

LOWER K- AND L-THEORY

BY

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For Gerda

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Introduction

The algebraic K -groups $K_*(A)$ and the algebraic L -groups $L_*(A)$ are the obstruction groups to the existence and uniqueness of geometric structures in homotopy theory, via Whitehead torsion and the Wall finiteness and surgery obstructions. In the topological applications the ground ring A is the group ring $\mathbb{Z}[\pi]$ of the fundamental group π . For K -theory a geometric structure is a finite CW complex, while for L -theory it is a compact manifold. The lower K - and L -groups are the obstruction groups to imposing such a geometric structure after stabilization by forming a product with the i -fold torus

$$T^i = S^1 \times S^1 \times \dots \times S^1 ,$$

arising algebraically as the codimension i summands of the K - and L -groups of the i -fold Laurent polynomial extension of A

$$A[\pi_1(T^i)] = A[z_1, (z_1)^{-1}, z_2, (z_2)^{-1}, \dots, z_i, (z_i)^{-1}] .$$

The object of this text is to provide a unified algebraic framework for lower K - and L -theory using chain complexes, leading to new computations in algebra and to further applications in topology.

The ‘fundamental theorem of algebraic K -theory’ of Bass [7] relates the torsion group K_1 of the Laurent polynomial extension $A[z, z^{-1}]$ of a ring A to the projective class group K_0 of A by a naturally split exact sequence

$$\begin{aligned} 0 \longrightarrow K_1(A) &\longrightarrow K_1(A[z]) \oplus K_1(A[z^{-1}]) \\ &\longrightarrow K_1(A[z, z^{-1}]) \longrightarrow K_0(A) \longrightarrow 0 . \end{aligned}$$

The lower K -groups $K_{-i}(A)$ of [7] were defined inductively for $i \geq 1$ to fit into natural split exact sequences

$$\begin{aligned} 0 \longrightarrow K_{-i+1}(A) &\longrightarrow K_{-i+1}(A[z]) \oplus K_{-i+1}(A[z^{-1}]) \\ &\longrightarrow K_{-i+1}(A[z, z^{-1}]) \longrightarrow K_{-i}(A) \longrightarrow 0 , \end{aligned}$$

generalizing the case $i = 0$.

The quadratic L -groups of polynomial extensions were first studied by Wall [84], Shaneson [72], Novikov [48] and Ranicki [57]. The free quadratic L -groups L_*^h of the Laurent polynomial extension $A[z, z^{-1}]$ ($\bar{z} = z^{-1}$) of a ring with involution A were related in [57] to the projective quadratic L -groups L_*^p of A by natural direct sum decompositions

$$L_n^h(A[z, z^{-1}]) = L_n^h(A) \oplus L_{n-1}^p(A) .$$

The lower quadratic L -groups $L_*^{\langle -i \rangle}(A)$ of [57] were defined inductively for $i \geq 1$ to fit into natural direct sum decompositions

$$L_n^{\langle -i+1 \rangle}(A[z, z^{-1}]) = L_n^{\langle -i+1 \rangle}(A) \oplus L_{n-1}^{\langle -i \rangle}(A)$$

with $L_*^{(0)} \equiv L_*^p$, generalizing the case $i = 0$ with $L_*^{(1)} \equiv L_*^h$.

An algebraic theory unifying the torsion of Whitehead [88], the finiteness obstruction of Wall [83] and the surgery obstruction of Wall [84] was developed in Ranicki [60]–[69] using chain complexes in any additive category \mathbb{A} . This approach is used here in the K - and L -theory of polynomial extensions and the lower K - and L -groups. Chain complexes offer the usual advantage of a direct passage from topology to algebra, avoiding preliminary surgery below the middle dimension. A particular feature of the exposition is the insistence on relating the geometric transversality properties of manifolds to the algebraic transversality properties of chain complexes.

The computation $Wh(\mathbb{Z}^i) = 0$ ($i \geq 1$) of Bass, Heller and Swan [8] gives the lower K - and L -groups of \mathbb{Z} , which are used (more or less explicitly) in Novikov's proof of the topological invariance of the rational Pontrjagin classes, the work of Kirby and Siebenmann on high-dimensional topological manifolds, Chapman's proof of the topological invariance of Whitehead torsion and West's proof that compact $ANRs$ have the homotopy type of finite CW complexes. A systematic treatment of homeomorphisms of compact manifolds requires the study of non-compact manifolds, using the controlled algebraic topology of spaces initiated by Chapman, Ferry and Quinn. In this theory topological spaces are equipped with maps to a metric space X , and the notions of maps, homotopy, cell exchange, surgery etc. are required to be small when measured in X . The original simple homotopy theory of Whitehead detects if a PL map is close to being a PL homeomorphism. The controlled simple homotopy theory detects if a continuous map is close to being a homeomorphism, by considering the size of the point inverses.

After an initial lull, the lower K - and L -groups have found many applications in the controlled and bounded topology of non-compact manifolds, stratified spaces and group actions on manifolds. The following alphabetic list of references is representative: Anderson and Hsiang [3], Anderson and Munkholm [4], Anderson and Pedersen [5], Bryant and Pacheco [13], Carlsson [16], Chapman [18], Farrell and Jones [25], Ferry and Pedersen [28], Hambleton and Madsen [30], Hambleton and Pedersen [31], Hughes [35], Hughes and Ranicki [36], Lashof and Rothenberg [42], Madsen and Rothenberg [45], Pedersen [51], Pedersen and Weibel [53], [54], Quinn [56], Ranicki and Yamasaki [70], [71], Siebenmann [75], Svensson [80], Vogell [81], Weinberger [86], Weiss and Williams [87], Yamasaki [89].

Karoubi [38] and Farrell and Wagoner [26] were motivated by the

proof of Bott periodicity using operators on Hilbert space and by the simple homotopy theory of infinite complexes (respectively) to describe the lower K -groups of a ring A as the ordinary K -groups of rings of infinite matrices which are locally finite

$$K_{-i}(A) = K_0(S^i A) = K_1(S^{i+1} A) \quad (i \geq 0) ,$$

with $S^i A$ the i -fold suspension ring: the suspension SA is the ring defined by the quotient of the ring of locally finite countable matrices with entries in A by the ideal of globally finite matrices. Gersten [29] used the ring suspension to define a non-connective spectrum $\mathbb{K}(A)$ with homotopy groups

$$\pi_i(\mathbb{K}(A)) = K_i(A) \quad (i \in \mathbb{Z}) .$$

The applications of the lower K - and L -groups to manifolds generalize the end invariant of Siebenmann [73], which interprets the Wall finiteness obstruction $[W] \in \tilde{K}_0(\mathbb{Z}[\pi])$ of an open n -dimensional manifold W with one tame end as the obstruction to closing the end, assuming that $n \geq 6$ and that $\pi = \pi_1(W)$ is also the fundamental group of the end. The following conditions on W are equivalent :

- (i) $[W] = 0 \in \tilde{K}_0(\mathbb{Z}[\pi])$,
- (ii) W is homotopy equivalent to a finite CW complex,
- (iii) W is homeomorphic to the interior of a closed n -dimensional manifold M ,
- (iv) the cellular chain complex $C(\widetilde{W})$ of the universal cover \widetilde{W} of W is chain equivalent to a finite f.g. free $\mathbb{Z}[\pi]$ -module chain complex.

The product $W \times S^1$ has end invariant

$$[W \times S^1] = 0 \in \tilde{K}_0(\mathbb{Z}[\pi \times \mathbb{Z}]) ,$$

so that $W \times S^1$ is homeomorphic to the interior of a closed $(n+1)$ -dimensional manifold N . However, if $[W] \neq 0 \in \tilde{K}_0(\mathbb{Z}[\pi])$ then N is not of the form $M \times S^1$ for a closed n -dimensional manifold M with interior homeomorphic to W .

Motivated by controlled and the closely related bounded topology, Pedersen [49], [50] expressed the lower K -group $K_{-i}(A)$ ($i \geq 0$) of a ring A both as the class group of the idempotent completion $\mathbb{P}_i(A)$ of the additive category $\mathbb{C}_i(A)$ of \mathbb{Z}^i -graded A -modules which are f.g. free in each grading, with bounded morphisms, and as the torsion group of $\mathbb{C}_{i+1}(A)$

$$K_{-i}(A) = K_0(\mathbb{P}_i(A)) = K_1(\mathbb{C}_{i+1}(A)) \quad (i \geq 0) .$$

These K -theory identifications are obtained here by direct chain complex constructions, and extended to corresponding identifications of the

lower L -groups of a ring with involution A as the L -groups of additive categories with involution

$$L_n^{(-i)}(A) = L_{n+i}(\mathbb{P}_i(A)) = L_{n+i+1}(\mathbb{C}_{i+1}(A)) \quad (i \geq 0) .$$

The method can also be used to express the lower L -groups as the L -groups of multiple suspensions

$$L_n^{(-i)}(A) = L_{n+i}^p(S^i A) = L_{n+i+1}^h(S^{i+1} A) \quad (i \geq 0) .$$

For $i = 0$ this is an unpublished result of Farrell and Wagoner (cf. Wall [84, p.251]).

The *open cone* of a subspace $X \subseteq S^k$ is the metric space

$$O(X) = \{tx \in \mathbb{R}^{k+1} \mid t \in [0, \infty), x \in X\} .$$

Open cones are especially important in the topological applications of bounded K - and L -theory, because (roughly speaking) the controlled algebraic topology of X is the bounded algebraic topology of $O(X)$.

