mathscr{C}}.$$

In other words if G is an abelian group and $\varphi : \operatorname{obj}\mathscr{C} \to G$ a map satisfying $K0$ and $K1$, then there exists a unique homomorphism of groups $\psi : K_0\mathscr{C} \to G$ such that $\varphi = \psi 0(\ )_{\mathscr{C}}$.

Clearly $K_0\mathscr{C}$ is unique. We can construct $K_0\mathscr{C}$ by reducing the free **10**
abelian group on the isomorphism classes of $\operatorname{obj}\mathscr{C}$ by relations forced by $K1$.

When $\mathscr{C}$ is clear from the context, we shall write ( ). instead of $(\ )_{\mathscr{C}}$.

Proposition 1.1. *(a) Every element of $K_0\mathscr{C}$ has the form $(A) - (B)$ for some $A, B \epsilon \operatorname{obj}\mathscr{C}$.*

(b) $(A) = (B) \Leftrightarrow$ *there exists $C \in \operatorname{obj}\mathscr{C}$ such that $A \perp C \approx B \perp C$.*

(c) If $F : \mathscr{C} \to \mathscr{C}'$ is a functor of categories with product, then the map

$$K_0F : K_0\mathscr{C} \to K_0\mathscr{C}',$$

given by $(A)_{\mathscr{C}} \longmapsto (FA)_{\mathscr{C}'}$ is well defined and makes K_0 a functor into abelian groups.

We defer the proof of this proposition, since we are going to prove it in a more general form (proposition 1.2 below).

Definition. *A* composition *on a category $\mathscr{C}$ with product is a sometimes defined composition $\circ$ of objects of $\mathscr{C}$, which satisfies the following condition: if $A \circ A'$ and $B \circ B'$ are defined ($A, A', B, B' \in \operatorname{obj}\mathscr{C}$), then so also is $(A \perp B) \circ (A' \perp B')$, and*

$$(A \perp B) \circ (A' \perp B') = (A \circ A') \perp (B \circ B').$$

When this structure is present, we shall require the functors to preserve it: $F(A \circ B) = (FA) \circ (FB)$.

11 **Definition.** *Let $\mathscr{C}$ be a category with product and composition.*

The *Grothendieck group* of $\mathscr{C}$ is defined to be an abelian group $K_0\mathscr{C}$, together with a map

$$(\ )_{\mathscr{C}} : \operatorname{obj}\mathscr{C} \to K_o\mathscr{C},$$

which is universal for maps into abelian groups satisfying $K0$, $K1$ and

$K2$. if $A \circ B$ is defined, then $(A \circ B)_{\mathscr{C}} = (A)_{\mathscr{C}} + (B)_{\mathscr{C}}$.

If composition is never defined, we get back the K_0 defined earlier.

As before we write ( ) instead of $(\ )_{\mathscr{C}}$ when $\mathscr{C}$ is clear from the context.

We shall now generalize proposition 1.1.

Proposition 1.2. *Let $\mathscr{C}$ be a category with product and composition.*

(a) Every element of $K_0\mathscr{C}$ has the form $(A) - (B)$ for some $A, B \in \operatorname{obj}\mathscr{C}$.

(b) $(A) = (B) \Leftrightarrow$ there exist $C, D_0, D_1, E_0, E_1 \in \operatorname{obj}\mathscr{C}$, such that $D_0 \circ D_1$ and $E_0 \circ E_1$ are defined, and

$$A \perp C \perp (D_0 \circ D_1) \perp E_0 \perp E_1 \approx B \perp C \perp D_0 \perp D_1 \perp (E_0 \circ E_1).$$

(c) If $F : \mathscr{C} \to \mathscr{C}'$ is a functor of categories with product and composition, then the map

$$K_0F : K_0\mathscr{C} \to K_0\mathscr{C}',$$

given by $(A)_{\mathscr{C}} \longmapsto (FA)_{\mathscr{C}'}$, is well defined and makes K_0 a functor into abelian groups.

Proof. (a) Any element of $K_0\mathscr{C}$ can be written as **12**

$$\sum_i (A_i) - \sum_j (B_j) = (\underset{i}{\perp} A_i) - (\underset{j}{\perp} B_j).$$

(b) Let us denote by $[A]$ the isomorphism class containing $A \epsilon \operatorname{obj} \mathscr{C}$, and by M the free abelian group generated by these classes. A relation $\sum[A_i] = \sum[B_j]$ in M implies an isomorphism $\perp A_i \approx \perp B_j$ in $\mathscr{C}$.

Now, if $(A) = (B)$, then we have a relation of the following type in M:

$$\begin{aligned} [A] - [B] = \sum &\{[C_{h0} \perp C_{h1}] - [C_{h0}] - [Ch_{h1}]\} \\ + \sum &\{[C'_{i0}] + [C'_{i1}] - [C'_{i0} \perp C'_{i1}]\} \\ + \sum &\{[D_{j0}] + [D_{j1}] - [D_{j0} \circ D_{j1}]\} \\ + \sum &\{[E_{l0} \circ E_{l1}] - [E_{l0}] - [E_{l1}]\}, \end{aligned}$$

or

$$\begin{aligned} &[A] + \sum\{[C_{h0}] + [C_{h1}]\} + \sum[C'_{i0} \perp C'_{i1}] \\ &+ \sum[D_{j0} \circ D_{j1}] + \sum\{[E_{l0}] + [E_{l1}]\} \\ &= [B] + \sum[C_{h0} \perp C_{h1}] + \sum\{[C'_{i1}] \\ &+[C'_{i1}]\} + \sum\{[D_{j0}] + [D_{j1}]\} + \sum[E_{l0} \circ E_{l1}]. \end{aligned}$$

This implies an isomorphism

$$A \perp C \perp (D_0 \circ D_1) \perp E_0 \perp E_1 \approx B \perp C \perp D_0 \perp D_1 \perp (E_0 \circ E_1).$$

where **13**

$$C = (\underset{h}{\perp} C_{h0}) \perp (\underset{h}{\perp} C_{h1}) \perp (\underset{i}{\perp} C'_{i0}) \perp (\underset{i}{\perp} C'_{i1}),$$

$$D_0 = \perp_j D_{j0}, \quad E_0 = \perp_l E_{10},$$
$$D_1 = \perp_j D_{j1}, \quad E_1 = \perp_l E_{11}.$$

The other implication is a direct consequence of the definition of $K_0\mathscr{C}$.

(c) The map $\mathrm{obj}\,\mathscr{C} \to K_0\mathscr{C}'$ given by $A \longmapsto (FA)_{\mathscr{C}'}$ satisfies $K0$, $K1$ and $K2$. This gives rise to the required homomorphism $K_0\mathscr{C} \to K_0\mathscr{C}'$. The rest is straightforward.

□

Now let $\mathscr{C}$ be simply a category with product. For $A \in \mathrm{obj}\,\mathscr{C}$, we write

$$G(A) = \mathscr{C}(A, A),$$

the group of automorphisms of A. (Recall that $\mathscr{C}$ is a groupoid.) If $f : A \to B$, we have a homomorphism

$$G(f) : G(A) \to G(B),$$

given by $G(f)(\alpha) = f\alpha f^{-1}$.

We shall now construct, out of $\mathscr{C}$, a new category $\Omega\mathscr{C}$. We take $\mathrm{obj}\,\Omega\mathscr{C}$ to be the collection of all automorphisms in $\mathscr{C}$. If $\alpha \in \mathrm{obj}\,\Omega\mathscr{C}$ is an automorphism of $A \in \mathscr{C}$, we shall sometimes write (A, α) instead of α, to make A explicit. A morphism $(A, \alpha) \to (B, \beta)$ in $\Omega\mathscr{C}$ is a morphism
14 $f : A \to B$ in $\mathscr{C}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is commutative, that is, $G(f)(\alpha) = \beta$. We define a product in $\Omega\mathscr{C}$ by setting $(A, \alpha) \perp (B, \beta) = (A \perp B, \alpha \perp \beta)$. There is a natural composition 0 in $\Omega\mathscr{C}$: if $\alpha, \beta \epsilon\, \mathrm{obj}\,\Omega\mathscr{C}$ are automorphisms of the same object in $\mathscr{C}$, then we take $\alpha \circ \beta$ to be the usual of morphisms. The compatibility of $\perp$ and 0 in $\Omega\mathscr{C}$ is the identity

$$(\alpha \perp \beta) \circ (\alpha' \perp \beta') = (\alpha \circ \alpha') \perp (\beta \circ \beta'),$$

which simply expresses the fact that $\perp$ is a functor (of two variables).

Definition. *If $\mathscr{C}$ is a category with product, we define*

$$K_1\mathscr{C} = K_0\Omega\mathscr{C}.$$

Let $F : \mathscr{C} \to \mathscr{C}'$ be a functor of categories with product. Then F induces $\Omega F : \Omega\mathscr{C} \to \Omega\mathscr{C}'$, preserving product and composition, so we obtain homomorphisms

$$K_iF : K_i\mathscr{C} \to K_i\mathscr{C}' \qquad i = 0, 1.$$

We propose now to introduce a relative group to connect the above into a 5-term exact sequence.

First we construct the relative category ΦF with respect to the functor F. Objects of ΦF are triples (A, α, B), A, $B \in \operatorname{obj}\mathscr{C}$ and $\alpha : FA \to FB$. A morphism $(A, \alpha, B) \to (A', \alpha', \beta')$ in ΦF is a pair (f, g) of morphisms $f \colon A \to A'$ and $g \colon B \to B'$ in $\mathscr{C}$ such that **15**

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha"'] & FA' \arrow[d, "\alpha'"] \\ FB \arrow[r, "Fg"'] & FB' \end{tikzcd}$$

is a commutative diagram. We define product and composition in ΦF, by setting

$$(A, \alpha, B) \perp (A', \alpha', B') = (A \perp A', \alpha \perp \alpha', B \perp B'),$$
$$(B, \beta, C) \circ (A, \alpha, B) = (A, \beta\alpha, C).$$

We shall see in §4 that under some restriction on F, the Grothendieck group of this relative category ΦF fits into an exact sequence involving the $K_i's$ of $\mathscr{C}$ and $\mathscr{C}'$.

We record here a few facts about $K_0\Phi F$ which we shall need later:

Remark 1.3. (a) $(A, 1_{FA}, A)_{\Phi F} = 0$ for any $A \in \operatorname{obj}\mathscr{C}$. This follows from the fact $(A, 1_{FA}, A) \circ (A, 1_{FA}, A) = (A, 1_{FA}, A)$ in ΦF.

(b) $(A, \alpha, B)_{\Phi F} = -(B, \alpha^{-1}, A)_{\Phi F}$ for any $(A, \alpha, B) \in \operatorname{obj}\Phi F$. This follows from (a) and the equation $(B, \alpha^{-1}, A)0(A, \alpha, B) = (A, 1_{FA}, A)$.

16 (c) Any element of $K_0\Phi F$ can be written as $(A, \alpha, B)_{\Phi F}$. For, by proposition 1.2 (a), any element of $K_0\Phi F$ can be written as $(A, \alpha, B)_{\Phi F} - (A', \alpha', B')_{\Phi F}$. But this equals $(A \perp B', \alpha \perp \alpha'^{-1}, B \perp A')_{\Phi F}$, in view of (b) above, and the axiom $K1$.

We close this section with a lemma about permutations that will be needed in §4. Consider a permutation s of $\{1, \ldots, n\}$. The axiom of commutativity for $\perp$ gives us, for any $A_1, \ldots, A_n$, a well defined isomorphism

$$A_1 \perp \cdots \perp A_n \xrightarrow{\approx} A_{s(1)} \perp \cdots \perp A_{s(n)},$$

which we shall also denote by s. If $\alpha_i : A_i \to B_i$, then the diagram

$$\begin{tikzcd}
A_1 \perp \ldots \perp A_n \arrow[r, "\alpha_1 \perp \ldots \perp \alpha_n"] \arrow[d, "s"'] & B_1 \perp \ldots \perp B_n \arrow[d, "s"] \\
A_{s(1)} \perp \ldots \perp A_{s(n)} \arrow[r, "\alpha_{s(1)} \perp \ldots \perp \alpha_{s(n)}"'] & B_{s(1)} \perp \ldots \perp B_{s(n)}
\end{tikzcd}$$

is commutative, that is

$$s(\alpha_1 \perp \cdots \perp \alpha_n) = (\alpha_{s(1)} \perp \ldots \perp \alpha_{s(n)})s. \tag{1.4}$$

Suppose now that $\alpha_i : A_i \to A_{i+1}$, $1 \leq i \leq n-1$, and $\alpha_n : A_n \to A_1$. Let $s(i) = i - 1(\mod n)$. and set $\alpha = \alpha_1 \perp \cdots \perp \alpha_n$. Then $(A_1 \perp \cdots \perp A_n, s_\alpha) \epsilon \operatorname{obj}\Omega\mathscr{C}$. If

$$\beta = (1_{A_1} \perp \alpha_1^{-1} \perp \cdots \perp (\alpha_{n-1} \cdots \alpha_1)^{-1}),$$
$$A_1 \perp A_2 \perp \cdots \perp A_n \to A_1 \perp \cdots \perp A_1.$$

17 then $\beta : (A_1 \perp \cdots \perp A_n, s\alpha) \to (A_1 \perp \cdots \perp A_1, \beta s\alpha\beta^{-1})$ in $\Omega\mathscr{C}$. Now $\alpha\beta^{-1} = (\alpha_1 \perp \alpha_2\alpha_1 \perp \cdots \perp (\alpha_n \cdots \alpha_1))$, and by (1.4) above, $\beta s = \alpha_1^{-1} \perp (\alpha_2\alpha_1)^{-1} \perp \cdots \perp (\alpha_{n-1} \cdots \alpha_1)^{-1} \perp 1_{A_1}$. Consequently:

Lemma 1.5. *Suppose* $\alpha_i : A_i \to A_{i+1}$, $1 \le i \le n-1$ *and* α_n*:* $A_n \to A_1$*. Let* s *denote the permutation* $s(i) = i - 1(\mod n)$*. Then in* $\Omega\mathscr{C}$

$$\begin{aligned}&(A_1 \perp \cdots \perp A_n, s(\alpha_1 \perp \cdots \perp \alpha_n))\\ \approx\ &(A_1 \perp \cdots \perp A_1,\ 1_{A_1} \perp \cdots \perp' A_1 \perp (\alpha_n - \alpha_1))\end{aligned}$$

In particular, if $\alpha : A \to B$ *and* $\beta : B \to C$*, then*

$$(A \perp B, t(\alpha \perp \alpha^{-1})) \approx (A \perp A, 1)$$

and

$$(A \perp B \perp C, s(\alpha \perp \beta \perp (\beta\alpha)^{-1})) \approx (A \perp A \perp A, 1)$$

in $\Omega\mathscr{C}$*, where* t *and* s *are the transposition and the three cycle, respectively.*

2 Directed categories of abelian groups

In the next section we shall see that $K_1\mathscr{C}$ can be calculated as a kind of generalized direct limit. We discuss in this section some necessary technical preliminaries.

In this section $\mathscr{G}$ will denote a category of abelian groups. Also, we shall assume that $\mathscr{G}$ is a set.

Definition. *A* direct limit *of* $\mathscr{G}$ *is an abelian group* $\underset{\rightarrow}{\mathscr{G}}$ *together with a family of homomorphisms* $f_A : A \to \underset{\rightarrow}{\mathscr{G}}$. $A \in \operatorname{obj}\mathscr{G}$*, such that the diagram* **18**

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "f_A"'] & & B \arrow[dl, "f_B"] \\ & \underset{\rightarrow}{\mathscr{G}} & \end{tikzcd}$$

is commutative for any morphism $f : A \to B$ *and* $\underset{\rightarrow}{\mathscr{G}}$ *is universal for this property.*

Clearly $\underset{\rightarrow}{\mathscr{G}}$ is unique. Also, it follows that $\underset{\rightarrow}{\mathscr{G}}$ is the sum of its subgroups $f_A(A)$. We can describe $\underset{\rightarrow}{\mathscr{G}}$ as the quotient of $\bigoplus_{A \in \operatorname{obj} \mathscr{G}} A$ by the subgroup generated by the elements of the type $f(a) - a$, where f is any morphism $A \to B$ and $a \in A$.

Lemma 2.1. *Let $\mathscr{G}$ be such that given two objects A, B, there exists an object C, with $\mathscr{G}(A,C)$ and $\mathscr{G}(B,C)$ non-empty. Then $\underset{\rightarrow}{\mathscr{G}} = \bigcup f_A(A)$.*

Proof. Any element of $\underset{\rightarrow}{\mathscr{G}}$ can be written as a finite sum $\sum f_{A_i}(a_i)$, $a_i \in A_i$. To establish our assertion, it is enough to express an element of the type $f_A(a) + f_B(b)$ as $f_C(c)$ for some C and $c \in C$. We choose C such that there are morphisms $g : A \to C$, $h : B \to C$. Then $c = g(a) + h(b)$ serves our purpose. □

It follows in particular that, if $\mathscr{G}$ has a "finial" object, that is, an object C such that $\mathscr{G}(A,C) \neq \phi$ for every $A \in \operatorname{obj} \mathscr{G}$, then $f_C : C \to \underset{\rightarrow}{\mathscr{G}}$ is surjective. Let N be the subgroup of C generated by all elements of
19 the type $f_1(a) - f_2(a)$, $f_i \in \mathscr{G}(A,C)$, $a \in A$. Clearly $N \subset \ker f_C$ and this induces a map $C/N \to \underset{\rightarrow}{\mathscr{G}}$. On the other hand, all morphisms $A \to C$ induce the same map $A \to C/N$, and the latter are clearly compatible with the morphisms $A \to B$. The universal mapping property gives now a map $\underset{\rightarrow}{\mathscr{G}} \to C/N$ which is easily checked to be the inverse of $C/N \to \underset{\rightarrow}{\mathscr{G}}$. Thus $C/N \to \underset{\rightarrow}{\mathscr{G}}$ is an isomorphism, that is, $N = \ker f_C$.

Definition. *$\mathscr{G}$ is called* directed, *if*

(1) given $A, B \in \operatorname{obj} \mathscr{G}$, there exists $C \in \operatorname{obj} \mathscr{G}$, such that $\mathscr{G}(A,C)$ and $\mathscr{G}(B,C)$ are both non-empty.

(2) given $f_i : A \to B$, $i = 1, 2$, there exists $g : B \to C$ such that $gf_1 = gf_2$.

We note that lemma 2.1 is valid for directed categories.

Lemma 2.2. *Let $\mathscr{G}$ be directed and let $f_A(a) = 0$ for some $A \in \operatorname{obj} \mathscr{G}$ and $a \in A$. There exists then a morphism $g : A \to B$ such that $g(a) = 0$.*

Proof. Since $f_A(a) = 0$, we have, in the direct sum of the $C's$, $a = \sum_i \pm(f_i(c_i) - c_i)$. Since only a finite number of terms appear in the relation, we can find a C into which all the intervening groups map. In particular, if $\mathscr{G}'$ is the full subcategory of $\mathscr{G}$ whose objects are those which have a map into C, then $\mathscr{G}'$ has C as a final object and we have $f'_A(a) = 0$, where $f'_A : A \to \underset{\rightarrow}{\mathscr{G}'}$. Now it follows from the last paragraph that if $f : A \to C$, then there is a family of pairs $f_{1i}, f_{2i} : B_i \to C$, and $b_i \in B_i$, $1 \le i \le m$, such that $f(a) = \sum_{i=1}^{m} f_{1i}(b_i) - f_{2i}(b_i)$. Since $\mathscr{G}$ is directed, it follows easily, by induction on m, that there exists an **20**
$h : C \to B$ such that $hf_{1i} = hf_{2i}$, $1 \le i \le m$. Then $hf(a) = 0$. $\square$

Definition. *A subcategory $\mathscr{G}'$ of a directed category $\mathscr{G}$ is called* dominating, *if*

(1) given $A \in \operatorname{obj}\mathscr{G}$, there exists $A' \in \operatorname{obj}\mathscr{G}'$ and a map $A \to A'$ in $\mathscr{G}$,

(2) given $f_i : A'_i \to B$ in $\mathscr{G}$, $i = 1, 2$, with $A'_i \in \mathscr{G}'$, there exists $g : B \to C'$ with $C' \in \operatorname{obj}\mathscr{G}'$ such that $gf_i \in \mathscr{G}'$, $i = 1, 2$.

We note first that $\mathscr{G}'$ is also directed. For given $A'_1, A'_2 \in \operatorname{obj}\mathscr{G}'$, we can find $f_i : A'_i \to B$ in $\mathscr{G}$. There exists then $ag : B \to C'$ with $C' \in \operatorname{obj}\mathscr{G}'$ such that gf_i is a morphism in $\mathscr{G}'$, $i = 1, 2$. Next suppose $f_1, f_2 : A' \to B'$ are maps in $\mathscr{G}'$. There exists $g : B' \to C$ in $\mathscr{G}$ such that $gf_1 = gf_2$. We can find $h : C \to D'$ with hg in $\mathscr{G}'$. Thus we have a morphism hg in $\mathscr{G}'$ with $(hg)f_1 = (hg)f_2$.

Proposition 2.3. *If $\mathscr{G}'$ is a dominating subcategory of the directed category $\mathscr{G}$, then the natural map $\varphi : \underset{\rightarrow}{\mathscr{G}'} \to \underset{\rightarrow}{\mathscr{G}}$ is an isomorphism.*

Proof. Write $f'_{A'} : A' \to \underset{\rightarrow}{\mathscr{G}'}$ and $f_A : A \to \underset{\rightarrow}{\mathscr{G}}$ for the canonical maps. If $A' \in \operatorname{obj}\mathscr{G}'$, $A \in \operatorname{obj}\mathscr{G}$ and $A' \overset{g}{\text{---}} A$ is a map in either direction in $\mathscr{G}$, then

$$\begin{tikzcd}
A' \arrow[r, "g"] \arrow[d, "f'_{A'}"'] \arrow[dr, "f'_{A'}"] & A \arrow[d, "f_A"] \\
\underset{\rightarrow}{\mathscr{G}'} \arrow[r, "\varphi"'] & \underset{\rightarrow}{\mathscr{G}}
\end{tikzcd}$$

is commutative. □

21 Given A, we can find $g : A \to A'$ by (1), so that $\operatorname{im} f_A \subset \operatorname{Im} f_{A'} = \operatorname{im} \varphi f_{A'} \subset \operatorname{im} \varphi$. It follows from lemma 2.1, that φ is surjective. Suppose now $x \in \ker \varphi$. Since $\mathscr{G}'$ is directed, lemma 2.1 is applicable to $\mathscr{G}'$ and we can write $x = f_{A'}(a')$. By lemma 2.2 there exists $g : A' \to A$ such that $g(a') = 0$. Choose $h : A \to B'$ in $\mathscr{G}$ such that hg is in $\mathscr{G}'$. Thus $hg(a') = 0$, so that $x = f'_{A'}a' = f'_{B'}hg(a') = 0$, which shows that φ is injective.

3 $K_1\mathscr{C}$ as a direct limit

Let $\mathscr{C}$ be a category with product. If A is an object of $\mathscr{C}$, we write $G(A)$ for its automorphism group, [A] for the isomorphism class of A, and $G[A]$ for the abelianization of $G(A)$, that is, the quotient of $G(A)$ by its commutator subgroup. This notation is legitimate because any two isomorphisms $A \to B$ induce the same isomorphism $G[A] \to G[B]$, since $G(A) \to G(B)$ is unique up to inner automorphisms.

We now propose to construct a directed category $\mathscr{G}$ of abelian groups, in the sense of §2. The objects of $\mathscr{G}$ are the $G[A]$, $A \in \operatorname{obj} \mathscr{C}$. As for morphisms in $\mathscr{G}$, we set $\mathscr{G}(G[A], G[B]) = \phi$ if there exists no A' with $A \perp A' \approx B$. Otherwise let $h : A \perp A' \to B$ be an isomorphism for some A'. We have a homomorphism $G(A) \to G(B)$ given by $\alpha \mapsto G(h)(\alpha \perp 1_{A'})$. This induces a homomorphism $f : G[A] \to G[B]$ which is independent of the isomorphism h chosen and depends only
22 on the isomorphism class $[A']$ of A'. The homomorphism f will be denoted by $G[A] \perp [A']$. Now we define $\mathscr{G}(G[A], G[B])$ to be the set of all homomorphisms $G[A] \to G[B]$ which are of the form $G[A] \perp [A']$ for some A' with $A \perp A' \approx B$.

We define composition of morphisms in $\mathscr{G}$ by

$$(G[B] \perp [B'])(G[A] \perp [A']) = G[A] \perp [A' \perp B'],$$

where $A \perp A' \approx B$.

Since $\mathscr{G}(G[A_i], G[A_1 \perp A_2])$ is not empty for $i = 1, 2$, $\mathscr{G}$ satisfies the condition (1) in the definition of a directed category. To verify (2), suppose given $f_1, f_2 : G[A] \to G[B]$,

$f_i = G[A] \perp [A_i']$. Then $[A \perp A_i'] = [B]$ so if we set $g = G[B] \perp [A] : G[B] \to G[B \perp A]$, we have

$$g f_i = (G[B] \perp [A])(G[A] \perp [A_i']) = G[A] \perp [A_i' \perp A] = G[A] \perp [B],$$

which is independent of i.

Definition. *A functor $F : \mathscr{C}' \to \mathscr{C}$ of categories with product is called* cofinal *if every $A \in \operatorname{obj}\mathscr{C}$ "divides" FB' for some $B' \in \operatorname{obj}\mathscr{C}'$, that is, if $A \perp A_1 \approx FB'$ for some object A_1 of $\mathscr{C}$. A subcategory $\mathscr{C}' \subset \mathscr{C}$ is called* cofinal *if the inclusion functor is.*

Theorem 3.1. *Let $\mathscr{C}$ be a category with product.*

(1) Let $\mathscr{G}$ be the directed category of abelian groups constructed above. There is a canonical isomorphism

$$\underset{\rightarrow}{\mathscr{G}} \xrightarrow{\approx} K_1\mathscr{C}.$$

(2) Let $\mathscr{C}'$ be a full cofinal subcategory of $\mathscr{C}$. The inclusion of $\mathscr{C}'$ in $\mathscr{C}$ induces an isomorphism

$$K_1\mathscr{C}' \xrightarrow{\approx} K_1\mathscr{C}.$$

Proof. (a) If $\alpha \in G(A)$ let (α) denote its image in $K_1^{\mathscr{C}}$. **23**

Since $(\alpha\beta) = (\alpha) + (\beta)$, the map $\alpha \mapsto (\alpha)$ is a homomorphism $G(A) \to K_1\mathscr{C}$ which, since $K_1\mathscr{C}$ is abelian, induces $g_{[A]} : G[A] \to K_1\mathscr{C}$. In particular, since $(1_{A'}) = 0$, we have $(\alpha \perp 1_{A'}) = (\alpha) + (1_{A'}) = (\alpha)$ and this implies that the $g_{[A]}$ actually define a map of the directed category $\mathscr{G}$ into $K_1\mathscr{C}$. Hence we have a homomorphism $\underset{\rightarrow}{\mathscr{G}} \to K_1\mathscr{C}$. To construct its inverse we need only observe the obvious fact that the map assinginig to each α its image ($via G(A) \to G[A] \to \underset{\rightarrow}{\mathscr{G}}$) in $\underset{\rightarrow}{\mathscr{G}}$ satisfies the axioms defining K_1, so that by universality, we get the desired homomorphism $K_1\mathscr{C} \to \underset{\rightarrow}{\mathscr{G}}$.

(b) If A' and B' are two objects of $\mathscr{C}' \subset \mathscr{C}$, the symbols $G(A'), G[A']$ and $G[A'] \perp [B']$ are unambiguous since $\mathscr{C}'$ is full in $\mathscr{C}$. Let $\mathscr{G}'$ be the directed category associated with $\mathscr{C}'$. Evidently $\mathscr{G}' \subset \mathscr{G}$, and we need only show that $\mathscr{G}'$ is a dominating subcategory of $\mathscr{G}$ (in the sense of §2), for then we can invoke proposition 2.3.

□

Given $G[A] \in \operatorname{obj} \mathscr{G}$, choose $A \perp B \approx C', B \in \operatorname{obj} \mathscr{C}, C' \in \operatorname{obj} \mathscr{C}'$. This is possible because $\mathscr{C}'$ is cofinal in $\mathscr{C}$. Then $G[A] \perp [B] : G[A] \to G[C']$, and $G[C'] \in \operatorname{obj} \mathscr{G}'$. This verifies condition 1) for $\mathscr{G}'$ to be dominating in $\mathscr{G}$. Condition 2) requires that if $f_1, f_2 : G[A'] \to G[B]$, $A' \in \operatorname{obj} \mathscr{C}'$, then there exists $g : G[B] \to G[C']$ such that $g f_i$ is a morphism in $\mathscr{G}'$, $i = 1, 2$. Let $f_i = G[A'] \perp [A_i]$.

24 Choose $D \in \operatorname{obj} \mathscr{C}$ so that $B \perp D \approx D' \in \operatorname{obj} \mathscr{C}'$. Set $C' = A' \perp D'$ and let $g = G[B] \perp [A' \perp D]$. Then $g f_i = (G[B] \perp [A' \perp D])(G[A'] + [A_i]) = G[A'] \perp [A_i \perp A' \perp D] = G[A'] \perp [B \perp D] = G[A'] \perp [D']$ which is a morphism in $\mathscr{G}$.

Definition. *An object A of $\mathscr{C}$ is called basic if the sequence $A^n = A \perp \cdots \perp A$ (n factors) is cofinal; that is, every $B \in \operatorname{obj} \mathscr{C}$ divides A^n for some n.*

If A is basic the full subcategory $\mathscr{C}'$ whose objects are the A^n, $n \geq 1$, is a full cofinal subcategory (with product) to which we may apply the last theorem. If we assume that $A^n \approx A^m \implies n = m$, then $\mathscr{G}'$ is an ordinary direct sequence of abelian groups. The groups are $G[A^n]$, $n \geq 1$, and there is a unique map, $G[A^n] \to G[A^{n+m}]$, namely $G[A^n] \perp [A^m]$, which is induced by $\alpha \mapsto \alpha 1_A m$. These are only non-identity morphisms in $\mathscr{G}'$.

In this case we can even make a direct system from the $G(A^n)$, by

$$G(A^n) \to G(A^{n+m}); \alpha \mapsto \alpha \perp 1_A m.$$

If we write

$$G(A^\infty) = \varinjlim G(A^n)$$

then it is clear that

$$\varinjlim G[A^n] = G(A^\infty)/[G(A^\infty), G(A^\infty].$$

Theorem 3.2. *Suppose that A is a basic object of $\mathscr{C}$, and that $A^n \approx A^m$* 25
implies $n = m$.

(a) *$K_1\mathscr{C}$ is the direct limit*

$$\begin{aligned} K_1\mathscr{C} &\approx \varinjlim(G[A^n]; G[A^n] \perp [A^m] : G[A^n] \to G[A^{n+m}]) \\ &\approx G(A^\infty)/[G(A^\infty, G(A^\infty)]. \end{aligned}$$

(b) *If $\alpha, \beta \in G(A^n)$, then $(\alpha) = (\beta)$ in $K_1\mathscr{C} \Leftrightarrow$ there exist $\gamma \in G(A^m)$ and $\delta_1, \delta_2, \varepsilon_1, \varepsilon_2 \in G(A^p)$, for some m and p, such that*

$$\alpha \perp \gamma(\delta_1\delta_2) \perp 1_{A^p} \perp \varepsilon_1 \perp \varepsilon_2$$

and

$$\beta \perp \gamma \perp \delta_1 \perp \delta_2 \perp (\varepsilon_1\varepsilon_2) \perp 1_{A^p}$$

are conjugate in $G(A^{n+m+4p})$.

(c) *$(\alpha) = 0$ in $K_1\mathscr{C} \Leftrightarrow$ there exists $\gamma \in G(A^m)$ for some m, such that*

$$\alpha \perp \gamma \perp \gamma^{-1}$$

is a commutator. Moreover, $\alpha^2 \perp 1_{A^{2m}}$ is a product of two commutators.

Proof. (a) Follows directly from Theorem 3.1 and the preceding remarks.

(b) The implication $\Leftarrow$ is clear.

For $\Rightarrow$, we apply Proposition 1.2 (b) to the category $\mathscr{C}'$ consisting 26
of A^n, and use part (a) above to obtain $\gamma, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2$ such that

$$\bar{\alpha} = \alpha \perp \gamma \perp (\delta_1\delta_2) \perp \varepsilon_1 \perp \varepsilon_2$$

and

$$\bar{\beta} = \beta \perp \gamma \perp \delta_1 \perp \delta_2 \perp (\varepsilon_1\varepsilon_2)$$

are isomorphic Write $n = n(\alpha)$ if $\alpha \in G(A^n)$, and similarly for $\beta, \gamma, \ldots$. Our hypothesis shows that $n(\alpha)$ is well defined and that

$$n(\alpha) + n(\gamma) + n(\delta_1\delta_2) + n(\varepsilon_1) + n(\varepsilon_2)$$

$$= n(\beta) + n(\gamma) + n(\delta_1) + n(\delta_2) + n(\varepsilon_1\varepsilon_2).$$

Since $n(\alpha) = n(\beta)$, $n(\varepsilon_1) = n(\varepsilon_2) = n(\varepsilon_1\varepsilon_2)$ and $n(\delta_1) = n(\delta_2) = n(\delta_1\delta_2)$, we conclude that $n(\delta_i) = n(\varepsilon_i)$; call this integer p, and write $m = n(\gamma)$.

□

Since $\bar{\alpha} \approx \bar{\beta}$, we have $\bar{\alpha} \perp 1_{A^p} \approx \bar{\beta} \perp 1_{Ap}$. Both of these are in $G(A^{n+m+4p})$, and we can conjugate by suitable permutations of the factors to obtain

$$\alpha' = \alpha \perp \gamma \perp (\delta_1\delta_2) \perp 1_{Ap} \perp \varepsilon_1 \perp \varepsilon_2$$

and

$$\beta' = \beta \perp \gamma \perp \delta_1 \perp \delta_2 \perp (\varepsilon_1\varepsilon_2) \perp 1_{Ap}.$$

27 Now, two elements of $G(A^{n+m+4p})$ are isomorphic if and only if they are conjugate (recall the definition, in §1, of isomorphism in $\Omega\mathscr{C}$). Therefore α' and β' are conjugates. This completes the proof of (b).

Moreover, $\beta'^{-1}\alpha' = (\beta^{-1}\alpha) \perp 1_{A^m} \perp \delta_2 \perp \delta_2^{-1} \perp \varepsilon_2^{-1} \perp \varepsilon_2$ is a commutator. Conjugating by a permutation of factors, we find that $(\beta^{-1}\alpha) \perp 1_{A^m} \perp (\delta_2 \perp \varepsilon_2) \perp (\delta_2 \perp \varepsilon_2)^{-1}$ is a commutator. Since we could have chosen $m = 2m'$, we could take $\gamma_1 = 1_{A^m} \perp \delta_2 \perp \varepsilon_2$, and a further conjugation shows that

$$(\beta^{-1}\alpha) \perp \gamma_1 \perp \gamma_1^{-1}$$

is a commutator. Assuming $\beta = 1_{A^n}$ we have proved the first part of $(c) : \alpha \perp \gamma_1 \perp \gamma_1^{-1}$ is a commutator. Since $\alpha \perp \gamma_1 \perp \gamma_1^{-1}$ is conjugate to $\alpha \perp \gamma_1^{-1} \perp \gamma_1$, their product $\alpha^2 \perp 1_A 2m_1, m_1 = n(\gamma_1)$, is a product of two commutators. This proves the last assertion in (c).