Given a filtered additive category \mathbb{A} and a metric space X let $\mathbb{C}_X(\mathbb{A})$ be the filtered additive category of X -graded objects in \mathbb{A} and bounded morphisms defined by Pedersen and Weibel [53], and let $\mathbb{P}_X(\mathbb{A})$ be the idempotent completion of $\mathbb{C}_X(\mathbb{A})$. Let $\mathbb{P}_0(\mathbb{A})$ denote the idempotent completion of \mathbb{A} itself, and let $\mathbb{K}(\mathbb{A})$ be the non-connective algebraic K -theory spectrum of \mathbb{A} , with homotopy groups

$$\begin{aligned} \pi_i(\mathbb{K}(\mathbb{A})) &= K_i(\mathbb{P}_0(\mathbb{A})) \quad (i \in \mathbb{Z}) \\ &= K_i(\mathbb{A}) \quad (i \neq 0) . \end{aligned}$$

The main result of Pedersen and Weibel [54] shows that the algebraic K -theory assembly map

$$H_*^{lf}(X; \mathbb{K}(\mathbb{A})) \longrightarrow K_*(\mathbb{P}_X(\mathbb{A}))$$

is an isomorphism for $X = O(Y)$ an open cone on a compact polyhedron $Y \subseteq S^k$, so that

$$K_*(\mathbb{P}_{O(Y)}(\mathbb{A})) = H_*^{lf}(O(Y); \mathbb{K}(\mathbb{A})) = \tilde{H}_{*-1}(Y; \mathbb{K}(\mathbb{A})) .$$

In particular, the algebraic K -groups $K_*(\mathbb{P}_{O(Y)}(\mathbb{A}))$ of the open cone of a union $Y = Y^+ \cup Y^- \subseteq S^k$ of compact polyhedra fit into a Mayer-Vietoris exact sequence

$$\begin{aligned} \dots &\longrightarrow K_i(\mathbb{P}_{O(Y^+ \cap Y^-)}(\mathbb{A})) \longrightarrow K_i(\mathbb{P}_{O(Y^+)}(\mathbb{A})) \oplus K_i(\mathbb{P}_{O(Y^-)}(\mathbb{A})) \\ &\longrightarrow K_i(\mathbb{P}_{O(Y)}(\mathbb{A})) \xrightarrow{\partial} K_{i-1}(\mathbb{P}_{O(Y^+ \cap Y^-)}(\mathbb{A})) \\ &\longrightarrow K_{i-1}(\mathbb{P}_{O(Y^+)}(\mathbb{A})) \oplus K_{i-1}(\mathbb{P}_{O(Y^-)}(\mathbb{A})) \\ &\longrightarrow K_{i-1}(\mathbb{P}_{O(Y)}(\mathbb{A})) \longrightarrow \dots . \end{aligned}$$

Carlsson [16] extended the methods of [54] to metric spaces other than open cones, obtaining a Mayer-Vietoris exact sequence for the algebraic

K -groups $K_*(\mathbb{P}_X(\mathbb{A}))$ of a union $X = X^+ \cup X^-$ of arbitrary metric spaces

$$\begin{aligned} \dots &\longrightarrow \varinjlim_b K_i(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \longrightarrow K_i(\mathbb{P}_{X^+}(\mathbb{A})) \oplus K_i(\mathbb{P}_{X^-}(\mathbb{A})) \\ &\longrightarrow K_i(\mathbb{P}_X(\mathbb{A})) \xrightarrow{\partial} \varinjlim_b K_{i-1}(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \\ &\longrightarrow K_{i-1}(\mathbb{P}_{X^+}(\mathbb{A})) \oplus K_{i-1}(\mathbb{P}_{X^-}(\mathbb{A})) \longrightarrow K_{i-1}(\mathbb{P}_X(\mathbb{A})) \longrightarrow \dots, \end{aligned}$$

with

$$\begin{aligned} \mathcal{N}_b(X^+, X^-, X) &= \mathcal{N}_b(X^+, X) \cap \mathcal{N}_b(X^-, X) \\ &= \{x \in X \mid d(x, x^+) \leq b \text{ for some } x^+ \in X^+, x^- \in X^-\} \end{aligned}$$

the intersection of the b -neighbourhoods of X^+ and X^- in X . This exact sequence will be obtained in §4 for $i = 1$ using elementary chain complex methods, with the connecting map

$$\partial : K_1(\mathbb{C}_X(\mathbb{A})) \longrightarrow \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}))$$

defined by sending the torsion $\tau(E)$ of a contractible finite chain complex E in $\mathbb{C}_X(\mathbb{A})$ to the projective class $[E^+]$ of a $\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ -finitely dominated subcomplex $E^+ \subseteq E$, corresponding to the end obstruction of Siebenmann [73], [75] of the part of E lying over a b -neighbourhood of X^+ in X . In the special case

$$X = X^+ \cup X^- = \mathbb{R}, \quad X^+ = [0, \infty), \quad X^- = (-\infty, 0]$$

the algebraic K -groups are such that

$$K_*(\mathbb{C}_{\mathbb{R}^\pm}(\mathbb{A})) = K_*(\mathbb{P}_{\mathbb{R}^\pm}(\mathbb{A})) = 0, \quad K_*(\mathbb{C}_{\mathbb{R}}(\mathbb{A})) = K_*(\mathbb{C}_1(\mathbb{A})).$$

The connecting map in this case is the isomorphism of Pedersen and Weibel [53]

$$\partial : K_1(\mathbb{C}_1(\mathbb{A})) \cong K_0(\mathbb{P}_0(\mathbb{A})).$$

The algebraic properties of modules and quadratic forms over a Laurent polynomial extension ring $A[z, z^{-1}]$ are best studied using algebraic transversality techniques which mimic the geometric transversality technique for the construction of fundamental domains of infinite cyclic covers of compact manifolds. The linearization trick of Higman [34] was the first such algebraic transversality result, leading to the method of Mayer-Vietoris presentations developed by Waldhausen [80]. In §10 this method is used to obtain a split exact sequence

$$\begin{aligned} 0 &\longrightarrow K_1(\mathbb{A}) \longrightarrow K_1(\mathbb{A}[z]) \oplus K_1(\mathbb{A}[z^{-1}]) \\ &\longrightarrow K_1(\mathbb{A}[z, z^{-1}]) \longrightarrow K_0(\mathbb{P}_0(\mathbb{A})) \longrightarrow 0 \end{aligned}$$

for any filtered additive category \mathbb{A} . The split projection $K_1(\mathbb{A}[z, z^{-1}])$

$\longrightarrow K_0(\mathbb{P}_0(\mathbb{A}))$ is induced by the embedding of $\mathbb{A}[z, z^{-1}]$ in $\mathbb{C}_1(\mathbb{A})$ as a subcategory with homogeneously \mathbb{Z} -graded objects. Let $\mathbb{P}_i(\mathbb{A})$ denote the idempotent completion $\mathbb{P}_0(\mathbb{C}_i(\mathbb{A}))$ of the bounded \mathbb{Z}^i -graded category $\mathbb{C}_i(\mathbb{A})$. The above sequence is used in §11 to recover the expression due to Pedersen and Weibel [53] of the lower K -groups of \mathbb{A} as

$$K_{-i}(\mathbb{A}) = K_0(\mathbb{P}_i(\mathbb{A})) = K_1(\mathbb{C}_{i+1}(\mathbb{A})) \quad (i \geq 1),$$

using polynomial extensions (in the spirit of Bass [7]) instead of the delooping machinery.

The quadratic L -groups $L_*(\mathbb{A})$ are defined in Ranicki [68] for any additive category \mathbb{A} with an involution, as the cobordism groups of quadratic Poincaré complexes in \mathbb{A} . The intermediate quadratic L -groups $L_*^J(\mathbb{A})$ are defined for a $*$ -invariant subgroup $J \subseteq K_0(\mathbb{A})$ to be the cobordism groups of quadratic Poincaré complexes in \mathbb{A} with the projective class required to belong to J . The algebraic transversality method is applied in §14 to obtain a Mayer-Vietoris exact sequence for the quadratic L -groups $L_*(\mathbb{C}_X(\mathbb{A}))$ of a union $X = X^+ \cup X^-$

$$\begin{aligned} \dots &\longrightarrow \varinjlim_b L_n^{J_b}(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \longrightarrow L_n(\mathbb{C}_{X^+}(\mathbb{A})) \oplus L_n(\mathbb{C}_{X^-}(\mathbb{A})) \\ &\longrightarrow L_n(\mathbb{C}_X(\mathbb{A})) \xrightarrow{\partial} \varinjlim_b L_{n-1}^{J_b}(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \longrightarrow \dots, \end{aligned}$$

with

$$J_b = \ker(K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_{X^+}(\mathbb{A})) \oplus K_0(\mathbb{P}_{X^-}(\mathbb{A}))).$$

In particular, the quadratic L -groups $L_*(\mathbb{C}_{O(Y)}(\mathbb{A}))$ of the open cone of a union $Y = Y^+ \cup Y^- \subseteq S^k$ fit into a Mayer-Vietoris exact sequence

$$\begin{aligned} \dots &\longrightarrow L_n^J(\mathbb{P}_{O(Y^+ \cap Y^-)}(\mathbb{A})) \longrightarrow L_n(\mathbb{C}_{O(Y^+)}(\mathbb{A})) \oplus L_n(\mathbb{C}_{O(Y^-)}(\mathbb{A})) \\ &\longrightarrow L_n(\mathbb{C}_{O(Y)}(\mathbb{A})) \xrightarrow{\partial} L_{n-1}^J(\mathbb{P}_{O(Y^+ \cap Y^-)}(\mathbb{A})) \longrightarrow \dots, \end{aligned}$$

with

$$J = \ker(K_0(\mathbb{P}_{O(Y^+ \cap Y^-)}(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_{O(Y^+)}(\mathbb{A})) \oplus K_0(\mathbb{P}_{O(Y^-)}(\mathbb{A}))).$$

In the important special case

$$Y^\pm = \{\pm 1\}, \quad Y = Y^+ \cup Y^- = S^0$$

the open cones are

$$O(Y^\pm) = \mathbb{R}^\pm, \quad O(Y) = \mathbb{R}$$

and

$$L_*(\mathbb{C}_{\mathbb{R}^\pm}(\mathbb{A})) = 0, \quad L_*(\mathbb{C}_{\mathbb{R}}(\mathbb{A})) = L_*(\mathbb{C}_1(\mathbb{A})), \quad J = K_0(\mathbb{P}_0(\mathbb{A})),$$

so that the connecting maps define isomorphisms

$$\partial : L_*(\mathbb{C}_1(\mathbb{A})) \cong L_{*-1}(\mathbb{P}_0(\mathbb{A})).$$

In §17 the lower L -groups of a filtered additive category with involution \mathbb{A} are defined inductively by

$$L_*^{(0)}(\mathbb{A}) = L_*(\mathbb{P}_0(\mathbb{A})) ,$$

$$L_*^{\langle -i \rangle}(\mathbb{A}[z, z^{-1}]) = L_*^{\langle -i \rangle}(\mathbb{A}) \oplus L_{*-1}^{\langle -i-1 \rangle}(\mathbb{A}) \quad (i \geq 1) .$$