4 The exact sequence

Throughout this section $F : \mathscr{C} \to \mathscr{C}'$ denotes a cofinal functor of categories with product.

We define

$$d : K_0\Phi F \to K_0\mathscr{C}$$

to be the homomorphism induced by the map $(A, \alpha, B) \mapsto (A)_{\mathscr{C}} - (B)_{\mathscr{C}}$ from obj ΦF to $K_0\mathscr{C}$. This is clearly additive with respect to $\perp$ to 0 **28**
in ΦF so it does define a homomorphism d. The composite of d and $K_0F : K_0\mathscr{C} \to K_0\mathscr{C}'$ sends $(A, \alpha, B)_{\Phi F}$ to $(FA)_{\mathscr{C}'} - (FB)_{\mathscr{C}'}$, which is zero, since FA and FB are isomorphic.

Suppose $(A)_{\mathscr{C}} - (B)_{\mathscr{C}} \in \ker K_0F$. Using Proposition 1.1, we can find a $C' \in \mathscr{C}'$ and an $\alpha : Fa \perp C' \to FB \perp C'$. Cofinality of F permits us to choose $C' = FC$ for some $C \in \operatorname{obj} \mathscr{C}$. Then d maps $(A \perp C, \alpha, B \perp C)$ into $(A)_{\mathscr{C}} - (B)_{\mathscr{C}}$. Thus we have proved that the sequence

$$K_0\Phi F \xrightarrow{d} K_0\mathscr{C} \xrightarrow{K_0F} K_0\mathscr{C}'$$

is exact.

Let $\mathscr{C}_1$ denote the full subcategory of $\mathscr{C}'$ whose objects are all FA, $A \in \operatorname{obj} \mathscr{C}$. By Theorem 3.1 (b), we have an isomorphism

$$\theta : K_1\mathscr{C}_1 \to K_1\mathscr{C}'.$$

Let

$$d_1 : K_1\mathscr{C}_1 \to K_0\Phi F$$

be the homomorphism induced by the map $(FA, \alpha) \mapsto (A, \alpha, A)_{\Phi F}$ from obj $\Omega\mathscr{C}_1$ to $K_0\Phi F$. This map is additive with respect to $\perp$ and 0 in $\Omega\mathscr{C}_1$, so that d_1 is well defined. The composite $d \circ d_1$ sends $(FA, \alpha)_{\Omega\mathscr{C}_1}$ to $(A)_{\mathscr{C}} - (A)_{\mathscr{C}} = 0$. Thus $d \circ d_1 = 0$.

We define now a homomorphism **29**

$$d' : K_1\mathscr{C}' \to K_0\Phi F$$

by setting $d' = d_1 \circ \theta^{-1}$.

Clearly $d \circ d' = 0$. Suppose $(A, \alpha, B)_{\Phi F} \in \ker d$. Then, by Proposition 1.1(b), there is an isomorphism $\beta : A \perp C \to B \perp C$ for some $C \in \operatorname{obj} \mathscr{C}$. We have then a commutative diagram

$$\begin{tikzcd}
FA \perp FC \arrow[r, "\alpha \perp 1_{FC}"] \arrow[d, "F1_{A \perp C}"'] & FB \perp FC \arrow[d, "F\beta^{-1}"] \\
FA \perp FC \arrow[r, "\alpha'"'] & FA \perp FC
\end{tikzcd}$$

for a suitable α', showing that the triples $(A \perp C, \alpha \perp 1_{FC}, B \perp C)$ and $(A \perp C, \alpha', A \perp C)$ are isomorphic. Thus

$$(A, \alpha, B)_{\Phi F} = (A \perp C, \alpha \perp 1_{FC}, B \perp C)_{\Phi F} = (A \perp C, \alpha', A \perp C)_{\Phi F}.$$

The third member is the image of $(F(A \perp C), \alpha')_{\Omega \mathscr{C}_1}$ by d_1.

Hence the sequence

$$K_1 \mathscr{C}' \xrightarrow{d'} K_0 \Phi F \xrightarrow{d} K_0 \mathscr{C}$$

is exact.

30 Next we note that $d' \circ K_1 F = 0$. This follows from the fact that $d' \circ K_1 F$ sends $(A, \alpha)_{\Omega \mathscr{C}}$ to $(A, F, \alpha, A)_{\Phi F}$ and the triples $(A, F\alpha, A)$ and $(A, 1_{FA}, A)$ are isomorphic in view of the commutative diagram

$$\begin{tikzcd}
FA \arrow[r, "F\alpha"] \arrow[d, "F\alpha"'] & FA \arrow[d, "F1_A"] \\
FA \arrow[r, "1_{FA}"'] & FA
\end{tikzcd}$$

Theorem 4.6. *If $F : \mathscr{C} \to \mathscr{C}'$ is a cofinal functor of categories with product, then the sequence*

$$K_1 \mathscr{C} \xrightarrow{K_1 F} K_1 \mathscr{C}' \xrightarrow{d'} K_0 \Phi F \xrightarrow{d} K_0 \mathscr{C} \xrightarrow{K_0 F} K_0 \mathscr{C}'$$

is exact.

We have only to show that $\ker d' \subset \operatorname{im} K_1 F$. For this we need an effective criterion for recognizing the triples (A, α, B) with the property $(A, \alpha, B)_{\Phi F} = 0$. This is given in Lemma 4.7 below, for which we now prepare.

In $\Omega \mathscr{C}'$ let $\mathscr{E}$ denote the smallest class of objects such that

(i) $\alpha \approx \beta, \alpha \in \mathscr{E} \Rightarrow \beta \in \mathscr{E}$

(ii) $\alpha, \beta \in \mathscr{E} \Rightarrow \alpha \perp \in \mathscr{E}$

(iii) $\alpha, \beta \in \mathscr{E}, \alpha \circ \beta$ defined $\Rightarrow \alpha \circ \beta \in \mathscr{E}$

(iv) $(FA, 1_{FA})$, $(FA \perp FA, t) \in \mathscr{E}$ for all A, t being the transposition.

31 These properties imply the following:

(v) If $\alpha \in \mathscr{E}$, then $(\alpha)_{\Omega\mathscr{C}'} \in \operatorname{im} K_1 F \subset \ker d'$.

We need only note in (*iv*), that $''t = Ft''$, with the obvious abuse of notation.

(vi) $(FA \perp \cdots \perp FA, s) \in \mathscr{E}$ for any permutation s.

Using (*i*), (*ii*), (*iii*) and (*iv*), this reduces easily to the fact that transpositions generate the symmetric group.

(vii) If $\alpha : FA \to FB$ and $\beta : FB \to FC$, then

$$(FA \perp FB, t)(\alpha \perp \alpha^{-1})) \in \mathscr{E}$$

and

$$(FA \perp FB \perp FC, s(\alpha \perp \beta(\beta\alpha)^{-1})) \in \mathscr{E},$$

where t and S are the appropriate transposition and 3-cycle respectively.

This statement follows from (*i*), (*iv*) and Lemma 1.5.

Now in ΦF we call an object of the form (A, α, A) an *automorphism*. We call it *elementary* if $(FA, \alpha) \in \mathscr{E}$. For any $\alpha = (A, \alpha, B)$, we write

$$\alpha \sim 1$$

if $\alpha \perp 1_{FC} \approx \varepsilon$ for some $c \in \mathscr{C}$ and some elementary automorphism ε.

We also write

$$\alpha \sim \beta$$

if and only if $\alpha \perp \beta^{-1} \sim 1$.

Lemma 4.7. *For $\alpha,\beta \in \Phi F$, $(\alpha)_{\Phi F} = (\beta)_{\Phi F} \Leftrightarrow \alpha \sim \beta$. In particular,* 32
$(\alpha)_{\Phi F} = 0 \Leftrightarrow \propto \sim 1$.

Before proving this lemma, let us use it to finish the

Proof of Theorem 4.6. Given $(FA, \propto)$ such that $(\alpha)_{\Omega\mathscr{C}'} \in \ker d'$ we have to show that $(\alpha)_{\Omega\mathscr{C}'} \in \operatorname{im} K_1F$. The hypothesis means that $(\alpha)_{\Phi F} = 0$, so that by Lemma 4.7, there is a $c \in \operatorname{obj}\mathscr{C}$, an elementary automorphism $\varepsilon = (E, \varepsilon, E)$, and an isomorphism $(f, g) : \alpha \perp 1_{FC} \to \varepsilon$. This means that the diagram

$$\begin{tikzcd}
FA \perp FC \arrow[r, "\alpha \perp 1_{FC}"] \arrow[d, "F\bar{f}"'] & FA \perp FC \arrow[d, "Fg"] \\
FE \arrow[r, "\varepsilon"'] & FE
\end{tikzcd}$$

is commutative. Hence $\alpha \perp 1_{FC} = Fg^{-1}Ff(Ff)^{-1}\varepsilon Ff = F(g^{-1}f)\varepsilon'$, where $\varepsilon' = (Ff)^{-1}\varepsilon Ff \approx \varepsilon$ in $\Omega\mathscr{C}'$. By properties (i) and (v) above, $(\varepsilon')_{\Omega\mathscr{C}'} \in \operatorname{im} K_1F$, so we have $(\alpha)_{\Omega\mathscr{C}'} = (\alpha \perp 1_{FC})_{\Omega\mathscr{C}'} = (F(g^{-1}f))_{\Omega\mathscr{C}'} + (\varepsilon)_{\Omega\mathscr{C}'} \varepsilon \operatorname{im} K_1F$, as required.

Proof of Lemma 4.7. If $\alpha \sim \beta$, then $(\alpha)_{\Phi F} = (\beta)_{\Phi F}$ by virtue of (v) above. For the converse, we will prove:

(a) $\sim$ is an equivalence relation

(b) $\perp$ induces a structure of abelian group on $M = \operatorname{obj}\Phi F/\sim$.

(c) $\alpha \circ \beta \sim \alpha \perp \beta$ whenever $\alpha \circ \beta$ is defined.

33 Once shown, these facts imply that the map obj $\Phi F \to M$ satisfies the axioms for $K_0\Phi F$, so it induces a homomorphism $K_0\Phi F \to M$, which is evidently surjective. Injectivity follows from the first part of the proof above

(1) If α and β are elementary automorphisms, then so are α^{-1}, $\alpha \perp \beta$, and $\alpha \circ \beta$ (if defined).

This is obvious.

(2) If $\beta \sim 1$ and $\alpha \perp \beta \sim 1$, then $\alpha \sim 1$.

For, by adding an identity to β, we can find elementary automorphisms $\varepsilon_1 = (E_1, \varepsilon_1, E_1)$ and $\varepsilon = (E, \varepsilon, E)$ and an isomorphism $(f, g) : \alpha \perp \varepsilon_1 \to \varepsilon$. Thus

$$\begin{tikzcd} FA \perp FE_1 \arrow[r, "\alpha\perp\varepsilon_1"] \arrow[d, "Ff"'] & FB \perp FE_1 \arrow[d, "Fg"] \\ FE \arrow[r, "\varepsilon"'] & FE \end{tikzcd}$$

commutes. Set $\varepsilon_2 = (A \perp E_1, 1_{FA} \perp \varepsilon_1^{-1}, A \perp E_1)$; $\varepsilon_2 = 1_{FA} \perp \varepsilon_1^{-1}$ is clearly elementary. Set $\varepsilon_2' = (E, Ff\varepsilon_2 Ff^{-1}, E)$. Since $Ff : (FA \perp FE_1, \varepsilon_2) \to (FE, \varepsilon_2')$ in $\Omega\mathscr{C}'$, ε_2' is also an elementary automorphism. Moreover, we have

$$(f, g) : (\alpha \perp \varepsilon_1) \circ \varepsilon_2 \to \varepsilon \circ \varepsilon_2',$$

clearly, and $(\alpha \perp \varepsilon_1) \circ \varepsilon_2 = \alpha \perp 1_{FE_1}$. Since $\varepsilon \circ \varepsilon_2'$ is elementary, we have shown $\alpha \sim 1$, as claimed.

If $\alpha = (A, \alpha, B)$ and $\beta = (B, \beta, C)$, then **34**

(4) $\alpha \perp \alpha^{-1} \sim 1$,

and

(5) $\alpha \perp \beta \perp (\beta\alpha)^{-1} \sim 1$.

For,

$$(1_{A\perp B}, t) : \alpha \perp \alpha^{-1} \to (A \perp B, t(\alpha \perp \alpha^{-1}), A \perp B)$$

and

$$\begin{gathered}(1_{A\perp B\perp C}, s) : \alpha \perp \beta \perp (\beta\alpha)^{-1} \\ \to (A \perp B \perp C, s(\alpha \perp \beta \perp (\beta\alpha)^{-1}, A \perp B \perp C),\end{gathered}$$

and the latter are elementary by property (vii) of $\mathscr{E}$.

Now, for the proof of (a), we note that (4) $\Rightarrow$ reflexivity, (1) $\Rightarrow$ symmetry, and (3) plus (4) $\Rightarrow$ transitivity. The statements (*b*) and (*c*) follow respectively from (1) and (5).

5 The category $\underset{=}{P}$

Let k be a commutative ring. We define $\underset{=}{P}(k)$ (or $\underset{=}{P}$) to be the category of finitely generated projective k-modules and their isomorphisms, with product $\oplus$.

The groups $K_i\underset{=}{P}$ are denoted in [K, §12] by $K_i(k)$. Strictly speaking, the definitions do not coincide since the $K_i(k)$ are defined in terms of exact sequences, and not just $\oplus$. Of course this makes no difference for K_0 since all sequences split. For K_1, however, the exact sequences of automorphisms $0 \to \alpha' \to \alpha \to \alpha'' \to 0$ need not split. In terms of matrices this means that α has the form

$$\alpha = \begin{pmatrix} \alpha' & \beta \\ 0 & \alpha'' \end{pmatrix}$$

35 It is clear that α can be written in the form $\alpha = (\alpha' \oplus \alpha'')\varepsilon'$, where ε is of the form $\left(\begin{smallmatrix} id & \gamma \\ 0 & id \end{smallmatrix}\right)$, and the equivalence of the two definitions results from the fact that $(\varepsilon)_{\Omega\underset{=}{P}} = 0$ in $K_1\underset{=}{P}$. The last fact is seen by adding a suitable identity automorphism to ε to put it in $GL(n,k)$, for some n, and then writing the result as a product of elementary matrices (see (5.3) below).

We summarize now some results from [K].

The tensor product $\oplus_k$ is additive with respect to $\oplus$ so that it induces on $K_0\underset{=}{P}$ a structure of commutative ring.

If $\mathscr{Y} \in \operatorname{spec}(k)$ and $P \in \underset{=}{P}$, then $P_{\mathscr{Y}}$ is a free $k_{\mathscr{Y}}$- module and its rank is denoted by $rk_p(\mathscr{Y})$. The map

$$rk_p : \text{ spec } (k) \to \mathbb{Z},$$

given by $\mathscr{Y} \to rk_p(\mathscr{Y})$, is continuous, and is called the *rank* of P. Since $rk_{P\oplus Q} = rk_P + rk_Q$ and $rk_{P\oplus Q} = rk_P rk_Q$, we have a *rank homomorphism*

$$rk : K_0\underset{=}{P} \to C,$$

where C is the ring of continuous functions spec $(k) \to \mathbb{Z}$.

(5. 1) The rank homomorphism rk is split by a ring homomorphism $C \to K_0\underset{=}{P}$, so that we can write

$$K_0\underset{=}{P} \approx \oplus \tilde{K}_0\underset{=}{P},$$

where $\tilde{K}_0\underline{\underline{P}} = \ker(rk)$. $\tilde{K}_0\underline{\underline{P}}$ is a nil ideal.

This result is contained in [K, Proposition 15.4]. **36**

(5.2) *Suppose* $\max(k)$ *the space of maximal ideals of k, is a noetherian space of dimension d. Then*

(a) If $x \in K_0\underline{\underline{P}}$ and $rk(x) \geq d$, then $x = (P)_{\underline{\underline{P}}}$ for some $P \in \underline{\underline{P}}$.

(b) If $rk(P)_{\underline{\underline{P}}} > d$ and if $(P)_{\underline{\underline{P}}} = (Q)_{\underline{\underline{P}}}$, then $P \approx Q$.

(c) $(\tilde{K}_0\underline{\underline{P}})^{d+1} = 0$.

Since k is a basic object for $\underline{\underline{P}}$ in the sense of §3, we deduce immediately from Theorem 3.2 and [K, Theorem 3.1 and Proposition 12.1], that

(5.3) There is a natural isomorphism

$$K_1\underline{\underline{P}} \approx GL(k)/[GL(K), GL(k)],$$

where $GL(k) = \varinjlim GL(n,k)(= AutK^n)$ with respect to the maps $\alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & I_m \end{pmatrix}$ from $GL(n,k)$ to $GL(n+m,k) \cdot [GL(k), GL(k)] = E(k)$, the group generated by all elementary matrices in $GL(k)$, we have also $E(k) = [E(k), E(k)]$. The determinant map $\det : GL(K) \to U(k)$ is split by $U(k) \to GL(k)$ (defined via $GL(1,K)$). Thus we have a natural decomposition

$$K_1\underline{\underline{P}} \approx U(k) \oplus SK_1\underline{\underline{P}},$$

where $SK_1\underline{\underline{P}} \approx SL(k)/E(k) = SL(k)/[SL(k), SL(k)]$.

We have also the following interesting consequence of Theorem 3.2. **37**

(5.4) If $\alpha \in [GL(n,k), GL(n,k)]$, then for some m and some $\gamma \in GL(m,k)$, $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma^{-1} \end{pmatrix}$ is a commutator in $GL(n+2m,k)$ and $\begin{pmatrix} \alpha^2 & 0 \\ 0 & I_{2m} \end{pmatrix}$ is a product of two commutators.

6 The category $\underline{\underline{FP}}$

Let k be a commutative ring.

Proposition 6.1. *The following conditions on a k-module P are equivalent:*

(a) P is finitely generated, projective, and has zero annihilator.

(b) P is finitely generated, projective, and has every where positive rank (that is $P_{\mathscr{Y}} \neq 0$ for all $\mathscr{Y} \in \operatorname{spec}(k)$).

(c) There exists a module Q and an $n > 0$ such that $P \otimes_k Q \approx k^n$.

Proof. The equivalence $(a) \Leftrightarrow (b)$ is well known. □

$(b) \Rightarrow (c)$. The module P is "defined over" a finitely generated subring k_0 of k. By this we mean that there exists a finitely generated projective k_0-module P_0 such that $P \approx k \otimes_{k_0} P_0$. To see this, we express P as the cokernel of an idempotent endo-morphism of a free k-module k^n. Let α be the matrix of this endomorphism with respect to the canonical
38 basis of k^n. We take for k_0, the subring of k generated by the entries of α. It is easily seen that P_0 can be takes to be the cokernel of the endomorphisms of k_0^n determined by α.

So we can assume that k is noetherian with $\dim\max(k) = d < \infty$. Let $x = (P)_{\underline{\underline{P}}} \in K_0\underline{\underline{P}}$. Then $rk(x)$ is a positive continuous functions spec $(k) \to \mathbb{Z}$, and it takes only finitely many values, since spec (k) is quasi - compact. Hence we can find $y \in C$ (in the notation of (5.1)) such that $rk(x)y = m > 0$ (the constant function m). Now $x = rk(x) - z'$ with z' nilpotent, so that $xy = m - z$ with $z = yz'$ nilpotent. It follows that $n = m^h = wxy$ for some $h > 0$ and $\omega \in K_0\underline{\underline{P}}$; for instance we can take $h \geq d + 1$, in view of (5.2) (c). By enlarging h we can make $rk(wy) > d$. Then we have $wy = (Q)_{\underline{\underline{P}}}$ for some Q by (5.2) (a), it follows that $P \otimes_k Q \approx k^n$.

$(c) \Rightarrow (a)$. Assume $P \otimes_k Q \approx k^n$. There is a finite set of elements $x_1, \ldots, x_p \in P$ such tht $P \otimes_k Q = \sum_{i=1}^{p} x_i \otimes Q$. We have then a homomorphisms $f : F \to P, F$ a free k-module of finite rank, such that $f \otimes 1_Q : F \otimes_K Q \to P \otimes_k Q$ is surjective and therefore splits. Hence $f \otimes 1_{Q\otimes P} : F \otimes Q \otimes P \to P \otimes Q \otimes P$ is surjective and splits. Thus $P \otimes k^n (\approx P \otimes Q \otimes P)$, being a direct summand of $F \otimes k^n (\approx F \otimes Q \otimes P)$, is finitely generated and projective. It follows that P is finitely generated and projective.

That P has zero annihilator is clear.

Remark. The argument in $(b) \Rightarrow (c)$ can be used to show, more precisely, that if P is a finitely generated projective $k-$module of constant rank $r > 0$, then $P \otimes_k Q \approx k^{r^{d+1}}$ for some projective k-module Q and some $d \geq 0$. If max(k) is a noeterian space of finite dimension, then this number can be chosen for d. 39

Modules satisfying (a), (b) and (c) above will be called *faithfully projective*. They are stable under $\otimes (= \otimes_k)$. The faithfully projective modules together with their isomorphisms form a category

$$\underline{\underline{FP}}(k) \quad (\text{or } \underline{\underline{FP}})$$

with product $\otimes$, in the sense of §1. Condition (c) in the proposition above shows that *the free modules are cofinal in* $\underline{\underline{FP}}$. We propose now to calculate the groups $K_i\underline{\underline{FP}}$.

We write

$$Q \otimes_z K_0\underline{\underline{P}} = (Q \otimes_{\mathbb{Z}} C) \oplus (Q \otimes_{\mathbb{Z}} \tilde{K}_0\underline{\underline{P}})$$

in the notation of (5.1). Thus $\mathbb{Q} \otimes_{\mathbb{Z}} C$ is the ring continuous functions from spec(k) (discrete) $\mathbb{Q}$. Let $U^+(\mathbb{Q} \otimes_{\mathbb{Z}} K_0\underline{\underline{P}})$ denote the unit whose "rank" (= projection on $\mathbb{Q} \otimes C$) is a positive function.

Theorem 6.2. $K_0F\underline{\underline{P}} \approx U^+(\mathbb{Q} \otimes_{\mathbb{Z}} K_0\underline{\underline{P}})$ 40

$$\approx U^+(\mathbb{Q} \otimes_{\mathbb{Z}} C) \oplus (\mathbb{Q} \otimes_{\mathbb{Z}} \tilde{K}_0\underline{\underline{P}}).$$

Example. Suppose spec(k) is connected, so that $C = \mathbb{Z}$. The

$$K_0\underline{\underline{FP}} \approx \text{ (positive rationals) } \oplus (\mathbb{Q} \otimes_{\mathbb{Z}} \tilde{K}_0\underline{\underline{P}}),$$

the direct sum of free abelian group and a vector space over $\mathbb{Q}$.

Proof. If P is faithfully projective, then $P \otimes Q \approx k^n$ for some $n > 0$, so that $(P)_P(Q)\underline{\underline{P}} = n$ in $K_0\underline{\underline{P}}$. It follows that $1 \otimes (P)_{\underline{\underline{P}}} \in U^+(\mathbb{Q} \otimes_{\mathbb{Z}} K_0\underline{\underline{P}})$, and this homomorphism, being multiplicative with respect to $\otimes$, defines a homomorphism

$$K_0\underline{\underline{FP}} \to U^+(\mathbb{Q} \otimes_{\mathbb{Z}} K_0\underline{\underline{P}}). \tag{6.3}$$

□

We first show that this map is surjective. Any element of the right hand side can be written as $\dfrac{1}{n} \otimes x$, $x \in K_0\underline{\underline{P}}$, and $rk(x)$ is a positive function of spec (k) into $\mathbb{Z}$. Since x is defined over a finitely generated subring of k, we can assume without loss of generality, that k is finitely generated with max(k) of dimension d, say. By increasing n by a multiple we can make $rk(x)$ exceed d, so that $x = (P)_{\underline{\underline{P}}}$ so some $P \in \underline{\underline{P}}$ by (5.2)(a). Clearly $P \in \underline{\underline{FP}}$. Thus $\dfrac{1}{n} \otimes x = (1 \otimes (k^n)_{\underline{\underline{P}}})^{-1}(1 \otimes (P)_{\underline{\underline{P}}})$ is in the image of (6.3).

41 Next we prove the injectivity of (6.3). Suppose $1 \otimes (P)_{\underline{\underline{P}}} = 1 \otimes (Q)_{\underline{\underline{P}}}$. Then, for some integer $n > 0$, $n((P)_{\underline{\underline{P}}} - (Q)_{\underline{\underline{P}}}) = 0$, so that $(k^n \otimes_k P)_{\underline{\underline{P}}} = (k^n \otimes_k Q)_{\underline{\underline{P}}}$. By choosing n large we can make rank $(k^n \otimes_k P)$ large and then invoke (5.2)(b) to obtain $k^n \otimes P \approx k^n \otimes Q$. Hence $(P)_{\underline{\underline{FP}}} = (Q)_{\underline{\underline{FP}}}$. This establishes the first isomorphism in the theorem.

To prove the second isomorphism, we note that

$$U^+(\mathbb{Q} \otimes_{\mathbb{Z}} K_0\underline{\underline{P}}) = U^+(\mathbb{Q} \otimes_{\mathbb{Z}} C) \times (1 + (\mathbb{Q} \otimes_{\mathbb{Z}} \tilde{K}_0\underline{\underline{P}})),$$

and, since $\mathbb{Q} \otimes_{\mathbb{Z}} \tilde{K}_0 P$ is a nil algebra over $\mathbb{Q}$, we have an isomorphism

$$\exp : \mathbb{Q} \otimes_{\mathbb{Z}} \tilde{K}_0 P \to 1 + (\mathbb{Q} \otimes_{\mathbb{Z}} \tilde{K}_0 P).$$

In order to compute $K_1\underline{\underline{FP}}$, we prove a general lemma about direct limits. Let

$$L = (W_n, f_{n,nm} : W_n \to W_{nm})_{n,m,\in\mathbb{N}}$$

be a direct system of abelian groups, indexed by the positive integers, ordered by divisibility. We introduce an associated direct system

$$L' = (W_n, f'_{n,\, nm} : W_n \to W_{nm}),$$

where $f'_{n,nm} = m f_{n,\,nm}$, and a homomorphism

$$(n.1_{W_n}) : L \to L'$$

of direct systems. For the latter we note that **42**

$$\begin{tikzcd} W_n \arrow[r, "n\cdot 1_{W_n}"] \arrow[d, "f_{n,nm}"'] & W_n \arrow[d, "m f_{n,nm} = f^1_{n,nm}"] \\ W_{nm} \arrow[r, "nm1_{W_{n,m}}"'] & W_{nm} \end{tikzcd}$$

is commutative. L' is a functor of L. We have an exact sequence of direct systems

$$L \to L' \to L'' \to 0, \tag{6.4}$$

where $L'' = (W_n/W_{nm}, f''_{n,nm})$ is the cokernel of $L \to L'$.

Lemma 6.5. *With the notation introduced above, the exact sequences*

$$\varinjlim L \to \varinjlim L' \to \varinjlim L'' \to 0$$

and

$$\varinjlim L \otimes (\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0)$$

are isomorphic. (here $\otimes = \otimes_{\mathbb{Z}}$*.).*

Proof. Let $E = (\mathbb{Z}_n, e_{n,nm})$ with $\mathbb{Z}_n = \mathbb{Z}$ and $e_{n,\,nm} = 1_{\mathbb{Z}}$ for all $n, m \in \mathbb{N}$. Evidently the exact sequence of direct systems

$$L \to L' \to L'' \to 0$$

and

$$L \otimes (E \to E' \to E'' \to 0)$$

are isomorphic. The lemma now follows from the fact that $\varinjlim E = \mathbb{Z}$, **43**
$\varinjlim E' = \mathbb{Q}$, and standard properties of direct limit. □

Theorem 3.1 allows us to compute $K_1\underline{\underline{FP}}$ using only the free modules. Let

$$W_n = GL(n,k)/[GL(n,k), GL(n,k)]$$

and let $f_{n,nm}$ and $g_{n,nm}$ be the homomorphisms $W_n \to W_{nm}$, induced respectively by $\alpha \mapsto \begin{pmatrix} \alpha_{I_n} & & 0 \\ & \ddots & \\ 0 & & In \end{pmatrix}$ and $\alpha \to \begin{pmatrix} \alpha & & 0 \\ & \ddots & \\ 0 & & \alpha \end{pmatrix}$ from $GL(n,k)$ to $GL(nm,k)$. Then it follows from theorem 3.1 that

$$K_1\underline{\underline{P}} = \varinjlim (W_n,\ f_{n,nm})$$

and

$$K_1\underline{\underline{FP}} = \varinjlim (W_n, g_{n,nm}).$$

Lemma 6.6. *If $\alpha \in GL(n,k)$ and if $nm \geq 3$, then*

$$\begin{pmatrix} \alpha^m & & & 0 \\ & I_n & & \\ & & \ddots & \\ 0 & & & I_n \end{pmatrix} \equiv \begin{pmatrix} \alpha & & & 0 \\ & \alpha & & \\ & & \ddots & \\ 0 & & & \alpha \end{pmatrix} \quad \text{mod } [GL(n,k), GL(n,k)].$$

(See [K, Lemma 1.7]).

It follows from lemma 6.6, that $g_{n,nm} = f'_{n,nm} = m f_{n,nm}$, and hence, using lemma 6.5, we have the following

44 **Theorem 6.7.** $K_1\underline{\underline{FP}} \approx \mathbb{Q} \otimes_{\mathbb{Z}} K_1\underline{\underline{P}}$

$$\approx (\mathbb{Q} \otimes_{\mathbb{Z}} U(k)) \oplus (\mathbb{Q} \otimes_{\mathbb{Z}} SK_1\underline{\underline{P}}).$$

If we pass to the limit before abelianizing, we obtain the groups

$$GL_{\otimes}(k) = \varinjlim (GL(n,k), \alpha \mapsto \alpha \otimes I_m)_{n,m\in\mathbb{N}},$$

which consists of matrices of the type

$$\begin{pmatrix} \alpha & & & 0 \\ & \ddots & & \\ & & \alpha & \\ 0 & & & \ddots \end{pmatrix}$$

where α is in $GL(n,k)$ for some n. The centre of this group consists of scalar matrices (the case $n = 1$) and is isomorphic to $U(k)$. We write

$$PGL(k) = GL_{\otimes}(k)/\text{ centre } = GL_{\otimes}(k)/U(k).$$

Now

$$K_1\underline{\underline{FP}} = GL_{\otimes}(k)/[GL_{\otimes}(k), GL_{\otimes}(k)] = (\mathbb{Q} \otimes_{\mathbb{Z}} U(k)) \oplus (\mathbb{Q} \otimes_{\mathbb{Z}} SK_1\underline{\underline{P}})$$

and we have projective on the first summand

$$\det' : K_1\underline{\underline{FP}} \to \mathbb{Q} \otimes_{\mathbb{Z}} U(k),$$

which is induced by the determinant. Explicitly, if $\alpha \in GL(n,k)$, then

$$\det' \begin{pmatrix} \alpha & & & 0 \\ & \ddots & & \\ & & \alpha & \\ 0 & & & \ddots \end{pmatrix} = \frac{1}{n} \otimes \det \alpha.$$

This evaluates det′, in particular, on elements of the centre (the case. **45**
$n = 1$); so we see easily that:

$$\operatorname{coker}\,(U(k) \to K_1\underline{\underline{FP}}) \tag{6.8}$$

$$\begin{aligned} &= (\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} U(k)) \oplus (\mathbb{Q} \otimes_{\mathbb{Z}} SK_1\underline{\underline{P}}) \\ &= PGL(k)/[PGL(k), PGL(k)] \\ &= \varinjlim PGL(n,k)/[PGL(n,k), PGL(n,k)], \end{aligned}$$

where the maps are induced by the homomorphisms $\alpha \mapsto \alpha \otimes I_m$ *from* $GL(n,k)$ *to* $GL(nm,k)$.

7 The category $\underline{\underline{\mathrm{Pic}}}$

$\underline{\underline{\mathrm{Pic}}}(k)$ (or $\underline{\underline{\mathrm{Pic}}}$) is the full subcategory of $\underline{\underline{FP}}$ consisting of projective k-modules of rank one, with $\otimes_k$ as product. We shall denote $K_0\underline{\underline{\mathrm{Pic}}}$ by Pic (k).

A module P in $\underline{\underline{\mathrm{Pic}}}$ satisfies

$$P \otimes_k P^* \approx k,$$

where $P^* = \mathrm{Hom}_k(P, k)$. So any object of $\underline{\underline{\mathrm{Pic}}}$, in particular k, is cofinal. Theorem 3.1 then shows that

$$K_1 \underline{\underline{\mathrm{Pic}}} \approx \text{ Aut } (k) \approx U(k).$$

The inclusion $\underline{\underline{\mathrm{Pic}}} \subset \underline{\underline{FP}}$ induces homomorphisms

$$\mathrm{Pic}(k) \to K_0 \underline{\underline{FP}} \tag*{$(7.1)_0$}$$

46 and

$$U(k) \to K_1 \underline{\underline{FP}}. \tag*{$(7.1)_1$}$$

The latter is induced by $U(k) = GL(1, k) \subset GL_\otimes(k)$, which identifies $U(k)$ with the centre of $GL_\otimes(k)$. So the co-kernel is $PGL(k)$. Thus, we have from (6.8),

$$\begin{aligned} \text{coker } (7.1)_1 &\approx (Q/\mathbb{Z} \otimes_\mathbb{Z} U(k)) \oplus (\mathbb{Q} \otimes_\mathbb{Z} SK_1 \underline{\underline{P}}) \\ &\approx PGL(k)/[PGL(k), PGL(k)], \end{aligned} \tag{7.2}$$

and

$$\begin{aligned} \ker(7.1)_1 = &\text{ the torsion subgroup of } U(k) \\ &\text{(that is, the roots of unity in } k). \end{aligned}$$

The last assertion follows from the fact that $(7.1)_1$ is the natural map $U(k) \to \mathbb{Q} \otimes_\mathbb{Z} U(k)$ followed by the inclusion of the latter into $K_1 \underline{\underline{FP}} = (\mathbb{Q} \otimes_\mathbb{Z} U(k)) \oplus (\mathbb{Q} \otimes_\mathbb{Z} SK_1 \underline{\underline{P}})$.

(7.3) *The kernel of the natural map* $(7.1)_0$; $\mathrm{Pic}(k) \to K_0 \underline{\underline{FP}}$ *is the torsion subgroup of* $\mathrm{Pic}(k)$.

Proof. If $(L)_{\underline{\underline{\mathrm{Pic}}}} \in \ker\, (7.1)_0$, then $L \otimes_k P \approx k \otimes_k P \approx P$ for some $P \in \underline{\underline{FP}}$. By (6.1)(*c*), we can choose P to be k^n, in which case we have $L \otimes \cdots \otimes L \approx k^n$. Taking n^{th} exterior powers, we get $L \otimes \cdots \otimes L \approx k$, so that $(L)_{\underline{\underline{\mathrm{Pic}}}}$ is a torsion element in $\mathrm{Pic}(k)$. □

Conversely, suppose $(L)_{\underline{\underline{\mathrm{Pic}}}}$ has order n, that is that $L \otimes \cdots \otimes L \approx k$.
47 We have to show that $(L)_{\underline{\underline{FP}}} = 0$. This amount to showing that $L \otimes P \approx P$ for some P in $\underline{\underline{FP}}$. It is immediate that we can take for P, the module $k \oplus L \oplus L^{\otimes 2} \oplus \cdots \oplus L^{\otimes(n-1)}$, where $L^{\otimes i}$ denotes the i-fold tensor product of L with itself.