The isomorphisms ∂ are used to obtain the expressions

$$L_n^{\langle -i \rangle}(\mathbb{A}) = L_{n+i}(\mathbb{P}_i(\mathbb{A})) = L_{n+i+1}(\mathbb{C}_{i+1}(\mathbb{A})) \quad (n \geq 0, i \geq 1) .$$

The lower L -groups of a ring with involution A are the lower L -groups of the additive category with involution of based f.g. free A -modules. The ultimate lower quadratic L -groups of \mathbb{A} are defined by

$$L_n^{\langle -\infty \rangle}(\mathbb{A}) = \varinjlim_i L_n^{\langle -i \rangle}(\mathbb{A}) \quad (n \in \mathbb{Z}) ,$$

and as in Ranicki [69] there is defined an algebraic L -theory spectrum $\mathbb{L}^{\langle -\infty \rangle}(\mathbb{A})$ with homotopy groups

$$\pi_*(\mathbb{L}^{\langle -\infty \rangle}(\mathbb{A})) = L_*^{\langle -\infty \rangle}(\mathbb{A}) .$$

The Mayer-Vietoris exact sequence of §14 shows that the algebraic L -theory assembly map of [69, Appendix C]

$$H_*^{lf}(X; \mathbb{L}^{\langle -\infty \rangle}(\mathbb{A})) \longrightarrow L_*^{\langle -\infty \rangle}(\mathbb{C}_X(\mathbb{A}))$$

is an isomorphism for $X = O(Y)$ the open cone of a compact polyhedron $Y \subseteq S^k$, so that

$$\begin{aligned} L_*^{\langle -\infty \rangle}(\mathbb{C}_{O(Y)}(\mathbb{A})) &= H_*^{lf}(O(Y); \mathbb{L}^{\langle -\infty \rangle}(\mathbb{A})) \\ &= \tilde{H}_{*-1}(Y; \mathbb{L}^{\langle -\infty \rangle}(\mathbb{A})) . \end{aligned}$$

In §20 the chain complex methods are used to provide an abstract treatment of the obstruction theory of Farrell [21], [22] and Siebenmann [76] for fibering a manifold over the circle S^1 .

In Ranicki and Yamasaki [70] the lower K -theory algebra is applied to the controlled topology of Chapman-Ferry-Quinn, obtaining systematic proofs of the results of Chapman and West on the topological invariance of Whitehead torsion and the homotopy finiteness of compact $ANRs$. The bounded surgery theory of Ferry and Pedersen [28] is the topological context for which the lower L -theory algebra presented here is most directly suited. However, in Ranicki and Yamasaki [71] the algebra will be applied to the controlled surgery theory of Quinn [56] and Yamasaki [89].

An earlier version of this text was issued as Heft 25 (1990) of the *Mathematica Gottingensis*. I should like to take this opportunity of thanking the Sonderforschungsbereich ‘Geometrie und Analysis’ in Göttingen for its manifold hospitality on various occasions since its inception in 1984.

§1. Projective class and torsion

This section is a brief recollection from Ranicki [64], [65] of the algebraic theory of finiteness obstruction and torsion in an additive category \mathbb{A} .

Give \mathbb{A} the split exact structure: a sequence in \mathbb{A}

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

is *exact* if there exists a morphism $k : C \longrightarrow B$ such that

- (i) $jk = 1 : C \longrightarrow C$,
- (ii) $(i k) : A \oplus C \longrightarrow B$ is an isomorphism in \mathbb{A} .

The *class group* $K_0(\mathbb{A})$ is the abelian group with one generator $[A]$ for each object A in \mathbb{A} , subject to the relations

- (i) $[A] = [A']$ if A is isomorphic to A' ,
- (ii) $[A \oplus B] = [A] + [B]$ for any objects A, B in \mathbb{A} .

A chain complex C in \mathbb{A} is *n-dimensional* if $C_r = 0$ for $r < 0$ and $r > n$

$$C : \dots \longrightarrow 0 \longrightarrow C_n \xrightarrow{d} C_{n-1} \longrightarrow \dots \longrightarrow C_1 \xrightarrow{d} C_0 .$$

A chain complex C in \mathbb{A} is *finite* if it is *n-dimensional* for some $n \geq 0$.

The *class* of a finite chain complex C in \mathbb{A} is the chain homotopy invariant defined by

$$[C] = \sum_{r=0}^{\infty} (-)^r [C_r] \in K_0(\mathbb{A}) .$$

The *k-fold suspension* of a chain complex C is the chain complex $S^k C$ defined for any $k \in \mathbb{Z}$ by a dimension shift $-k$

$$d_{S^k C} = d_C : (S^k C)_r = C_{r-k} \longrightarrow (S^k C)_{r-1} = C_{r-k-1} .$$

If C is *n-dimensional* and $n+k \geq 0$ the *k-fold suspension* $S^k C$ is $(n+k)$ -dimensional, with class

$$[S^k C] = (-)^k [C] \in K_0(\mathbb{A}) .$$

The *idempotent completion* $\mathbb{P}_0(\mathbb{A})$ of \mathbb{A} is the additive category with objects (A, p) defined by the objects A of \mathbb{A} together with a projection $p = p^2 : A \longrightarrow A$. A morphism $f : (A, p) \longrightarrow (B, q)$ in $\mathbb{P}_0(\mathbb{A})$ is a morphism $f : A \longrightarrow B$ in \mathbb{A} such that $qfp = f : A \longrightarrow B$. The full embedding

$$\mathbb{A} \longrightarrow \mathbb{P}_0(\mathbb{A}) ; A \longrightarrow (A, 1)$$

will be used to identify \mathbb{A} with a subcategory of $\mathbb{P}_0(\mathbb{A})$. The *reduced class group* of $\mathbb{P}_0(\mathbb{A})$ is defined by

$$\widetilde{K}_0(\mathbb{P}_0(\mathbb{A})) = \text{coker}(K_0(\mathbb{A}) \longrightarrow K_0(\mathbb{P}_0(\mathbb{A}))) .$$

Given a ring A let $\mathbb{B}^f(A)$ be the additive category of based f.g. free A -modules. The idempotent completion $\mathbb{P}_0(\mathbb{B}^f(A))$ is isomorphic to the additive category $\mathbb{P}(A)$ of f.g. projective A -modules, and

$$\begin{aligned} K_0(\mathbb{P}_0(\mathbb{B}^f(A))) &= K_0(\mathbb{P}(A)) = K_0(A) , \\ \tilde{K}_0(\mathbb{P}_0(\mathbb{B}^f(A))) &= \tilde{K}_0(A) . \end{aligned}$$

Let $(\mathbb{B}, \mathbb{A} \subseteq \mathbb{B})$ be a pair of additive categories, with \mathbb{A} full in \mathbb{B} . A chain complex in \mathbb{B} is *homotopy \mathbb{A} -finite* if it is chain equivalent to a finite chain complex in \mathbb{A} . An *\mathbb{A} -finite domination* (D, f, g, h) of a chain complex C in \mathbb{B} is a finite chain complex D in \mathbb{A} together with chain maps $f : C \rightarrow D$, $g : D \rightarrow C$ and a chain homotopy $h : gf \simeq 1 : C \rightarrow C$. The *projective class* of an \mathbb{A} -finitely dominated chain complex C in \mathbb{B} is the class of any finite chain complex (D, p) in $\mathbb{P}_0(\mathbb{A})$ which is chain equivalent to $(C, 1)$ in $\mathbb{P}_0(\mathbb{B})$

$$[C] = [D, p] \in K_0(\mathbb{P}_0(\mathbb{A})) .$$

See Ranicki [64] for an explicit construction of such a (D, p) from an \mathbb{A} -finite domination of C . The reduced projective class is such that $[C] = 0 \in \tilde{K}_0(\mathbb{P}_0(\mathbb{A}))$ if and only if C is homotopy \mathbb{A} -finite.

A *chain homotopy projection* (D, p) is a chain complex D together with a chain map $p : D \rightarrow D$ such that there exists a chain homotopy

$$p \simeq p^2 : D \rightarrow D .$$

A *splitting* (C, f, g) of (D, p) is a chain complex C together with chain maps $f : C \rightarrow D$, $g : D \rightarrow C$ such that $gf \simeq 1 : C \rightarrow C$, $fg \simeq p : D \rightarrow D$. The *projective class* of a chain homotopy projection (D, p) with D a finite chain complex in \mathbb{A} was defined in Lück and Ranicki [44] by

$$[D, p] = [C] \in K_0(\mathbb{P}_0(\mathbb{A}))$$

for any splitting (C, f, g) of (D, p) in $\mathbb{P}_0(\mathbb{A})$. See [44] for an explicit construction of an object (D_ω, p_ω) in $\mathbb{P}_0(\mathbb{A})$ such that

$$[D, p] = [D_\omega, p_\omega] - [D_{\text{odd}}] \in K_0(\mathbb{P}_0(\mathbb{A})) ,$$

with

$$D_{\text{even}} = D_0 \oplus D_2 \oplus D_4 \oplus \dots ,$$

$$D_{\text{odd}} = D_1 \oplus D_3 \oplus D_5 \oplus \dots ,$$

$$D_\omega = D_{\text{even}} \oplus D_{\text{odd}} = D_0 \oplus D_1 \oplus D_2 \oplus \dots .$$

The reduced projective class is such that $[D, p] = 0 \in \tilde{K}_0(\mathbb{P}_0(\mathbb{A}))$ if and only if (D, p) has a splitting (C, f, g) with C a finite chain complex in \mathbb{A} .

A finite chain complex C in \mathbb{A} is *round* if

$$[C] = 0 \in K_0(\mathbb{A}) .$$

The projective class of an \mathbb{A} -finitely dominated chain complex C is such that $[C] = 0 \in K_0(\mathbb{P}_0(\mathbb{A}))$ if and only if C is chain equivalent to a round finite chain complex in \mathbb{A} . The reduced projective class is such that $[C] = 0 \in \tilde{K}_0(\mathbb{P}_0(\mathbb{A}))$ if and only if C is homotopy \mathbb{A} -finite.

PROPOSITION 1.1 *If in an exact sequence of chain complexes in \mathbb{B}*

$$0 \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow 0$$

any two of C, D, E are \mathbb{A} -finitely dominated then so is the third, and the projective classes are related by the sum formula

$$[C] - [D] + [E] = 0 \in K_0(\mathbb{P}_0(\mathbb{A})) .$$

□

The *torsion group* of an additive category \mathbb{A} is the abelian group $K_1(\mathbb{A})$ with one generator $\tau(f)$ for each automorphism $f : M \longrightarrow M$, subject to the relations

- (i) $\tau(gf : M \longrightarrow M \longrightarrow M) = \tau(f : M \longrightarrow M) + \tau(g : M \longrightarrow M) ,$
- (ii) $\tau(i^{-1}fi : L \longrightarrow M \longrightarrow M \longrightarrow L) = \tau(f : M \longrightarrow M) ,$
- (iii) $\tau(f \oplus f' : M \oplus M' \longrightarrow M \oplus M')$
 $= \tau(f : M \longrightarrow M) + \tau(f' : M' \longrightarrow M') .$

A *stable isomorphism* $[f] : L \longrightarrow M$ between objects in \mathbb{A} is an equivalence class of isomorphisms $f : L \oplus X \longrightarrow M \oplus X$ in \mathbb{A} , under the equivalence relation

$$(f : L \oplus X \longrightarrow M \oplus X) \sim (f' : L \oplus X' \longrightarrow M \oplus X')$$

if the automorphism

$$\alpha = (f'^{-1} \oplus 1)(f \oplus 1) : L \oplus X \oplus X' \longrightarrow L \oplus X \oplus X'$$

has torsion $\tau(\alpha) = 0 \in K_1(\mathbb{A})$.

The composite of stable isomorphisms

$$[f] : L \longrightarrow M , [g] : M \longrightarrow N$$

represented by

$$f : L \oplus X \longrightarrow M \oplus X , g : M \oplus Y \longrightarrow N \oplus Y$$

is the stable isomorphism $[gf] : L \longrightarrow N$ represented by the composite

$$(g \oplus 1)(f \oplus 1) : L \oplus X \oplus Y \longrightarrow N \oplus X \oplus Y .$$

A stable automorphism $[f] : L \longrightarrow L$ has a well-defined torsion

$$\tau([f]) = \tau(f : L \oplus X \longrightarrow M \oplus X) \in K_1(\mathbb{A}) ,$$

with

$$\tau([g][f]) = \tau([f]) + \tau([g]) , \quad \tau([f \oplus f']) = \tau([f]) + \tau([f']) \in K_1(\mathbb{A}) .$$

The stable automorphism group of any object M in \mathbb{A} is isomorphic to $K_1(\mathbb{A})$.

An additive functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ of additive categories induces morphisms of algebraic K -groups

$$F_0 : K_0(\mathbb{A}) \longrightarrow K_0(\mathbb{B}) ; [M] \longrightarrow [F(M)] ,$$

$$F_1 : K_1(\mathbb{A}) \longrightarrow K_1(\mathbb{B}) ;$$

$$\tau(g : M \longrightarrow M) \longrightarrow \tau(F(g) : F(M) \longrightarrow F(M)) .$$

The *relative K_1 -group* $K_1(F)$ is the abelian group of equivalence classes $[M, N, g]$ of triples (M, N, g) consisting of objects M, N in \mathbb{A} and a stable isomorphism $[g] : F(M) \longrightarrow F(N)$ in \mathbb{B} , subject to the equivalence relation

$$(M, N, g) \sim (M', N', g') \text{ if there exists a stable isomorphism}$$

$$[h] : M \oplus N' \longrightarrow M' \oplus N \text{ in } \mathbb{A} \text{ such that}$$

$$\tau([g'] [F(h)] [g]^{-1} : F(M \oplus M') \longrightarrow F(M \oplus M')) = 0 \in K_1(\mathbb{B}) ,$$

with addition by

$$[M_1, N_1, g_1] + [M_2, N_2, g_2] = [M_1 \oplus M_2, N_1 \oplus N_2, g_1 \oplus g_2] \in K_1(F) .$$

The relative K_1 -group fits into an exact sequence

$$K_1(\mathbb{A}) \xrightarrow{F_1} K_1(\mathbb{B}) \longrightarrow K_1(F) \xrightarrow{\partial} K_0(\mathbb{A}) \xrightarrow{F_0} K_0(\mathbb{B})$$

with

$$K_1(\mathbb{B}) \longrightarrow K_1(F) ; \tau(g : M \longrightarrow M) \longrightarrow [0, 0, g] ,$$

$$\partial : K_1(F) \longrightarrow K_0(\mathbb{A}) ; [M, N, g] \longrightarrow [M] - [N] .$$

The full embedding

$$\mathbb{A} \longrightarrow \mathbb{P}_0(\mathbb{A}) ; M \longrightarrow (M, 1)$$

induces an isomorphism of torsion groups

$$K_1(\mathbb{A}) \longrightarrow K_1(\mathbb{P}_0(\mathbb{A})) ; \tau(f : M \longrightarrow M) \longrightarrow \tau(f : (M, 1) \longrightarrow (M, 1))$$

which is used to identify

$$K_1(\mathbb{A}) = K_1(\mathbb{P}_0(\mathbb{A})) .$$

The following additivity property of torsion will be used in §4.

LEMMA 1.4 *Let $f : M \longrightarrow M$ be an automorphism in \mathbb{A} . Given a resolution of M by an exact sequence in \mathbb{A}*

$$0 \longrightarrow K \xrightarrow{i} L \xrightarrow{j} M \longrightarrow 0$$

and a resolution of f by an isomorphism of exact sequences in \mathbb{A}

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \oplus K & \xrightarrow{1_A \oplus i} & A \oplus L & \xrightarrow{0 \oplus j} & M & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow g & & \downarrow f & & \\ 0 & \longrightarrow & B \oplus K & \xrightarrow{1_B \oplus i} & B \oplus L & \xrightarrow{0 \oplus j} & M & \longrightarrow & 0 \end{array}$$

the stable isomorphisms $[g] : A \longrightarrow B$, $[h] : A \longrightarrow B$ are such that

$$\tau(f) = \tau([h]^{-1}[g] : A \longrightarrow A) \in K_1(\mathbb{A}) .$$

PROOF The stable automorphism $[h]^{-1}[g] : A \longrightarrow A$ is represented by the isomorphism

$$(h \oplus 1_L)^{-1}(g \oplus 1_K) : A \oplus K \oplus L \longrightarrow B \oplus K \oplus L \longrightarrow A \oplus K \oplus L .$$

Choosing a splitting morphism $k : M \longrightarrow L$ for $j : L \longrightarrow M$ there is defined an isomorphism

$$s = (i \ k) : K \oplus M \longrightarrow L$$

such that

$$(s \oplus 1_B)^{-1}g(s \oplus 1_A) = \begin{pmatrix} h & e \\ 0 & f \end{pmatrix} : (A \oplus K) \oplus M \longrightarrow (B \oplus K) \oplus M$$

for some morphism $e : M \longrightarrow B \oplus K$. Thus

$$\begin{aligned} (h^{-1} \oplus 1_M)(s \oplus 1_B)^{-1}g(s \oplus 1_A) &= \begin{pmatrix} 1 & h^{-1}e \\ 0 & f \end{pmatrix} \\ &: (A \oplus K) \oplus M \longrightarrow (A \oplus K) \oplus M , \\ (s \oplus 1_{A \oplus K})^{-1}(h \oplus 1_L)^{-1}(g \oplus 1_K)(s \oplus 1_{A \oplus K}) &= \begin{pmatrix} 1 & h^{-1}e & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &: (A \oplus K) \oplus M \oplus K \longrightarrow (A \oplus K) \oplus M \oplus K , \end{aligned}$$

and

$$\begin{aligned} \tau(f) &= \tau((s \oplus 1_{A \oplus K})^{-1}(h \oplus 1_L)^{-1}(g \oplus 1_K)(s \oplus 1_{A \oplus K})) \\ &= \tau([h]^{-1}[g]) \in K_1(\mathbb{A}) . \end{aligned}$$

□

§2. Graded and bounded categories

The bounded X -graded category $\mathbb{C}_X(\mathbb{A})$ of Pedersen and Weibel [53] is defined for a metric space X and a filtered additive category \mathbb{A} , as a subcategory of the following (unbounded) X -graded category.

DEFINITION 2.1 Given a set X and an additive category \mathbb{A} define the X -graded category $\mathbb{G}_X(\mathbb{A})$ to be the additive category in which an object

is a collection $\{M(x) \mid x \in X\}$ of objects in \mathbb{A} indexed by X , written as a direct sum

$$M = \sum_{x \in X} M(x) ,$$

and a morphism

$$f = \{f(y, x)\} : L = \sum_{x \in X} L(x) \longrightarrow M = \sum_{y \in X} M(y)$$

is a collection $\{f(y, x) : L(x) \longrightarrow M(y) \mid x, y \in X\}$ of morphisms in \mathbb{A} such that for each $x \in X$ the set $\{y \in X \mid f(y, x) \neq 0\}$ is finite. It is convenient to regard f as a matrix with one column $\{f(y, x) \mid y \in X\}$ for each $x \in X$ (containing only a finite number of non-zero entries) and one row $\{f(y, x) \mid x \in X\}$ for each $y \in X$. The composite of morphisms $f : L \longrightarrow M, g : M \longrightarrow N$ in $\mathbb{G}_X(\mathbb{A})$ is the morphism $gf : L \longrightarrow N$ defined by

$$(gf)(z, x) = \sum_{y \in X} g(z, y)f(y, x) : L(x) \longrightarrow N(z) ,$$

where the sum is actually finite. □

Note that if X is a finite set the functor

$$\mathbb{G}_X(\mathbb{A}) \longrightarrow \mathbb{A} ; M \longrightarrow \sum_{x \in X} M(x)$$

is an equivalence of additive categories.