Chapter 2

Categories of modules and their equivalences

In this chapter we first characterize (up to equivalence) categories of 48
modules as abelian categories with arbitrary direct sums and having a faithfully projective object. Then we show that any equivalence from the category A-mod of left modules over a ring A into the category B-mod for another ring B, is of the form $P\otimes_A$, where P is a B-A-bimodule, unique up to isomorphism. We deduce a number of consequence of the existence of such an equivalence, and we characterize the modules P that can arise in this manner. A detailed account of the Wedderburn structure theory for semi-simple algebras is obtained in this context, Finally, for an algebra A over a commutative ring k, the study of autoequivalences of A-mod leads to the introduction of a group $\mathrm{Pic}_k(A)$ for a k-algebra A, which generalizes the usual Picard group $\mathrm{Pic}(k) = \mathrm{Pic}_k(k)$.

Most of this material is folklore. The main sources are Gabriel [1] and Morita [1]. I have borrowed a great deal from an unpublished exposition of S.Chase and S.Schanuel.

1 Categories of modules; faithfully projective modules

Let $\mathscr{A}$ and $\mathscr{B}$ be two categories. We recall that $\mathscr{A}$ and $\mathscr{B}$ are said to be *equivalent* if there exist functors $T : \mathscr{A} \to \mathscr{B}$ and $S : \mathscr{B} \to \mathscr{A}$ such that ST and TS are isomorphic to the identity functors of $\mathscr{A}$ and $\mathscr{B}$ respectively. By abuse of language we shall say that T is an equivalence.

49 We call a functor $T : \mathscr{A} \to \mathscr{B}$ *faithful* (resp. *full*, *fully faithful*) if the map

$$T : \mathscr{A}(X, Y) \to \mathscr{B}(TX, TY) \tag{1.1}$$

is injective (resp surjective, bijective) for all X, $Y \in \operatorname{obj} \mathscr{A}$, where $\mathscr{A}(X, Y)$ denotes the set of morphisms from X into Y. If T is an equivalence, then obviously it is fully faithful; also, given $Y \in \operatorname{obj} \mathscr{B}$, there exists $X \in \operatorname{obj} \mathscr{A}$, such that $TX \approx Y$. Conversely, these two conditions together imply that T is an equivalence. This gives us a

(1.2) *Criterion for equivalence: Let $T : \mathscr{A} \to \mathscr{B}$ be a functor satisfying the following conditions:*

(i) *T is fully faithful*

(ii) *Given $Y \in \operatorname{obj} \mathscr{B}$, there exists $X \in obj \mathscr{A}$ with $TX \approx Y$.*

Then T is an equivalence.

Proof. Using (ii) we can choose, for each $Y \in \operatorname{obj} \mathscr{B}$, an $SY \in \operatorname{obj} \mathscr{A}$ and an isomorphism

$$f(Y) : Y \to TSY.$$

These induce bijections $\mathscr{B}(Y, Y') \to \mathscr{B}(TSY, TSY')$, and by (i), we have bijections $\mathscr{A}(SY, SY') \to \mathscr{B}(TSY, TSY')$. The first map, followed by the inverse of the second, defines a bijection

$$S : \mathscr{B}(Y, Y') \to \mathscr{A}(SY, SY').$$

It is easy to see that S, so defined, is a functor satisfying our requirements. □

50 We shall now consider abelian categories. We shall discuss them only provisionally, mainly for the purpose of characterizing categories of modules. Definitions can be found in Gabriel [1], Freyd [1], and Mitchell [1].

A functor $T : \mathscr{A} \to \mathscr{B}$ between abelian categories is called *additive* if the maps (1.1) are homomorphisms. T is *left exact* if it preserves kernels, *right exact* if it preserves cokernels, and *exact* if it does both. We call T *faithfully exact* if it is faithful, exact, and preserves arbitrary direct sums. We shall often call direct sums *coproducts*, and use the symbol $\coprod$ in place of the more familiar $\oplus$.

Let P be an object of the abelian category $\mathscr{A}$. Then

$$h^p = \mathscr{A}(P,\ )$$

defines a functor from $\mathscr{A}$ to the category of abelian groups. We call P a *generator of* $\mathscr{A}$ if h^P is faithful, *projective* if h^P is exact, and *faithfully projective* if h^P is faithfully exact.

Lemma 1.3. *Let $\mathscr{A}$ be an abelian category with arbitrary direct sums.*

(a) An object P of $\mathscr{A}$ is a generator of $\mathscr{A}$ $\Leftrightarrow$ every object of $\mathscr{A}$ is a quotient of a direct sum of copies of P.

(b) A class of objects of $\mathscr{A}$ which contains a generator is suitable under arbitrary direct sums, and which contains the co-kernel of any morphism between its members, is the whole of obj $\mathscr{A}$.

Proof. (a). $\Rightarrow$. Let X be any object of $\mathscr{A}$ and let $S = \coprod_{f \in \mathscr{A}(P,X)} P_f$, 51
where $P_f = p$, with inclusion $i_f : P \to S$. There is a morphism $F : S \to X$ such that $Fi_f = f$ for all f. Let $g : X \to \operatorname{coker} F$. We want to show that $g = 0$, and, by hypothesis, if suffices to show that $h^p(g) = \mathscr{A}(P, g) = 0$. But $h^p(g)(f) = gf = gFi_f = 0$.

(a) $\Leftarrow$ Suppose $g : X \to Y$ be a non-zero morphism. We want $h^p(g) \neq 0$, i.e. $gf \neq 0$ for some $f : P \to X$. Choose a surjection $F : S \to X$ with $S = \coprod_i P_i$, each $P_i = P$. The morphism F is defined by a family of morphisms $f_i : P \to X$, and since $gF \neq 0$, we must have $gf_i \neq 0$ for some i.

(b) is a trivial consequence of (a).

□

The theorem below gives a characterization of categories of modules. We shall denote by

$$A - \quad \text{mod (resp.} \quad \text{mod} \ - A)$$

the category of left (resp. right) modules over a ring A.

Theorem 1.4 (See Gabriel [1] of Mitchell [1]). *Let $\mathscr{A}$ be an abelian category with arbitrary direct sums. Suppose $\mathscr{A}$ has a faithfully projective object P. Let $A = \mathscr{A}(P, P)$. Then*

$$h^p = \mathscr{A}(P.\ ) : \mathscr{A} \to \quad \text{mod} \ - A$$

is an equivalence of categories, and $h^p(P) = A$.

Proof. Clearly $h^p(P) = A$, and since h^p is faithful,

$$h^p : \mathscr{A}(X, Y) \to \mathrm{Hom}_A(h^p X, h^p Y) \tag{1.5}$$

52 is a monomorphism. Using the criterion for equivalence (1.2), it remains to show that

(i) h^p is full (that is, that (1.5) is surjective), and

(ii) each A-module is isomorphic to some $h^p X$.

□

For $X = P$ we see easily that (1.5) is the standard isomorphism $h^p(Y) \to \mathrm{Hom}_A(A, h^P Y)$. As contravariant functors in X, the two side of (1.5) are both left exact and convert direct sums into direct products. This follows for the functor on the right, because h^p is faithfully exact. It follows from these remarks and the 5-lemma that the collection of X for which (1.5) is an isomorphism satisfies the hypothesis of (1.3)(b), and hence is the whole of obj $\mathscr{A}$. This proves (i).

If M is an A-module, there is an exact sequence $F_1 \xrightarrow{d} F_0 \to M \to 0$ with F_i free. Up to isomorphism we can write $F_i = h^P G_i$, with G_i a direct sum of copies of P. By (i), we can write $d = h^P g$ for some $s : G_1 \to G_0$. Then, from exactness, $M \approx \operatorname{coker} h^P g \approx h^P \operatorname{coker} g$.

Proposition 1.5. *Let P be a right module over a ring A. The following statements are equivalent:*

(i) P is faithfully projective.

(ii) P is finitely generated, projective, and is a generator of $\operatorname{mod} A$.

generator of $\operatorname{mod} - A$.

Proof. In view of the definition of faithful projectivity, we have only **53**
to show if P is *projective*, then P is finitely generated if and only the functor $\operatorname{Hom}_A(P, \ )$ preserves coproducts. □

Suppose P is finitely generated. Any homomorphism of P into a coproduct has its image in a finite coproduct (a finite number of factors is enough for catching the non-zero coordinates of the images of a finite system of generators of P). Thus such a homomorphism is a (finite) sum of a homomorphisms of P into the factors.

Conversely, suppose $\operatorname{Hom}_A(P, \ )$ preserves coproducts. Consider a homomorphism $e : P \to \coprod_i A_i$ (each $A_i = A$) with a left inverse (such a map exists since P is projective). By hypothesis, e is a finite sum of homomorphisms $e_i : P \to A_i$, $i \in S$, S a finite set. Thus P is a direct summand of $\coprod_{i \in S} A_i$ and hence finitely generated.

Remark. If P is not projective, then finite generation is no longer equivalent with $\operatorname{Hom}_A(P, \ )$ preserving coproducts. For, we have obviously,

(1.6) P is finitely generated $\Leftrightarrow$ the proper submodules of P are inductively ordered by inclusion.

On the other hand

(1.7) $\operatorname{Hom}_A(P, \ )$ *preserves coproducts* $\Leftrightarrow$ *the union of any ascending sequence of proper submodules of P is a proper submodule.*

If P is the maximal ideal of a valuation ring, where the value group **54**
has a suitably pathological order type, then P will satisfy (1.7) but not (1.6).

Proof of (1.7) $\Leftarrow$. If $f : P \to \coprod_{i \in I} M_i$ is a homomorphism such that $f(P)$ is not contained in a finite direct sum of the $M_i's$, then we can choose a countable subset J of I such that if $g : \coprod_{i \in I} M_i \to \coprod_{j \in J} M_j$ is the projection, then $gf(P)$ is like wise not in a finite sum. Letting S expand through a sequence of finite subsets of J, with J as their union, we find that the submodules $(gf)^{-1}(\sum_{j \in S} M_j)$ violate the assumed chain condition on P.

$\Rightarrow$. Suppose $P_1 \subset P_2 \subset \cdots \subset P_n \subset \cdots$ are proper sub - modules of P with $\bigcup_{n \geq 1} P_n = P$. The projections $f_n : P \to P/P_n$ define a map of $f : P \to \prod_n P/P_n$, whose image is clearly in $\coprod P/P_n$, but not in a finite sum of the P/P_n.

2 k-categories and k-functors

Let A be a ring and let M be a right A-module. For an element $a \in$ centre A, the homothetie $h(a)_M : M \to M$ (defined by $h(a)_M(x) = xa$) is A-liner. These homomorphisms define an endomorphism of the identity functor $\mathrm{Id}_{\mathrm{mod}-A}$.

Proposition 2.1. *The homothetie map*

$$h : \text{ centre } A \to \text{ End } (\mathrm{Id}_{\mathrm{mod}-A})$$

is an isomorphism of rings.

55 *Proof.* If $h(c) = 0$, then $c = h(c)_A(1) = 0$. Let f be an endomorphism of the functor $\mathrm{Id}_{\mathrm{mod}-A}$. f_A is the left multiplication in A by $c = f_A(1)$. The element c belongs to the centre of A. This follows from the fact that f_A commutes with all left multiplications in A, since f is a natural transformation. Set $f' = f - h(c)$. We shall show that $f' = 0$. Let M be a right A-module. For an $x \in M$, consider the A-linear map $t : A \to M$ given by $t(a) = xa$. We have $f'_M \circ t = t \circ f'_A$. It follows that $f'_M(x) = 0$. Thus $f' = 0$. □

The proposition suggests the definition

$$\text{centre } \mathscr{A} = \text{ End } (\mathrm{Id}_{\mathscr{A}})$$

for any abelian category $\mathscr{A}$. Let k be a commutative ring and $k \to$ centre $\mathscr{A}$ a homomorphism. This converts the $\mathscr{A}(X, Y)$ into k-modules so that the composition is k-bilinear. Conversely, given the latter structure, we can clearly reconstruct the unique homomorphism $k \to$ centre $\mathscr{A}$ which induces it. An abelian category $\mathscr{A}$ with a homomorphism $k \to$ centre $\mathscr{A}$ will be called a *k- category*. A functor $T : \mathscr{A} \to \mathscr{B}$ between two such categories will be called a *k-functor* if the maps (1.1) are k-linear. The k-functors forms a category, which we shall denote by k-Funct $(\mathscr{A}, \mathscr{B})$.

If A is a k-algebra, then by virtue of (2.1), mod-A is a k-category. Let A and B be k-algebras and suppose M is a left $A-$, right B-module. If B-module. If $t \in k$ and $x \in M$, then tx and xt are both defined. The following statement is easily checked:

(2.2) $tx = xt$ for all $t \in k$, $x \in<\Leftrightarrow \otimes_A M : \mod -A \to \mod -B$ **56**
is a k-functor.

This condition simply means that M can be viewed as left module over $A \underset{k}{\otimes} B^0$. We will often follow the Cartan-Eilenberg convention of writing ${}_AM_B$ to denote the fact that M is left $A-$, right B-bimodule, and when a ground ring k is fixed by the context, it will be understood that M satisfies (2.2).

Proposition 2.3. *$h({}_AM_B) \otimes_A M : \mod - A \to \mod - B$ defines a fully faithful functor*

$$h : (A \otimes_k B^0) - \mod \to k - Funct(\mod - A, \mod - B).$$

In particular, ${}_AM_B \approx_A N_B$ as bimodules $\Leftrightarrow \otimes_A M \approx \otimes_A N$ as functors from $\mod - A$ to $\mod - B$.

Proof. If $f : {}_AM_B \to {}_AN_B$ is a bimodule homomorphism, then $h(f) = \otimes_A f$ is a morphism of functors. Thus h is a functor. If $h(f) = 0$, then $1_A \otimes_A f = h(f)(1_A) = 0$, i.e, $f = 0$. So h is faithful. ◻

Suppose $t : hM \to hN$ is a natural transformation. We will conclude by showing that $t = h(f)$, where f is the unique B-morphisms rendering

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\approx"'] & N \arrow[d, "\approx"] \\ A \otimes_A M \arrow[r, "t_A"'] & A \otimes_A N \end{tikzcd}$$

57 commutative. The vertical maps are bimodule isomorphisms. Since left multiplications in A are right A-linear, t_A must respect it, by naturality. Thus t_A, and hence also f, is a bimodule homomorphism, so $h(f)$ is defined. Let $s = t - h(f) : hM \to hN$. The class $\mathscr{C}$ of X in obj mod $-A$ for which $s_X = 0$ contains A. Since hM and hN are right exact and preserve coproducts, it follows from (1.3) (b), that $\mathscr{C} =$ obj mod $-A$.

3 Right continuous functors

We will here describe the image of the functor of proposition 2.3. Functors of the type $\otimes_A M :$ mod $-A \to$ mod $-B$ are (i) right exact, and (ii) preserve arbitrary coproducts. It follows that they also preserve direct limits. A functor satisfying (i) and (ii) will be called *right continuous.* The next theorem says that they are all tensor products.

Theorem 3.1 (Eilenberge-Watts)**.** *The correspondence* $_AM_B \mapsto \otimes_A M$ *induces a bijection from the isomorphism classes of left* $A \otimes_k B^0$*- modules to the isomorphism classes of right continuous k-functors from* mod $-$ *A to* mod $-B$*. In the situation* $({}_AM_B, {}_BN_C)$*,* $_A(M \otimes_B N)_C$ *corresponds to the composite of the respective functors.*

Proof. The last statement follows from

$$\begin{aligned} ((\otimes_B N) \circ (\otimes_A M))(X) &= (\otimes_B N)(X \otimes_A M) \\ &= (X \otimes_A M) \otimes_B N \\ &= X \otimes_A (M \otimes_B N) \\ &= \otimes_A (M \otimes_B N)(X). \end{aligned}$$

58 Injectivity is just the last part of proposition 2.3. $\square$

Let $T : \mod - A \to \mod - B$ be a right continuous k-functor. The composite

$$A \to \mathrm{Hom}_A(A, A) \to \mathrm{Hom}_B(TA, TA),$$

where the first map is given by left multiplications, is a homomorphism of k-algebras. This makes $M = TA$ into a left $A \otimes_k B^0$ - module. We will conclude by showing that the functors T and $\otimes_A M$ are isomorphic. If X is a right A-module, we have maps

$$X \xrightarrow{\approx} \mathrm{Hom}_A(A, X) \xrightarrow{T} \mathrm{Hom}_B(TA, TX) = \mathrm{Hom}_B(M, TX),$$

and the composite f_X is A-linear (for the action of A on M just constructed). Now, there is a canonical isomorphism

$$\mathrm{Hom}_A(X, \mathrm{Hom}_B(M, TX)) \approx \mathrm{Hom}_B(X \otimes_A M, TX),$$

and f_X is an element of the first member. Let g_X be the corresponding element in the second member. The homomorphisms g_X define a natural transformations of functors $g : \otimes_A M \to T$. For $X = A$, we have g_A as the obvious isomorphism $A \otimes_A M \to TA = M$. Using the right continuity of T and $\otimes_A M$, we now see that the class of objects x for which g_x is an isomorphism, satisfies the conditions of (1.3) (b). Thus g is an isomorphism of functors.

Definition 3.2. *We shall call a bimodule* ${}_A M_B$ invertible, *if the functor* $\otimes_A M : \mod - A \to \mod - B$ *is an equivalence.*

This equivalence is evidently right continuous (indeed, any equiv- 59
alence is). It therefore follows from theorem 3.1. that the invertibility of M is equivalent to the existence of a bimodule ${}_B N_A$ such that $M \otimes_B N \approx A$ and $N \otimes_A M \approx B$ as bimodules (over appropriate rings). This shows that the definition of in vertibility is left-right symmetric. In particular, $M \otimes_B : B - \mod \to A - \mod$ is also equivalence.

4 Equivalences of categories of modules

We have just seen that an equivalence is, up to isomorphism, tensoring with an invertible bimodule. We now summarize.

Proposition 4.1. *Let A and B be a k-algebras and suppose*

$$\mathrm{mod}\ -A \underset{S}{\overset{T}{\rightleftarrows}} \mathrm{mod} - B$$

are k-functors such that ST and TS are isomorphic to the identity functors of $\mathrm{mod}\ -A$ and $\mathrm{mod}\ -B$ respectively. Set $P = TA$ and $Q = SB$. Then we are in the situation $({}_AP_B, {}_BQ_A)$, and :

(1) $T \approx \otimes_A P$*, and* $S \approx \otimes_B Q$*.*

(2) There are bimodule isomorphisms

$$f : P \otimes_B Q \to A \text{ and } g : Q \otimes_A P \to B.$$

(3) f and g may be chosen to render the diagrams

$$\begin{tikzcd} P \otimes_B Q \otimes_A P \arrow[r, "f\otimes 1_P"] \arrow[d, "1_P\otimes g"'] & A \otimes_A P \arrow[d] \\ P \otimes_B B \arrow[r] & P \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Q \otimes_A P \otimes_B Q \arrow[r, "g\otimes 1_Q"] \arrow[d, "1_Q\otimes f"'] & B \otimes_B Q \arrow[d] \\ Q \otimes_A A \arrow[r] & Q \end{tikzcd}$$

commutative.

60 *Proof.* Statements (1) and (2) follow immediately from theorem 3.1, since an equivalence is automatically right continuous. To prove the statement (3), we first note that all the intervening maps are isomorphisms of bimodules. If $a : A \otimes_A P \to P$ and $b : P \otimes_B B \to P$ are the natural maps, then we have $b(1 \otimes g) = ua(f \otimes 1)$ for some $A - B$ - automorphism u of P. In particular, $u \in \mathrm{Hom}_B(P, P) = \mathrm{Hom}_B(TA, TA) \approx \mathrm{Hom}_A(A, A) = A$. So u is a left multiplication by a unit in A, which we shall denote by the same letter u. Since u is also an A-homomorphism, we must have $u \in$ centre A. Now, evidently $ua = a(u \otimes 1_P)$. So if we replace f by uf we have made the first square commutative. Assume that this has been done. □

Write $f(p \otimes q) = pq$ and $g(q \otimes p) = qp$ for $p \in P$, $q \in Q$. We have arranged that $(pq)p' = p(qp')$, and we will prove that the desired equality $(qp)q' = q(pq')$ follows automatically. For, if p, $p' \in P$, q, $q' \in Q$ we have

$$\begin{aligned}((qp)q')p' &= (qp)(q'p') && (g \text{ is left } B\text{-linear})\\ &= q(p(q'p')) && (g \text{ is right } B\text{-linear})\\ &= q((pq')p') && (\text{ by assumption})\\ &= (q(pq')p') && (q \otimes ap = qa \otimes p, a \in A).\end{aligned}$$

Hence, if $d = (qp)q' - q(pq')$, then $dp' = 0$ for all $p' \in P$. Let $h : A \to Q$ be defined by $h(a) = da$. Then $h \otimes 1_p : A \otimes_A P \to Q \otimes_A P$ followed by the isomorphism g is zero. So $h \otimes 1_P = 0$. But $\otimes_A P$ is a fully faithful functor. Therefore $h = 0$, that is $d = 0$.

Definition 4.2. A set of pre-equivalence data (A, B, C, P, f, g) *consists* 61
of k-algebras A and B, bimodules ${}_AP_B$ and ${}_BQ_A$, bimodule homomorphisms

$$f : P \otimes_B Q \to A \text{ and } g : Q \otimes_A P \to B,$$

which are "associative" in the following sense: Writing $f(p \otimes q) = pq$ and $g(q \otimes p) = qp$, we require that

$$(pq)p' = p(qp') \text{ and } (qp)q' = q(pq') \;\; p, p' \in P, q, q' \in Q.$$

We call it a set of equivalence data *if f and g are isomorphisms.*

Theorem 4.3. *Let (A, B, P, Q, f, g) be a set of pre-equivalence data. If f is surjective, then*

(1) f is an isomorphism

(2) P and Q are generators as A-modules

(3) P and Q are finitely generated and projective as B-modules.

(4) g induces bimodule isomorphisms

$$P \approx \mathrm{Hom}_B(Q, B) \text{ and } Q \approx \mathrm{Hom}_B(P, B)$$

(5) The k-algebra homomorphisms

$$\mathrm{Hom}_B(P,P) \leftarrow A \to \mathrm{Hom}_B(Q,Q)^0$$

induced by the bimodule structures, are isomorphisms.

Proof. The hypothesis on f means that we can write

$$1 = \sum p_i q_i \text{ in } A.$$

62 (1) Suppose $\sum p'_j \otimes q'_j \in \ker f$. Then

$$\begin{aligned}\sum p'_j \otimes q'_j &= \sum_{j,i} (p'_j \otimes q'_j) p_i q_i = \sum_{j,i} p'_j \otimes ((q'_j p_i) q_i) = \\ &= \sum_{j,i} (p'_j (q'_j p_i)) \otimes q_i = (\sum_{j,i} (p'_j q'_j)(p_i \otimes q_i) \\ &= (\sum_j p'_j q'_j)(\sum_i p_i q_i) = 0, \text{ since } \sum p'_j q'_j = 0.\end{aligned}$$

(2) We have A-linear maps $h_i : P \to A_i$ given by $h_i(p) = pq_i$. These define an A-linear map $h : \coprod_i P_i \to A$ (each $P_i = P$), which is surjective. It follows by (1.3) (a), that P is a generator of $A-$ mod , since A is so. The argument for Q is similar.

(3) Define $P \underset{h}{\overset{e}{\rightleftarrows}} \coprod_i B_i$ (each $B_i = B$), by $e(p) = (q_i p)$ and $h((b_i)) = \sum p_i b_i$. Then $he(p) = \sum_i p_i(q_i p) = (\sum_i p_i q_i)p = p$. Thus P is finitely generated and projective. Similarly Q also is finitely generated and projective.

(4) g induces an A-B-bimodule homomorphism $h : P \to \mathrm{Hom}_B(Q, B)$, given by $h(p)(q) = qp$. If $h(p) = 0$, then $p = \sum_i (p_i q_i) p = \sum_i p_i (q_i p) = 0$. If $f : Q \to B$ is B-linear, then $f(q) = f(\sum_i q(p_i q_i)) = f(\sum_i (qp_i) q_i) = \sum_i (qp_i) f(q_i) = \sum_i q(p_i f(q_i))$, so $f = h(\sum_i p_i f(q_i))$. Similarly $Q \approx \mathrm{Hom}_B(P, B)$.

63 (5) Define $h : A \to \mathrm{Hom}_B(P,P)$ by $h(a)p = ap$. If $h(a) = 0$, then $a = \sum\limits_i a(p_i q_i) = \sum\limits_i (ap_i)q_i = 0$. If $f : P \to P$ is a B-linear, then $f(p) = f(\sum\limits_i (p_i q_i)p) = f(\sum\limits_i p_i(q_i p)) = (\sum\limits_i f(p_i)q_i)p$, so that $f = h(\sum\limits_i f(p_i)q_i)$. Similarly $A \approx \mathrm{Hom}_B(Q,Q)^0$ via right multiplication.

□

Theorem 4.4. *Let (A, B, P, Q, f, g) be a set of equivalence data (see definition 4.2). Then*

(1) The functors $P\times_B, \otimes_A P, Q\otimes_A$, and $\otimes_B Q$ are equivalences between the appropriate categories of A-modules and B-modules.

(2) P and Q are faithfully projective both as A-modules and B-modules.

(3) f and g induce bimodule isomorphisms of P and Q with each others duals with respect to A and to B.

(4) The k-algebra homomorphisms

$$\mathrm{Hom}_B(P,P) \leftarrow A \to \mathrm{Hom}_B(Q,Q)^0$$

and

$$\mathrm{Hom}_A(P,P)^0 \leftarrow B \to \mathrm{Hom}_A(Q,Q),$$

induced by the bimodule structures on P and Q, are isomorphisms.

(5) The bimodule endomorphism rings of A, B, P and Q are all isomorphic to the centres of A, B $\mod -A$ and $\mod -B$.

(6) The lattice of right A-ideals is isomorphic, via $\mathscr{U} \mapsto \mathscr{U}P$, with the lattice of B-submodules of P, the two sided ideals corresponding to $A-B$-submodels, or equivalently, to fully invariant B-submodules. Similar conclusions apply with appropriate permutations of (A, B), (P, Q), (left, right). In particular, by symmetry, A and B have isomorphic lattices of two-sided ideals. 64

Proof. (1) is immediate.

(2), (3) and (4) follow immediately from (2), (3), (4) and (5) of theorem 4.3.

We have isomorphisms

$$\text{centre}\quad A \approx \mathrm{Hom}_{A-A}(A, A) \xrightarrow[\approx]{\otimes_A P} \mathrm{Hom}_{A-B}(P, P),$$
$$\text{centre}\quad B \approx \mathrm{Hom}_{B-B}(B, B) \xrightarrow[\approx]{P\otimes_B} \mathrm{Hom}_{A-B}(P, P),$$

and similarly for Q also. The statement (5) follows from these isomorphisms plus proposition 2.1. □

We now prove (6). Since P is A-projective, the canonical map $\mathscr{U} \otimes_A P \to \mathscr{U}P$ is an isomorphism. That $\mathscr{U} \mapsto \mathscr{U}P$ is an isomorphism of the lattice of right ideals of A onto the lattice of B-submodules of P, now follows from the fact that $\otimes_A P : \mod - A \to \mod - B$ is an equivalence. The fully invariant right A-submodules of A, i.e., the two-sided ideals of A, correspond to the fully invariant $B-$submodules of P, which, by virtue of (4), are just the $A - B$- submodules of P.

65 The remaining assertions in (6) are clear. The isomorphism between the lattices of two-sided ideals of A and B can be made explicit: $\mathscr{U} \leftrightarrow \mathfrak{b}$ if $\mathscr{U}P = P\mathfrak{b}$, where $\mathscr{U}$ and $\mathfrak{b}$ are two-sided ideals in A and B respectively. The conclusion above show that given $\mathscr{U}$, the ideal $\mathfrak{b}$ exists and is unique.

5 Faithfully projective modules

Let B be a $k-$alegbra and let P be right B-module. From B and P we will construct a set of pre-equivalence data and then determine in terms of B and P alone, what it means for them to be equivalence data.

We set

$$A = \mathrm{Hom}_B(P, P),$$

and

$$Q = \mathrm{Hom}_B(P, B).$$

Then A is a $k-$algebra and P is an $A-B$-bimodule, that is, a left $A\otimes_k B^0$-module. Moreover, Q is a $B-A$ - bimodule with the following prescription:

$$(bq)p = b(qp) \tag{5.1}$$

and

$$(qa)p = q(ap), \tag{5.2}$$

$a \in A$, $b \in B$, $p \in P$, $q \in Q$. Next we define $pq \in A$ for $p \in P$ and $q \in Q$, by requiring that

$$(pq)p' = p(qp'), \qquad p' \in P \tag{5.3}$$

This permits us to define a homomorphism of $A-A$-bimodules

$$f_p : P \otimes_B Q \to A, \text{ by } f_P(p \otimes q) = pq,$$

and a homomorphism of $B-B$-bimodules **66**

$$g_p : Q \otimes_A P \to B, \text{ by } g_p(q \otimes p) = qp.$$

Finally, we claim that

$$(qp)q' = q(pq'), \tag{5.4}$$

for $p \in P$, $q, q' \in Q$. Since these are linear maps $P \to B$, we need only show that they have the same value at any $p' \in P$. But

$$\begin{aligned} ((qp)q')p' &= (qp)(q'p') && \text{by (5.1)} \\ &= q(p(q'p')) && \text{by B - linearity of q} \\ &= q((pq')p') && \text{by (5.3)} \\ &= (q(pq')p') && \text{by (5.2).} \end{aligned}$$

We have now proved

Proposition 5.5. *Let B be a k-algebra, P a right B-module, and f_p and g_p be as constructed above. Then*

$$(\mathrm{Hom}_B(P, P), B, P, \mathrm{Hom}_B(P, B), f_p.g_p)$$

is a set of pre-equivalence data.

Example. Let $P = eB$, where e is an idempotent. Then $B = P \oplus (1-e)B$. Any B-linear map $f : P \to B$ can be extended to a $B-$linear map $\bar{f} : B \to B$ by setting $\bar{f}(1-e) = 0$. Thus we have inclusions $\mathrm{Hom}_B(P, P) \subset \mathrm{Hom}_B(P, B) \subset \mathrm{Hom}_B(B, B)$. With this identification, $\mathrm{Hom}_B(P, P) = eBe$ and $\mathrm{Hom}_B(P, B) = Be$.

67 **Proposition 5.6.** *In the notation of proposition 5.5:*

(a) $\mathrm{im}\, f_P = \mathrm{Hom}_B(P, P) \Leftrightarrow$ *P is a finitely generated projective B-module, in which case f_p is an isomorphism.*

(b) $\mathrm{im}\, g_p = B \Leftrightarrow$ *P is a generator of* $\mod -B$*, in which case g_p is an isomorphism.*

(c) $(\mathrm{Hom}_B(P, P), B, P, \mathrm{Hom}_B(P, B), f_p, g_p)$ *is a set of equivalence data* $\Leftrightarrow$ *P is faithfully projective.*

Proof. (c) follows from (a), (b) and proposition 1.5.

In view of theorem 4.3, it remains only to show the implications $\Leftarrow$ in (a) and (b).

Suppose P is a finitely generated projective $B-$module. We can find a free B-module $\coprod_{e_i} B$ with a basis $e_1, \ldots, e_n$, and $B-$linear maps $P \xrightarrow{h_1} \coprod e_i B \xrightarrow{h_2} P$ such that $h_2 h_1 = 1_P$. If $q_i : P \to B$ denotes the composite of h_1 and the i^{th} coordinate linear form on $\coprod e_i(B)$, we can write $h_1(p) = \sum e_i(q_i p)$. Let $p_i = h_2(e_i)$. Then $p = h_2 h_1 p = h_2(\sum e_i(q_i\, p)) = \sum p_i(q_i p) = (\sum p_i q_i)p$. So $1_p = \sum p_i q_i \in \mathrm{im}\, f_p$, and the latter is a two-sided ideal in $\mathrm{Hom}_B(P, P)$. Hence $\mathrm{im}\, f_p = \mathrm{hom}_B(P.P)$.

Next, suppose P is a generator of $\mod -B$. Then B is a quotient of a sum (which we mau take finite) of copies of P. This means that we can find $q_i \in \mathrm{Hom}_B(P, B)$ such that $\sum q_i P = B$. Hence g_p is surjective. $\square$

68 **Lemma 5.7.** *A right $B-$module P is projective $\Leftrightarrow$ there exist $p_i \in P$, $q_i \in \mathrm{Hom}_B(P, B)$, $i \in I$, such that*

(i) given $p \in P$, $q_i p = 0$ for almost all i, and

(ii) $\sum_i p_i(q_i p) = p$, $p \in P$.

The family (p_i) *which arise in this manner are precisely the generating systems of* P. *If* $\mathscr{U} = \operatorname{im} g_p$, *then* $\mathscr{U}$ *is generated, as a two-sided ideal, by the* $q_i p_j$. *Moreover,* $P\mathscr{U} = P$ *and* $\mathscr{U}^2 = \mathscr{U}$.

Proof. Projectivity of P is equivalent to the existence of a free B-module $\coprod_{i\in I} e_i B$ and B-linear maps $P \xrightarrow{h_1} \coprod_{i\in I} e_i B \xrightarrow{h_2} P$ such that $h_2 h_1 = 1_P$. The latter condition, in turn, is equivalent to the existence of the p_i and q_i. For, given p_i and q_i, one can construct h_1 and h_2 in an obvious fashion. On the other hand, given h_1 and h_2, we can take p_i to be $h_2(e_i)$, and q_i to be the composite of h_1 with the i^{th} coordinate linear form on $\coprod_{i\in I} e_i B$. If P is projective, it is clear that families (p_i) are precisely the systems of generators for P. □

Setting $Q = \operatorname{Hom}_B(P, B)$ we can write $\mathscr{U} = QP$ (the set of sums of elements of the form qp, $q \in Q$, $p \in P$). But $qp = q\sum_i p_i(q_i p) = \sum_{i,j} q(p_j(q_j p_i))(q_i p) = \sum_{i,j}(qp_j)(q_j p_i)(q_i p)$, which shows that $\mathscr{U}$ is generated, as a two-sided ideal, by the $q_j p_i$. Moreover (ii) shows that $P = P\mathscr{U} = PQP$, and therefore $\mathscr{U} = QP = QPQP = \mathscr{U}^2$.