EXAMPLE 2.2 Let $\mathbb{A} = \mathbb{B}^f(A)$ be the additive category of based f.g. free A -modules for a ring A . For any set X write

$$\mathbb{G}_X(\mathbb{A}) = \mathbb{G}_X(A) .$$

An object in $\mathbb{G}_X(A)$ is a based free A -module which is a direct sum

$$M = \sum_{x \in X} M(x)$$

of based f.g. free A -modules $M(x)$ ($x \in X$). The morphisms in $\mathbb{G}_X(A)$ are the A -module morphisms. □

An additive category \mathbb{A} is *filtered* if to each morphism $f : L \longrightarrow M$ there is assigned a *filtration degree* $\delta(f)$, a non-negative integer such that

- (i) $\delta(f + g) \leq \max(\delta(f), \delta(g))$ for any $f, g : L \longrightarrow M$,
- (ii) $\delta(gf) \leq \delta(f) + \delta(g)$ for any $f : L \longrightarrow M, g : M \longrightarrow N$,
- (iii) $\delta(-1 : L \longrightarrow L) = \delta((1 \ 0) : L \oplus M \longrightarrow L)$
 $= \delta\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} : L \longrightarrow L \oplus M\right) = 0 ,$

(iv) $\delta(f) = 0$ for any coherence isomorphism f .

The morphisms $f : L \longrightarrow M$ of filtration degree $\leq b$ define a subgroup $F_b(L, M)$ of $\text{Hom}_{\mathbb{A}}(L, M)$ such that

$$F_0(L, M) \subseteq F_1(L, M) \subseteq \dots \subseteq \bigcup_{b=0}^{\infty} F_b(L, M) = \text{Hom}_{\mathbb{A}}(L, M) .$$

DEFINITION 2.3 Let X be a metric space, and let \mathbb{A} be a filtered additive category.

(i) An object M in $\mathbb{G}_X(\mathbb{A})$ is *bounded* if for each $x \in X$ and each number $r > 0$ there is only a finite number of $y \in X$ with $d(x, y) < r$ and $M(y)$ non-zero.

(ii) A morphism $f : L \longrightarrow M$ in $\mathbb{G}_X(\mathbb{A})$ is *bounded* if there exists a number $b \geq 0$ such that

$$(a) \delta(f(y, x)) \leq b \text{ for all } y \in X ,$$

$$(b) f(y, x) = 0 \text{ for } d(x, y) > b .$$

(iii) The *bounded X -graded category* $\mathbb{C}_X(\mathbb{A})$ is the subcategory of $\mathbb{G}_X(\mathbb{A})$ consisting of bounded objects and bounded morphisms. The idempotent completion of $\mathbb{C}_X(\mathbb{A})$ is written

$$\mathbb{P}_X(\mathbb{A}) = \mathbb{P}_0(\mathbb{C}_X(\mathbb{A})) .$$

□

An X -graded CW complex (K, ρ_K) is a finite-dimensional CW complex K together with a continuous map $\rho_K : K \longrightarrow X$. For each r -cell $e = e^r \subset K$ let $\chi_e : D^r \longrightarrow K$ be the characteristic map, and let

$$\rho_e = \rho_K \chi_e : D^r \longrightarrow X , \quad \gamma(e) = \rho_e(0) \in X .$$

The X -graded CW complex (K, ρ_K) is *bounded* if $\rho_K : K \longrightarrow X$ is proper, and the maps

$$\chi_e : (D^r, \rho_e) \longrightarrow (K, \rho_K)$$

are uniformly bounded, i.e. there exists an integer $b \geq 0$ such that

$$d(\rho_e(x), \rho_e(y)) < b \text{ for each cell } e^r \subset K \text{ and all } x, y \in D^r .$$

Given a ring A let $\mathbb{B}^f(A)$ be the additive category of based f.g. free A -modules with the trivial filtered structure $F_0 \text{Hom} = \text{Hom}$, and write

$$\mathbb{C}_X(\mathbb{B}^f(A)) = \mathbb{C}_X(A) .$$

EXAMPLE 2.4 Given an X -graded CW complex (K, ρ_K) and a regular covering \tilde{K} of K with group of covering translations π consider the cellular based free $\mathbb{Z}[\pi]$ -module chain complex $C(\tilde{K})$ as a finite chain complex in the X -graded category $\mathbb{G}_X(\mathbb{Z}[\pi])$ with $C(\tilde{K})_r(x)$ the based

f.g. free $\mathbb{Z}[\pi]$ -module with one base element for each r -cell $e \subseteq K$ such that $\gamma(e) = x$. If (K, ρ_K) is bounded then $C(\tilde{K})$ is defined in $\mathbb{C}_X(\mathbb{Z}[\pi])$. \square

A *proper eventually Lipschitz map* $f : X \longrightarrow Y$ of metric spaces is a function (not necessarily continuous) satisfying

- (i) the inverse image of a bounded set is a bounded set,
- (ii) there exist numbers $r, k > 0$ depending only on f such that for all $s > r$ and all $x, y \in X$ with $d(x, y) < s$ it is the case that $d(f(x), f(y)) < ks$.

Given an object $M = \{M(x) \mid x \in X\}$ in $\mathbb{C}_X(\mathbb{A})$ and a proper eventually Lipschitz map $f : X \longrightarrow Y$ let

$$f_*M = \{f_*M(y) \mid y \in Y\}$$

be the object in $\mathbb{C}_Y(\mathbb{A})$ defined by

$$f_*M(y) = \sum_{x \in f^{-1}(y)} M(x) .$$

If $f : X \longrightarrow Y$ is a homotopy equivalence of metric spaces in the proper eventually Lipschitz category then $f_* : \mathbb{C}_X(\mathbb{A}) \longrightarrow \mathbb{C}_Y(\mathbb{A})$ is an equivalence of filtered additive categories.

A *subobject* $N \subseteq M$ of an object M in $\mathbb{G}_X(\mathbb{A})$ is an object N in $\mathbb{G}_X(\mathbb{A})$ such that for each $x \in X$ either $N(x) = M(x)$ or $N(x) = 0$. Given an object M in $\mathbb{G}_X(\mathbb{A})$ and a subset $Y \subseteq X$ let $M(Y)$ be the subobject of M defined by

$$M(Y) = \sum_{y \in Y} M(y) .$$

The evident projection is denoted by

$$p_Y : M \longrightarrow M(Y) .$$

Given also a subset $Z \subseteq X$ and a morphism $f : L \longrightarrow M$ in $\mathbb{G}_X(\mathbb{A})$ write

$$f(L(Y)) \subseteq M(Z)$$

to signify that

$$f(x, y) = 0 : L(y) \longrightarrow M(x) \text{ for all } x \in X - Z, y \in Y ,$$

or equivalently

$$p_{X-Z} f p_Y = 0 : L \longrightarrow M(X - Z) .$$

Let $(X, Y \subseteq X)$ be a pair of metric spaces. For any integer $b \geq 0$ define the *b-neighbourhood of Y in X* to be

$$\mathcal{N}_b(Y, X) = \{x \in X \mid d(x, y) \leq b \text{ for some } y \in Y\} .$$

The *b-neighbourhoods* are such that

- (i) $Y = \mathcal{N}_0(Y, X) \subseteq \mathcal{N}_1(Y, X) \subseteq \dots \subseteq \bigcup_{b=0}^{\infty} \mathcal{N}_b(Y, X) = X$,
- (ii) the inclusion $Y \longrightarrow \mathcal{N}_b(Y, X)$ is a homotopy equivalence in the proper eventually Lipschitz category, with a homotopy inverse

$$\mathcal{N}_b(Y, X) \longrightarrow Y ; x \longrightarrow y$$

defined by sending $x \in \mathcal{N}_b(Y, X)$ to any $y \in X$ with $d(x, y) \leq b$.

The induced isomorphisms of algebraic K -groups

$$K_*(\mathbb{C}_Y(\mathbb{A})) \cong K_*(\mathbb{C}_{\mathcal{N}_b(Y, X)}(\mathbb{A}))$$

$$K_*(\mathbb{P}_Y(\mathbb{A})) \cong K_*(\mathbb{P}_{\mathcal{N}_b(Y, X)}(\mathbb{A}))$$

will be used as identifications.

DEFINITION 2.5 (i) The *germ away from Y* $[f]$ of a morphism $f : L \longrightarrow M$ in $\mathbb{G}_X(\mathbb{A})$ is the equivalence class of f with respect to equivalence relation \sim defined on the morphisms $L \longrightarrow M$ by

$$f \sim g \text{ if } (f - g)(L(X \setminus \mathcal{N}_b(Y, X))) \subseteq M(\mathcal{N}_c(Y, X)) \text{ for some } b, c \geq 0 .$$

(ii) The (X, Y) -graded category $\mathbb{G}_{X, Y}(\mathbb{A})$ is the additive category with:

- (a) the objects of $\mathbb{G}_{X, Y}(\mathbb{A})$ are the bounded objects $M = \sum_{x \in X} M(x)$ of

$$\mathbb{G}_X(\mathbb{A}),$$

- (b) the morphisms $[f] : L \longrightarrow M$ in $\mathbb{G}_{X, Y}(\mathbb{A})$ are the germs away from Y of morphisms $f : L \longrightarrow M$ in $\mathbb{G}_X(\mathbb{A})$.

□

A morphism $f : L \longrightarrow M$ in $\mathbb{G}_X(\mathbb{A})$ with a factorization

$$f : L \longrightarrow N \longrightarrow M$$

through an object N in $\mathbb{G}_{\mathcal{N}_b(Y, X)}(\mathbb{A})$ has germ

$$[f] = [0] : L \longrightarrow M .$$

DEFINITION 2.6 The *bounded (X, Y) -graded category* $\mathbb{C}_{X, Y}(\mathbb{A})$ is the subcategory of $\mathbb{G}_{X, Y}(\mathbb{A})$ with the same objects, and morphisms the germs $[f]$ away from Y of morphisms $f : L \longrightarrow M$ in $\mathbb{C}_X(\mathbb{A})$.

□

$\mathbb{C}_{X, Y}(\mathbb{A})$ is the category denoted $\mathbb{C}_X^{>Y}(\mathbb{A})$ by Hambleton and Pedersen [31].