Lemma 5.8. *Let* B *be a commutative ring,* M *a finitely generated* B*-module, and* $\mathscr{U}$ *an ideal of* B *such that* $M\mathscr{U} = M$. *Then* $M(1-a) = 0$ *for some* $a \in \mathscr{U}$. **69**

Proof. If $x_1, \dots, x_n$ generate M, we can find $a_{ij} \in \mathscr{U}$ such that $x_i = \sum_j x_i a_{ij}$, that is, $\sum_j x_i(\delta_{ij} - a_{ij}) = 0$, $i = 1, \dots, n$. It follows by a well-known argument, that $x_i \det(\delta_{ij} - a_{ij}) = 0$, that is, $M \det(\delta_{ij} - a_{ij}) = 0$. But $\det(\delta_{ij} - a_{ij})$ is of the form $1 - a$ for some $a \in \mathscr{U}$. □

Proposition 5.9. *Let* B *be a commutative ring and* P *a projective* B*-module. If either* B *is noetherian or* P *is finitely generated, the ideal im* g_p *of* B *is generated by an idempotent* e*, and ann* $P = (1-e)B$. *Hence* P *is a generator of* $\mod - B$ *if and only if* P *is faithful (i.e., ann*$P = 0$*).*

Proof. The hypotheses guarantee that $\mathscr{U} = img_p$ is a finitely generated ideal of B, using (5.7) in the second alternative. From (5.7) we also have $P\mathscr{U} = P$ and $\mathscr{U}^2 = \mathscr{U}$. Taking $M = \mathscr{U}$ in (5.8) we find an $e \in \mathscr{U}$ such that $\mathscr{U}(1-e) = 0$. So $\mathscr{U} = \mathscr{U}e$ and $e^2 = e$. Moreover, $P = Pe$ so $P(1-e) = 0$. If $Pa = 0$, then, since $e = \sum q_j p_j$, we have $ea = \sum q_j p_j a = 0$ and thus $a = (1-e)a$. Hence ann $P = (1-e)B$. Finally, P is a generator $\Leftrightarrow$ im $g_p = B \Leftrightarrow e = 1 \Leftrightarrow annP = 0$. $\square$

The following corollary shows that for a *commutative* ring, B, the concept of a faithfully projective object of $\mathrm{mod} - B$ is the same as that of faithfully projective B- modules (as defined in §6 of Chapter 1).

70 **Corollary 5.10.** *Let P be a module over a commutative ring, B. Then P is a faithfully projective object of $\mathrm{mod} - B \Leftrightarrow P$ is finitely generated, projective, and faithful.*

Example 1. Let k be a field and let B be the ring of matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $a, b, c \in k$. Let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The right ideal $P = eB$ is a finitely generated, projective, faithful B–module. However, $\operatorname{im} g_p = P \neq B$, so P not a generator of $\mathrm{mod} - B$. Of course, B is not commutative.

Example 2 (Kaplansky)**.** Let B be the (commutative) ring of continuous real valued functions on the interval $[0, 1]$, and let P be the ideal of all functions vanishing in a neighbourhood of 0. It is known that P is projective, and clearly it is faithful. However, it is easy to show that $\operatorname{im} g_p \neq B$, so P is not a generator of $\mathrm{mod} - B$. Of course, P is not finitely generated.

6 Wedderburn structure theory

Given a ring, B, we shall denote by $\mathbb{M}_n(B)$, the ring of $n \times n$ matrices with entries in B. If P is a B-module, we shall write $p^{(n)}$ for the direct sum of n copies of P. There is a natural isomorphism

$$\mathrm{Hom}_B(P^{(n)}, P^{(n)}) \approx \mathbb{M}_n(\mathrm{Hom}_B(P, P)).$$

71 We recall

Schur's lemma. *A homomorphism from a simple module over a ring into another simple module is either an isomorphism or the zero map.*

Theorem 6.1. *Let P be a faithfully projective right module over a ring B. Suppose further that P is simple (This is rare !). Then*

(1) $A = \mathrm{Hom}_B(P, P)$ *is a division ring*

(2) P *is a finite dimensional left vector space over A, say $P \approx A^{(n)}$.*

(3) $B \approx \mathrm{Hom}_A(P, P)^0 \approx \mathbb{M}_n(A)$ *(via right multiplication).*

(4) B *is a simple ring whose lattice of left ideals is isomorphic, via* $\mathfrak{b} \mapsto P\mathfrak{b}$, *to the lattice of A-subspaces of P.*

(5) *Centre* $B \approx \mathrm{Hom}_{A-B}(P, P) \approx$ *centre A, and these are fields.*

(6) $P \otimes_B : B - \ \mathrm{mod} \ \to A - \ \mathrm{mod}$ *is an equivalence of categories.*

Conversely, if $P \neq 0$ is a finite dimensional left vector space over a division ring A, and if $B = \mathrm{Hom}_A(P, P^0)$, then P is a faithfully projective simple right B-module, and $A \approx \mathrm{Hom}_B(P, P)$ (via left multiplication).

Proof. (1) follows from Schur's lemma (2), (3), (4), (5) and (6) follows from theorem 4.4 and proposition 5.6 (c).

If $P \neq 0$ is a finite dimensional left vector space over a division ring **72**
A, then evidently P is a finitely generated projective generator of A-mod, that is, a faithfully projective A-module. Moreover, $B = \mathrm{Hom}_A(P, P)^0$ operates transitively on the non-zero elements of P, so that P is a simple B-module. It follows, as before, form theorem 4.4(4) and proposition 5.6 (c), that $A \approx \mathrm{Hom}_B(P, P)$.

We now describe the classical method for finding a P as above. □

Lemma 6.2. *If P is a minimal right ideal in a ring B, and if $P^2 \neq 0$, then $P = eB$ for some idempotent e.*

Proof. Since $P^2 \neq 0$, there exists $x \in P$ such that $xP \neq 0$. Schur's lemma then implies that $P \xrightarrow{x} P$ (left multiplication by x) is an isomorphism, so that $x = xe$ for a unique $e \in P$. But this implies $x = xe^2$, so $e^2 = e$. In particular, $0 \neq eB \subset P$, and thus $P = eB$. □

Proposition 6.3. *Let B be a ring having no idempotent two-sided ideals other than 0 and B, and let P be a minimal right ideal such that $P^2 \neq 0$. Then P is a faithfully projective and simple B–module, so we have the consequences of theorem 6.1.*

Proof. P is finitely generated projective thanks to lemma 6.2. Moreover $0 \neq P \subset \operatorname{im} g_p$ is, according to lemma 5.7, an idempotent two sided ideal. The hypothesis therefore implies that $\operatorname{im} g_p = B$, that is, P is a generator of mod-B. Thus P is faithfully projective. Also P is simple by hypothesis. □

73 **Example.** A right artinian ring B having no two-sided ideals other than 0 and B satisfies the hypothesis of the above proposition. For, it has a minimal right ideal $P \neq 0$ and P^2 cannot be zero (otherwise the two-sided ideal $BP \neq 0$ would be distinct from B since it would be nilpotent).

We now generalize these results to the semi-simple case. Recall that a module is called *semi-simple* if it is a direct sum of simple modules.

Lemma 6.4. *Suppose a module M is the sum of a submodule N and a family $(S_i)_{i \in I}$ of simple submodules. Then there is a subset J of I such that the map*

$$f_J : N \coprod (\coprod_{j \in J} S_j) \to M,$$

induced by inclusions, is an isomorphism.

Proof. Among the subsets J for which f_j is a monomorphism, we can choose a maximal one, say J_0, by Zorn's lemma. If f_{J_0} is not surjective, there exists $j \in I - J_0$ such that $S_j \not\subset \operatorname{im} f_{J_0}$. Since S_j is simple, im $f_{J_0} \cap S_j = 0$. Thus $J_0 \cup \{j\}$ contradicts the maximality of J_0. □

Corollary. *A submodule of a semi-simple module is a direct summand.*

Proposition 6.5. *Suppose B has a faithfully projective right B– module P which is semi-simple. Then*

$$P \approx S_1^{(n_1)} \oplus \cdots \oplus S_r^{(n_r)},$$

74 *where $S_1, \ldots, S_r$ are a complete set of non-isomorphic simple B– mod-*

ules, and each $n_i > 0$. If $D_i = \mathrm{Hom}_B(S_i, S_i)$, then D_i is a division ring, and

$$\mathrm{Hom}_B(P, P) \approx \prod_{1 \le i \le r} \mathbb{M}_{n_i}(D_i).$$

Moreover, B is itself a semi-simple $B-$ module.

Proof. Since P is finitely generated and semi-simple, it is a finite direct sum of simple modules, and we can write $P \approx S_1^{(n_1)} \oplus \cdots \oplus S_r^{(n_r)}$, where each S_i is simple, S_i not isomorphic to S_j for $i \neq j$, and each $n_i > 0$. If S is any simple module, then S is a quotient of a coproduct of copies of P and this clearly implies that S is isomorphic to some S_i. Since $\mathrm{Hom}_B(S_i, S_j) = 0$ for $i \neq j$ (Schur's lemma), we have $\mathrm{Hom}_B(P, P) \approx \prod_{1 \le i \le n} \mathrm{Hom}_B(S_i^{(n_i)}, S_i^{(n_i)}) \approx \prod_{1 \le i \le r} \mathbb{M}_{n_i}(D_i)$. Since B is a quotient, and hence a direct summand of a coproduct of copies of P, B is also semi-simple. □

Proposition 6.6. *Let B be right artinian and let B have no nilpotent two-sided ideals $\neq 0$. Then B is a semi-simple right $B-$ module. As a ring, B is a finite direct product of full matrix rings over division rings. In particular, the center of B is a finite product of fields.*

Proof. Once we know that B is a semi-simple right $B-$ module, the remaining conclusions follow from (6.5), since B is obviously faithfully $B-$projective and the ring of endomorphisms of the right B-module B is isomorphic to B. □

If $\mathfrak{b}$ is a minimal (i.e. simple) right ideal of B, then $\mathfrak{b} = eB$ with $e^2 =$ 75
e. This follows from lemma 6.2, provided $\mathfrak{b}^2 \neq 0$. But $\mathfrak{b}^2 = 0$ implies that $B\mathfrak{b} \neq 0$ is a nilpotent two-sided ideal contradicting our hypothesis. We note that $\mathfrak{b}$, being a direct summand of B, is a direct summand of any right ideal which contains $\mathfrak{b}$.

Now, if B is not semi-simple we can find a right ideal $\mathscr{O}$ minimal with the property that $\mathscr{O}$ is not semi-simple. Choose a simple right ideal $\mathfrak{b}$ in $\mathscr{O}$. Then $\mathscr{O} = \mathfrak{b} + \mathscr{O}'$ (direct sum) for some right ideal $\mathscr{O}' \underset{\neq}{\subset} \mathscr{O}$. Then $\mathscr{O}'$ is semi-simple and thus $\mathscr{O}$ also is semi-simple, which is a contradiction.

Proposition 6.7. *B is a semi-simple $B-$module $\Leftrightarrow$ every $B-$ module is projective.*

Proof. $\Rightarrow$ Let P be a right $B-$ module. Then P is a quotient of a free right $B-$module F which is semi-simple by assumption. It follows from the corollary to (6.4), that P a direct summand of F. ◻

$\Leftarrow$ Let $\mathscr{O}$ be the sum of all simple right ideals of B. Then $\mathscr{O}$ is semi-simple, by (6.4). By hypothesis, $B/\mathscr{O}$ is projective, so that $B = \mathscr{O} \oplus \mathfrak{b}$ for some right ideal $\mathfrak{b}$. If $\mathfrak{b} \neq 0$, then, being finitely generated, it has a simple quotient module and hence a simple submodule (because the simple quotient is projective). This contradicts the defining property of $\mathscr{O}$, and hence $\mathscr{O} = B$. Thus B is semi-simple.

76 **Definition 6.8.** *We call a ring B* semi-simple *if it is semi simple as a right module over itself.*

The results above show that is equivalent to B being a finite product of matrix rings over division rings. In particular, the definition of semi-simplicity of a ring is left-right symmetric.

7 Autoequivalence classes; the Picard group

If $\mathscr{A}$ is a $k-$category, k a comutative ring, we define

$$\mathrm{Pic}_k(\mathscr{A})$$

to be the group of isomorphism classes (T) of $k-$equivalences $T : \mathscr{A} \to \mathscr{A}$. The group law comes from composition of functors.

If A is a $k-$algebra, we define

$$\mathrm{Pic}_k(A)$$

to be the group of isomorphism classes (P) of invertible $A - A-$ bimodules (see definition 3.2) with law of composition induced by tensor product: $(P)(Q) = (P \otimes_A Q)$. It follows from proposition 4.1 and theorem 4.4(3), that this is indeed a group with $(P)^{-1} = (\mathrm{Hom}_A(P, A))$. In the latter we can use either the left or the right $A-$module structure of P.

77 According to theorem 3.1:

Proposition 7.1. $(P) \mapsto (P \otimes_A)$ *and* $(T) \mapsto (TA)$ *define inverse isomorphisms*

$$\mathrm{Pic}_k(A) \rightleftarrows \mathrm{Pic}_k(A - \quad \mathrm{mod}\ ).$$

Let P be an invertible $A - A$ bimodule. If $\alpha,\ \beta\ \in\ Aut_k(A)$ are $k-$algebra automorphisms, write

$$\alpha^P\beta$$

for the bimodule with additive group P and with operations

$$a \cdot p = \alpha(a)p, p \cdot a = p\beta(a) \quad (p \in P, a \in A).$$

Thus $P =_1 P_1$.

Suppose $f : P \to Q$ is a *left* $A-$isomorphism of invertible $A - A-$bimodules. Since, via right multiplication, $A = \mathrm{Hom}_A({}_AP, {}_AP)^0$, we can define $\alpha \in Aut_k(A)$ by

$$p\alpha(a) = f^{-1}(f(p)a)$$

or

$$f(p\alpha(a)) = f(p)a, \qquad p\varepsilon P, a \in A.$$

Then $f :_1 P_\alpha \to Q$ is a bimodule isomorphism. This proves, in particular, the statement (4) in the following

Lemma 7.2. *For* $\alpha, \beta, \gamma \in Aut_k(A)$ *we have* **78**

(1) $\alpha^A\beta \approx \gamma\alpha^A\gamma\beta$

(2) $1^A\alpha \otimes_A 1^A\beta \approx 1^A\alpha\beta$

(3) ${}_1A_\alpha \approx_1 A_1 \Leftrightarrow \alpha \in$ *In Aut (A), the group of inner automorphisms of A.*

(In all cases above the symbol ≈ denotes bimodule isomorphism.)

(4) If P is an invertible A−A−bimodule and if P ≈ A as left A− modules, then $P \approx_1 A_\alpha$ *as bimodules for some* $\alpha \in Aut_k(A)$.

Proof. (1) The map ${}_{\alpha}A_{\beta} \to_{\gamma\alpha} A_{\gamma\beta}$ given by $x \mapsto \gamma(x)$ is the required isomorphism.

(2) Using (1) we have ${}_{1}A_{\alpha} \otimes_{A} {}_{1}A_{\beta} \approx {}_{\alpha^{-1}}A_{1} \otimes_{A1} A_{\beta} \approx {}_{\alpha-1\beta} \approx {}_{1}A_{\alpha\beta}$.

(3) If $f : {}_{1}A_{\alpha} \to_{1} A_{1}$ is a bimodule isomorphism, then as a left A-automorphism $f(x) = xu$, where $u = f(1)$ is a unit in A. Moreover, $f(\alpha(a)) = f(1.a) = f(1)a$, which gives $\alpha(a)u = ua$, that is, $\alpha(a) = uau^{-1}$ for all $a \in A$.

□

Conversely, if $\alpha(a) = uau^{-1}$ for some unit $u \in A$, then $f(x) = xu$ defines a bimodule isomorphism ${}_{1}A_{\alpha} \to_{1} A_{1}$.

The group $\mathrm{Pic}_k(A-\ \ \mathrm{mod}\ ) \approx \mathrm{Pic}_k(A)$ operates on the isomorphism classes of faithfully projective left $A-$modules. We now describe the stability group of a faithfully projective module under this action.

79 **Proposition 7.3.** *Let Q be a faithfully projective left $A-$module, and let B denote the $k-$algebra* $\mathrm{Hom}_a(Q,Q)^0$. *Then there is an exact sequence*

$$1 \to \ In\ Aut\ (B) \to \ Aut_k(B) \xrightarrow{\varphi_Q} \mathrm{Pic}_k(A) \qquad (*)$$

with

$$\mathrm{im}\,\varphi_Q = \{(P) \in \mathrm{Pic}_k(A) \big| P \otimes_A Q \approx Q \textit{ as left } A\textit{- modules}\}.$$

Proof. Suppose first that $Q = A$, so that $B = A$. Define $\varphi_A(\alpha) = ({}_{1}A_{\alpha})$. Lemma 7.2 tells us that this is a homomorphism with kernel InAut (A), and with the indicated image. □

In the general case, we set $Q^* = \mathrm{Hom}_A(Q,A)$. Then the functor $T = \mathrm{Hom}_A(Q,\ ) \approx Q^* \otimes_A\ : A-\ \ \mathrm{mod}\ \to B-\ \ \mathrm{mod}$ is an equivalence, with $TQ = B$. This induces an isomorphism $\mathrm{Pic}_k(A-\ \mathrm{mod}\ ) \to \mathrm{Pic}_k(B-\ \ \mathrm{mod}\ )$. By proposition 7.1, we obtain an isomorphism $\mathrm{Pic}_k(A) \to \mathrm{Pic}_k(B)$, and this maps $(P) \in \mathrm{Pic}_k(A)$ into $(Q^* \otimes_A P \otimes_A Q) \in \mathrm{Pic}_k(B)$.

We define now $\varphi_Q : Aut_k(B) \to \mathrm{Pic}_k(A)$ as the composite $Aut_k(B) \to \mathrm{Pic}_k(B) \xrightarrow{\approx} \mathrm{Pic}_k(A)$, where the first map is defined as in the special case treated in the beginning, and the second is the inverse of the isomorphism just mentioned. The exactness of $(*)$ follows from the special case. Also, if $(P) \in \mathrm{Pic}_k(A)$, then $P \otimes_A Q \approx Q$ as left $A-$ modules $\Leftrightarrow Q^* \otimes_A P \otimes_A Q \approx Q^* \otimes_A Q$ as left $B-$modules. Since $Q^* \otimes_A Q \approx B$ as $B - B-$bimodules, the last statement in the proposition follows from the special case.

Let C = center A. If P is an invertible $A - A-$ bimodule, we can define a map **80**

$$\alpha_P : C \to C$$

by requiring that

$$pt = \alpha_P(t)p, \quad p \in P, t \in C.$$

This is possible because, the map $p \mapsto pt$, being a bimodule endomorphism of P, is the left multiplication by a unique element in the centre. Now α_P is a $k-$algebra homomorphism ($tp = pt$ for $t \in k$). If $p \otimes q \in P \otimes_A Q$ and $t \in C$, then $(p \otimes q)t = p \otimes \alpha_Q(t)q = p\alpha_Q(t) \otimes q = \alpha_p\alpha_Q(t)(p \otimes q)$. Thus

$$\alpha_{P\otimes Q} = \alpha_P\alpha_Q.$$

Since, evidently $\alpha_A = \mathrm{Id}_C$, it follows from the invertibility of P, that α_P is an automorphism of C, and that $(P) \mapsto \alpha_P$ is a homomorphism $\mathrm{Pic}_k(A) \to Aut_k(C)$. The kernel is clearly $\mathrm{Pic}_C(A)$. Summarization gives

Proposition 7.4. *If A is a k-algebra with center C, then there is an exact sequence*

$$0 \to \mathrm{Pic}_C(A) \to \mathrm{Pic}_k(A) \to Aut_k(C).$$

If A is commutative, then

$$0 \to \mathrm{Pic}_A(A) \to \mathrm{Pic}_k(A) \to Aut_k(A) \to 1$$

is exact and splits.

Proof. The map $\alpha \mapsto ({}_1A_\alpha)$ (see lemma 7.2) gives the required splitting **81**
$Aut_k(A) \to \mathrm{Pic}_k(A)$. $\square$

Example. Let A be the ring of integers in an algebraic number field k, and let $G(k/\mathbb{Q})$ be the group of automorphisms of k. (k need not be

Galois.) Evidently $Aut_{\mathbb{Z}}(A) \approx G(k/\mathbb{Q})$, and $\mathrm{Pic}_A(A)$ is just the ideal class group of A. Thus $\mathrm{Pic}_{\mathbb{Z}}(A)$ is the semi-direct product of the ideal class group of A with $G(k/\mathbb{Q})$, which operates on the ideal group, and hence on $\mathrm{Pic}_A(A)$. This is also the group of autoequivalences of the category A−mod. In particular, $\mathrm{Pic}_{\mathbb{Z}}(A)$ is finite (finiteness of class number) and $\mathrm{Pic}_{\mathbb{Z}}(\mathbb{Z}) = \{1\}$. Thus any autoequivalence $\mathbb{Z} - \mod \to \mathbb{Z} - \mod$ is isomorphic to the identity functor.

Chapter 3

The Brauer group of a commutative ring

In this chapter we prove the fundamental theorem on Azumaya algebras, following largely the paper of Auslander Goldman [1]. In §4 we obtain Rosenberg and Zelinsky's generalization of the Skolem-Noether theorem (see [1]). Finally we introduce the Brauer group $Br(k)$ of a commutative ring k. The functor End : $\underline{\underline{FP}} \to \underline{\underline{Az}}$ is cofinal, in the sense of chapter 1, and we obtain an exact sequence 82

$$K_1\underline{\underline{FP}} \to K_1\underline{\underline{Az}} \to K_o\Phi \text{ End } \to K_0\underline{\underline{FP}} \to K_0\underline{\underline{Az}} \to Br(k) \to 0.$$

We have computed the groups $K_i\underline{\underline{FP}}$ in chapter 1, and we further show here that $K_0\Phi$ End $\approx$ Pic(k). The final result is that the functors $\underline{\underline{\text{Pic}}} \to \underline{\underline{FP}} \to \underline{\underline{Az}}$ yield an exact sequence

$$U(k) \to K_1\underline{\underline{FP}} \to K_1\underline{\underline{Az}} \to \text{Pic}(k) \to K_0\underline{\underline{FP}} \to K_0\underline{\underline{Az}} \to Br(k) \to 0,$$

from which we can extract a short exact sequence

$$0 \to (\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} U(k)) \oplus (\mathbb{Q} \otimes_{\mathbb{Z}} sK_1P) \to K_1\underline{\underline{Az}} \to^t \text{Pic}(k) \to 0,$$

the last group being the torsion subgroup of Pic(k). This gives a fairly effective calculation of $K_1\underline{\underline{Az}}$.

1 Separable Algebras

Let k be a commutative ring. If A is a $k-$algebra, we write A^0 for the
83 opposite algebra of A, and $A^e = A \otimes_k A^0$. A two-sided $A-$ module M can be viewed as a left $A \otimes_k A^0$ module: We define the scalar multiplication by

$$(a \otimes b)x = axb, \quad x \in M, a, b \in A.$$

In particular, A is a left A^e- module, in a natural manner, and we have an exact sequence

$$0 \to J \to A^e \to A \to 0 \tag{1.1}$$

of A^e- linear maps (where $a \otimes b \in A^e$ goes to $ab \in A$). If needed, we shall make the notation more explicit by writing

$$A^e = (A/k)^e,$$

and

$$J = J(A) = J(A/k).$$

We define $k-$linear map

$$\delta : A \to J,$$

by setting $\delta(a) = a \otimes 1 - 1 \otimes a$.

Lemma 1.2. *Im δ generates J as a left ideal, and δ satisfies $\delta(ab) = a(\delta b) + (\delta a)b$.*

Proof. Clearly $\operatorname{im}\delta \subset J$. If $x = \sum a_i \otimes b_i \in J$, that is, if $\sum a_i b_i = 0$, then
84 $x = \sum a_i \otimes b_i - \sum a_i b_i \otimes 1 = \sum (a_i \otimes 1)((1 \otimes b_i) - (b_i \otimes 1)) = -\sum a_i \delta b_i$. Finally, $\delta(ab) = ab \otimes 1 - 1 \otimes ab = (a \otimes 1)(b \otimes 1 - 1 \otimes b) + (a \otimes 1)(1 \otimes b) - (1 \otimes b)(1 \otimes a) = a(\delta b) + (1 \otimes b)\delta a = a(\delta b) + (\delta a)b$. ◻

Corollary 1.3. *If M is a left A^e- module and N is a right A^e-module, there are natural isomorphisms*

$$\operatorname{Hom}_A e(A, M) \approx \{x \in M \big| ax = xa, \text{ for all } a \in A\},$$

and

$$N \otimes_{A^e} A \approx N/ \text{ (Submodule generated by } ax - xa, a \in A, x \in N).$$

Proof. Since $A \approx A^e/J$, $\mathrm{Hom}_{A^e}(A, M) \approx \{x \in M \big| Jx = 0\} = \{x \in M \big| (\delta a)x = 0 \forall a \in A\} = \{x \in M \big| ax = xa \forall a \in A\}$. The other part is trivial. $\square$

For a two-sided A−module M we shall denote the subgroup $\{x \in M \big| ax = xa \forall a \in A\}$ by M^A. Note that if A is a subalgebra of a k−algebra B, then B^A is just the centralizer of A in B. In particular, $A^A =$ centre A.

We denote by $\mathrm{Der}_k(A, M)$ the k−module of all k−derivations of A into M, that is, k−linear maps $d : A \to M$ satisfying $d(ab) = ad(b) + (da)b$, $a, b \in A$. If $f : M \to N$ is A^e−linear, then $d \mapsto fd$ defines a k−linear map $\mathrm{Der}_k(A, M) \to Der_k(A, N)$.. For example, if $x \in M$ and if $f : A^e \to M$ is defined by $f(1) = x$, then the composite

$$A \xrightarrow{\delta} J \hookrightarrow A^e \xrightarrow{f} M$$

is a derivation, called the *inner derivation* d_x defined by x. Thus, if $a \in A$, $d_x(a) = (\delta a)x = ax - xa$.

Proposition 1.4. *For an A^e-module M, the map $f \mapsto f\delta$ defines an isomorphism* **85**

$$\mathrm{Hom}_{A^e}(J, M) \to \mathrm{Der}_k(A, M),$$

with inner derivations corresponding to those f which can be extended to A^e.

Proof. Since im δ generated J, we have $f\delta = 0 \Rightarrow f = 0$. $\square$

Suppose $d \in \mathrm{Der}_k(A, M)$. We can define a k−linear map $f : A^e \to M$ by setting $f(\sum a_i \otimes b_i) = -\sum a_i d(b_i)$. This satisfies $f\delta a = f(a \otimes 1 - 1 \otimes a) = -ad(1) + 1d(a)$ for all $a \in A$. But $d(1) = d(1^2) = 1d(1) + d(1)1 = 2d(1)$ so that $d(1) = 0$. Thus $f\delta = d$. It remains to show that f/J is A^e−linear. If $x = \sum a_i \otimes b_i \in J$, we must show that $f((a \otimes b)x) = (a \otimes b)f(x)$. But $f((a \otimes b)x) = f(\sum aa_i \otimes b_i b) = -\sum aa_i d(b_i b) = -\sum aa_i(b_i db + d(b_i)b) = (a \otimes b)f(x)$.

The derived functors of $M \mapsto M^A$ are called the *Hochschild cohomology groups* of A with coefficients in M. We denote them by

$H^i(A, M)$. By virtue of (1.3), $H^i(A, M) \approx \mathrm{Ext}^i_{A^e}(A, M)$. The exact sequence (1.1) gives us an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{A^e}(A, M) \to \mathrm{Hom}_{A^e}(A^e, M) \\ &\to \mathrm{Hom}_{A^e}(J, M) \to \mathrm{Ext}^1_{A^e}(A, M) \to 0, \end{aligned}$$

which we can rewrite, using (1.3) and (1.4), to obtain:

Proposition 1.5. *There is an exact sequence*

$$0 \to M^A \to M \to Der_k(A, M) \to H^1(A, M) \to 0.$$

86 *so that $H^1(A, M)$ = the $k-$module of $k-$derivations of A into M, modulo the $k-$submodule of inner derivations.*

If $C = A^A$ = centre A, then $C \otimes 1 \subset$centre A^e, so we can view the above exact sequence as a sequence of $C-$modules and $C-$linear maps.

Proposition and Definition 1.6. *A $k-$algebra A is called separable, if it satisfies the following conditions, which are equivalent:*

(1) *A is a projective A^e-module.*

(1)$_{\text{bis}}$ *$M \mapsto M^A$ is an exact functor on A^e- modules.*

(1)$_{\text{ter}}$ *$(A^e)^A \to A^A \to 0$ is exact.*

(2) *If M is an A^e-module, then every $k-$derivation $A \to M$ is inner.*

(2)$_{\text{bis}}$ *the derivation $\delta : A \to J$ is inner.*

Proof. Since $M^A \approx \mathrm{Hom}_{A^e}(A, M)$, the implications (1) $\Leftrightarrow$ (1)$_{\text{bis}}$ $\Rightarrow$ (1)$_{\text{ter}}$ are clear. If $\mathrm{Hom}_{A^e}(A, A^e) \to \mathrm{Hom}_{A^e}(A, A) \to 0$ is exact, then 1_A factors through A^e, so that A is A^e projective, thus proving (1)$_{\text{ter}}$ $\Rightarrow$ (1). □

(1) $\Leftrightarrow$ (2) by virtue of the identifications $\mathrm{Ext}^1_{A^e}(A, M) = H^1(A, M)$ = derivations modulo inner derivations. Also the implication (2) $\Rightarrow$ (2)$_{\text{bis}}$ is obvious. Finally, proposition 1.4 shows that δ is inner $\Leftrightarrow$ 1_J extends to a homomorphism $A^e \to J$ that is, $\Leftrightarrow$ the exact sequence (1.1) splits. This proves (2)$_{\text{bis}}$ $\Rightarrow$ (1).

87 **Corollary 1.7.** *If A/k is separable with centre C, then for an A^e-module M, there is a split exact sequence of $C-$models,*

$$0 \to M^A \to M \to Der_k(A, M) \to 0.$$

In particular, C is a $C-$direct summand of A.

This follows directly from (1.5) and the definition above.

Corollary 1.8. *If $A \to B$ is an epimorphism of $k-$algebras, with A/k separable, then B/k is separable, and centre $B=$ image of centre A.*

Proof. If M is a two-sided $B-$module, then evidently $M^B = M^A$, so that $M \mapsto M^B$ is an exact functor, that is, B is separable. Also, $A^A \to B^A = B^B \to 0$ is exact. ☐

2 Assorted lemmas

The reader is advised to skip this section and use it only for references.

Lemma 2.1 (Schanuel's lemma)**.** *If $0 \to N_i \to P_i \xrightarrow{f_i} M \to 0$ are exact with P_i projective, $i = 1, 2$, then $P_1 \oplus N_2 \approx P_2 \oplus N_1$.*

Proof. If $P_1 \prod\limits_M P_2 = \{(x_1, x_2) \in P_1 \oplus P_2 \big| f_1(x_1) = f_2(x_2)\}$, then the coordinate projections give us maps $P_1 \prod_M P_2 \to P_i$, $i = 1, 2$, and a 88
commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & N_2 \arrow[r, equal] \arrow[d] & N_2 \arrow[d] & \\
0 \arrow[r] & N_1 \arrow[r] \arrow[d, equal] & P_1\Pi_M P_2 \arrow[r] \arrow[d] & P_2 \arrow[r] \arrow[d, no head] & 0 \\
0 \arrow[r] & N_1 \arrow[r] & P_1 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns, as is easily checked. Since the P_i are projective, we conclude that $P_1 \oplus N_2 \approx P_1 \prod\limits_M P_2 \approx P_2 \oplus N_1$. □

Let A be a ring. An $A-$ module M is called *finitely presented*, if there exists an exact sequence

$$F_1 \to F_0 \to M \to 0$$

of $A-$linear maps with F_i a finitely generated free $A-$ module, $i = 0, 1$.

Corollary 2.2. *(a) If* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence of $A-$modules, with M and M'' finitely presented, then M' is finitely generated.*

89 *(b) If A is commutative, and M and N are finitely presented $A-$modules, then so is $M \otimes_A N$.*

(c) If A is an algebra over a commutative ring k, and if A is finitely presented as a $k-$module, then A is finitely presented as an A^e-module.

Proof. (a) Case I. Suppose M is projective. Then the result follows easily from the definition and Schanuel's lemma.

General Case. Let $f : P \to M$ be surjective with P finitely generated and projective, and let $f'' : P \to M''$ be the epimorphism obtained by composing f with $M \to M''$. We have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & P \arrow[r, equal] \arrow[d, "f"] & P \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The exact sequence

$$0 = \ker f' \to \ker f \to \ker f'' \to \text{ coker } f' = M' \to \text{ coker } f = 0$$

shows that M' is finitely generated, since by case I, $\ker f''$ is.

(b) follows easily from right exactness.

(c) If A is finitely presented as a $k-$ module, then A^e is finitely presented as a $k-$module, This implies that J is finitely generated as a $k-$module and a fortiori as an A^e- module.

□

Lemma 2.3. *Let K_i be a commutative $k-$ algebra and let M_i, N_i be* **90**
K_i-modules, $i = 1, 2$. There is a natural isomorphism

$$(M_1 \otimes_k M_2) \otimes_{K_1 \otimes_k K_2} (N_1 \otimes_k N_2) \xrightarrow{\approx} (M_1 \otimes_{K_1} N_1) \otimes_k (M_2 \otimes_{K_2} N_2)$$

given by $(m_1 \otimes m_2) \otimes (n_1 \otimes n_2) \mapsto (m_1 \otimes n_1) \otimes (m_2 \otimes n_2)$. If the M_i and N_i are K_i-algebras, then the above map is an isomorphism of $K_1 \otimes_k K_2-$algebras.

Proof. Straightforward. □

Corollary 2.4. *(a) If K_i is a commutative k-algebra, and A_i a K_i-algebra, $i = 1, 2$, then (2.3) defines a natural isomorphism*

$$(A_1/K_1)^e \otimes_k (A_2/K_2)^e \approx (A_1 \otimes_k A_2/K_1 \otimes_k K_2)^e$$

(b) If K and A are $k-$algebras, K commutative, then $(K \otimes_k A/K)^e \approx K \otimes_k (A/k)^e$.

Proof. (a) In (2.3) we take $M_i = A_i$ and $N_i = A_i^0$. Evidently $(A_1 \otimes_k A_2)^0 = A_1^0 \otimes_k A_2^0$.

(b) Set $A_1 = K_1 = K$, $A_2 = A$ and $K_2 = k$ in (a).

□

Lemma 2.5. *Let K_i be a commutative k-algebra, let A_i be a K_i-algebra, and let M_i and N_i be A_i-modules, $i = 1, 2$. The k-bilinear map $(f_1, f_2) \mapsto f_1 \otimes f_2$ defines a $K_1 \otimes_k K_2$-homomorphism*

$$\mathrm{Hom}_{A_1}(N_1, M_1) \otimes_k \mathrm{Hom}_{A_2}(N_2, M_2) \to \mathrm{Hom}_{A_1 \otimes_k A_2}(N_1 \otimes_k N_2, M_1 \otimes_k M_2).$$

It is an isomorphism in either of the following situations: **91**

(i) N_i is a finitely generated projective A_i-module, $i = 1, 2$.

(ii) *N_1 and M_1 are finitely generated projective A_1−modules, A_1 is k−flat, and N_2 is a finitely presented A_2−module.*

Proof. The first assertion is clear.

(i) By additivity we are reduced to the case $N_i = A_i$, $i = 1, 2$, and then the assertion is clear.

(ii) By additivity again we can assume that $N_1 = M_1 = A_1$.

Write $SN_2 = A_1 \otimes_k \mathrm{Hom}_{A_2}(N_2, M_2)$, $TN_2 = \mathrm{Hom}_{A_1\otimes_k A_2}(A_1 \otimes_k N_2, A_1 \otimes_k M_2)$. We have a map $SN_2 \to TN_2$. This is a isomorphism for $N_2 = A_2$, and therefore, for $N_2 = A_2^{(n)}$. Let now $A_2^{(n)} \to A_2^{(m)} \to N_2 \to 0$ be an exact sequence. S and T being left exact contravariant functors in N_2, we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & SN_2 \arrow[r] \arrow[d] & SA_2^{(m)} \arrow[r] \arrow[d] & SA_2^{(n)} \arrow[d] \\
0 \arrow[r] & TN_2 \arrow[r] & TA_2^{(m)} \arrow[r] & TA_2^{(n)}
\end{tikzcd}$$

with exact rows. The second and third vertical maps being isomorphisms, if follows that the first one is also an isomorphism. □

92 **Corollary 2.6.** *If K_i is a commutative k−algebra, and if A_i/K_i is a separable algebra, $i = 1, 2$, then $A_1 \otimes_k A_2/K_1 \otimes_k K_2$ is a separable algebra with centre = (centre A_1) $\otimes_k$ (centre A_2). More generally, if M_i is an $(A_i/K_i)^e$ module, then the natural map*

$$M_1^{A_1} \otimes_k M_2^{A_2} \to (M_1 \otimes_k M_2)^{A_1\otimes_k A_2}$$

is an isomorphism.

Proof. We have, by hypothesis, (2.4), and (2.5), an isomorphism

$$\begin{aligned}
M_1^{A_1} \otimes_k M_2^{A_2} &= \mathrm{Hom}_{(A_1/K_1)}\, e(A_1, M_1) \otimes_k \mathrm{Hom}_{(A_2/K_2)}\, e(A_2, M_2) \to \\
&\quad \mathrm{Hom}_{(A_1/K_1)^e\otimes_k(A_2/K_2)^e}(A_1 \otimes_k A_2, M_1 \otimes_k M_2) \\
&= (M_1 \otimes_k M_2)^{A_1\otimes_k A_2}
\end{aligned}$$

Applying this to $(A_1/K_1)^e \otimes_k (A_2/K_2)^e = (A_1 \otimes_k A_2/K_1 \otimes_k K_2)^e \to A_1 \otimes_k A_2 \to 0$ we get a commutative diagram

$$\begin{tikzcd}
((A_1 \otimes_k A_2/K_1 \otimes_k K_2)^e)^{A_1\otimes_k A_2} \arrow[r] & (A_1 \otimes_k A_2)^{A_1\otimes_k A_2} \\
((A_1/k_1)^e)^{A_1} \otimes_k ((A_2/K_2)^e)^{A_2} \arrow[r] \arrow[u] & (A_1^{A_1} \otimes_k A_2^{A_2}) \arrow[u]
\end{tikzcd}$$

in which the vertical maps are isomorphisms and the lower horizontal map is surjective, by hypothesis and right exactness of $\otimes_k$. It follows that the upper map is also surjective, and this finishes the proof, using criterion $(1.6)(1)_{\text{ter}}$ for separability. □

Corollary 2.7. *If A_i/k is a separable algebra, $i = 1, 2$, then $(A_1 \otimes_k A_2)/k$ is separable with centre $A_1 \otimes_k$ centre A_2 as its centre.* 93

Corollary 2.8. *Suppose K and A are k-algebras. Suppose further that K is k-flat.*

(a) If M and N are A-modules with N finitely presented, then

$$K \otimes_k \operatorname{Hom}_A(N, M) \to \operatorname{Hom}_{k\otimes_k A}(K \otimes_k N, K \otimes_k M)$$

is an isomorphism.

(b) If K is commutative, and if A is finitely presented as anA^e-module then for an A^e-module M, the map

$$K \otimes_k (M^A) \to (K \otimes_k M)^{K\otimes_k A}$$

is an isomorphism.

Proof. The statement (a) follows from (2.5) (ii) with $N_1 = M_1 = A_1 = K_1 = K$, and $K_2 = k$. The statement (b) follows from (a) by substituting A^e for A, and A for N. □

Corollary 2.9. *Let K and A be k-algebras, with K commutative.*

(a) $A/_k$ separable $\Rightarrow$ $K \otimes_k A/K$ is separable with centre $(K \otimes_k A) = K \otimes_k$ (centre A).

(b) *If K is faithfully k-flat and if A is a finitely presented A^e-module, then $(K \otimes_k A)/K$ separable $\Rightarrow A/k$ separable.*

94 *Proof.* (a) is a special case of (2.6).

(b) Suppose $K \otimes_k A/K$ is separable. Corollary 2.4 implies that $(K \otimes_k A/K)^e \to K \otimes_k A$ is isomorphic to $K \otimes_k ((A/k)^e \to A)$. Then (2.8)(b) further implies that $((K \otimes_k A/k)^e)^{K\otimes_k A} \to (K \otimes_k A)^{K\otimes_k A}$ is isomorphic to $K \otimes_k (((A/k)^e)^A \to A^A)$. Therefore, by hypothesis, $K \otimes_k (((A/k^e)^A \to A^A)$ is surjective. Since K is faithfully k-flat, this implies that $((A/k)^e)^A \to A^A$ is surjective, so that A/k is separable (see (1.6)(1) ter).

□

Example. If K is a noetherian local ring, in (2.9)(b) we can take K to be completion of k.

Corollary 2.10. *If A/k is a finitely A^e-presented k-algebra, then A/k is separable $\Leftrightarrow A_{\mathscr{M}}/k_{\mathscr{M}}$ is separable for all maximal ideals $\mathscr{M}$ of k.*

Proof. Take $K = \prod_{\mathscr{M}} k_{\mathscr{M}}$ in (2.9)(b). Alternatively, repeat the proof of (2.9)(b) and at the end use the fact that a k-homomorphism f is surjective $\Leftrightarrow f_{\mathscr{M}}$ is surjective for all $\mathscr{M}$. □

Corollary 2.11. *Suppose A_i is a k-algebra and that P_i is a finitely generated projective A_i-module, $i = 1, 2$. Then*

$$\mathrm{End}_{A_1}(P_1) \otimes_k \mathrm{End}_{A_2}(P_2) \to \mathrm{End}_{A_1\otimes_k A_2}(P_1 \otimes_k P_2)$$

is an algebra isomorphism.

Proof. Set $N_i = M_i = P_i$ and $K_i = k$ in (2.5) (i). □

95 **Corollary 2.12.** *Suppose P_1 and P_2 are finitely generated projective k-modules. Then*

$$\mathrm{End}_k(P_1) \otimes_k \mathrm{End}_k(P_2) \to \mathrm{End}_k(P_1 \otimes_k P_2)$$

is an algebra isomorphism.

Proof. Set $A_i = k$ in (2. 11). $\square$

Proposition 2.13. *Let P be a finitely generated projective k-module. Then $A = \mathrm{End}_k(P)$ is a separable algebra with centre $k/annP$.*

Proof. Both centre A and B ann P commute with localization, and hence we can use (2.10) to reduce to the case when P is free, say $P \approx k^{(n)}$, so that $A \approx \mathbb{M}_n(k)$. Denoting the standard matrix algebra basis by (e_{ij}), we set $e = \sum\limits_{i=1}^{n} e_{i1} \otimes e_{1i} \epsilon A^e$. Then $(e_{rs} \otimes 1)e = \sum e_{rs}e_{i1} \otimes e_{1i} = e_{r1} \otimes e_{1s}$ and $(1 \otimes e_{rs})e = \sum e_{i1} \otimes e_{1i}e_{rs} = e_{r1} \otimes e_{1s}$. Hence $e \epsilon (A^e)^A$. Under $(A^e)^A \to A^A$, e maps into $\sum e_{i1}e_{1i} = \sum_{e_{ii}} = 1$. Hence $(A^e)^A \to A^A$ is surjective. Thus A is separable, by $(1.6)(1)_{\text{ter}}$. Theorem (6.1)(5) of chapter 2 implies that centre $\mathbb{M}_n(k) = k$. $\square$

Lemma 2.14. *Let $f : M \to M$ be a k-endomorphism, k a commutative ring. Suppose that M is either noetherian or finitely generated and projective. Then, if f is surjective, it is an automorphism.*

Proof. If $\ker f \neq 0$, then f surjective implies that $\ker f^n$ is a strictly ascending chain of submodules, an impossibility if M is noetherian, If M is projective, then $M \approx M \oplus \ker f$ and localization shows that $\ker f = 0$ **96**
if M is finitely generated. $\square$

Proposition 2.15. *Let k be a local ring with maximal ideal $\mathscr{M}$, and let A be a k-algebra, finitely generated as a k-module. Suppose that either k is noetherian or that A is k-projective. Then if $A/\mathscr{M}A$ is a separable $(k/\mathscr{M})$-algebra, A is a separable k-algebra.*

Proof. Consider $\delta : A \to J = J(A/k)$. Let k' denote reduction modulo $\mathscr{M}$; e. g. $k' = k/\mathscr{M}$. Then δ induces a k'-derivation $\delta' : A' \to J'$, where J' is a two-sided A'-module. By hypothesis and criterion (1.6)(2), δ' must be inner, $\delta'(a') = a'e' - e'a' = ae' - e'a = \delta(a)e'$, for some $e' \epsilon J'$, coming from say $e \epsilon J$. It follows that $\delta(a)e \equiv \delta(a) \mod \mathscr{M}J$, so (1.2) implies $J = Je + \mathscr{M}J$. The exact sequence

$$0 \to J \to A^e \to A \to 0$$

shows that J is noetherian if k is, and that J is k-projective and finitely generated if A is. Hence we can apply lemma 2.14 to the k-homomorphism $J \xrightarrow{e} J$, provided the latter is surjective. But this follows from Nakayama's lemma, since $J = Je + \mathscr{M} J$.

Now the composite $J \hookrightarrow A \xrightarrow{e} J$ is an automorphism of J, so that J is an A^e-direct summand of A^e. This proves that A is A^e -projective, as required. □

Lemma 2.16. *Let $f : P \to M$ be a k-homomorphism with P finitely*
97 *generated and projective. Denote the functor* $\mathrm{Hom}_k(,k)$ *by* *. *Then f has a left inverse $\Leftrightarrow$ $f^* : M^* \to P^*$ is surjective. If M is finitely presented, then (coker $f^*)_{\mathscr{M}}$ = coker $(f_{\mathscr{M}})^*$, so that f has a left inverse $\Leftrightarrow$ $f_{\mathscr{M}}$ does for all maximal ideals $\mathscr{M}$.*

Proof. f left invertible $\Rightarrow$ f^* right invertible $\Rightarrow$ f^* surjective $\Rightarrow$ f^* right invertible (because P^* is projective) $\Rightarrow$ $f^{**} : P^{**} \to M^{**}$ left invertible. The commutative square

$$\begin{array}{ccc} P & \xrightarrow{f} & M \\ {\scriptstyle\approx}\downarrow & & \downarrow \\ P^{**} & \xrightarrow[f^{**}]{} & M^{**} \end{array}$$

shows that f^{**} left invertible $\Rightarrow$ f left invertible. □

The natural homomorphism

$$(M^*)_{\mathscr{M}} = (\mathrm{Hom}_k(M,K))_{\mathscr{M}} \to (M_{\mathscr{M}})^* = \mathrm{Hom}_{k_{\mathscr{M}}}(M_{\mathscr{M}}, k_{\mathscr{M}})$$

is an isomorphism for M finitely presented, by (2.8)(a). Hence since P is also finitely presented, we have $(f^*)_{\mathscr{M}} \approx (f_{\mathscr{M}})^*$ in this case, so that by exactness of localization, (coker $f^*)_{\mathscr{M}} \approx$ coker $(f_{\mathscr{M}})^*$.

Corollary 2.17. *If A is a faithfully k-projective k-algebra, then k is a direct summand of A.*

Proof. We want $k \to A$ to have a left inverse, and (2. 16) plus our
98 hypothesis makes it sufficient to prove this for k local, say with maximal ideal $\mathscr{M}$. Then $1 \epsilon A/\mathscr{M}A$ is a part of a $k/\mathscr{M}$-basis for $A/\mathscr{M}A$, so Nakayama's lemma implies that $1 \epsilon A$ is a part of a k-basis of A. □

Proposition 2.18. *Let A and B be k-algebras with A a faithfully projective k-module. Then $A \otimes_k B/k$ separable $\Rightarrow B/k$ separable.*

Proof. A is k-projective implies that A^e is k-projective. Hence $(A \otimes_k B)^e \approx A^e \otimes_k B^e$ is B^e-projective. Thus, if $A \otimes_k B$ is $(A \otimes_k B)^e$-projective, then it is B^e-projective. Corollary 2.17 and our hypothesis implies $A \approx k \oplus A'$ as a k-module, so that $A \otimes_k B \approx B \oplus (A' \otimes_k B)$ as a B^e-module. Thus B^e-projectivity of $A \otimes_k B \Leftarrow B^e$-projectivity of B. □

Proposition 2.19. *Suppose A is a K-algebra and that K is a k-algebra. Then*

(1) A/K and K/k separable $\Rightarrow A/k$ separable.

(2) A/k separable $\Rightarrow A/k$ separable.

If A faithfully K-projective, then A/k separable $\Rightarrow K/k$ separable.

Proof. (1) K/k separable means that

$$0 \to J(K/k) \to K^e \to K \to 0$$

splits as an exact sequence of K^e-modules. Hence

$$0 \to (A/k)^e \otimes_{K^e} J(K/k) \to (A/k)^e \to (A/k)^e \otimes_{k^e} K \to 0$$

splits as an exact sequence of $(A/k)^e$-modules, so that $(A/k)^e \otimes_{K^e} K$ 99
is a projective $(A/k)^e$-module. it follows easily from corollary 1.3 that $(A \otimes_k A^o) \otimes_{K^e} K \approx A \otimes_K A^o = (A/K)^e$. Hence if we further assume that A is $(A/K)^e$-projective, it follows from the projectivity of $(A/K)^e$ over $(A/k)^e$, remarked above, that A is $(A/k)^e$-projective.

(2) In the commutative diagram with exact rows and columns

$$\begin{tikzcd}
(A/k)^e \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
(A/K)^e \arrow[r] \arrow[d] & A \arrow[r] & 0 \\
0 & &
\end{tikzcd}$$

if the top splits, then so much the bottom. Suppose A is faithfully K-projective. Then $(A/k)^e$ is $(K/k)^e$-projective, so that A is $(K/k)^e$-projective, assuming that (A/k) is separable. By corollary 2.17 K is a K-direct summand, of A, hence a $(K/k)^e$-direct summand, so we conclude that K is $(K/k)^e$-projective, as claimed.

□

Proposition 2.20. *(a) If A_1 and A_2 are k-algebras, then $A_1 \times A_2/k$ is separable $\Leftrightarrow$ A_1/k and A_2/k are.*

(b) If A_i is a k_i-algebra, $i = 1, 2$, then $A_1 \times A_2/k_1 \times k_2$ is separable $\Leftrightarrow$ A_1/k_1 and A_2 and A_2/k_2 are.

100 *Proof.* (a) $(A_1 \times A_2)^e = (A_1 \otimes_k A_1^0) \times (A_2 \otimes_k A_2^0) \times (A_1 \otimes_k A_2^0) \times (A_2 \otimes_k A_1^0) = A_1^e \times A_2^e \times B$, and $A_1 \times A_2$ is an $(A_1 \times A_2)^e$-module annihilated by B. As such it is the direct sum of the A_i^e-modues A_i. Thus $A_1 \times A_2$ is $(A_1 \times A_2)^e$ -projective $\Leftrightarrow$ A_i is A_i^e-projective

(b) Any $k_1 \times k_2$-module or algebra splits canonically into a product of one over k_1 and one over k_2. In particular $(A_1 \times A_2/k_1 \times k_2)^e = (A_1/k_1)^e \times (A_2/k_2)^e$, so $A_1 \times A_2$ is $(A_1 \times A_2/k_1 \times k_2)^e$-projective $\Leftrightarrow$ A_i is $(A_i/k_i)^e$-projective.

□

3 Local criteria for separability

Theorem 3.1. *Let A be a k-algebra, finitely generated as a k-module. Suppose either that k is noetherian or that A is a projective k-module. Then the following statements are equivalent:*

(1) A/k is separable.

(2) For each maximal ideal $\mathscr{M}$ of k, $A/\mathscr{M}A$ is a semi-simple $k/\mathscr{M}$-algebra whose centre is a product of separable field extensions of $k/\mathscr{M}$.

(3) For any homomorphism $k \to L$, L a field, $L \otimes_k A$ is a semi-simple algebra.

We will deduce this form the following special case:

Theorem 3.2. *Let A be a finite dimensional algebra over a field k. The following statements are equivalent:*

(1) A/k separable

(2) $L \otimes_k A$ is semi-simple for all field extensions L/k.

(3) For some algebraically closed field L/k, $L \otimes_k A$ is ε product of full matrix algebras over L. **101**

(4) A is semi-simple and centre A is a product of separable field extensions of k.

We first prove that (3.2) $\Rightarrow$ (3.1):

(1) $\Rightarrow$ (3). If $k \to L$, then $L \otimes_k A/L$ is separable, by (2.9), and we now apply (1) $\Rightarrow$ (4) of (3.2).

(3) $\Rightarrow$ (2). Apply (2) $\Rightarrow$ (4) of (3.2), where L ranges over field extensions of $k/\mathscr{M}$.

(2) $\Rightarrow$ (1). From (4) $\Rightarrow$ (1) of (3.2) we know that $A/\mathscr{M}A$ is a separable $(k/\mathscr{M})$-algebra so the hypothesis on A and proposition 2.15 imply that $A_{\mathscr{M}}/k_{\mathscr{M}}$ is separable, for all $\mathscr{M}$. (1) now follows from corollary 2.10.

Proof of Theorem 3.2. (1) $\Rightarrow$ (2)$'$. Since A/k separable implies that $(L \otimes_k A)/L$ is separable, it suffices to show that A/k separable $\Rightarrow$ A is semi-simple. Let M, N be left A-modules. Then $\mathrm{Hom}_m(M, N)$ is a two-sided A-module, i.e., an A^e-module, and $Hom_{A^e}(A, Hom_k(M, N)) = \mathrm{Hom}_A(M, N)$ clearly (see (1.3)). Since k is a field, $\mathrm{Hom}_k(M, \ )$ is an exact functor. Since A is A^e-projective (by assumption), $\mathrm{Hom}_{A^e}(A, \ )$ is exact. Hence $\mathrm{Hom}_A(M, \ )$ is an exact functor, so every A-module is projective. Proposition 6.7 of Chapter 2 now implies that A semi-simple.

102 (2) $\Rightarrow$ (3). This follows from the structure of semi-simple rings plus the fact that there are no non-trivial finite dimensional division algebras over an algebraically closed field.

(3) $\Rightarrow$ (1). By assumption, $L \otimes_k A$ is a product of full matrix algebras over L. Proposition 2.13 and 2.20 imply that $L \otimes_k A/L$ is separable. Since L is faithfully k-flat (k is a field!), (2.9)(b) implies that A/k is separable.

The proof of (1) $\Leftrightarrow$ (4) will be based upon the next two lemmas, which are special cases of the theorem.

Lemma 3.3. *A finite field extension C/k is separable as a k-algebra $\Leftrightarrow$ it is separable as a field extension of k.*

Proof. If $k \subset K \subset C$, then C/k is separable, in either sense $\Leftrightarrow$ C/K and K/k are, in the same sense. This follows from proposition 2.18 in one case, and from field theory in the other. An induction on degree therefore reduces the lemma to the case $C = k[X]/(f(X))$. Let L be an algebraic closure of k, and write $f(X) = \prod_i (X - a_i)^{e_i}$ in $L[X]$, with a_i's distinct. Then C is a separable field extension $\Leftrightarrow$ $L \otimes_k C = L[X]/f(X)L[X]$ has no nilpoint elements $\Leftrightarrow$ $L \otimes_k C$ is a product of copies of L $\Leftrightarrow$ $L \otimes_k C/L$ is separable $\Leftrightarrow$ C is a separable algebra over k, by (1) $\Leftrightarrow$ (3) of (3.2), which we have already proved. $\square$

We now prove the implication (1) $\Leftrightarrow$ (4) of (3.2). We have already proved that A/k is separable implies that A is semi-simple. Hence the

103 centre C of A must be a finite product of field extension of k. in particular A is a faithfully projective C-module, so by proposition 2.19, A/k separable $\Rightarrow$ C/k separable. The last part of (4) now follows from proposition 2.20 and lemma 3.3 above.

Lemma 3.4. *Let k be a field, and suppose that A is a finite dimensional k-algebra, simple and central (i.e.centre $A = k$). If B is any k-algebra, every two-sided ideal of $A \otimes_k B$ is of the form $A \otimes_k J$ for some two sided ideal J of B.*

Proof. According to theorem 6.1 chapter 2 there is a division algebra D and an $n > 0$ such that $A \approx \mathbb{M}_n(D) = D \otimes_k \mathbb{M}_n(k)$. Theorem 4.4 of chapter 2 contains the lemma when $A = \mathbb{M}_n(k)$, in which case $A \otimes_k B = \mathbb{M}_n(B) = \mathrm{End}_B(B^{(n)})$. It therefore suffices to prove the lemma for $A = D$, a division algebra. If (e_i) is a k-basis for B, then $(1 \otimes e_i)$ is a left D-basis for $D \otimes_k (B)$. Let I be a two-sided ideal of $D \otimes_k B$. Then I is a D-subspace of $D \otimes_k B$, and it is (clearly) generated by the "primordial" elements of I with respect to the basis $(1 \otimes e_i)$, i.e. by those elements $x = \sum(d_i \otimes 1)(1 \otimes e_i) \neq 0$ of I such that $S(x) = \{i | d_i \neq 0\}$ does not properly contain $S(y)$ for any $y \neq 0$ in I, and such that the least one $d_i = 1$. If x such an element and if $d \neq 0$ is in D, then $x(d \otimes 1) \in I$, because I is a two-sided ideal. Now $x(d \otimes 1) = \sum(d_i \otimes e_i)(d \otimes 1) = \sum d_i d \otimes e_i$ so $S(x(d \otimes 1)) = S(x)$. Subtracting $(d' \otimes 1)x$ from $x(1 \otimes d)$ will therefore render $S((d' \otimes 1)x - x(1 \otimes d))$ a proper subset of $S(x)$, for a suitable $d' \epsilon D$. □

Since x is primordial, this implies $(d' \otimes 1)x = x(d \otimes 1)$, i.e. that **104**
$\sum d' d_i \otimes e_i = \sum d_i d \otimes e_i$. Some $d_i = 1$ so we have $d' = d$. Moreover, $d_i d = d d_i$ for all i. By assumption centre $D = k$, so $x \epsilon k \otimes B = 1 \otimes B$. Setting $1 \otimes J = I \cap (1 \otimes B)$, we therefore have $I = D \otimes J$.

We shall now prove the implication (4) $\Rightarrow$ (1) of theorem 3.2. Let C = centre A. To show that A/k is separable Proposition 2.19 makes it sufficient to show that A/C and C/k are separable. In each case, moreover, proposition 2.20 reduces the problem to the case when C is a field, Separability of C/k then results from the hypothesis and lemma 3.3. Let L be an algebraic closure of C. Then it follows from lemma 3.4 that $L \otimes_C A$ is simple, hence a full matrix algebra over L. Separability of A/C now follows from the implication (3) $\Rightarrow$ (1) which we have proved. Thus the proof of the theorem 3.2 is complete.

4 Azumaya algebras

Theorem and Definition 4.1. *An azumaya algebra is a k-algebra A satisfying the following conditions, which are equivalent:*

(1) A is a finitely generated k-module and A/k is central and separable.

(2) *A/k is central and A is a generator as an A^e-module.*

(3) *A is a faithfully projective k-module, and the natural representation $A^e \to \mathrm{End}_k(A)$ is an isomorphism.*

105 (4) *The bimodule ${}_{A^e}A_k$ is invertible (in the sense of definition 3.2 of chapter 2), i.e. the functors*

$$\begin{array}{c} (N \longmapsto A \otimes_k N) \\ k-\mathrm{mod} \rightleftarrows A^e-\mathrm{mod} \\ (M^A \longleftarrow\!\shortmid M) \end{array}$$

are inverse equivalences of categories.

(5) *A is a finitely generated projective k-module, and for all maximal ideals $\mathscr{M}$ of k, $A/\mathscr{M}A$ is a central simple $k/\mathscr{M}$ -algebra.*

(6) *There exists a k-algebra B and a faithfully projective k-module P such that $A \otimes_k B \approx \mathrm{End}_k(P)$.*

Proof. (1) $\Rightarrow$ (2). Let $\mathfrak{M}$ be a maximal two-sided ideal of A, and set $\mathscr{M} = \mathfrak{M} \cap k$. According to (1.8) and our hypothesis $A/\mathfrak{m}$ is a separable k-algebra with centre $k/\mathscr{M}$. Since $A/\mathfrak{M}$ does not have two-sided ideals, its centre is a field. Thus $\mathscr{M}$ is a maximal ideal of k, so $A/\mathscr{M}A$ is a central separable algebra over $k/\mathscr{M}$, and it follows from theorem 3.2 that $A/\mathscr{M}A$ is simple. Consequently $m = \mathscr{M}A$. Applying this to A^e, which, by (2.7), is also a separable k-algebra, we conclude that every maximal two-sided ideal of A^e is of the form $\mathscr{M}A^e$ for some maximal ideal $\mathscr{M}$ of k. □

106 Viewing A as a left A^e-module we have the pairing g_A: $A \otimes_k \mathrm{Hom}_{A^e}(A, A^e) \to A^e$, and its image is a two -sided ideal which equals $A^e \Leftrightarrow A$ is a generator as an A^e-module (see(5.6) of chapter 2). If $img_A \neq A^e$, then im g_A is contained in some maximal two-sided ideal of A^e, so, according to the paragraph above, im $g_A \subset \mathscr{M}A^e$, for some maximal ideal $\mathscr{M}$ of k. Now lemma 5.7 of chapter 2, plus our hypothesis that A is A^e-projective,

imply that $A = (img_A)A \subset \mathscr{M}A$. But from (1.7), k = centre A is a direct summand of A. So $A = \mathscr{M}A \Rightarrow k = \mathscr{M}$, which is a contradiction.

(2) $\Rightarrow$ (3). A is a generator of $A^e -$ mod , so the pairing

$$g_A : \mathrm{Hom}_{A^e}(A, A^e) \otimes_k A \to A^e$$

is surjective. It follows now from theorem 4.3 and proposition 5.6 of chapter 2 that A is a finitely generated projective k-module, and that $A^e \to \mathrm{End}_k(A)$ is an isomorphism. Since A is a faithful k-module (k being centre of A), corollary 5.10 of chapter 2 implies that it is faithfully projective.

(3) $\Rightarrow$ (4). This follows directly from proposition 5.6 and definition 3.2 of chapter 2.

(4) $\Rightarrow$ (1). is trivial once we note that centre $A = \mathrm{Hom}_{A^e}(A, A)$.

(1) $\Rightarrow$ (5) follows from (1) $\Rightarrow$ (3) of theorem 3.1.

(5) $\Rightarrow$ (1). Theorem 3.1 shows that A/k is separable.

Let C =centre A. Then C is a C-direct summand of A, and hence a finitely generated projective k-module, since A is so. We have a ho- 107
momorphism $k \to C$, and (1.8) implies that $k/\mathscr{M} \to C/\mathscr{M}C$ is an isomorphism for all maximal ideals $\mathscr{M}$ of k. This implies that $k \to C$ is surjective, since the cokernel is zero modulo all maximal ideals of k, and hence it splits, because C is projective. The kernel of $k \to C$ is also zero, since it is zero modulo all maximal ideals of k. Thus $k \to C$ is an isomorphism.

(3) $\Rightarrow$ (6). Take $B = A^0$ and $P = A$.

(6) $\Rightarrow$ (1). $\mathrm{End}_k(P)$ is faithfully projective, since P is. Since $A \otimes_k B \approx \mathrm{End}_k(P)$, it follows from proposition 6.1 of chapter 1 that B is faithfully projective. Proposition 2.13 says that $\mathrm{End}_k(P)/k$ is central and separable, so that, by proposition 2.18, A/k is separable. Similarly B/k is separable. It follows from (2.7), that (centre A)$\otimes_k$(centre B) = centre $\mathrm{End}_k(P) = k$. Hence centre A has rank 1, as a projective k-module, and so centre $A = k$, since k is a direct summand of $A \otimes_k B$ and therefore of centre A.

Corollary 4.2. *If A/k is an azumaya algebra, then $\mathscr{U} \mapsto \mathscr{U}A$ is a bijection from the ideals of k to the two-sided ideals of A.*

Proof. This follows from theorem 4.4 of chapter 2, since two-sided ideals of A are simply A^e-submodules of A. □

Corollary 4.3. *Let $A \subset B$ be k-algebras with A azumaya. Then the natural map $A \otimes_k B^A \to B$ is an isomorphism.*

Proof. This is a special case of the statement (1) of theorem 4.1. □

108 **Corollary 4.4.** *Every endomorphism of an azumaya algebra is an automorphism.*

Proof. Suppose $f \colon A \to A$ is an endomorphism of an azumaya algebra A/k. By (4.2), $\ker f = \mathscr{U} A$ for some ideal $\mathscr{U}$ of k and hence $\ker f = 0$. Therefore (4.3) implies $A \approx f(A) \otimes_k A^{f(A)}$. Counting ranks we see that $A^{f(A)} = k$. □

Corollary 4.5. *The homomorphism*

$$\mathrm{Pic}_k(k) \longrightarrow \mathrm{Pic}_k(A),$$

induced by $L \mapsto A \otimes_k L$, is an isomorphism.

Proof. This follows (see (4) of (4.1)) from the fact that $A \otimes_k : k-\text{mod} \to A^e - \text{mod}$ is an equivalence which converts $\otimes_k$ into $\otimes_A$; the latter is just the identity

$$(A \otimes_k M) \otimes_A (A \otimes_k N) \approx (A \otimes_A A) \otimes_k (M \otimes_k N) \approx A \otimes_k (M \otimes_k N).$$

□

Corollary 4.6 (Rosenberg-Zelinsky). *If A/k is an azumaya algebra, then there is an exact sequence*

$$0 \to In\mathrm{Aut}(A) \to \mathrm{Aut}_{k-alg}(A) \xrightarrow{\varphi_A} \mathrm{Pic}(k),$$

where $im\varphi_A = \left\{ (L) | A \otimes_k L \approx A \text{ as a left } A\text{-module} \right\}$

Proof. This follows immediately from (4.5) and proposition 7.3 of chapter 2 □

Corollary 4.7. *If A/k is an azumaya algebra of rank r as a projective*
109 *k-module, then $Aut_{k-alg}(A)/InAut(A)$ is an abelian group of exponent r^d for some $d > 0$.*

Proof. Let $A \otimes_k L \approx A$ as a left A-module, hence as a k-module. The remark following proposition 6.1 of chapter 1 provides us with a k-module Q such that $Q \otimes_k A \approx k^{(r^d)}$ for some $d > 0$. So $L^{(r^d)} \approx k^{(r^d)}$. Taking r^dth exterior powers we have $L^{\otimes r^d} \approx k$. By virtue of (4.6), the corollary is now proved. ◻

Corollary 4.8 (Skolem-Noether)**.** *If $Pic\,(k) = 0$, then all automorphisms of an azumaya k-algebra are inner.*

Corollary 4.9. *If A is a k-algebra, finitely generated as a module over its centre C, then A/k is separable $\Leftrightarrow$ A/C and C/k are separable.*

Proof. In view of (2.19) it is enough to remark that, if A/k is separable, then A is faithfully C-projective. This follows from (1) $\Leftrightarrow$ (3) of theorem 4.1. ◻

Proposition and Definition 4.10. *Call two azumaya k-algebras A_1 and A_2 similar, if they satisfy the following conditions, which are equivalent:*

(1) $A_1 \otimes_k A_2^o \approx \mathrm{End}_k(P)$ *for some faithfully projective k-module P.*

(2) $A_1 \otimes_k \mathrm{End}_k(P_1) \approx A_2 \otimes_k \mathrm{End}_k(P_2)$ *for some faithfully projective k modules P_1 and P_2.*

(3) $A_1 - \ \mathrm{mod}$ *and* $A_2 - \ \mathrm{mod}$ *are equivalent k-categories.*

(4) $A_1 \approx \mathrm{End}_{A_2}(P)$ *for some faithfully projective right A_2-module P.*

Proof. (1) $\Rightarrow$ (2). $A_2 \otimes_k \mathrm{End}_k(P) \approx A_1 \otimes_k A_2 \otimes_k A_2^o \approx A_1 \otimes_k \mathrm{End}_k(A_2)$. 110

(2) $\Rightarrow$ (3). Since $A_i \otimes_k \mathrm{End}_k(P_i) \approx \mathrm{End}_{A_i}(A_i \otimes_k (P_i)$ (see (2.8) (a)), and since $A_i \otimes_k P_i$ is a faithfully projective A_i-module, it follows from theorem 4.4 and proposition 5.6 of chapter 2, that $A_i - \ \mathrm{mod}$ is k-equivalent to $(A_i \otimes_k \mathrm{End}_k(P_i) - \ \mathrm{mod}$, $i = 1, 2$.

(3) $\Rightarrow$ (4) follows proposition 4.1 and theorem 4.4 of chapter 2.

(4) $\Rightarrow$ (1). $A_1 \otimes_k A_2^0 \approx \mathrm{End}_{A_2}(P) \otimes_k A_2^o \approx \mathrm{End}_{A_1 \otimes_k A_2^0}(P \otimes_k A_2^0)$. Now $P \otimes_k A_2^0$ is faithfully projective $A_2 \otimes A_2^0$-module, so $P \otimes_k A_2^0 \approx A_2 \otimes_k Q$, where $Q = (P \otimes_k A_2^o)^{A_2}$ is faithfully projective k-module. Hence $A_1 \otimes_k A_2^0 \approx \mathrm{End}_{A_2^e}(A_2 \otimes_k (Q)$, since $A_2 \otimes_k$: $k -\ \ \mathrm{mod} \to A_2^e -\ \ \mathrm{mod}$ is an equivalence. □

It follows from this proposition that similarly is an equivalence relation between azumaya algebras, and that $\otimes_k$ induces a structure of abelian group on the set of similarity classes of azumaya k-algebras. We shall call this group the *Brauer group of* k, and hence denote it by $Br(k)$. The identity element in $Br(k)$ is the class of k, and the inverse of the class of an azumaya k-algebra A is the class of A^0.

If K is a commutative k-algebra, then $A \mapsto K \otimes_k A$ induces a homomorphism $Br(k) \to Br(K)$, by virtue of (2.9)(a), and this makes Br a functor from commutative rings to abelian groups.

5 Splitting rings

111 If P is a projective k-module, denote its rank by $[P : k]$. This is a function spec $(k) \to \mathbb{Z}$. If L is a commutative k-algebra, denote by φ_L the natural map spec $(L) \to$spec (k). Then, for a projective k-module P, we have $\varphi_L o[P : k] = [P \otimes_k L : L]$. If A/k is an Azumaya algebra, denote its class in $Br(k)$ by (A).

Theorem 5.1. *Let A/k be an azumaya algebra.*

(a) If $L \subset A$ is a maximal commutative subalgebra, then $A \otimes_k L \approx \mathrm{End}_L(A)$ as L-algebras, viewing A as a right L-module. Hence if A is L-projective, then $(A) \in \ker(BR(k) \to Br(L))$, and $\varphi_L o[A : k] = [A : L]^2$. If also L is k-projective, then $\varphi_L o[L : k] = [A : L]$. If L/k is separable, A is automatically L-projective.

(b) Suppose L is a commutative faithfully k-projective k-algebra, and suppose $(A) \in \ker(Br(k) \to Br(L))$. Then there is an algebra B, similar to A, which contains L as a maximal commutative subalgebra. If $\mathrm{End}_k(L)$ is projective as a right L-module, then so is B.