The germ $[f]$ away from Y of a morphism $f : L \longrightarrow M$ in $\mathbb{C}_X(\mathbb{A})$ consists of the morphisms $g : L \longrightarrow M$ in $\mathbb{C}_X(\mathbb{A})$ such that

$$(f - g)(L) \subseteq M(\mathcal{N}_b(Y, X))$$

for some $b \geq 0$.

Given a chain complex C in $\mathbb{C}_X(\mathbb{A})$ let $[C]$ be the chain complex

defined in $\mathbb{C}_{X,Y}(\mathbb{A})$ by

$$d_{[C]} = [d_C] : [C]_r = C_r \longrightarrow [C]_{r-1} = C_{r-1} .$$

PROPOSITION 2.7 *For every finite chain complex C in $\mathbb{C}_{X,Y}(\mathbb{A})$ there exists a finite chain complex D in $\mathbb{C}_X(\mathbb{A})$ such that*

$$C = [D] .$$

PROOF Let C be n -dimensional. For any representatives in $\mathbb{C}_X(\mathbb{A})$

$$d_C : C_r \longrightarrow C_{r-1} \quad (1 \leq r \leq n)$$

there exists a sequence $b = (b_0, b_1, \dots, b_n)$ of integers $b_r \geq 0$ such that

$$(i) \quad d_C(C_r(\mathcal{N}_{b_r}(Y, X))) \subseteq C_{r-1}(\mathcal{N}_{b_{r-1}}(Y, X)) ,$$

$$(ii) \quad (d_C)^2(C_r) \subseteq C_{r-2}(\mathcal{N}_{b_{r-2}}(Y, X)) .$$

(Start with $b_n = 0$ and work downwards, $b_{n-1}, b_{n-2}, \dots, b_0$). For any such sequence b let D be the chain complex defined in $\mathbb{C}_X(\mathbb{A})$ by

$$\begin{aligned} d_D &= \begin{pmatrix} 0 & 0 \\ 0 & d_C| \end{pmatrix} : \\ D_r &= C_r = C_r(\mathcal{N}_{b_r}(Y, X)) \oplus C_r(X \setminus \mathcal{N}_{b_r}(Y, X)) \\ \longrightarrow D_{r-1} &= C_{r-1} = C_{r-1}(\mathcal{N}_{b_{r-1}}(Y, X)) \oplus C_{r-1}(X \setminus \mathcal{N}_{b_{r-1}}(Y, X)) . \end{aligned}$$

□

DEFINITION 2.8 (i) The *support* of an object M in $\mathbb{C}_X(\mathbb{A})$ is the subspace

$$\text{supp}(M) = \{x \in X \mid M(x) \neq 0\} \subseteq X .$$

(ii) The *neighbourhood category* of $\mathbb{C}_Y(\mathbb{A})$ in $\mathbb{C}_X(\mathbb{A})$ is the full subcategory

$$\mathbb{N}_Y(\mathbb{A}) = \bigcup_{b \geq 0} \mathbb{C}_{\mathcal{N}_b(Y, X)}(\mathbb{A}) \subseteq \mathbb{C}_X(\mathbb{A})$$

with objects M which are in $\mathbb{C}_{\mathcal{N}_b(Y, X)}(\mathbb{A})$ for some $b \geq 0$, i.e. such that

$$\text{supp}(M) \subseteq \mathcal{N}_b(Y, X) .$$

□

The inclusion

$$F : \mathbb{C}_Y(\mathbb{A}) \longrightarrow \mathbb{N}_Y(\mathbb{A})$$

is an equivalence of additive categories. In order to define an inverse equivalence $F^{-1} = G$ choose for each $b \geq 0$ a strong deformation retraction

$$g_b = \text{inclusion}^{-1} : \mathcal{N}_b(Y, X) \longrightarrow Y$$

in the proper eventually Lipschitz category, and set

$$G : \mathbb{N}_Y(\mathbb{A}) \longrightarrow \mathbb{C}_Y(\mathbb{A}) ; M \longrightarrow (g_b)_* M$$

for any $b \geq 0$ such that M is defined in $\mathbb{C}_{\mathcal{N}_b(Y,X)}(\mathbb{A})$. The idempotent completions are also equivalent, allowing the identifications

$$K_0(\mathbb{P}_0(\mathbb{N}_Y(\mathbb{A}))) = K_0(\mathbb{P}_{\mathcal{N}_b(Y,X)}(\mathbb{A})) = K_0(\mathbb{P}_Y(\mathbb{A})) \quad (b \geq 0) .$$

The objects in $\mathbb{C}_{X,Y}(\mathbb{A})$ isomorphic to 0 are the objects in $\mathbb{N}_Y(\mathbb{A})$, and $\mathbb{C}_{X,Y}(\mathbb{A})$ is the localization of $\mathbb{C}_X(\mathbb{A})$ inverting the subcategory $\mathbb{N}_Y(\mathbb{A})$. The equivalence type of the filtered additive category $\mathbb{C}_{X,Y}(\mathbb{A})$ depends only on the homotopy type of the complement $X \setminus Y$ in the proper eventually Lipschitz category.

§3. End invariants

The algebraic theories of finiteness obstruction and torsion of §1 are applied to the graded and bounded categories of §2, to obtain an abstract account of the end invariant of Siebenmann [73] for closing a tame end of an open manifold.

Let $\mathbb{A} \subseteq \mathbb{B}$ be an embedding of additive categories (meaning a functor which is injective on the object sets). A morphism $f : L \rightarrow M$ in \mathbb{A} is a \mathbb{B} -*isomorphism* if there exists an inverse $f^{-1} : M \rightarrow L$ in \mathbb{B} . A chain complex C in \mathbb{A} is \mathbb{B} -*contractible* if there exists a chain contraction $\Gamma : 0 \simeq 1 : C \rightarrow C$ in \mathbb{B} .

Let $\mathbb{A}_0 \subseteq \mathbb{A}_1 \subseteq \mathbb{A}_2$ be embeddings of additive categories. A chain complex C in \mathbb{A}_1 is $(\mathbb{A}_2, \mathbb{A}_0)$ -*finitely dominated* if there exists a domination

$$(D, f : C \rightarrow D, g : D \rightarrow C, h : gf \simeq 1 : C \rightarrow C)$$

of C in \mathbb{A}_2 by a finite chain complex D in \mathbb{A}_0 .

In the following result the embeddings $\mathbb{A}_0 \subseteq \mathbb{A}_1 \subseteq \mathbb{A}_2$ are given by $\mathbb{N}_Y(\mathbb{A}) \subseteq \mathbb{C}_X(\mathbb{A}) \subseteq \mathbb{G}_X(\mathbb{A})$ for a pair of metric spaces $(X, Y \subseteq X)$ and a filtered additive category \mathbb{A} .

PROPOSITION 3.1 *A finite chain complex C in $\mathbb{C}_X(\mathbb{A})$ is $\mathbb{G}_{X,Y}(\mathbb{A})$ -contractible if and only if it is $(\mathbb{G}_X(\mathbb{A}), \mathbb{N}_Y(\mathbb{A}))$ -finitely dominated.*

PROOF Let C be n -dimensional

$$C : \dots \rightarrow 0 \rightarrow C_n \xrightarrow{d_C} C_{n-1} \rightarrow \dots \rightarrow C_1 \xrightarrow{d_C} C_0 .$$

A contraction $[\Gamma] : 0 \simeq 1 : C \rightarrow C$ of C in $\mathbb{G}_{X,Y}(\mathbb{A})$ is represented by a collection of morphisms in $\mathbb{G}_X(\mathbb{A})$

$$\Gamma = \{ \Gamma : C_r \rightarrow C_{r+1} \mid 0 \leq r < n \}$$

such that for some integers $c_0, c_1, \dots, c_n \geq 0$

$$(1 - d_C \Gamma - \Gamma d_C)(C_r) \subseteq C_r(\mathcal{N}_{c_r}(Y, X)) \quad (0 \leq r \leq n) .$$

Starting with $b_n = c_n$ there exists a sequence b_n, b_{n-1}, \dots, b_0 of integers

$b_r \geq c_r$ such that

$$d_C(C_r(\mathcal{N}_{b_r}(Y, X))) \subseteq C_{r-1}(\mathcal{N}_{b_{r-1}}(Y, X)) \quad (1 \leq r \leq n) .$$

Let $b = \max\{b_0, b_1, \dots, b_n\}$ and let D be the finite chain complex in $\mathbb{C}_{\mathcal{N}_b(Y, X)}(\mathbb{A})$ defined by

$$d_D = d_C| : D_r = C_r(\mathcal{N}_{b_r}(Y, X)) \longrightarrow D_{r-1} = C_{r-1}(\mathcal{N}_{b_{r-1}}(Y, X)) .$$

Define a $(\mathbb{G}_X(\mathbb{A}), \mathbb{C}_{\mathcal{N}_b(Y, X)}(\mathbb{A}))$ -domination of C

$$(f : C \longrightarrow D, g : D \longrightarrow C, h : gf \simeq 1 : C \longrightarrow C)$$

by

$$f = (1 - d_C \Gamma - \Gamma d_C) : C_r \longrightarrow D_r ,$$

$$g = \text{inclusion} : D_r \longrightarrow C_r ,$$

$$h = \Gamma : C_r \longrightarrow C_{r+1} .$$

Conversely, suppose given a $(\mathbb{G}_X(\mathbb{A}), \mathbb{C}_{\mathcal{N}_b(Y, X)}(\mathbb{A}))$ -domination of C

$$(f : C \longrightarrow D, g : D \longrightarrow C, h : gf \simeq 1 : C \longrightarrow C)$$

for some $b \geq 0$. Passing to the germs away from Y there is defined a chain contraction of C in $\mathbb{G}_{X, Y}(\mathbb{A})$

$$[\Gamma] = [h] : [gf] = 0 \simeq 1 : C \longrightarrow C .$$

□

By 2.7 every finite chain complex C in $\mathbb{C}_{X, Y}(\mathbb{A})$ is of the type $C = [D]$, for some finite chain complex D in $\mathbb{C}_X(\mathbb{A})$. In view of 3.1 the following conditions are equivalent:

- (i) C is $\mathbb{G}_{X, Y}(\mathbb{A})$ -contractible,
- (ii) D is $\mathbb{G}_{X, Y}(\mathbb{A})$ -contractible,
- (iii) D is $(\mathbb{G}_X(\mathbb{A}), \mathbb{N}_Y(\mathbb{A}))$ -finitely dominated.