Proof. (a) $\mathrm{End}_L(A)$ is the centralizer B in $\mathrm{End}_k(A) = A \otimes_k A^0$ of $l \otimes L \subset A \otimes A^0$. Since $A = A \otimes l \subset A \otimes L \subset B$, it follows from (4.3), that $B = A \otimes_k B^A$. Now $B^A \subset (A \otimes_k A^0)^A = l \otimes A^0$, and B^A commutes with $l \otimes L$, a maximal commutative subalgebra of $l \otimes A^0$. Hence $B^A = l \otimes L$, so $B = A \otimes_k L$, as claimed.

If A is L-projective, then $\varphi_L \circ [A : k] = [A \otimes_k L : L] = [\mathrm{End}_L(A) : L] = [A : L]^2$. If, further, L is k-projective, $\varphi_L \circ [A : k] = [A \otimes_k L : L] = [A \otimes_L (L \otimes_k L) : L] = [A : L].[L \otimes_k L : L] = [A : L].(\varphi_L \circ [L : k])$, so that $[A : L] = \varphi_L \circ [L : k]$. **112**

Suppose $E = (0 \to J \to L^e \to L \to 0)$ splits. Then $A \otimes_L E$ also splits. So $A = A \otimes_L L$ is projective over $A \otimes_L (L \otimes_k L^0) = A \otimes_k L$. But $A \otimes_k L/L$ is an azumaya algebra, so $A \otimes_k L$ is projective over L. Hence A is L-projective.

(b) If $(A) \epsilon \ker(Br(k) \to Br(L))$, then $A^0 \otimes_k L \approx \mathrm{End}_L(P)$ for some faithfully projective L-module P. Using the isomorphism to identity, we have $A^o \otimes_k L = \mathrm{End}_L(P) \subset \mathrm{End}_k(P) = D$. Let $B = D^{A^o}$. Then (4.3) implies that $D = A^0 \otimes_k B$. Since L is faithfully k-projective, so also is P. So D/k is a trivial azumaya algebra and $(B) = (A)$ in $Br(k)$. Clearly $L = l \otimes L \subset B$. Since $B = D^{A^o}$, $B^L = D^{A_o} \cap D^L$. Further, $D^L = \mathrm{End}_k(P)^L = \mathrm{End}_L(P)$, so $B^L = \mathrm{End}_L(P)^{A_o}$. Since $\mathrm{End}_L(P) = A^o \otimes_k L$, we have $\mathrm{End}_L(P)^{A_0} = $ centre $\mathrm{End}_L(P) = L$. Now $D = A^o \otimes_k B$ with $L \subset B$, so D is locally (with respect to k) a direct sum of copies of B as an L-module. Hence B is L-projective as soon as we show that $D = \mathrm{End}_k(P)$ is. Again we localize (with respect to k), whereupon P becomes L-free and I becomes k-free. Then We can write $P = P_0 \otimes_k L$, P_0 a free k-module, and we have $D = \mathrm{End}_k(P) = \mathrm{End}_k(P_0) \otimes_k \mathrm{End}_k(L)$. By hypothesis, $\mathrm{End}_k(L)$ is right L-projective, so D is L-projective.

□

6 The exact sequence

We now make out of the azumaya k-algebras, a category with product, **113** in the sense of chapter 1. We write

$$\underline{\underline{Az}} = \underline{\underline{Az}}(k)$$

for the category whose objects are azumaya k-algebras, whose morphisms are algebra isomorphisms, and with product $\otimes_k$.

Recall that the category $\underline{\underline{FP}} = \underline{\underline{FP}}(k)$ (see §6 of chapter 1) of faithfully projective k-modules also has $\otimes_k$ as product. More over, (2.12) says that functor

$$\text{End} = \text{End}_k : \underline{\underline{FP}} \to \underline{\underline{Az}}$$

preserves products. (If $f : P \to Q$ in $\underline{\underline{FP}}$, then f is an isomorphism, and End $(f) : \text{End}(P) \to \text{End}(Q)$ is defined by End $(f)(e) = fef^{-1}$.) Theorem 4.1 (6) asserts that End is a cofinal functor, so we have the five term exact sequence from theorem 4.6 of chapter 1:

$$K_1\underline{\underline{FP}} \xrightarrow{K_1 \text{End}} K_1\underline{\underline{Az}} \to K_o\Phi\,\text{End} \to K_c\underline{\underline{FP}} \xrightarrow{K_o \text{End}} K_o\underline{\underline{Az}}. \qquad (6.1)$$

It follows immediately from (4.10)(2), that

$$\text{coker}\,(K_0\,\text{End}) = Br(k). \qquad (6.2)$$

Consider the composite functor

$$\underline{\underline{\text{Pic}}} \overset{I}{\hookrightarrow} \underline{\underline{FP}} \xrightarrow{\text{End}} \underline{\underline{Az}}, \qquad (6.3)$$

which sends every object of $\underline{\underline{\text{Pic}}}$ to the algebra $k \in \underline{\underline{Az}}$. Hence the com-
114 posites $(K_i\,\text{End}) \circ (K_i I) = 0$ for $i = 0, 1$. We will now construct a connecting homomorphism $K_1\underline{\underline{Az}} \to K_0\underline{\underline{\text{Pic}}} = \text{Pic}(k)$ and use it to identity (6.1) with the sequence we will thus obtain from (6.3).

Recall that $K_1\underline{\underline{Az}}$ is derived from the category $\Omega\underline{\underline{Az}}$, whose objects are pairs (A, α), $A \in \underline{\underline{Az}}$, $\alpha \in Aut_{k-alg}(A)$. Let ${}_1A_\alpha$ denote the invertible two-sided A-module constructed in lemma 7.2 of chapter 2. We have ${}_1A_\alpha \approx A \otimes_k L_\alpha$, where $L_\alpha = ({}_1A_\alpha)^A$, according to theorem (4.1)(4). In this way we have a map

$$\text{obj}\,\Omega\underline{\underline{Az}} \to \text{obj}\,\underline{\underline{\text{Pic}}},$$

given by $(A,\alpha) \mapsto L_\alpha = ({}_1A_\alpha)^A$. If $f : (A,\alpha) \to (B,\beta)$ is an isomorphism in $\Omega\underline{\underline{Az}}$, then f induces (by restriction) an isomorphism $L_\alpha \to L_\beta$, thus extending the map above to a functor. If $\alpha, \beta \in \mathrm{Aut}_{k-alg}(A)$, then we have from (II, (7.2)(2)) a natural isomorphism

$$ {}_1A_{\alpha\beta} \approx {}_1A_\alpha \otimes_{A^1} A_\beta. $$

Since $A\otimes_k : k - \mathrm{mod} \to A - \mathrm{mod}$ converts $\otimes_k$ into $\otimes_A$, it follows that $L_{d\beta} \approx L_\alpha \otimes_k L_\beta$. Finally, given (A,α) and (B,β), we have

$$ \begin{aligned} L_{\alpha\otimes\beta} &= ({}_1(A\otimes B)_{\alpha\otimes\beta})^{A\otimes_k B} \\ &= ({}_1A_\alpha \otimes_{k} {}_1B_\beta)^{A\otimes_k B} \\ &= ({}_1A_\alpha)^A \otimes_k ({}_1B_\beta)^B \\ &= L_\alpha \otimes_k L_\beta. \end{aligned} $$

We have thus proved: **115**

Proposition 6.4. $(A,\alpha) \mapsto L_\alpha = ({}_1A_\alpha)^A$ *defines a functor*

$$ J : \Omega\underline{\underline{Az}} \to \underline{\underline{\mathrm{Pic}}} $$

of categories with product, and it satisfies, for $\alpha, \beta \in Aut_{k-\log}(A)$, $A \in \underline{\underline{Az}}$,

$$ L_{\alpha\beta} \approx L_\alpha \otimes_k L_\beta. $$

Now we define a functor.

$$ T : \underline{\underline{\mathrm{Pic}}} \to \Phi\,\mathrm{End} $$

by setting $T(L) = (L, \alpha_L, k)$, where α_L is the unique k-algebra isomorphism $\mathrm{End}_k(L) \approx k \to \mathrm{End}_k(k) \approx k$. *Clearly T preserves products.*

Suppose $(P,\alpha,Q) \in \Phi$ End. Thus $\alpha : A = \mathrm{End}(P) \to B = \mathrm{End}(Q)$ *is an algebra isomorphism. α permits us to view left B-modules as left A-modules. Since $P\otimes_k : k -$* mod *$A -$* mod *is an equivalence, the inverse being* $\mathrm{Hom}_A(P, \ ) : A - \mathrm{mod} \to k - \mathrm{mod}$, *we can apply this functor to the $B-$, hence A-module, Q and obtain a k-module*

$$ L = \mathrm{Hom}_A(P, Q) $$

such that $Q \approx P \otimes_k L$ *as a left* A*-module. It follows that* $L_\alpha \in \underline{\underline{\mathrm{Pic}}}$. *If* $(f, g)\colon (P, \alpha, Q) \to (P', \alpha', Q')$ *is in* $\Phi\,\mathrm{End}$, *then the map* $\mathrm{Hom}_{\mathrm{End}(P)}$ $(P, Q) \to \mathrm{Hom}_{\mathrm{End}(P')}(P', Q')$, *given by* $e \mapsto gef^{-1}$, *is in* $\underline{\underline{\mathrm{Pic}}}$, *thus giving* 116
us a functor

$$S : \Phi\,\mathrm{End} \to \underline{\underline{\mathrm{Pic}}}.$$

Moreover, S *preserves products because*

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{End}(P \otimes_k P')}(P \otimes_k P', Q \otimes_k Q') \\
&\approx \mathrm{Hom}_{\mathrm{End}(P) \otimes_k \mathrm{End}(P')}(P \otimes_k P', Q \otimes_k Q') && \text{by (2.12)} \\
&\approx \mathrm{Hom}_{\mathrm{End}(P)}(P, Q) \otimes_k \mathrm{Hom}_{\mathrm{End}(P')}(P', Q). && \text{by (2.5)(i).}
\end{aligned}$$

If $L \in \underline{\underline{\mathrm{Pic}}}$, *then* $STL = S(L, \alpha_L, L) = \mathrm{Hom}_k(k, L) \approx L$.

We have now proved all but the last statement of

Proposition 6.5. *There are product-preserving functors*

$$\underline{\underline{\mathrm{Pic}}} \underset{S}{\overset{T}{\rightleftarrows}} \Phi\,\mathrm{End}$$

defined by $TL = (k, \alpha_L, L)$ *and* $S(P, \alpha, Q) = \mathrm{Hom}_{\mathrm{End}(P)}(P, Q)$, *such that* $ST \approx \mathrm{Id}_{\underline{\underline{\mathrm{Pic}}}}$. *If* $(P, \alpha, R) \in \Phi\,\mathrm{End}$, *then* $S(P, \beta\alpha, R) \approx S(Q, \beta, R) \otimes_k S(P, \alpha, Q)$.

Proof. The prove the last statement we note that composition defines a homomorphism

$$\mathrm{Hom}_{\mathrm{End}(Q)}(Q, R) \otimes_k \mathrm{Hom}_{\mathrm{End}(P)}(P, Q) \to \mathrm{Hom}_{\mathrm{End}(P)}(P, R).$$

117 The module above are projective and finitely generated over k. Therefore it is enough to check that the map is an isomorphism over residue class fields $k/\mathscr{M}$. □

Proposition 6.6. S *and* T *define inverse isomorphisms*

$$\mathrm{Pic}(k) = K_0\underline{\underline{\mathrm{Pic}}} \leftrightarrows K_0\Phi\underline{\underline{\mathrm{End}}}.$$

Proof. $ST \approx \mathrm{Id}_{\underline{\underline{\mathrm{Pic}}}}$ so it suffices to show that K_0S is injective. If K_0S $(P, \alpha, Q) = k$, then $Q \approx P \otimes_k k = P$ as a left $\mathrm{End}(P)$-module. Let $f : Q \to P$ be such an isomorphism. This means that for all $e \in \mathrm{End}(P)$ and $q \in Q$, $f(\alpha(e)q) = ef(q)$, that is, that

$$\begin{tikzcd}
\mathrm{End}(P) \arrow[r, "\alpha"] \arrow[d, "\mathrm{End}(1_P)"'] & \mathrm{End}(Q) \arrow[d, "\mathrm{End}(f)"] \\
\mathrm{End}(P) \arrow[r, "1_{\mathrm{End}(P)}"'] & \mathrm{End}(P)
\end{tikzcd}$$

commutes. Thus $(1_P, f) : (P, \alpha, Q) \to (P, 1_{\mathrm{End}(p)}, P)$ in $\Phi\,\mathrm{End}$, so $(P, \alpha, Q)_{\Phi\,\mathrm{End}} = 0$ in $K_0\Phi\,\mathrm{End}$. □

Theorem 6.7. *The sequence of functors*

$$\Omega\underline{\underline{\mathrm{Pic}}} \xrightarrow{\Omega I} \Omega\underline{\underline{FP}} \xrightarrow{\Omega\,\mathrm{End}} \Omega\underline{\underline{Az}} \xrightarrow{J} \underline{\underline{\mathrm{Pic}}} \xrightarrow{I} \underline{\underline{FP}} \xrightarrow{\mathrm{End}} \underline{\underline{Az}}$$

of categories with product defines an exact sequence which is the top row of the following commutative diagram:

$$\begin{tikzcd}
U(k) \arrow[r] & K_1\underline{\underline{FP}} \arrow[r] \arrow[d, equal] & K_1\underline{\underline{Az}} \arrow[r] \arrow[d, equal] & \mathrm{Pic}(k) \arrow[r] \arrow[d, shift left, "K_0T"] & K_0\underline{\underline{FP}} \arrow[r] \arrow[d, equal] & K_0\underline{\underline{Az}} \arrow[r] \arrow[d, equal] & Br(k) \arrow[r] & 0 \\
 & K_1\underline{\underline{FP}} \arrow[r] & K_1\underline{\underline{Az}} \arrow[r] & K_0\Phi\,\mathrm{End} \arrow[r] \arrow[u, shift left, "K_0S"] & K_0\underline{\underline{FP}} \arrow[r] & K_0\underline{\underline{Az}} & &
\end{tikzcd}$$

The bottom row is the exact sequence of theorem 4.6 of chapter 1 for the functor End : $\underline{\underline{FP}} \to \underline{\underline{Az}} \cdot K_0S$ *and* K_0T *are the isomorphisms of proposition 6.6.*

Proof. We first check commutativity: If $L \in \underline{\underline{\mathrm{Pic}}}$, then $K_0I(L)_{\underline{\underline{\mathrm{Pic}}}} =$ **118**
$(L)_{\underline{\underline{FP}}}$, while $K_0T(L)_{\underline{\underline{\mathrm{Pic}}}} = (L, \alpha_L, L)_{\Phi\,\mathrm{End}}$ is sent to $(L)_{\underline{\underline{FP}}}(k)^{-1}_{\underline{\underline{FP}}} = (L)_{\underline{\underline{FP}}}$. □

Now that the diagram commutes, exactness of the top row follows from that of the bottom row, wherever the isomorphisms imply this. At

$K_0\underline{\underline{Az}}$ and $Br(k)$ exactness has already been remarked in (6.2) above. At $K_1\underline{\underline{FP}}$ the composite is clearly trivial. So it remains only to show that $\ker(K_1 \text{ End}) \subset \operatorname{Im} K_1 I$.

Now we know that the free modules k^n are cofinal in $\underline{\underline{FP}}$, and hence (by cofinality of End) that the matrix algebras $\mathbb{M}_n(k) = \text{End}(k^n)$ are cofinal in $\underline{\underline{Az}}$. Hence we may use them to compute K_1 as a direct limit (theorem 3.1 of chapter 1).

Write $GL_n(\underline{\underline{FP}}) = GL(n,k) = Aut_k(k^n)$, and $GL_n(\underline{\underline{Az}}) = Aut_{k-alg}$ $(\mathbb{M}_n(k))$. We have the "inner automorphism homomorphism"

$$f_n : GL_n(\underline{\underline{FP}} \to GL_n(\underline{\underline{Az}})$$

with $\ker f_n = \text{centre } GL_n(\underline{\underline{FP}} = GL_1(\underline{\underline{FP}}) = U(k))$, and $\operatorname{im} f_n \approx PGL$ (n,k).

Tensoring with an identity automorphism defines maps

$$GL_n(\underline{\underline{FP}}) \to GL_{nm}(\underline{\underline{FP}})$$

119 and

$$GL_n(\underline{\underline{Az}}) \to GL_{nm}(\underline{\underline{Az}})$$

making $(f_n)_{n\in\mathbb{N}}$ a map of directed systems. Writing

$$GL(\underline{\underline{FP}}) = \varinjlim GL_n(\underline{\underline{FP}}),$$
$$GL(\underline{\underline{Az}}) = \varinjlim GL_n(\underline{\underline{Az}}),$$

and

$$f = \varinjlim f_n : GL(\underline{\underline{FP}}) \to GL(\underline{\underline{Az}}),$$

we see that K_1 End is just the ablianization of f. In §6 of chapter 1 we computed

$$\begin{aligned} K_1\underline{\underline{FP}} &= GL(\underline{\underline{FP}})/[GL(\underline{\underline{FP}}), GL(\underline{\underline{FP}})] \\ &= (\mathbb{Q} \otimes_{\mathbb{Z}} U(k)) \oplus (\mathbb{Q} \otimes_{\mathbb{Z}} S\,K_1\underline{\underline{P}}). \end{aligned}$$

Moreover $K_1 I$ is induced by the inclusion $U(k) = GL_1(\underline{\underline{FP}}) \to GL(\underline{\underline{FP}})$. $K_1 I$ is the map $K_1\underline{\underline{\text{Pic}}} = U(k) - \mathbb{Z} \otimes_{\mathbb{Z}} U(k) \to \mathbb{Q} \otimes_{\mathbb{Z}} U(k) \subset K_1\underline{\underline{FP}}$, so coker

$(K_1 I) = (\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} U(k)) \otimes (\mathbb{Q} \otimes_{\mathbb{Z}} SK_1 P = PGL(k)/[PGL(k), PGL(k)]$. Thus exactness at $K_1\underline{\underline{FP}}$ means that: the inclusion $PGL(k) \subset GL(\underline{\underline{Az}})$ induces a monomorphism

$$PGL(k)/[PGL(k), PGL(k)] \to GL(\underline{\underline{Az}})/[GL(\underline{\underline{Az}}), GL(\underline{\underline{Az}})].$$

Suppose $\alpha, \beta \in GL_n(\underline{\underline{Az}}) = Aut_{k-alg}(\mathbb{M}_n(k))$. Let $\tau : k^n \otimes k^n \to$
$k^n \to k^n$ be the transposition. Write $E(\tau)$ for the corresponding inner **120**
automorphism of $\mathbb{M}_{N^2}(K)$. Then $E(\tau)(\alpha \otimes 1_{\mathbb{M}_n(k)})E(\tau)^{-1} = 1_{\mathbb{M}_n(k)} \otimes \alpha$ commutes with $\beta \otimes 1_{\mathbb{M}_n(k)}$. Now τ is just a permutation of the basis of $k^n \otimes_k k^n$, so it is a product of elementary matrices, provided it has determinant $+1$ (which happens when $\frac{1}{2}n(n-1)$ is even). For example, if we restrict our attention to values of n divisible by 4, then τ is a product of elementary matrices, hence lies in $[GL_{n^2}, GL_{n^2}(k)]$, so $E(\tau) \in [PGL_{n^2}(k), PGL_{n^2}(k)]$. It follows that, for n divisible by 4, the image of $GL_n(\underline{\underline{Az}})$ in $GL_{n^2}(\underline{\underline{Az}})/[PGL_{n^2}(k), PGL_{n^2}(k)]$ is abelian. Note that $PGL_{n^2}(k) = \text{In Aut}\,(\mathbb{M}_n(k))$ is normal in $GL_n(\underline{\underline{Az}})$, hence so also is $[PGL^2_n(k), PGL^2_n(k)]$, so the factor group above is defined. Finally, since the n divisible by 4 are cofinal $\mathbb{N}$ we can pass to the limit to obtain

$$[GL(\underline{\underline{Az}}), GL(\underline{\underline{Az}})] \subset [PGL(k), PGL(k)],$$

are required. Q.E.D.

Proposition 6.8. *In the exact sequence of theorem 6.7,*

$$\ker(U(k) \to K_1\underline{\underline{FP}}) = \textit{ the torsion subgroup of } U(k),$$
$$\ker(\mathrm{Pic}(k) \to K_0\underline{\underline{FP}}) = \textit{ the torsion subgroup of } \mathrm{Pic}(k),$$

Hence there is an exact sequence **121**

$$0 \to (\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} U(k)) \oplus (\mathbb{Q} \otimes_{\mathbb{Z}} SK_1\underline{P}) \to K_1\underline{\underline{Az}} \to \begin{pmatrix} \textit{the torsion} \\ \textit{subgroup of} \\ \mathrm{Pic}(k) \end{pmatrix} \to 0.$$

This sequence splits (not naturally) as sequence of abelian groups.

Proof. For $K_1 I : U(k) = K_1 \underline{\underline{\mathrm{Pic}}} \to K_1 \underline{\underline{FP}}$ we have from chapter 1, §7,

$$\ker K_1 I = \text{ the torsion subgroup of } U(k),$$

and

$$\operatorname{coker} K_1 I = (\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} U(k)) \oplus (\mathbb{Q} \otimes_{\mathbb{Z}} S K_1 \underline{\underline{P}}).$$

From the same source we have

$\ker K_0 I =$ the torsion subgroup of $\mathrm{Pic}(k)$.

The last assertions now follow from the exact sequence plus the fact that the left hand term is divisible, hence an injective $\mathbb{Z}$-module. □

Chapter 4

The Brauer-Wall group of graded Azumaya algebras

This chapter contains only a summary of results, without proofs. They 122
are included because of their relevance to the following chapter, on Clifford algebras.

1 Graded rings and modules

All graded objects here are graded by $\mathbb{Z}/2\mathbb{Z}$. A ring $A = A_0 \oplus A_1$ is graded if $A_i A_j \subset A_{i+j} (i, j \in \mathscr{Z}/2\mathscr{Z})$, and an A-module $M = M_0 \oplus M_1$ is graded if $A_i M_j \subset M_{i+j}$. (We always assume modules to be left modules unless otherwise specified.) If S is a subset of a graded object, hS will denote the homogeneous elements of S, and $\partial x =$ degree of x, for $x \in hS$.

If A is a graded ring, then $|A|$ will denote the underlying ungraded ring. If A is ungraded, then (A) denotes the graded ring with A concentrated in degree zero. An A-module is graded or not according as A is or is not. If M is an A-module (A graded) we write $|M|$ for the underlying $|A|$-module. If A is not graded, we write (M) for the (A)-module with M concentrated in degree zero.

Let A be a graded ring. For A-modules M and N,

$$\mathrm{HOM}_A(M, N)$$

is the graded group of additive maps from M to N defined by: $f \in hHOM_A(M,N) \Leftrightarrow (i) f$ is homogeneous of degree ∂f (i.e. $f(M_i) \subset N_{i+\partial f}$); and (ii) fax $= (-1)^{\partial f \partial a} afx (a \in hA, x \in M)$.

123 The degree zero term of $\mathrm{HOM}_A(M,N)$ is denoted by $\mathrm{Hom}_A(M,N)$.
Let A' denote the graded group A with new multiplication

$$a \cdot b = (-1)^{\partial a \partial b} ab \qquad (a, b, \in hA).$$

If M is an A-module let M' denote the A'-module with M as the underlying graded group and operators defined as

$$a \cdot x = (-1)^{\partial a \partial x} ax \qquad (a \in hA, x \in hM).$$

Then it is straightforward to verify that

$$\mathrm{HOM}_A(M,N) = \mathrm{Hom}_{|A'|}(|M'|, |N'|),$$

an equality of graded groups.

A-mod refers to the category with A-modules as objects and homomorphisms of degree zero (i.e. $\mathrm{Hom}_A(,)$) as morphisms.

Lemma 1.1. *The following conditions on an $A-$module P are equivalent:*

(1) $\mathrm{Hom}_A(P, \ )$ *is exact on* $A -$ mod .

(2) $\mathrm{Hom}_{|A|}(|P|, \ )$ *is exact on* $|A| -$ mod .

(3) $\mathrm{Hom}_A(P, \ )$ *is exact on* $A -$ mod .

(4) P is a direct summand of $A^{(I)} \oplus (\tau A)^{(J)}$ for some I and J, where τA is the A-module A with grading shifted by one.

This lemma tells us that the statement "P is A-projective" is unambiguous.

124 If $S \subset A$ (graded), we define the *centralizer* of S in A to be graded subgroup C such that $c \in hC \Leftrightarrow cs = (-1)^{\partial c \partial s} sc$ for all $s \in hS$. It is easy to see that C is actually a subring of A. We say that two subrings of A *commute*, if each lies in the centralizer of the other. If B_1 and B_2

are subrings generated by sets S_1 and S_2, respectively, of homogeneous elements, then B_1 and B_2 commute $\Leftrightarrow$ S_1 and S_2 commute. We write

A^A = CENTRE (A) = centralizer of A in A.

The degree zero term will be denote centre A. One must not confuse centre A, centre $|A|$, and CENTRE (A). They are all distinct in general.

Let k be a commutative ring which is graded, but concentrated in degree zero. Even though k and $|k|$ are not essentially different, $k-$ mod and $|k|$-mod are. A k-algebra is a graded ring A and a homomorphism $k \to A$ of graded rings such that the image $k1$ lies in A^A, and hence in centre A.

If A^1 and A^2 are k-algebras and if M^i is an A^i-module $i = 1, 2$ we define the k-module $M^1 \otimes M^2$ by

$$(M^1 \otimes M^2)_n = (M^1_0 \otimes M^2_n) \oplus (M^1_1 \otimes M^2_{n+1}).$$

We define an action of $A^1 \otimes A^2$ on $M^1 \otimes M^2$ by

$$(a_1 \otimes a_2)(x_1 \otimes x_2) = (-1)^{\partial a_2 \partial x_1} a_1 x_1 \otimes a_2 x_2$$

for $a_i \in hA^i$, $x_i \in hM^i$, $i = 1, 2$. This makes $A^1 \otimes A^2$ a k-algebra (for $M^i = A^i$) and $M^1 \otimes M^2$ an $A^1 \otimes A^2$-module. The subalgebras $A^1 \otimes 1$ and $1 \otimes A^2$ commute, and the pair of homomorphisms $A^i \to A^1 \otimes A^2$ is **125**
universal for pairs of homomorphisms $f^i : A^i \to B$ of k-algebras such that im f^1 and im f^2 commute. In practice it is useful to observe that: if A^i is generated by homogeneous elements S^i, and if $f^1(S^1)$ and $f^2(S^2)$ commute, then $f^1(A^1)$ and $f^2(A^2)$ commute.

2 Separable algebras

In this section also k denotes a commutative ring concentrated in degree zero. If A is a k-algebra, then A^0 denotes the opposite algebra, and we write

$$A^* = (A')^0 = (A^0)'$$

for the algebra with graded group A and multiplication

$$a \times b = (-1)^{\partial a \partial b_{ba}} \qquad (a, b \in hA).$$

A right A-module M will be considered a left A^*-module by setting

$$ax = (-1)^{\partial a \partial x} xa \qquad (a \in hA^*, x \in hM).$$

If M is a left A-, right B-module such that $(ax)b = a(xb)$ and $tx = xt$ for all $a \in A$, $x \in M$, $b \in B$, $t \in k$, then we view M as an $A \otimes_k B^*$-module by

$$(a \otimes b)x = (-1)^{\partial b \partial x} axb \qquad (a \in hA, b \in hB, x \in hM).$$

In particular, two-sided A-modules will be identified with modules over

$$A^e = A \otimes_k A^*$$

126 We have an exact sequence

$$0 \to J \to A^e \to A \to 0$$
$$(a \otimes b) \mapsto ab$$

of A^e-modules. We call A a *separable* k-algebra if A is A^e-projective. This means that the functor

$$A^e - \text{mod} \to k - \text{mod}$$
$$M \mapsto M^A = \text{HOM}_{A^e}(A, M)$$

is exact.

The stability of separability and CENTRES under base change and tensor products all hold essentially as in the ungraded case. In particular $\text{END}_k(P) = \text{HOM}_k(P, P)$ is separable with CENTRE $k/annP$, for P a finitely generated projective k-module. Moreover :

Proposition 2.1. *Let A be finitely generated as a $k-$module and suppose either that k is noetherian or that A is $k-$projective. Then A is separable $\Leftrightarrow$ $(A/\mathscr{M}A)/(k/\mathscr{M})$ is separable for all maximal ideals $\mathscr{M}$ of k.*

Suppose now that k is a field. If $a \in k$. write $k < a >= k[X]/(X^2 - a)$, with grading $k.1 \oplus k \cdot x$, $x^2 = a$. It can be shown that if char $k \neq 2$ and if $a \neq 0$, then $k < a >$ is separable with CENTRE k. Moreover

$$k < a > \otimes_k k < b > \approx \left(\frac{a, b}{k}\right),$$

the k-algebra with generators α, β of degree one defined by relations: $\alpha^2 = a, \beta^2 = b, \alpha\beta = -\beta\alpha$.

127 **Theorem 2.2.** *Let A be a finite dimensional k-algebra, k a field. Then the following conditions are equivalent:*

(1) *A/k is separable.*

(2) *$A = \Pi A_i$, where A_i is a simple (graded) k-algebra and $A_i^{A^i}$ is a separable field extension of k, concentrated in degree zero.*

(3) *For some algebraically closed field $L \supset k$, $L \otimes_k A$ is a product of algebras of the types*

(i) *$END_L(P)$, P a finite dimensional L-module, and*

(ii) *$L < 1 > \otimes_L \mathrm{END}_L(P)$, P a finite dimensional L-module with $P_1 = 0$.*

If char $k = 2$, then type (ii) does not occur.

Corollary 2.3. *Let k be any commutative ring and A a k-algebra finitely generated as a k-module. Suppose either k is noetherian or that A is k-projective. Then if A/k is separable, $|A|$, $|A_0|$, $|A^A|$, and $|A^{A_0}|$ are separable $|k|$-algebras.*

3 The group of quadratic extensions

A *quadratic extension* of k is a separable k-algebra L which is a finitely generated projective k-module of rank 2. By localizing and extending 1 to a k-basis of L we see that $|L|$ is commutative.

Proposition 3.1. *If L/k is a quadratic extension, then there is a unique k-algebra automorphism $\sigma = \sigma(L)$ of L such that $L^\sigma = k$.*

Proposition 3.2. *If L^1 and L^2 are quadratic extensions of k, then so also* 128
is

$$L^1 * L^2 = (L^1 \otimes_k L^2)^{\sigma_1 \otimes \sigma_2},$$

where $\sigma_i = \sigma(L^i)$. Further, $*$ induces on the isomorphism classes of quadratic extensions the structure of an abelian group,

$$Q_2(k).$$

If we deal with $|k|$-algebras, then we obtain a similar group,

$$Q(k)$$

of ungraded quadratic extensions. Each of these can be viewed as a graded quadratic extension of k, concentrated in degree zero, and this defines an exact sequence

$$0 \to Q(k) \to Q_2(k) \to T,$$

where T = continuous functions spec $(k) \to \mathbb{Z}/2\mathbb{Z}$, and right hand map is induced by $L \mapsto [L_1 : k]$ = the rank of the degree one term, L_1, of L. In particular, if Spec (k) is connected, we have

$$0 \to Q(k) \to Q_2(k) \to \mathbb{Z}/2\mathbb{Z},$$

and the right hand map is surjective $\Leftrightarrow$ $2 \in U(k)$. In this case $L = k < u >$ is a quadratic extension for $u \in U(k)$. $L = k \cdot 1 \oplus k \cdot x$ with $x^2 = u$, and $\sigma(x) = -x$ for $\sigma = \sigma(L)$. If u_1, $u_2 \in U(k)$, then $k < u_1 > \otimes_k k < u_2 >= \left(\dfrac{u_1, u_2}{k}\right)$ has k-basis 1, x_1, x_2, $x_3 = x_1 x_2 = -x_2 x_1$ with $x_1^2 = u_1$, $x_2^2 = u_2$, $x_3^2 = -u_1 u_2$. If $\sigma_i = \sigma(k < u_i >)$, then $\sigma_1 \otimes \sigma_2$ sends $x_1 \mapsto -x_1$, $x_2 \mapsto -x_2$, $x_3 \mapsto x_3$.

129 It follows that

$$k < u_1 > * k < u_2 >= k[-u_1 u_2],$$

where $k[u] = k[X]/(X^2 - u)$, concentrated in degree zero.

Proposition 3.3. *Suppose* $2 \in U(k)$. *Then*

(a) the sequence $0 \to Q(k) \to Q_2(k) \to \mathbb{Z}/2\mathbb{Z} \to 0$ *is exact; and*

(b) there is an exact sequence

$$U(k) \xrightarrow{2} U(k) \to Q(k) \to \mathrm{Pic}(k) \xrightarrow{2} \mathrm{Pic}(k),$$

where the map in the middle are defined by $u \mapsto k[u]$ *and* $L \mapsto (L/k)$*, respectively.*

Next suppose that char $k = 2$.

Proposition 3.4. *Suppose k is a commutative ring of characteristic 2. Then if $a \in k$, $k[a] = k[X]/(X^2 + X + a)$ is a quadratic extension, concentrated in degree zero. $k[a] = k \cdot 1 + k \cdot x$ with $x^2 + x + a = 0$, and $\sigma(x) = x + 1$. $Q_2(k) \approx Q[k]$ and there is an exact sequence*

$$k \xrightarrow{\wp} k \to Q(k) \to 0,$$

where $\wp(a) = a^2 + a$, and $a \mapsto k[a]$ induces $k \to Q[k]$.

4 Azumaya algebras

k is a commutative ring concentrated in degree zero.

Theorem and Definition 4.1. *A is an azumaya k- algebra if it satisfies the following conditions, which are equivalent:*

(1) A is a finitely generated k- module, $A^A = k$, and A/k is separable.

(2) $A^A = k$ and $|A|$ is a generator as an $|A^e|$-module.

(3) A is a faithfully projective k-module and $A^e \to END_k(A)$ is an isomorphism. **130**

(4) The functors

$$\begin{array}{ccc} (M & \longmapsto & M^A) \\ A^e - \mathrm{mod} & \rightleftarrows & k - \mathrm{mod} \\ (A \otimes_k N) & \longleftarrow\!\!\shortmid & N) \end{array}$$

are inverse equivalences of categories.

(5) For all maximal ideals $\mathscr{M} \subset k$, $A/\mathscr{M}A$ is simple, and CENTRE $(A/\mathscr{M}A) = k/\mathscr{M}$.

(6) There exists a k-algebra B and a faithfully projective k-module P such that $A \otimes_k B \approx END_k(P)$.

Corollary 4.2. *Let A and B be k-algebras with A azumaya. Then $\mathfrak{b} \mapsto A \otimes \mathfrak{b}$ is a bijection from two-sided ideals of B to those of $A \otimes_k B$.*

Corollary 4.3. *If $A \subset B$ are k-algebras with A azumaya, then $B \approx A \otimes_k B^A$.*

Call two azumaya algebras A and B *similar* if $A \otimes_k B^* \approx END_K(P)$ for some faithfully projective module P. With multiplication induced by $\otimes_k$, the similarity classes form a group, denoted

$$Br_2(k),$$

and called the *Brauer-Wall group* of k.