DEFINITION 3.2 The *end invariant* of a finite $\mathbb{G}_{X, Y}(\mathbb{A})$ -contractible chain complex C in $\mathbb{C}_{X, Y}(\mathbb{A})$ is the projective class

$$[C]_+ = [D] \in K_0(\mathbb{P}_{\mathcal{N}_b(Y, X)}(\mathbb{A})) = K_0(\mathbb{P}_Y(\mathbb{A}))$$

of any $(\mathbb{G}_X(\mathbb{A}), \mathbb{N}_Y(\mathbb{A}))$ -finitely dominated chain complex D in $\mathbb{C}_X(\mathbb{A})$ such that $[D] = C$. The *reduced end invariant* of C is the image of the end invariant in the reduced class group

$$[C]_+ \in \tilde{K}_0(\mathbb{P}_Y(\mathbb{A})) = \text{coker}(K_0(\mathbb{C}_Y(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_Y(\mathbb{A}))) .$$

□

EXAMPLE 3.3 Let $(X, Y) = (\mathbb{Z}^+, \{0\})$. For any object M in \mathbb{A} let $M[z]$ be the object in $\mathbb{C}_X(\mathbb{A})$ defined by

$$M[z](k) = z^k M \quad (k \geq 0) ,$$

with $z^k M$ a copy of M graded by $k \in X$. For any object (M, p) in $\mathbb{P}_0(\mathbb{A})$ the 1-dimensional chain complex C defined in $\mathbb{C}_X(\mathbb{A})$ by

$$d_C = \begin{pmatrix} 1 & 0 & 0 & \dots \\ -p & 1 & 0 & \dots \\ 0 & -p & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$: C_1 = M[z] = \sum_{k=0}^{\infty} z^k M \longrightarrow C_0 = M[z] = \sum_{k=0}^{\infty} z^k M$$

is $\mathbb{G}_{X,Y}(\mathbb{A})$ -contractible, since d_C fits into the direct sum system in $\mathbb{P}_0(\mathbb{G}_X(\mathbb{A}))$

$$M[z] \begin{array}{c} \xrightarrow{d_C} \\ \xleftarrow{\Gamma} \end{array} M[z] \begin{array}{c} \xleftarrow{i} \\ \xrightarrow{j} \end{array} (M, p)$$

with

$$\Gamma = \begin{pmatrix} 1-p & -p & -p & \dots \\ 0 & 1-p & -p & \dots \\ 0 & 0 & 1-p & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$: M[z] = \sum_{k=0}^{\infty} z^k M \longrightarrow M[z] = \sum_{k=0}^{\infty} z^k M ,$$

$$i = (p \ p \ p \ \dots) : M[z] = \sum_{k=0}^{\infty} z^k M \longrightarrow M ,$$

$$j = \begin{pmatrix} p \\ 0 \\ 0 \\ \vdots \end{pmatrix} : M \longrightarrow M[z] = \sum_{k=0}^{\infty} z^k M .$$

The direct sum system includes a $(\mathbb{G}_X(\mathbb{A}), \mathbb{C}_Y(\mathbb{A}))$ -domination (M, i, j, Γ) of C with $ij = p : M \longrightarrow M$, so that C has end invariant

$$[C]_+ = [M, p] \in K_0(\mathbb{P}_Y(\mathbb{A})) = K_0(\mathbb{P}_0(\mathbb{A})) .$$

□

Given a subobject $N \subseteq M$ of an object M in $\mathbb{C}_X(\mathbb{A})$ define the *quotient object* M/N to be the object in $\mathbb{C}_X(\mathbb{A})$ with

$$(M/N)(x) = \begin{cases} M(x) & \text{if } N(x) = 0 \\ 0 & \text{if } N(x) = M(x) . \end{cases}$$

As usual, there is defined an exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0 .$$

PROPOSITION 3.4 *The end invariants of a pair $(C, D \subseteq C)$ of n -dimensional $\mathbb{G}_{X,Y}(\mathbb{A})$ -contractible chain complexes in $\mathbb{C}_{X,Y}(\mathbb{A})$ with C/D defined in $\mathbb{C}_{\mathcal{N}_b(Y,X),Y}(\mathbb{A})$ for some $b \geq 0$ are related by*

$$\begin{aligned} [C] - [D] &= [C/D] \\ &= \sum_{r=0}^n (-)^r [(C/D)_r] \in \text{im}(K_0(\mathbb{C}_Y(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_Y(\mathbb{A}))) . \end{aligned}$$

The reduced invariants are such that

$$[C] - [D] = 0 \in \tilde{K}_0(\mathbb{P}_Y(\mathbb{A})) .$$

PROOF Immediate from the sum formula 1.1 for projective classes, identifying

$$K_0(\mathbb{C}_{\mathcal{N}_b(Y,X)}(\mathbb{A})) = K_0(\mathbb{C}_Y(\mathbb{A})) .$$

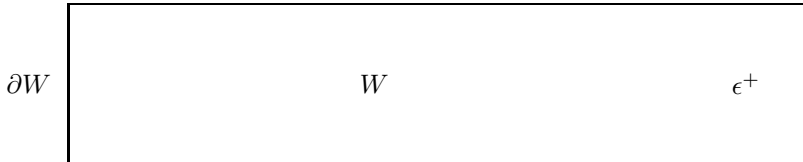
□

Thus the reduced end invariant $[C] \in \tilde{K}_0(\mathbb{P}_Y(\mathbb{A}))$ depends only on the part of C away from $\mathcal{N}_b(Y, X) \subseteq X$, for any $b \geq 0$.

EXAMPLE 3.5 Let $(X, Y) = (\mathbb{R}^+, \{0\})$, so that

$$\mathcal{N}_b(Y, X) = [0, b] \subset X = \mathbb{R}^+ = [0, \infty) \quad (b \geq 0) .$$

Let W be a connected open n -dimensional manifold with compact boundary ∂W , with one tame end ϵ^+ .



Use a handle structure on W and a proper map $p : (W, \partial W) \longrightarrow (X, Y)$ to give $(W, \partial W)$ the structure of a bounded (X, Y) -graded n -dimensional CW pair. Let

$$\mathbb{A} = \mathbb{B}^f(\mathbb{Z}[\pi_1(W)]) .$$

The chain complex $C(\tilde{W})$ of the universal cover \tilde{W} of W is an n -dimensional $(\mathbb{G}_X(\mathbb{A}), \mathbb{N}_Y(\mathbb{A}))$ -finitely dominated chain complex in $\mathbb{C}_X(\mathbb{A})$. The reduced end invariant of $C(\tilde{W})$ is the finiteness obstruction of Wall [83] for the finitely dominated CW complex W

$$[C(\tilde{W})]_+ = [W] \in \tilde{K}_0(\mathbb{P}_0(\mathbb{A})) = \tilde{K}_0(\mathbb{Z}[\pi_1(W)]) ,$$

with $[W] = 0$ if and only if W is homotopy equivalent to a finite CW complex. The fundamental group of ϵ^+ is

$$\pi_1(\epsilon^+) = \varprojlim_b \pi_1(W_b) ,$$

with

$$W_b = p^{-1}(X \setminus \mathcal{N}_b(Y, X)) = p^{-1}(b, \infty) \subset W \quad (b \geq 0)$$

a cofinal family of neighbourhoods of ϵ^+ . The end invariant of Siebenmann [73]

$$[\epsilon^+] = \varprojlim_b [C(\widetilde{W}_b)] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\epsilon^+)])$$

is such that $[\epsilon^+] = 0$ if (and for $n \geq 6$ only if) there exist a compact cobordism $(V; \partial_- V, \partial_+ V)$ with $\partial_- V = \partial W$ and a homeomorphism

$$(W, \partial W) \cong (V \cup \partial_+ V \times [0, \infty), \partial_- V)$$

which is the identity on ∂W , or equivalently such that

$$(W, \partial W) \cong (V \setminus \partial_+ V, \partial_- V) .$$

Working as in [73] it is possible to express W as a union of adjoining cobordisms,

$$W = V_0 \cup V_1 \cup \dots \cup V_{n-4} \cup W_{n-3}$$

with V_0, V_1, \dots, V_{n-4} compact and W_{n-3} open, such that

$$\partial_- V_0 = \partial W \quad , \quad \partial_+ V_r = \partial_- V_{r+1} \quad , \quad \partial_+ V_{n-r} = \partial W_{n-3} \quad ,$$

$$V_r = \partial_- V_r \times I \bigcup \bigcup D^{r+1} \times D^{n-r-1}$$

$$(= \text{trace of surgeries on } S^r \times D^{n-r-1} \subset \partial_- V_r) \quad (0 \leq r \leq n-4) \quad ,$$

and such that each

$$U_r = \bigcup_{i=r}^{n-4} V_i \cup W_{n-3} \quad (0 \leq r \leq n-3)$$

is an ' r -neighbourhood' of the end ϵ^+ with the inclusion

$$\partial U_r = \partial_- V_r \longrightarrow U_r$$

r -connected.