Theorem 4.4. *If A is an azumaya algebra, define $L(A) = A^{A_0}$. Then $L(A)$ is a quadratic extension of k, and $L(A \otimes_k B) \approx L(A) * L(B)$. $A \mapsto L(A)$ induces an exact sequence*

$$0 \to Br(k) \to Br_2(k) \to Q_2(k) \to 0.$$

5 Automorphisms

131 If A is a k-algebra $a = a_0 + a_1 \in A$ write $\sigma(a) = a' = a_0 - a_1$. Let $U(A)$ denote the group of units in A, and $hU(A)$ the subgroup of homogeneous units. If $u \in hU(A)$, we define the *inner automorphism*, α_u, by

$$\alpha_u(a) = u\sigma^{\partial u}(a)u^{-1}.$$

This is clearly an algebra automorphism of A, and a simple calculation shows that $\alpha_{uv} = \alpha_u \alpha_v$. Thus we have a homomorphism

$$hU(A) \to Aut_{k-alg}(A); u \mapsto \alpha_u.$$

The kernel consists of those u such that $u\sigma^{\partial u}(a) = au$ for all $a \in A$. Taking a homogeneous, $\sigma(a) = (-1)^{\partial a}a$, so the condition becomes $(-1)^{\partial u \partial a}ua = au$, for all $a \in hA$, i.e. $u \in \mathrm{CENTRE}(A) = A^A$.

Thus we have an exact sequence

$$1 \to hU(A^A) \to hU(A) \to Aut_{k-alg}(A). \tag{5.1}$$

Now just as in the ungraded case one can prove:

Theorem 5.2. *Let A be an azumaya k-algebra. If $\alpha \in Aut_{k-alg}(A)$, let $1^A\alpha$ denote the A^e-module A with action $a \cdot x \cdot b = ax\alpha(b)$ for $a, x, b \in A$. Then $L_\alpha = ({}_1A_\alpha)^A$ is an invertible k-module, and $\alpha \mapsto L_\alpha$ induces a homomorphism g making the sequence $1 \to U(k) \to U(A_0) \to Aut_{k-alg}(A) \xrightarrow{g} \mathrm{Pic}(k)$ exact.* $\operatorname{im} g = \{(L) | A \otimes_k L \approx A$ *as left A-modules*$\}$

Here $U(k) = U(A^A) \subset U(A_0) \subset hU(A)$, and the left hand portion of 132
the sequence is induced by (5.1) above. Pic (k) is the group of "graded invertible k-modules". If u is a unit of degree one in A then L_{α_u} is just k, but concentrated in degree one. This explains why we have $U(A_0)$, and not $hU(A)$, in the exact sequence.

This theorem will be applied, in Chapter 5, §4, to the study of orthogonal groups. We conclude with the following corollary:

Corollary 5.3. *$Aut_{k-alg}(A)/$ (inner automorphisms) is a group of exponent r^d for some $d > 0$, where $r = [A : k]$.*

Chapter 5

The structure of the Clifford Functor

In this chapter we introduce the category, $\underline{\underline{\mathrm{Quad}}}(k)$ of quadratic forms on **133**
projective k-modules, and the hyperbolic functor, $\mathbb{H} : \underline{\underline{P}} \to \underline{\underline{\mathrm{Quad}}}$. This satisfies the conditions of chapter 1 to yield an exact sequence,

$$K_1\underline{\underline{P}} \to K_1\underline{\underline{\mathrm{Quad}}} \to K_0\Phi\mathbb{H} \to K_0\underline{\underline{P}} \to K_0\underline{\underline{\mathrm{Quad}}} \to \mathrm{Witt}(k) \to 0,$$

where Witt (k) = coker $(K_0\mathbb{H})$ is the classical “Witt ring” over k.

The Clifford algebra is constructed as a functor from $\underline{\underline{\mathrm{Quad}}}$ to k-algebras, graded mod 2, and the main structure theorem (§3) asserts that the Clifford algebra are (graded) azumaya algebras, in the sense of Chapter 4 and that the diagram

$$\begin{tikzcd}
\underline{\underline{P}} \arrow[r, "\mathbb{H}"] \arrow[d, "\wedge"'] & \underline{\underline{\mathrm{Quad}}} \arrow[d, "C1"] \\
\underline{\underline{FP}}_2 \arrow[r, "\mathrm{END}"'] & \underline{\underline{Az}}_2
\end{tikzcd}$$

commutes up to natural isomorphism. Here $\wedge$ denotes exterior algebra, graded mod 2 by even and odd degrees. The proof is achieved by a simple adaptation of arguments in Bourbaki [2].

This commutative diagram leads to a map of exact sequences

$$\begin{tikzcd}
K_1\underset{=}{P} \arrow[r] \arrow[d] & K_1\underline{\underline{\mathrm{Quad}}} \arrow[r] \arrow[d] & K_0\Phi\mathbb{H} \arrow[r] \arrow[d] & K_0\underset{=}{P} \arrow[r] \arrow[d] & K_0\underline{\underline{\mathrm{Quad}}} \arrow[r] \arrow[d] & \mathrm{Witt}(k) \arrow[r] \arrow[d] & 0 \\
K_1\underline{\underline{FP}}_2 & K_1\underline{\underline{Az}}_2 \arrow[r] & K_0\Phi END \arrow[r] & K_0\underline{\underline{FP}}_2 \arrow[r] & K_0\underline{\underline{Az}}_2 \arrow[r] & Br_2(k) \arrow[r] & 0
\end{tikzcd}$$

134 This map of exact sequences is the "generalized Hasse-Wall invariant."

In §4 we indicate briefly what the construction of the spinor norm looks like in this generality.

1 Bilinear modules

We shall consider modules over a fixed commutative ring k, and we shall abbreviate,

$$\otimes = \otimes_k, \mathrm{Hom} = \mathrm{Hom}_k, M^* = \mathrm{Hom}(M, k).$$

Bil $(P \times Q)$ denotes the module of k-bilinear maps, $P \times Q \to k$.

Let P be a k-module. If $x \in P$ and $y \in P^*$ write

$$\langle y, x\rangle_P = y(x).$$

If $f : P \to Q$ then $f^* : Q^* \to P^*$ is defined by

$$\langle f^* y, x\rangle_Q = \langle y, fX\rangle_P \ (x \in P, y \in Q^*).$$

There are natural isomorphisms

$$\begin{gathered}
\mathrm{Hom}(P, Q^*) \overset{s}{\leftarrow} \mathrm{Bil}(P \times Q) \overset{d}{\to} \mathrm{Hom}(Q, P^*) \\
s_B \longleftarrow B \longrightarrow d_B
\end{gathered}$$

defined by

$$\langle S_B x, y\rangle_Q = B(x, y) = \langle d_B y, x\rangle_P (x \in P, y \in Q).$$

Applying this to the natural pairing

$$\langle\ ,\ \rangle_P : P^* \times P \to k,$$

we obtain the natural homomorphism

$$d_P : P \to P^{**}; \langle d_P x, y\rangle_{P*} = \langle y, x\rangle_P.$$

We call P *reflexive* if d_P is an isomorphism, and we will then often identify P and P^{**} via d_P. 135

Suppose $B \epsilon \operatorname{Bil}(P \times Q)$, $x \in P$, and $y \epsilon Q$. Then

$$\langle d_B y, x\rangle_P = \langle s_B x, y\rangle_Q = \langle d_Q y, s_B x\rangle_{Q^*} = \langle s_B^* d_Q y, x\rangle_P.$$

From this and the dual calculation we conclude:

$$d_B = s_B^* d_Q \text{ and } s_B = d_B^* d_P. \tag{1.1}$$

We call B *non-singular* if d_B and s_B are isomorphisms. In view of (1.1) this implies P and Q are reflexive. Conversely, if P and Q are reflexive, and if d_B is an isomorphism, then (1.1) shows that s_B is also.

A pair (P, B), $B \in \operatorname{Bil}(P \times P)$, is called a *bilinear module*. $f : P_1 \to P_2$ is a *morphism* $(P_1, B_1) \to (P_2, B_2)$ if $B_2(fx, fy) = B_1(x, y)$ for x, $y \epsilon P_1$. We define

$$(P_1, B_1) \perp (P_2, B_2) = (P_1 \oplus P_2, B_1 \perp B_2)$$

and

$$(P_1, B_1) \otimes (P_2, B_2) = (P_1 \otimes P_2, B_1 \otimes B_2)$$

by $(B_1 \perp B_2)((x_1, x_2), (y_1, y_2)) = B_1(x_1, y_1) + B_2(x_2, y_2)$, and $(B_1 \otimes B_2)(x_1 \otimes x_2, y_1 \otimes y_2) = B_1(x_1, y_1)B_2(x_2, y_2)$. Identifying $(P_1 \oplus P_2)^* = P_1^* \oplus P_2^*$ we have $d_{B_1 \perp B_2} = d_{B_1} \oplus d_{B_2}$. Moreover, $d_{B_1 \otimes B_2}$ is $d_{B_1} \otimes d_{B_2}$ followed by the natural map $P_1^* \otimes P_2^* \to (P_1 \otimes P_2)^*$. The latter is an isomorphism if one of the P_i is finitely generated and projective.

If (P, B) is a bilinear module we shall write $B^*(x, y) = B(y, x)$. If P is reflexive and we identify $P = P^{**}$ then (1.1) shows that $d_{(B^*)} = s_B = (d_B)^*$. We call (P, B) or B *symmetric* if $B = B^*$. For any B, $B + B^*$ is clearly symmetric. 136

If (P, B) is a symmetric bilinear module we have a notion of *orthogonality*. Specifically, if U is a subset of P, write

$$P^U = \{x \in P | B(x, y) = 0 \ \forall y \in U\}.$$

When P is fixed by the context we will sometimes write

$$U^\perp = P^U.$$

The following properties are trivial to verify:

$$\begin{aligned} &U^\perp \text{ is a submodule of } P.\\ &U \subset V \Rightarrow V^\perp \subset U^\perp\\ &U \subset U^{\perp\perp}\\ &U^\perp = U.^{\perp\perp\perp} \end{aligned}$$

We say U and V are *orthogonal* if $U \subset V^\perp$, and we call a submodule U *totally isotropic* if $U \subset U^\perp$, i.e. if $B(x, y) = 0$ for all x, $y \epsilon U$. The expression $P = U \perp V$ denotes the fact that P is the direct sum of the orthogonal submodules U and V.

Lemma 1.2. *Let (P, B) be a non-singular symmetric bilinear module. If U is a direct summand of P then $U^\perp$ is also a direct summand, and B induces a non-singular pairing on $U \times (P/U^\perp)$.*

137 *Proof.* Since $0 \to U \to P \to P/U \to 0$ splits so does $0 \to (P/U)^* \to P^* \to U^* \to 0$. By hypothesis $d_B : P \to P^*$ is an isomorphism, so $U^\perp = d_B^{-1}(P/U)^*$ is a direct summand of P. Moreover the composite $P \xrightarrow{d_B} P^* \to U^*$ is surjective, with kernel $U^\perp$, so B induces an isomorphism $(P/U^\perp) \to U^*$. Since U and $(P/U^\perp)$ are reflexive this implies the pairing on $U \times (P/U^\perp)$ is non-singular (see(1.1)). ◻

Lemma 1.3. *Let $f : (P_1, B_1) \to (P_2, B_2)$ be a morphism of symmetric bilinear modules, and suppose that (P_1, B_1) is non-singular. Then f is a monomorphism, and*

$$P_2 = fP_1 \perp P_2^{(fP_1)}$$

Proof. If $x\epsilon \ker f$ then $0 = B_2(fx, fy) = B_1(x,y)$ for all $y \in P_1$ so $x = 0$ because B_1 is non-singular. Now use f to identify $P_1 \subset P_2$ and $B_1 = B_2|P_1 \times P_1$. Then $P_1 \cap P_2^{P_1} = 0$ because B_1 is non-singular. If $x \in P_2$ define $h : P_1 \to k$ by $h(y) = B_2(x \cdot y)$. Since B_1 is non-singular $h(y) = B_1(x_1, y)$ for some $x_1 \epsilon P_1$, and then we have $x = x_1 + (x - x_1)$ with $x - x_1 \in P_2^{P_1}$. □

Lemma 1.4. *Let (P, B) be a non-singular symmetric bilinear module and suppose that U is a totally isotropic direct summand of P.*

(a) We can write $P = U^\perp \oplus V$, and, for any such V, $W = U \oplus V$ is a non-singular bilinear submodule of P. Hence $P = W \perp P^W$.

(b) $V \approx U^*$, so if U is finitely generated and projective then so is W, and $[W : k] = 2[U : k]$.

(c) If $B = B_0 + B_0^*$ and if $B_0(x, x) = 0$ for all $x\epsilon U$ then we can choose V above so that $B_0(x, x) = 0$ for all $x\epsilon V$ also.

Proof. (a) According to Lemma 1.2, $P = U^\perp \oplus V$, and for any such 138
V, B induces a non-singular form on $U \times V$. Thus B induces isomorphisms $f : U \to V^*$ and $g : V \to U^*$. If $B_1 = B|W \times W$ then $d_{B_1} : U \oplus V \to U^* \oplus V^*$ is represented by a matrix $\begin{pmatrix} 0 & g \\ f & d_{B_2} \end{pmatrix}$, where $B_2 = B|V \times V$. Evidently d_{B_1} is an isomorphism. Lemma 1.3 now implies $P = W \perp P^W$.

(b) is clear

(c) Identifying $U = U^{**}$ and $V = V^{**}$, the symmetry of B implies $f^* = g$. Let $B_3 = B_0|V \times V$, where $B = B_0 + B_0^*$ (by hypothesis), and set $k = f^{-1} d_{B_3} : V \to U$. Then for $v \in V$ we have

$$B(v, hv) = \langle fhv, v\rangle_V = \langle f f^{-1} d_{B_3} v, v\rangle_V = B_3(v, v) = B_0(v, v).$$

□

Let $t : V \to U \oplus V$ by $t(v) = v - h(v)$. Then if $V_1 = tV$ it is still clearly true that $P = U^\perp \oplus V_1$ (in fact, $W = U \oplus V_1$). We conclude the proof by showing that $B_0(v, v) = 0$ for $v\epsilon V_1$. Suppose $v\epsilon V$. Then

$$B_0(tv, tv) = B_0(v - hv, v - hv) = B_0(v, v) + B_0(hv, hv) -$$

$$- B_0(v, hv) - B_0(hv, v).$$

Since $hv \in U$ and $B_0(x,x) = 0$ for $x \in U$, by hypothesis, and since $B = B_0 + B_0^*$, we have $B_0(tv, tv) = B_0(v, v) - B_0(v, hv) - B_0^*(v, hv) = B_0(v, v) - B(v, hv)$. This vanishes according to the calculation above, so lemma 1.4 is proved.

Let P be a module and $B \in \mathrm{Bil}(P \times P)$. We define the function

$$q = q_B : P \to k; q(x) = B(x, x).$$

139 q has the following properties:

$$q(ax) = a^2 x \quad (a \in k, x \in P), \tag{1.5}$$

If $B_q(x, y) = q(x + y) - q(x) - q(y)$, then $B_q \in \mathrm{Bil}(P \times P)$. Indeed, direct calculation shows that $B_q = B + B^*$.

Lemma 1.6. *Suppose P is finitely generated and projective, and that q: $P \to k$ satisfies (1.5). Then there is a $B \in \mathrm{Bil}(P \times P)$ such that $q = q_B$. In particular, $B_q = B + B^*$.*

Proof. If P is free with basis $(e_i)_{1 \le i \le n}$ then $q(\sum\limits_i a_i e_i) = \sum\limits_i a_i^2 q(e_i) + \sum\limits_{i<j} a_i a_j B_q(e_i, e_j)$. Set $b_{ii} = q(e_i)$, $b_{ij} = B_q(e_i, e_j)$ for $i < j$, and $b_{ij} = 0$ for $i > j$. Then $q(\sum\limits_i a_i e_i) = \sum\limits_{i,j} a_i a_j b_{ij} = B(\sum\limits_i a_i e_i, \sum\limits_i a_i e_i)$, where $B(\sum\limits_i a_i e_i, \sum\limits_i c_i e_i) = \sum\limits_{i,j} a_i c_j b_{ij}$.

In the general case choose P' so that $F = P \oplus P'$ is free and extend q to q_1 on F by $q_1(x, x') = q(x)$ for $(x, x') \in P \oplus P'$. If $q_1 = q_{B_1}$ then $q = q_B$ where $B = B_1 \big| P \times P$.

We define a *quadratic form* on a module P to be a function of the form q_B for some $B \in \mathrm{Bil}(P \times P)$. B is then uniquely determined modulo "alternating forms," i.e. those B such that $B(x, x) = 0$ for all $x \in P$. We shall call the pair (P, q) *a quadratic module*, and we call it *non-singular* if B_q is non-singular. $f : P_1 \to P_2$ is a *morphism* $(P_1, q_1) \to (P_2, q_2)$ of quadratic modules if $q_2(fx) = q_1(x)$ for all $x \epsilon P_1$. Evidently f then induce a morphism $(P_1, B_{q_1}) \to (P_2, B_{q_2})$ of the associated bilinear modules.

140 If f is an isomorphism we call f an *isometry*. If $q_i = q_{B_i}$ then we define $q_1 \perp q_2 = q_{B_1 \perp B_2}$ on $P_1 \oplus P_2$, and $q_1 \otimes q_2 = q_{B_1 \otimes B_2}$ on $P_1 \otimes P_2$. It is easily checked that these definition are unambiguous. □

2 The hyperbolic functor

Let P be a k-module and define

$$B_0^P \in \mathrm{Bil}((P \oplus P^*) \times (P \oplus P^*)) \text{ by } B_0^P((x_1, y_1), (x_2, y_2)) = \langle y_1, x_2 \rangle_P,$$

and let $q^P = q_{B_0^P}$ be the induced quadratic form:

$$q^P(x, y) = \langle y, x \rangle_P \qquad (x \in P, y \in P^*).$$

Let $B^P = B_0^P + (B_0^P)^*$ be the associated bilinear form, $B^P = B_{q^P}$. Then

$$B^P((x_1, y_1), (x_2, y_2)) = \langle y_1, x_2 \rangle_P + \langle y_2, x_1 \rangle_P.$$

If $d_P : P \to P^{**}$ is the natural map then it is easily checked that

$$d_{B^P} : P \oplus P^* \to (P \oplus P^*)^* = P^* \oplus P^{**}$$

is represented by the matrix

$$\begin{pmatrix} 0 & 1_{P^*} \\ d_P & 0 \end{pmatrix}.$$

Consequently, *B^P is non-singular if and only if P is reflexive*. If, in this case, we identify $P = P^{**}$ then the matrix above becomes $\left(\begin{smallmatrix} 0 & 1_{P^*} \\ 1_P & 0 \end{smallmatrix}\right)$.

We will write 141

$$\mathbb{H}(P) = (P \oplus P^*, q^P)$$

and call this quadratic module the *hyperbolic form* on P.

Suppose $f : P \to Q$ is an isomorphism of k-modules. Define

$$\mathbb{H}(f) = f \oplus (f^*)^{-1} : \mathbb{H}(P) \to \mathbb{H}(Q).$$
$$q^Q(\mathbb{H}(f)(x, y)) = q^Q(fx, (f^*)^{-1}y) = \langle (f^{-1})^* y, fx \rangle_Q$$

$$= \langle y, f^{-1} f x\rangle_P = q^P(x, y), \text{ so } \mathbb{H}(f) \text{ is an isometry.}$$

If we identify $(P_1 \oplus P_2)^* = P_1^* \oplus P_2^*$ so that

$$\langle (y_1, y_2), (x_1, x_2)\rangle_{P_1 \oplus P_2} = \langle y_1, x_1\rangle_{P_1} + \langle y_2, x_2\rangle_{P_2}$$

then the natural homomorphism

$$f : \mathbb{H}(P_1) \perp \mathbb{H}(P_2) \to \mathbb{H}(P_1 \oplus P_2),$$

$f((x_1, y_1), (x_2, y_2)) = ((x_1, x_2), (y_1, y_2))$. is an isometry.

Summarizing the above remarks, $\mathbb{H}$ *is a product preserving functor* (in the sense of chapter 1) *from (modules, isomorphisms*, $\oplus$) to (*quadratic modules, isometries*, $\perp$). We now characterize non-singular hyperbolic forms.

Lemma 2.1. *A non-singular quadratic module* (P, q) *is hyperbolic if and only if* P *has a direct summand* U *such that* $q|U = 0$ *and* $U = U^\perp$. *In this case* $(P, q) \approx \mathbb{H}(U)$ *(isometry).*

142 *Suppose* P *is finitely generated and projective. If* U *is a direct summand such that* $q|U = 0$ *and* $[P : k] \leq 2[U : k]$ *then* $(P, q) \approx \mathbb{H}(U)$.

Proof. If $(P, q) \approx \mathbb{H}(U) = (U \oplus U^*, q^U)$ then the non-singularity of (P, q) implies U is reflexive, and it is easy to check that $U \subset U \oplus U^*$ satisfies $q^U|U = 0$ and $U = U^\perp$. □

Conversely, suppose given a direct summand U of P such that $q|U = 0$ and $U = U^\perp$. Write $q = q_{B_0}$, so that $B_q = B_0 + B_0^*$. According to Lemma 1.4 we can write $P = U^\perp \oplus V = U \oplus V$ and B_q induces a non-singular pairing on $U \times V$. Moreover we can arrange that $B_0(v, v) = 0$ for all $v \in V$, i.e. that $q|V = 0$. Let $d : V \to U^*$ be the isomorphism induced by B_q; $\langle dv, u\rangle_U = B_q(v, u)$ for $u \in U$, $v \in V$.

Let

$$f = 1_U \oplus d : P = U \oplus V \to U \oplus U^*.$$

This is an isomorphism, and we want to check that

$$q^U((u, dv)) = q(u, v) \text{ for } u \epsilon U, v \epsilon V. q^U((u, dv)) = \langle dv, u\rangle_U = B_q(v, u),$$

while $q(u,v) = q(u) + q(v) + B_q(u,v) = B_q(u,v)$, since $q/U = 0$ and $q/V = 0$.

The last assertion reduces to the preceding ones we show that $U = U^\perp$. Lemma 1.2 shows that $U^\perp$ is a direct summand of rank $[U^\perp : k] = [P : k] - [U : k] \leq [U : k]$, because, by assumption, $[P : k] \leq 2[U : k]$. But we also have $q/U = 0$ so $U \subset U^\perp$, and therefore $U = U^\perp$, as claimed.

Lemma 2.2. *A quadratic module* (P, q) *is non-singular if and only if*

$$(P, q) \perp (P, -q) \approx \mathbb{H}(P),$$

provided P is reflexive.

Proof. P reflexive implies $\mathbb{H}(P)$ is non-singular, and hence likewise for any orthogonal summand. **143**

Suppose now that (P, q) is non-singular. Then so is $(P, q) \perp (P, -q) = (P \oplus P, q_1 = q \perp (-q))$.

Let $U = \{(x, x)\epsilon P \oplus P | x \in P\}$. Then $q_1/U = 0$, and U is a direct summand of $P \oplus P$, isomorphic to P. If $U \subsetneq U^\perp$ we can find a $(0, y)\epsilon U^\perp$, $y \neq 0$. Then, for all $x \in P$,

$$\begin{aligned} 0 = B_{q_1}((x, x), (0, y)) &= q_1(x, x + y) - q_1(x, x) - q_1(0, y) \\ &= q(x) - q(x + y) + q(y) \\ &= -B_q(x, y). \end{aligned}$$

Since B_q is non-singular this contradicts $y \neq 0$. Now the Lemma follows from Lemma 2.1. ◻

Lemma 2.3. *Let P be a reflexive module and let* (Q, q) *be a non-singular quadratic module with Q finitely generated and projective. Then*

$$\mathbb{H}(P) \otimes (Q, q) \approx \mathbb{H}(P \otimes Q).$$

Proof. The hypothesis on Q permits us to identify $(P \otimes Q)^* = P^* \otimes Q^*$, so it follows that $(W, q_1) = \mathbb{H}(P) \otimes (Q, q)$ is non-singular. We shall apply Lemma 2.1 by taking

$U = P \otimes Q \subset W = (P \otimes Q) \oplus (P^* \otimes Q)$. If $\sum x_i \otimes y_i \epsilon U$, then $q_1(\Sigma x_i \otimes y_i) = \Sigma q^P(x_i) q(y_i) + \sum_{i<j} B_{q_1}(x_i \otimes y_i, x_j \otimes y_j) = \sum_{i<j} B^P(x_i, x_j) B_q(y_i, y_j) = 0$, because $q^P/P = 0$ in $\mathbb{H}(P)$. Thus $U \subset U^\perp$, and to show equality it suffices clearly to show that $(P^* \otimes Q) \cap U^\perp = 0$. If $\Sigma x_i \otimes y_i \in U$ and $\Sigma w_j \otimes z_j \in (P^* \otimes Q) \cap U^\perp$ then $0 = B_{q_1}(\Sigma x_i \otimes y_i, \Sigma w_j \otimes z_j) = \sum\limits_{i,j} B^P(x_i, w_j) B_q(y_i, z_j)$. □

144 Since $(P^* \otimes Q)^* = P \otimes Q^*$ (P is reflexive) the non-singularity of q guarantees that all linear functionals on $P^* \otimes Q$ have the form $\sum_i B^P(x_i, \ ) B_q(y_i, \ )$, so $\Sigma w_j \otimes z_j$ is killed by all linear functionals, hence is zero. We have now shown $U = U^\perp$ so the lemma follows from Lemma 2.1.

A *quadratic space* is a non-singular quadratic module (P, q) with P finitely generated and projective, i.e. $P \epsilon \operatorname{obj} \underset{=}{P}$, the category of such modules. We define the category

$$\underline{\underline{\mathrm{Quad}}} = \underline{\underline{\mathrm{Quad}}}(k)$$

with

objects : quadratic spaces
morphisms : isometries
product : $\perp$

The discussion at the beginning of this section shows that

$$\mathbb{H} : \underset{=}{P} \to \underline{\underline{\mathrm{Quad}}}$$

is a product preserving functor of categories with product (in the sense of chapter 1), and Lemma 2.1 shows that $\mathbb{H}$ is cofinal. We thus obtain an exact sequence from Theorem 4.6 of chapter 1. We summarize this:

Proposition 2.4. *The hyperbolic functor*

$$\mathbb{H} : \underset{=}{P} \to \underline{\underline{\mathrm{Quad}}}$$

is a cofinal functor of categories with product. It therefore induces (Theorem 4.6 of chapter 1) an exact sequence

$$K_1 \underset{=}{P} \to K_1 \underline{\underline{\mathrm{Quad}}} \to K_0 \Phi \mathbb{H} \to K_0 \underset{=}{P} \to K_0 \underline{\underline{\mathrm{Quad}}} \to \mathrm{Witt}(k) \to 0,$$

where we define Witt (k) = *coker* $(K_0\mathbb{H})$.

We close this section with some remarks about the multiplicative structures. Tensor products endow $K_0\underline{\underline{\mathrm{Quad}}}$ with a commutative multiplication, and Lemma 2.3 shows that the image of $K_0\mathbb{H}$ is an ideal, so Witt (k) also inherits a multiplication. The difficulty is that, if 2 is not invertible in k, then these are rings without identity elements. For the identity should be represented by the form $q(x) = x^2$ on k. But then $B_q(x, y) = 2xy$ is not non-singular unless 2 is invertible. **145**

Here is one natural remedy. Let $\underline{\underline{\mathrm{Symbil}}}$ denote the category of non-singular symmetric bilinear forms, (P, B) with $P \epsilon \operatorname{obj} \underline{\underline{P}}$. If $(P, B) \in \underline{\underline{\mathrm{Symbil}}}$ and $(Q, q) \epsilon \underline{\underline{\mathrm{Quad}}}$ define

$$(P, B) \otimes (Q, q) = (P \otimes Q, B \otimes q), \tag{2.5}$$

where $B \otimes q$ is the quadratic form $q_{B \otimes B_0}$, for some $B_0 \epsilon \operatorname{Bil}(Q \times Q)$ such that $q = q_{B_0}$. It is easy to see that $B \otimes q$ does not depend on the choice of B_0. Moreover, the bilinear form associated to $B \otimes q$ is $(B \otimes B_0) + (B \otimes B_0)^* = (B \otimes B_0) + (B^* \otimes B_0^*) = B \otimes (B_0 \otimes B_0^*) = B \otimes B_q$, because $B = B^*$. Since B and B_q are non-singular so is $B \otimes B_q$ so $(P \otimes Q, B \otimes q) \in \underline{\underline{\mathrm{Quad}}}$.

If $a \epsilon k$ write $\langle a \rangle$ for the bilinear module (k, B) with $B(x, y) = axy$ for $x, y \epsilon k$. If a is a unit then $\langle a \rangle \epsilon \underline{\underline{\mathrm{Symbil}}}$.

Tensor products in $\underline{\underline{\mathrm{Symbil}}}$ make $K_0\underline{\underline{\mathrm{Symbil}}}$ a commutative ring, with identity $\langle 1 \rangle$, and (2.5) makes $K_0\underline{\underline{\mathrm{Quad}}}$ a $K_0\underline{\underline{\mathrm{Symbil}}}$−module. The "forgetful" functor $\underline{\underline{\mathrm{Quad}}} \to \underline{\underline{\mathrm{Symbil}}}$, $(P, q) \longmapsto (P, B_q)$, induces a K_0 $\underline{\underline{\mathrm{Symbil}}}$-homomorphism $K_0\underline{\underline{\mathrm{Quad}}} \to K_0\underline{\underline{\mathrm{Symbil}}}$, so its image is an ideal. The hyperbolic forms generate a $K_0\underline{\underline{\mathrm{Symbil}}}$ submodule, image $K_0\mathbb{H}$), of $K_0\underline{\underline{\mathrm{Quad}}}$, so Witt (k) is a $K_0\underline{\underline{\mathrm{Symbil}}}$-module. This follows from an analogue of Lemma 2.3 for the operation (2.5) **146**

Similarly, the hyperbolic forms, $(P \oplus P^*, B^P)$, generate an ideal in $K_0\underline{\underline{\mathrm{Symbil}}}$ which annihilates $Witt(k)$. Lemma 2.2 says that $\langle 1 \rangle \perp \langle -1 \rangle$ also annihilates Witt (k).

3 The Clifford Functor

If P is a k-module we write

$$T(P) = (k) \oplus (P) \oplus (P \otimes P) \oplus \ldots \oplus (P^{\otimes n}) \oplus \ldots$$

for its tensor algebra. If (P, q) is a quadratic module then its *Cliford algebra* is

$$Cl(P, q) = T(P)/I(q),$$

where $I(q)$ is the two sided ideal generated by all $x \otimes x - q(x)(x \in P)$. If we grade $T(P)$ by even and odd degree (a $(\mathbb{Z}/2\mathbb{Z})$-grading) then $x \otimes x - q(x)$ is homogeneous of even degree, so

$$Cl(P, q) = Cl_0(P, q) \oplus Cl_1(P, q)$$

is a graded algebra in the sense of chapter 4, *We will consider $Cl(P, q)$ to be a graded algebra*, and this must be borne in mind when we discuss tensor products.

147 The inclusion $P \subset T(P)$ induces a k-linear map

$$C_P : P \to Cl(P, q)$$

such that $C_P(x)^2 = q(x)$ for all $x \in P$, and C_P is clearly universal among such maps of P into a k-algebra.

Cl evidently defines a functor from quadratic modules, and their morphisms, to graded algebras, and their homomorphisms (of degree zero). Moreover it is easy to check that both T and Cl commute with base change, $k \to K$. The next lemma says Cl is "product preserving," in an appropriate sense.

Lemma 3.1. *There is a natural isomorphism of (graded) algebras,*

$$Cl((P_1, q_1) \perp (P_2, q_2)) \approx Cl(P_1, q_1) \otimes Cl(P_2, q_2).$$

Proof. $P_i \to P_1 \otimes P_2 \xrightarrow{C_{P_1 \oplus P_2}} Cl((P_1, q_1) \perp (P_2, q_2))$ induces an algebra homomorphism,

$$f_i : Cl(P_i, q_i) \to Cl((P_1, q_1) \perp (P_2, q_2)), i = 1, 2.$$

$\square$

If $x_i \in P_i$ then $(f_1x_1 + f_2x_2)^2 = (q_1 \perp q_2)(x_1, x_2) = q_1x_1 + q_2x_2 = (f_1x_1)^2 + (f_2x_2)^2$, so f_1x_1 and f_2x_2, being of odd degree, commute (in the graded sense). Therefore so also do the algebras they generate, im f_1 and im f_2. Hence f_1 and f_2 induce an algebra homomorphism

$$F : Cl(P_1, q_1) \otimes Cl(P_2, q_2) \to Cl((P_1, q_1) \perp (P_2, q_2)),$$

and this is clearly natural. To construct its inverse let

$$g : P_1 \oplus P_2 \to Cl((P_1, q_1) \perp (P_2, q_2)) \text{ by } g(x_1, x_2) = C_1x_1 \otimes 1 + 1 \otimes C_2x_2,$$

where $C_i = C_{P_i}$. If g extends to an algebra homomorphism from $Cl \perp ((P_1, q_1) \perp (P_2, q_2))$ it will evidently be inverse to F, since this is so on the generators, $C_{P_1 \oplus P_2}(P_1 \oplus P_2)$, and $C_1P_1 \otimes 1 + 1 \otimes C_2P_2$, respectively. To show that g extends we have to verify that $g(x_1, x_2)^2 = (q_1 \perp q_2)(x_1, x_2)$. $g(x_1, x_2)^2 = (C_1x_1)^2 \otimes 1 + 1 \otimes (C_2x_2)^2 + C_1x_1 \otimes C_2x_2 + (-1)^{(\deg C_1x_1)(\deg C_2x_2)}(C_1x_1 \otimes C_2x_2) = q_1x_1 + q_2x_2 = (q_1 \perp q_2)(x_1, x_2)$. 148

Examples 3.2. If q is the quadratic form $q(x) = ax^2$ on k, denote this quadratic module by (k, a). Then $C1(k, a) = k\langle a\rangle = k1 \oplus kx$, $x^2 = a$. Thus $C1(\mathbb{R}, -1) \approx \mathbb{C} = \mathbb{R}1 \oplus \mathbb{R}i$, for example.

$$C1((k, a) \perp (k, b)) \approx C1(k, a) \otimes C1(k, b) \approx (\frac{a, b}{k}). \tag{3.3}$$

The latter denotes the k-alegbra with free k-basis 1, x_a, x_b, y, where $\deg x_a = \deg x_b = 1$, $x_a^2 = a$, $x_b^2 = b$, $y = x_ax_b = -x_bx_a$. The degree zero component is

$$\left(\frac{a, b}{k}\right)_0 = k[y], y^2 = -ab.$$

For example, as a graded $\mathbb{R}$-algebra, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \approx (\frac{-1 - 1}{k})$, the standard quaternion algebra (plus grading).

$$\mathbb{H}(k) = (k \oplus k^*, q^k). \tag{3.4}$$

Let e_1 be a basis for k (e.g. $e_1 = 1$) and e_2 the dual basis for k^*. Writing $q = q^k$ we have $q(a_1e_1 + a_2e_2) = a_1a_2$. Hence $C1(\mathbb{H}(k))$ is

generated by elements x_1 and x_2 (the images of e_1 and e_2) of degree 1 with the relations $x_1^2 = 0 = x_2^2$ and $x_1x_2 + x_2x_1 = 1$. In $\mathbb{M}_2(k)$ the matrices $y_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ satisfy these relations, so there is a **149**
homomorphism

$$\begin{gathered} Cl(\mathbb{H}(k)) \to \mathbb{M}_2(k) \\ x_i \mapsto y_i \end{gathered}$$

It is easy to check that this is an isomorphism. This isomorphism is the simplest case of Theorem below, which we prepare for in the following lemmas.

Lemma 3.5. *Let P be a k-module. There is a k-linear map $P^* \to \mathrm{Hom}_k(T(P), T(P))$, $f \mapsto d_f$, where d_f is the unique map of degree -1 on $T(P)$ such that $d_f(x \otimes y) = f(x)y - x \otimes d_f(y)$ for $x \epsilon P$, $y \in T(P)$. Moreover $d_f^2 = 0$ and $d_f d_g + d_g d_f = 0$ for $f, g \epsilon P^*$. If q is a quadratic form on P then $d_f I(q) \subset I(q)$, so d_f induces a k-linear map, also denoted d_f, of degree one on $Cl(P, q)$.*

Proof. d_f is defined on $P^{\otimes(n+1)} = P \otimes P^{(\otimes n)}$ by induction on n, from the formula given. This shows uniqueness, and that

$$d_f(x_0 \otimes \ldots \otimes x_n) = \sum_{0 \le i \le n} (-1)^i f x_i (x_0 \otimes \ldots \hat{i} \ldots x_n).$$

□

Hence

$$\begin{aligned} d_f^2(x_0 \otimes \ldots \otimes x_n) &= \sum_{0 \le j < i \le n} (-1)^{i+j} (f x_i)(f x_j)(x_0 \otimes \ldots \hat{j} \ldots \hat{i} \ldots \otimes x_n) \\ &\quad + \sum_{0 \le j < i \le n} (-1)^{i+j-1} (f x_i)(f x_j)(x_0 \otimes \ldots \hat{i} \ldots \hat{j} \ldots \otimes x_n) \\ &= 0. \end{aligned}$$

150 It is easy to check that $f \mapsto d_f$ is k-linear, so we have $0 = (d_{f+g})^2 = (d_f + d_g)^2$, and hence $d_f d_g + d_g d_f = 0$ for $f, g \epsilon P^*$.

The formula above shows that if x, $x, y \in hT(P)$ (the set of homogeneous elements) then

$$d_f(x \otimes y) = d_f x \otimes y + (-1)^{\partial x} x \otimes d_f y.$$

If q is a quadratic form on P write $u(x) = x \otimes x - q(x)$ for $x \epsilon P$. Since $u(x)$ has even degree the formula above shows that $d_f(u(x) \otimes y) = d_f u(x) \otimes y + u(x) \otimes d_f y$, and $d_f(u(x)) = f(x)x - f(x)x = 0$, so $d_f(u(x) \otimes y) \in I(q)$. If $v \in hT(P)$ then $d_f(v \otimes u(x) \otimes y) = d_f v \otimes u(x) \otimes y + (-1)^{\partial v} v \otimes d_f(u(x) \otimes y) \in I(q)$. Since $I(q)$ is additively generated by all such $v \otimes u(x) \otimes y$ it follows that $d_f I(q) \subset I(q)$.

Lemma 3.6. *If $B \in \mathrm{Bil}(P \times P)$ there is a unique k-linear map $\lambda_B : T(P) \to T(P)$ satisfying*

(i) $\lambda_B(1) = 1$

(ii) $\lambda_B L_x = (L_x + d_{B(x,\ )})\lambda_B$ *for $x \in P$.*

(Here L_x denotes left multiplication by x in $T(P)$.)λ_B also has the following properties:

(a) *λ_B preserves the ascending filtration on $T(P)$ and induces the identity map on the associated graded module.*

(b) *For $f \epsilon P^*$, $\lambda_B d_f = d_f \lambda_B$.*

(c) *$\lambda_0 = 1_{T(P)}$ and $\lambda_{B+B'} = \lambda_B \circ \lambda_{B'}$ for $B, B' \epsilon$ Bill $(P \times P)$.*

(d) *If q is a quadratic form on P, then $\lambda_B I(q) = I(q - q_B)$, and λ_B induces an isomorphism $Cl(P, q) \to Cl(P, q - q_B)$ of filtered modules.* **151**

Proof. Writing xy in place of $x \otimes y$ in $T(P)$, (ii) reads:

$$\lambda_B(xy) = x\lambda_B(y) + d_{B(x,\ )}(\lambda_B(y))(x \epsilon P, y \epsilon T(P)).$$

Starting with $\lambda_B(1) = 1$ this gives an inductive definition of λ_B on $P^{(\otimes n)}$, since the right side is k-bilinear in x and y. Moreover (a) follows also from this by induction on n.

(b) We prove that $\lambda_B d_f = d_f \lambda_B$ by induction on n, the case $n = 0$ being clear (from (a)). For $x \in P$, $y \in hT(P)$,

$$\begin{aligned} \lambda_B d_f(xy) &= \lambda_B((fx)y - x(d_f y)) \\ &= (fx)(\lambda_B y) - (x(\lambda_B(d_f y) + d_{B(x,\ )}(\lambda_B(d_f y))) \\ d_f \lambda_B(xy) &= d_f(x(\lambda_B y) + d_{B(x,\ )}(\lambda_B y)) \\ &= (fx)(\lambda_B y) - x(d_f(\lambda_B y)) + d_f d_{B(x,\ )}(\lambda_B y) \end{aligned}$$

Their equality follows from $d_f \lambda_B y = \lambda_B d_f y$ (induction) and the fact (Lemma 3.5) that $d_f d_{B(x,\ )} = -d_{B(x,\ )} d_f$.

(c) If $B = 0$ then $d_{B(x,\ )} = 0$ for all x so (ii) reads $\lambda_0 L_x = L_x \lambda_0$, and $1_{T(P)}$ solves this equation for λ_0. We prove $\lambda_B 0 \lambda'_B = \lambda_{B+B'}$ by checking (i) (which is clear) and (ii):

$$\begin{aligned} \lambda_B \circ \lambda_{B'}(xy) &= x(\lambda_B \circ \lambda_{B'} y) + d_{B+B'(x,\ )}(\lambda_B \circ \lambda_{B'} y). \\ \lambda_B \lambda_{B'}(xy) &= \lambda_B(x(\lambda_{B'} y) + d_{B'(x,\ )}(\lambda_{B'} y)) \\ &= x\lambda_B \lambda_{B'} y + d_{B(x,\ )}(\lambda_B \lambda_{B'} y) + d_{B'(x,\ )}(\lambda_B \lambda_{B'} y) \\ &= x(\lambda_B \lambda_{B'} y) + (d_{B(x,\ )} + d_{B'(x,\ )})(\lambda_B \lambda_{B'} y). \end{aligned}$$

and $d_{B(x,\ )} + d_{B'(x,\ )} = d_{(B+B')(x,\ )}$.

152 (d) Let $I = \{u \in T(P) | \lambda_B(u) \epsilon I(q - q_B)\}$. $\lambda_B(xu) = x(\lambda_B u) + d_{B(x,\ )}(\lambda_B u)$, so, thanks to Lemma 3.5, I is a left ideal. $\lambda_B((X^2 - (qx))y) = x\lambda_B(xy) + d_{B(x,\ )}\lambda_B(xy)$

$$\begin{aligned} &- (qx)(\lambda_B y) = x(x\lambda_B y + d_{B(x,\ )}(\lambda_B x)) + d_{B(x,\ )}(x\lambda_B y + d_{B(x,\ )}\lambda_B y) \\ &- (qx)(\lambda_B y) = x^2(\lambda_B y) + xd_{B(x,\ )}(\lambda_B y) + B(x,x)(\lambda_B y) - xd_{B(x,\ )}(\lambda_B y) \\ &- (qx)(\lambda_B y) \quad \text{(we have used } d^2_{B(x,\ )} = 0;\ \text{Lemma 3.5)} = \\ &= (x^2 - (qx - q_B x))\lambda_B y \in I(q - q_B). \end{aligned}$$

Thus I is a left ideal containing all $(x^2 - qx)y$, so it contains $I(q)$. We have proved

$$\lambda_B I(q) \subset I(q - q_B) = \lambda_B \lambda_{-B} I(q - q_B) \subset \lambda_B(I(q - q_B - q_{-B}) = \lambda_B I(q),$$

using (c). Now (a) implies λ_B induces an isomorphism $Cl(P,q) \to Cl(P, q - q_B)$ of filtered modules. □

Corollary 3.7. *Giving $Cl(P,q)$ the filtration induced by the ascending filtration on $T(P)$, the structure of $Cl(P,q)$ as a filtered module is independent of q. In particular, taking $q = 0$, we have an isomorphism*

$$Cl(P,q) \approx \Lambda(P)$$

of filtered modules.

Proof. Writing $q = q_B$ for some $B \in \mathrm{Bil}\ (P \times P)$ we obtain an isomorphism $Cl(P,q) \to Cl(P,0) = \Lambda(P)$, induced by λ_B. ◻

Corollary 3.8. $C_P : P \to Cl(P,q)$ *is a monomorphism. If U is a direct summand of P then the map*

$$Cl(U, q/U) \to Cl(P,q),$$

induced by the inclusion $U \subset P$, is a monomorphism.

Proof. The first assertion follows from the commutativity of **153**

$$\begin{tikzcd} Cl(P,q) \arrow[rr, "\lambda_B"] & & \wedge(P) \\ & P \arrow[ul, "C_P)"] \arrow[ur] & \end{tikzcd}$$

and the fact that $P \to \wedge(P)$ is a monomorphism. Let $B' = B/U \times U$, so $q/U = q'_B$. Then it is easily checked that

$$\begin{tikzcd} Cl(P,q) \arrow[r, "\lambda_B"] & \wedge(P) \\ Cl(U,q') \arrow[u] \arrow[r, "\lambda_{B'}"'] & \wedge(U) \arrow[u] \end{tikzcd}$$

is commutative, so the second assertion follows since $\wedge(U) \to \wedge(P)$ is injective. ◻

$\wedge(P) = T(P)/I(), I(0)$ being the (homogeneous) ideal generated by all $x \otimes x$, $x \epsilon P$.

$$\wedge(P) = k \oplus \wedge^1 P \oplus \wedge^2 P \oplus \ldots$$

and $\wedge^1 P \approx P$. Lemma 3.1 givens a natural isomorphism

$$\wedge(P \oplus Q) \approx \wedge(P) \oplus \wedge(Q)$$

of $(\mathbb{Z}/2\mathbb{Z})$- graded algebras. $\wedge$ therefore defines a product preserving functor

$$\wedge : \underline{\underline{P}} \to \underline{\underline{FP}}_2,$$

if, for P finitely generated and projective, we view $\wedge(P)$ as a faithfully projective module, graded modulo 2. Similarly, by virtue of Lemma 3.1, the Clifford algebra defines a product preserving functor,

$$Cl : \underline{\underline{Quad}} \to (\text{graded algebras}, \otimes)$$

154 **Theorem 3.9.** *If $(P, q) \in \operatorname{obj} \underline{\underline{Quad}}$, then $Cl(P, q) \in \operatorname{obj} \underline{\underline{Az}}_2$, i.e. it is a graded azumaya algebra. The resulting functor $Cl : \underline{\underline{Quad}} \to \underline{\underline{Az}}_2$ renders the diagram*

$$\begin{tikzcd}
\underline{\underline{P}} \arrow[r, "\mathbb{H}"] \arrow[d, "\wedge"'] & \underline{\underline{Quad}} \arrow[d, "Cl"] \\
\underline{\underline{FP}}_2 \arrow[r, "END"'] & \underline{\underline{Az}}_2
\end{tikzcd}$$

commutative up to natural isomorphism, i.e. for P finitely generated and projective,

$$Cl(\mathbb{H}(P)) \approx END(\wedge(P))$$

as graded algebras.

Corollary 3.10. *There is a natural map of exact sequences.*

$$\begin{tikzcd}
K_1\underline{\underline{P}} \arrow[r] \arrow[d] & K_1\underline{\underline{Quad}} \arrow[r] \arrow[d] & K_0\Phi\mathbb{H} \arrow[r] \arrow[d] & K_0\underline{\underline{P}} \arrow[r] \arrow[d] & K_0\underline{\underline{Quad}} \arrow[r] \arrow[d] & Witt(k) \arrow[r] \arrow[d] & 0 \\
K_1\underline{\underline{FP}}_2 \arrow[r] & K_1\underline{\underline{Az}}_2 \arrow[r] & K_0\Phi END \arrow[r] & K_0\underline{\underline{FP}}_2 \arrow[r] & K_0\underline{\underline{Az}}_2 \arrow[r] & Br_2(k) \arrow[r] & 0
\end{tikzcd}$$

In Theorem 4.6 of Chapter 4 we exhibited an exact sequence

$$0 \to Br(k) \to Br_2(k) \to Q_2(k) \to 0,$$

where $Q_2(k)$ was "the group of graded quadratic extensions of k." The map above assigns to the class of (P,q) in Witt (k) and element β of $Br_2(k)$. The projection of β in $Q_2(k)$ corresponds, in the classical case when k is a field, to the discriminant, if char $k \neq 2$, and the Arf invariant if char $k = 2$. The remaining contribution from $Br(k)$ is essentially the Hasse invariant.

Proof of 3.9. We want to construct a natural isomorphism 155

$$\varphi_P : Cl(\mathbb{H}(P)) \to \text{ END } (\wedge(P)).$$

for $P \in \text{obj}\, \underline{P}$. Suppose this is done. Then if $(P,q) \in \text{obj}\, \underline{Quad}$, we have $(P,q) \perp (P,-q) \approx \mathbb{H}(P)$ (Lemma 2.2), so $Cl(P,q) \otimes \overline{Cl(P,-q)} \approx Cl(P,q) \perp (P,-q))$ (Lemma 3.1) $\approx Cl(\mathbb{H}(P)) = END(\wedge(P))$, by assumption. Therefore, by criterion (6) of Theorem 4.1, Chapter 4, $Cl(P,q)$ is a graded azumaya algebra. Thus we only have to construct φ_P.

$\mathbb{H}(P) = (P \oplus P^*, q^P)$ with $q^P(x,y) = \langle y, x\rangle_P$
$= y(x)$ for $(x,y) \in P \otimes P^*$. Define

$$P \otimes P^* \to END(\wedge(P))$$

by $(x,y) \mapsto L_x + d_y$. Then, using Lemma 3.5, $(L_x + d_y)^2 = L_{x^2} + L_x d_y + d_y L_x + d_y^2 = L_x d_y + d_y L_x$, because $x^2 = 0$ in $\wedge(P)$ and $d_y^2 = 0$. If $u \in \wedge(P)$ then $(L_x d_y + d_y L_x)u = x d_y(u) + d_y(xu) = x d_y(u) + y(x)u - x d_y(u) = y(x)u$. Thus $(L_x + d_y)^2$ is multiplication by $y(x) = q^P(x,y)$ on $\wedge(P)$, i.e. $(L_x + d_y)^2 = q^P(x,y)$ in END $(\wedge(P))$. Thus we have defined an algebra homomorphism

$$\varphi_P : Cl(\mathbb{H}(P)) \to END(\wedge(P)),$$

and since $L_x + d_y$ has degree 1, it is a homomorphism of graded algebras.

Suppose $f : P_1 \to P_2$ is an isomorphism. Then on $\wedge(P_2)$, $L_{f(x)} = \wedge(f) L_x \wedge (f)^{-1}$ and $d_{f^*} - 1_y(x_2) = (f^{*-1}y)(x_2) = y(f^{-1}x_2)$, so $\wedge(f) d_y \wedge (f)^{-1} x_2) = \wedge(f) d_y(f^{-1}x_2) = \wedge(f) y(f^{-1}x_2) = d_{f*-1_y}(x_2)$ for $x_2 \in P_2$, because $y(f^{-1}x_2)$ has degree zero in $\wedge(P_2)$. Therefore $\wedge(f)(L_x + d_y) \wedge$ 156
$(f)^{-1} = L_{f(x)} + d_{f*-1_{(y)}}$ so it follows that φ_P is natural, recalling that $\mathbb{H}(f) = f \oplus f^{*-1}$.

Next we will show that, for $P = P_1 \otimes P_2$ the following diagram is commutative:

$$\begin{tikzcd}
Cl(\mathbb{H}(P_1 \oplus P_2)) \arrow[r, "\varphi_{P_1 \oplus P_2}"] \arrow[d, "\approx"'] & END(\wedge(P_1 \oplus P_2)) \arrow[d, "\approx"] \\
Cl(\mathbb{H}(P_1) \perp \mathbb{H}(P_2)) \arrow[d, "\approx"'] & END(\wedge P_1 \otimes \wedge P_2) \\
Cl(\mathbb{H}(P_1)) \otimes Cl(\mathbb{H}(P_2)) \arrow[r, "\varphi_{P_1} \otimes \varphi_{P_2}"'] & END(\wedge P_1) \otimes END(\wedge P_2) \arrow[u, "\approx"']
\end{tikzcd}$$

To see this, we trace the images of $((x_1, x_2), (y_1, y_2))$

$$\in (P_1 \otimes P_2) \oplus (P_1^* \otimes P_2^*) \subset Cl(\mathbb{H}(P_1 \oplus P_2)):$$

$$\begin{tikzcd}
((x_1, x_2), (y_1, y_2)) \arrow[r, mapsto] \arrow[d, mapsto] & L_{(x_1,x_2)} + d_{(y_1,y_2)} \arrow[d, mapsto] \\
((x_1, y_1), (x_2, y_2)) \arrow[d, mapsto] & \begin{array}{c} L_{x_1 \otimes 1} + L_{1 \otimes x_2} \\ + d_{y_1} \otimes 1_{\wedge P_2} + 1_{\wedge P_1} \otimes d_{y_2} \end{array} \\
((x_1, y_1) \otimes 1) + (1 \otimes (x_2, y_2)) \arrow[r, mapsto] & \begin{array}{c} ((L_{X_1} + d_{y_1}) \otimes 1_{\wedge P_2}) \\ (1_{\wedge P_1} \otimes (L_{x_2} + d_{y_2})) \end{array} \arrow[u, mapsto]
\end{tikzcd}$$

157 Since all of these algebras are faithfully projective k-modules we conclude that $\varphi_{P_1 \oplus P_2}$ is an isomorphism $\Leftrightarrow$ $\varphi_{P_1} \oplus \varphi_{P_2}$ is an isomorphism $\Leftrightarrow$ φ_{P_1} and φ_{P_2} are isomorphisms. (In Chapter 2 we showed that the functor $Q\otimes$ is faithfully exact for Q faithfully projective.)

Now given P_1 we choose P_2 so that $P_1 \oplus P_2 \approx k \oplus \cdots \oplus k$, and then the problem of showing that φ_{P_1} is an isomorphism reduces to the special case $P_1 = k$. We do this case now by a direct calculation.

$\mathbb{H}(k) \approx (ke_1 \oplus ke_2, q)$ with $q(a_1 e_1 + a_2 e_2) = a_1 a_2$. Here $ke_2 = (ke_1)^*$ and e_2 is the dual basis to e_1, i.e. $e_2(e_1) = 1$. Therefore $\wedge(ke_1) = k[e_1] = k1 \oplus ke_1$ with $e_1^2 = 0$, and $d_{e_2}(1) = 0$, $d_{e_2}(e_1) = 1$. Moreover, $L_{e_1}(1) = e_1$ and $L_{e_1}(e_1) = 0$.

$Cl(\mathbb{H}(ke_1)) = k[e_1, e_2]$ with $e_1^2 = 0 = e_2^2$, and $e_1e_2 + e_2e_1 = 1$, because $1 = q(e_1 + e_2) = (e_1 + e_2)^2$.

$$\varphi_k : Cl(\mathbb{H}(ke_1)) \to END(\wedge(ke_1))$$

is defined by $\varphi_k(e_1) = L_{e_1}$ and $\varphi_k(e_2) = d_{e_2}$. With respect to the basis $1, e_1$ for $\wedge(ke_1)$ these endomorphisms are represented by the matrices $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, respectively, and these clearly generate $\mathbb{M}_2(k)$. Thus φ_k is surjective. On the other hand Corollary 3.7 says that, as a module, $Cl(\mathbb{H}(k)) \approx \wedge(k \oplus k^*)$, a free module of rank four (because $\wedge(k \oplus k^*) \approx \wedge(k) \otimes (k^*)$). A surjective homomorphism of free modules of the same finite rank must be an isomorphism, so φ_k is an isomorphism as claimed.

4 The orthogonal group and spinor norm

We assume here that spec (k) is connected. Suppose (P, q) is a quadratic **158**
space (i.e. $\in$ obj $\underline{\underline{\mathrm{Quad}}}\,(k)$) and that $[P : k] = n$. If n is odd then $2 \in U(k)$; otherwise reduce (P, q) modulo a maximal ideal containing $2k$, and we contradict the fact that non-singular forms over fields of char 2 have even dimension. We propose to use the Clifford algebra,

$$A = Cl(P, q) = A_\circ \oplus A_1$$

to study the *orthogonal group*

$$\Omega = \Omega(P, q),$$

i.e. the group of isometries of (P, q).

We take the position from Chapter 4 that everything is graded. Thus

$$\mathrm{Pic}(k) = \mathrm{Pic}\,|k| \oplus \mathbb{Z}/2\mathbb{Z} \tag{4.1}$$

is the group of invertible k-modules. The first summand describes the underlying ungraded module ($|k|$-module) and the $\mathbb{Z}/2\mathbb{Z}$ summand designates the degree (0 or 1) in which it is concentrated.

Write

$$G(A) = Aut_{k-alg}(A),$$

and $hU(A)$ for the homogeneous units of A. Recall that if $u \in hU(A)$ then $\alpha_u \in G(A)$ is defined by $\alpha_u(a) = \begin{cases} uau^{-1} \text{ if } \partial u = 0 \\ ua'v^{-1} \text{ if } \partial u = 1 \end{cases}$. Here, for $a = a_0 + a_1$, $a' = a_0 - a_1$. Thus, for homogeneous a, we can write this as

$$\alpha_u(a) = (-1)^{\partial u \partial a} uau^{-1} (u \in hU(A), a \in hA).$$

159 According to Theorem 3.9 A is azumaya k-algebra. Therefore Theorem 5.2 of chapter 4 gives us an exact sequence

$$\begin{gathered} 1 \to U(k) \to U(A_\circ) \to G(A) \to Pic(k) \\ u \to \alpha_u \end{gathered} \tag{4.2}$$

To apply this we first embed Ω in $G(A)$. Indeed, since the Clifford algebra is a functor of (P, q) there is a canonical homomorphism, $\alpha \mapsto C(\alpha)$, of Ω into $G(A)$. If we identify $P \subset A$ (in fact $P \subset A_1$) then $C(\alpha)$ *is the unique algebra automorphism of A such that* $C(\alpha)(x) = \alpha(x)$ for $x \in P$. For example, the automorphism $a \mapsto a'$ described above is just $C(-1_P)$. We will use this monomorphism to identify Ω with a subgroup of $G(A)$. We can characterise it:

$$\Omega = \{\alpha \in G(A) \big| \alpha P \subset P\}.$$

For if $\alpha P \subset P$ then for $x \in P$ we have $q(\alpha x) = (\alpha x)^2 = \alpha(x^2) = \alpha(qx) = qx$, so α induces an isometry, $\alpha' : P \to P$. Evidently then $\alpha = C(\alpha')$.

Next we introduce the *Clifford group*

$$\Gamma = \{u \in hU(A) | \alpha_u \in \Omega\}.$$

and the *special Clifford group*

$$\Gamma_\circ = \Gamma \cap A_\circ = \{u \in U(A_\circ) \big| \alpha_u \in \Omega\}.$$

If $u \in U(k) \subset U(A_\circ)$ then $\alpha_u = 1$, so $U(k) \subset \Gamma_\circ$. Therefore the exact sequence (4.2) induces a sub-exact sequence,

$$\begin{tikzcd} 1 \arrow[r] & U(k) \arrow[r] & U(A_\circ) \arrow[r] & G(A) \arrow[r] & \mathrm{Pic}(k) \\ 1 \arrow[r] & U(k) \arrow[r] \arrow[u, equal] & \Gamma_\circ \arrow[r] \arrow[u, phantom, "\cup"] & \Omega \arrow[r] \arrow[u, phantom, "\cup"] & \mathrm{Pic}(k) \arrow[u, equal] \end{tikzcd} \tag{4.3}$$

160 Now $\mathrm{Pic}(k) = \mathrm{Pic}\,|k| \otimes \mathbb{Z}/2\mathbb{Z}$ (see (4.1)), so we obtain homomorphisms

$$\Omega \to \mathrm{Pic}(k)$$

and

$$\Omega \to \mathbb{Z}/2\mathbb{Z},$$

the second being the first followed by projection on the second factor. We shall write

$$\begin{array}{c} S\,\Omega = \ker(\Omega \to \mathbb{Z}/2\mathbb{Z}) \\ \bigcup \\ VS\,\Omega = \ker(\Omega \to \mathrm{Pic}(k)), \end{array}$$

the *special*, and *very special orthogonal groups* (of (P, q)), respectively. With this notation we can extract from (4.3) an exact sequence

$$1 \to U(k) \to \Gamma_\circ \to vS\,\Omega \to 1. \tag{4.4}$$

If $x \in P$ then $x^2 = qx$ in A, therefore also in A°, so the identity map on P extends to an isomorphism $A \to A^\circ$, or, in other words, an antiautomorphism of A. We shall denote it by $a \mapsto \tilde{a}$. All $\alpha \in \Omega$ commute with this antiautomorphism; just check it on P. For a$\in A$ we will define its *conjugate*, $\bar{a}$, by

$$\bar{a} = \tilde{a}' = \tilde{a'}$$

and write

$$Na = a\bar{a}.$$

The last remark shows that $\alpha(\bar{a}) = \overline{\alpha(a)}$ for $\alpha \varepsilon \Omega$.

Let 161

$$\mathfrak{n} = \{a \in A \,|\, Na \in k\}.$$

If $x \in P$ then $\bar{x} = \tilde{x'} = x' = -x$ so $Nx = -x^2 = -q(x) \in k$, and $P \subset \mathfrak{n}$.

Suppose $a, b \in \mathfrak{n}$. Then $N(ab) = (ab)\overline{ab} = ab\bar{b}\bar{a} = aN(b)\bar{a} = a\bar{a}N(b) = N(a)N(b)$, because $N(b) \in U(k)$.

$$a, b \in \mathfrak{n} \Rightarrow ab \in \mathfrak{n} \text{ and } N(ab) = N(a)N(b).$$

Next suppose $u \in \Gamma$. Then for $x \in P$ we have $\alpha_u(x) = (-1)^{\partial u} u x u^{-1}$, or $u'x = \alpha_u(x)u$. Therefore $\overline{u\alpha_u(x)} = \bar{x}\bar{u}'$, and $\overline{\alpha_u(x)} = \alpha_u(\bar{x})$ with $\bar{x} = -x$. Setting $y = \alpha_u(\bar{x})$ we have $\bar{u}y = \alpha_u^{-1}(y)\bar{u}'$, so, by definition, $\bar{u} \in \Gamma$ and $\alpha_{\bar{u}} = \alpha_u^{-1}$. In particular $\alpha_{u\bar{u}} = \alpha_u \alpha_{\bar{u}} = 1$ so $N(u) = u\bar{u} \in \ker(\Gamma \to G(A)) = U(k)$. Summarizing, we have proved:

If $u \in \Gamma$ then $\bar{u} \in \Gamma$ and $\alpha_{\bar{u}} = \alpha_u^{-1}$.

Moreover $N(u) \in U(k)$ (i.e. $\Gamma \subset \mathfrak{n}$) so

$$u^{-1} = N(u)^{-1}\bar{u}.$$

Thus N defines a homomorphism

$$N : \Gamma \to U(k).$$

We now introduce the groups

$$\mathrm{Pin} = \ker(\Gamma \xrightarrow{N} U(k))$$

162 and

$$\mathrm{Spin} = \ker(\Gamma_\circ \xrightarrow{N} U(k)).$$

If $u \in U(k)$ then $\bar{u} = u$ so $N(u) = u^2$. Therefore if we apply N to the exact sequence (4.4) we obtain a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & {}_2U(k) \arrow[r] \arrow[d] & \mathrm{Spin} \arrow[r] \arrow[d] & VS\Omega' \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & U(k) \arrow[r] \arrow[d] & \Gamma_0 \arrow[r] \arrow[d, "N"] & VS\Omega \arrow[r] \arrow[d, "\sigma"] & 1 \\
1 \arrow[r] & U(k)^2 \arrow[r] \arrow[d] & U(k) \arrow[r] & U(k)/U(k)^2 \arrow[r] & 1 \\
 & 1 & & &
\end{tikzcd} \tag{4.5}$$

Here ${}_2U(k)$ denotes units of order 2 (square roots of 1). $\sigma : VS\Omega \to U(k)/U(k)^2$ is called the *spinor norm*, and its kernel, $VS\Omega'$, the *spinorial kernel*.

So far we have the following subgroups, with indicated successive quotients, of Ω:

$$\begin{array}{l} \Omega \\ \Big| \Big\} \subset \mathrm{Pic}(k) = \mathrm{Pic}\,|k| \oplus \mathbb{Z}/2\mathbb{Z} \\ VS\Omega \\ \Big| \Big\} \xrightarrow{\sigma} U(k)/U(k)^2 \\ VS\Omega' \\ \Big| \Big\} \approx \Gamma_\circ/U(k) \subset U(A_\circ)/U(k). \\ 1 \end{array} \tag{4.6}$$

Of course the bulk of the group is the bottom layer. We shall now investigate this for small values of $n = [p : k]$. **163**

$n = 1$ (so $2 \in U(k)$). $A_\circ = k$, $A_1 = P$, and $\Gamma_\circ = U(k)$. $\Omega = \{\pm 1\}$ in this case.

$n = 2$ $A_1 = P$ so $\Gamma_\circ = U(A_\circ)$. $A_\circ$ is a quadratic extension of k (in the sense of chapter 4, §3,) so $\Gamma_\circ$ is abelian group. If $(P, q) = \mathbb{H}(k)$ then $A_\circ = k \times k$ so $\Gamma_\circ = U(k) \times U(k)$ and $VS\Omega \approx U(k)$.

$n = 3$ (so $2 \in U(k)$). Then $A_1 = P \oplus L_1$, where L_1 is the degree one term of $L = |A|^{|A|}$ the centre of the ungraded algebra A. $A_\circ$ is a "quaternion $|k|$-algebra," i.e. azumaya $|k|$-algebra of rank 4, and $N(a) \in k$ for all $a \in A_\circ$. If $u \in U(A_\circ)$ then. Since $N(u) = u\bar{u} \in U(k)$, we have **164**
$u^{-1} = N(u)^{-1}\bar{u}$. Therefore, for $a \in A$,

$$\begin{aligned} \overline{\alpha_u(a)} = \overline{uau^{-1}} &= \bar{u}^{-1}\bar{a}\,\bar{u} = (N(u)u^{-1})^{-1}\bar{a}(N(u)u^{-1}) \\ &= u\bar{a}u^{-1} = \alpha_u(\bar{a}). \end{aligned}$$

Consequently α_u leaves invariant the eigenspaces of $^-$; these behave nicely because $\bar{\bar{a}} = a$ and $2 \in U(k)$.

Now $\bar{x} = -x$ for $x \in P$. If we localize k then P has an orthogonal basis, e_1, e_2, e_3, and it is easy to see that $L_1 = ke_1e_2e_3$, $\overline{e_1e_2e_3} =$

$(-1)^3 e_3e_2e_1 = (-1)^{3+2+1}e_1e_2e_3$. Therefore, under the action of $^-$, $A_1 = P \oplus L_1$ is the eigenspace decomposition. In summary we have observed that $u \in U(A_\circ) \Rightarrow Nu \in U(k) \Rightarrow \alpha_u$ leaves the eigenspaces of $^-$ invariant $\Rightarrow \alpha_u P \subset p$, i.e. $u \in \Gamma_\circ$. Therefore

$$\Gamma_\circ = U(A_\circ) \text{ and } VS\Omega = U(A_\circ)/U(k).$$

In case $A_\circ = \mathbb{M}_2(k)$ we have $\Gamma_\circ = GL_2(k)$, the norm N is just the determinant, and

$$VS\Omega = PGL_2(k) = GL_2(k)/U(k).$$

$n = 4$. In this case $L = A_\circ^{A_\circ}$ is a quadratic extension of k (in the sense of Chapter 4, §3), and $A_\circ$ is a quaternion L-algebra. The norm N takes values in L. In case $2 \in U(k)$ a calculation like that for the case $n = 3$ (localize k and diagonalize (P, q) first) shows that

$$\Gamma_\circ = \{u \in U(A_\circ) \big| Nu \in U(k)\}.$$

and this is probably in general. In case $A_\circ \mathbb{M}_2(L)$ then $N : U(A_\circ) = GL_2(L) \to U(L)$ is just the determinant. Hence $SL_2(L) \subset \Gamma_0$ and $\Gamma_\circ/SL_2(L) \approx U(k)$, in this case. $VS\Omega = \Gamma_\circ/_{U(k)} \supset SL_2(L)/_2U(k)$,
165 i.e. modulo element of order 2 in the centre, and modulo this subgroup $VS\Omega$ lands in $U(k)/U(k)^2$. Note that if $L = k \times k$ then $SL_2(L) = SL_2(k) \times SL_2(k)$.

Suppose k happens to be a Dedekind ring of arithmetic type in a global field. Then one knows that Pic(k) is finite (finiteness of class number) and $U(k)$ is finitely generated (Dirichlet unit theorem). Hence $VS\Omega = \Gamma_0/U(k)$ is of finite index in Ω. The discussion above shows, therefore, that the finite generation of Ω is equivalent to the finite generation of Γ_0, and that for $n \leq 4$ this is "usually" equivalent to the finite generation of $U(A_0)$. The point is that $U(A_0)$ is often an easily recognized linear group.

One can similarly use this procedure to reduce the study of normal subgroups of Ω to those of $U(A_0)$, at least in many cases, for $n \leq 4$.

Bibliography

[1] Auslander M. and Goldman O. -[1] The Brauer group of a commutative ring, Trans. A.M.S., 97 (1960), pp. 367-409. **166**

[2] Azumaya G. -[1] On maximally central algebras, Nagoys Math. J., Vol. 2 (1951), pp. 119-150.

[3] Bass H. -[K] K-theory and stable algebra I.H.E.S, n^0 22 (1964), pp. 489-544.

-[2] The Morita theorems, mimeographed notes.

[4] Bourbaki N. -[1] Algébre, Chapitre VIII, Hermann, Paris, 1958.

-[2] Algébre, Chapitre IX, Hermann, Paris, 1959.

[5] Freya P. -[1] Abelian categories, Harper and Row, New York, 1964.

[6] Gabriel P. -[1] Des catégories abélinnes, Bull. Soc. Math. France, 90(1962), pp. 323-448.

[7] Heller A. -[1] Some exact sequences in algebraic K-theory, Topology, Vol.3 (1965), pp.389-408.

[8] Maclane S. -[1] Natural associativity and commutativity, Rice University Studies, 49(1963), pp.28-46.

[9] Mitchell B. -[1] The full imbedding theorem, Amer. J. Math., Vol. 86 (1964), pp. 619-637.

[10] Morita K. -[1] Duality for modules and its applications to the theory of rings with minimum condition. Science reports of the Tokyo Kyoiku Daigaku, Vol. 6 Sec A (1958), pp. 83-142.

[11] O'Meara O.T. -[1] Introduction to quadratic forms, Springer-Verlag,1963.

[12] Rosenberg A. and Zelinsky D. -[1] Automorphisms of separable algebras, Pacific J. Math., Vol. 11 (1961), pp. 1109-1117.

[13] Wall C.T.C -[1] Graded Brauer groups, Crelle's J., Vol. 213 (1964), pp. 187-199.

[14] Witt E. -[1] Theorie der quadratischen Formen in beliebigen Körpern, Crelle's J., Vol.176 (1937), pp. 31-44.