V_0	V_1	V_2	
-------	-------	-------	--

$$\partial W = \partial_- V_0 \quad \partial_+ V_0 = \partial_- V_1 \quad \partial_+ V_1 = \partial_- V_2 \quad \partial_+ V_2 = \partial_- V_3$$

The $(n-3)$ -neighbourhood $(U_{n-3}, \partial U_{n-3}) = (W_{n-3}, \partial W_{n-3})$ is such that

$$H_i(\widetilde{W}_{n-3}, \partial \widetilde{W}_{n-3}) = 0 \quad (i \neq n-2) \quad ,$$

and $P = H_{n-2}(\widetilde{W}_{n-3}, \partial \widetilde{W}_{n-3})$ is a f.g. projective $\mathbb{Z}[\pi_1(\epsilon^+)]$ -module such that

$$[\epsilon^+] = [P] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\epsilon^+)]) .$$

$[P] = 0$ if and (for $n \geq 6$) only if there exists an expression

$$W_{n-3} = V_{n-3} \cup W_{n-2}$$

for some compact cobordism $(V_{n-3}; \partial_- V_{n-3}, \partial_+ V_{n-3})$ with

$$\partial_- V_{n-3} = \partial W_{n-3} \quad , \quad \partial_+ V_{n-3} = \partial W_{n-2} \quad ,$$

$$V_{n-3} = \partial_- V_{n-3} \times I \bigcup \cup D^{n-2} \times D^2$$

such that W_{n-2} is an $(n-2)$ -neighbourhood of ϵ^+ , in which case the inclusion $\partial W_{n-2} \longrightarrow W_{n-2}$ is a homotopy equivalence and

$$(W_{n-2}, \partial W_{n-2}) \cong \partial W_{n-2} \times ([0, \infty), \{0\})$$

by the invertibility of h -cobordisms. Let $\pi_1(\epsilon^+) = \pi$. The map $\tilde{K}_0(\mathbb{Z}[\pi]) \longrightarrow \tilde{K}_0(\mathbb{Z}[\pi_1(W)])$ induced by the inclusions $W_b \longrightarrow W$ sends the end obstruction $[\epsilon^+]$ to the finiteness obstruction $[W]$. The end has a finitely dominated open neighbourhood which is the infinite cyclic cover \overline{M} of a compact n -dimensional manifold M with $\pi_1(M) = \pi \times \mathbb{Z}$, $\pi_1(\overline{M}) = \pi$. The product $\overline{M} \times (\mathbb{R}^+, \{0\})$ has a preferred $(\mathbb{R}^+, \{0\})$ -bounded finite structure, and there is defined an $(\mathbb{R}^+, \{0\})$ -bounded homotopy equivalence $f : (W, \partial W) \longrightarrow \overline{M} \times (\mathbb{R}^+, \{0\})$ with $(\mathbb{R}^+, \{0\})$ -bounded torsion

$$\tau(f) = [\epsilon^+] - [\overline{M}] = (-)^n [\epsilon^+]^* \in Wh(\mathbb{C}_{\mathbb{R}^+, \{0\}}(\mathbb{Z}[\pi])) = \tilde{K}_0(\mathbb{Z}[\pi]) \quad .$$

(See §7 for the isomorphism $Wh(\mathbb{C}_{\mathbb{R}^+, \{0\}}(\mathbb{Z}[\pi])) \cong \tilde{K}_0(\mathbb{Z}[\pi])$, and §13 for duality.)

□

DEFINITION 3.6 The *end invariant* of an isomorphism $[f] : L \longrightarrow M$ in $\mathbb{G}_{X,Y}(\mathbb{A})$ is the end invariant

$$[f]_+ = [C]_+ \in K_0(\mathbb{P}_Y(\mathbb{A}))$$

of the 1-dimensional contractible chain complex C in $\mathbb{G}_{X,Y}(\mathbb{A})$ defined by

$$d = [f] : C_1 = L \longrightarrow C_0 = M \quad .$$

□

The end invariant of isomorphisms has the following algebraic properties:

PROPOSITION 3.7 (i) *For any isomorphisms $[f] : L \longrightarrow M$, $[f'] : L' \longrightarrow M'$ in $\mathbb{G}_{X,Y}(\mathbb{A})$*

$$[f \oplus f' : L \oplus L' \longrightarrow M \oplus M']_+$$

$$= [f : L \longrightarrow M]_+ + [f' : L' \longrightarrow M']_+ \in K_0(\mathbb{P}_Y(\mathbb{A})) \quad .$$

(ii) *For any isomorphisms $[f] : L \longrightarrow M$, $[g] : M \longrightarrow N$ in $\mathbb{G}_{X,Y}(\mathbb{A})$*

$$[gf : L \longrightarrow N]_+ = [f : L \longrightarrow M]_+ + [g : M \longrightarrow N]_+ \in K_0(\mathbb{P}_Y(\mathbb{A})) \quad .$$

(iii) For any $\mathbb{G}_X(\mathbb{A})$ -isomorphism $f : L \longrightarrow M$ in $\mathbb{C}_X(\mathbb{A})$

$$[f]_+ = 0 \in K_0(\mathbb{P}_Y(\mathbb{A})) .$$

(iv) For any morphism $[e] : M \longrightarrow L$ in $\mathbb{G}_{X,Y}(\mathbb{A})$

$$\left[\begin{pmatrix} 1 & [e] \\ 0 & 1 \end{pmatrix} : L \oplus M \longrightarrow L \oplus M \right]_+ = 0 \in K_0(\mathbb{P}_Y(\mathbb{A})) .$$

PROOF (i) Projective class is additive for direct sum of chain complexes.

(ii) Let C, D be the 1-dimensional $\mathbb{G}_{X,Y}(\mathbb{A})$ -contractible chain complexes defined in $\mathbb{C}_X(\mathbb{A})$ by

$$\begin{aligned} d_C &= f : C_1 = L \longrightarrow C_0 = M , \\ d_D &= gf : D_1 = L \longrightarrow D_0 = N , \end{aligned}$$

so that

$$[f]_+ = [C]_+ , [gf]_+ = [D]_+ \in K_0(\mathbb{P}_0(\mathbb{C}_Y(\mathbb{A}))) .$$

The 2-dimensional $\mathbb{G}_{X,Y}(\mathbb{A})$ -contractible chain complex E defined in $\mathbb{C}_X(\mathbb{A})$ by

$$d_E = \begin{cases} \begin{pmatrix} -1 \\ f \end{pmatrix} : E_2 = L \longrightarrow E_1 = L \oplus M \\ (gf \ g) : E_1 = L \oplus M \longrightarrow E_0 = N \end{cases}$$

has end invariant

$$[E]_+ = [g]_+ \in K_0(\mathbb{P}_Y(\mathbb{A})) .$$

It follows from the short exact sequence

$$0 \longrightarrow D \xrightarrow{i} E \xrightarrow{j} SC \longrightarrow 0$$

with i the inclusion and j the projection that

$$[E]_+ = [D]_+ + [SC]_+ = [gf]_+ - [f]_+ \in K_0(\mathbb{P}_Y(\mathbb{A})) ,$$

and so

$$[gf]_+ = [f]_+ + [g]_+ \in K_0(\mathbb{P}_Y(\mathbb{A})) .$$

(iii) If $f : L \longrightarrow M$ is a $\mathbb{G}_X(\mathbb{A})$ -isomorphism in $\mathbb{C}_X(\mathbb{A})$ then the 1-dimensional chain complex C defined in $\mathbb{C}_X(\mathbb{A})$ by

$$d_C = f : C_1 = L \longrightarrow C_0 = M$$

is $\mathbb{G}_X(\mathbb{A})$ -contractible, and

$$[f]_+ = [C] = 0 \in K_0(\mathbb{P}_Y(\mathbb{A}))$$

by the chain homotopy invariance of projective class.

(iv) The automorphisms

$$\begin{aligned} [f] &= \begin{pmatrix} 1 & [e] \\ 0 & 1 \end{pmatrix} : L \oplus M \longrightarrow L \oplus M, \\ [g] &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : L \oplus M \oplus L \longrightarrow L \oplus M \oplus L, \\ [h] &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & [e] & 1 \end{pmatrix} : L \oplus M \oplus L \longrightarrow L \oplus M \oplus L \end{aligned}$$

are such that

$$[f] \oplus 1_L = [g][h][g]^{-1}[h]^{-1} : L \oplus M \oplus L \longrightarrow L \oplus M \oplus L.$$

Applying (ii) and (iii) gives

$$\begin{aligned} [f]_+ &= [f \oplus 1_L]_+ = [[g][h][g]^{-1}[h]^{-1}]_+ \\ &= [g]_+ + [h]_+ - [g]_+ - [h]_+ = 0 \in K_0(\mathbb{P}_Y(\mathbb{A})). \end{aligned}$$

□

PROPOSITION 3.8 *Let*

$$0 \longrightarrow C \xrightarrow{f} D \xrightarrow{g} E \longrightarrow 0$$

be an exact sequence of $\mathbb{G}_{X,Y}(\mathbb{A})$ -contractible finite chain complexes in $\mathbb{C}_{X,Y}(\mathbb{A})$. The end invariants of C, D, E are related by

$$[D]_+ = [C]_+ + [E]_+ + \sum_{r=0}^{\infty} (-)^r [(f \ h) : C_r \oplus E_r \longrightarrow D_r]_+ \in K_0(\mathbb{P}_Y(\mathbb{A}))$$

with $h : E_r \longrightarrow D_r$ splitting morphisms in $\mathbb{G}_{X,Y}(\mathbb{A})$ for $g : D_r \longrightarrow E_r$ ($r \geq 0$).

PROOF Consider first the special case when $E = 0$, so that $f : C \longrightarrow D$ is an isomorphism of contractible finite chain complexes in $\mathbb{G}_{X,Y}(\mathbb{A})$. The sum formula in this case

$$[D]_+ = [C]_+ + \sum_{r=0}^{\infty} (-)^r [f : C_r \longrightarrow D_r]_+ \in K_0(\mathbb{P}_Y(\mathbb{A}))$$

is proved by repeated application of the sum formula of 3.7 (ii).

Returning to the general case consider the exact sequence of contractible finite chain complexes in $\mathbb{G}_{X,Y}(\mathbb{A})$

$$0 \longrightarrow C \xrightarrow{f'} D' \xrightarrow{g'} E \longrightarrow 0$$

defined by

$$d_{D'} = \begin{pmatrix} d_C & k \\ 0 & d_E \end{pmatrix} : D'_r = C_r \oplus E_r \longrightarrow D'_{r-1} = C_{r-1} \oplus E_{r-1} ,$$

$$f' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : C_r \longrightarrow D'_r = C_r \oplus E_r ,$$

$$g' = (0 \ 1) : D'_r = C_r \oplus E_r \longrightarrow E_r ,$$

with $k : E_r \longrightarrow C_{r-1}$ ($r \geq 1$) the unique morphisms in $\mathbb{C}_X(\mathbb{A})$ such that

$$fk = d_D h - h d_E : E_r \longrightarrow D_{r-1} .$$

It is immediate from the definitions that

$$[D']_+ = [C]_+ + [E]_+ \in K_0(\mathbb{P}_Y(\mathbb{A})) .$$

There is defined an isomorphism $(f \ h) : D' \longrightarrow D$ of contractible finite chain complexes in $\mathbb{G}_{X,Y}(\mathbb{A})$, and by the special case

$$[D]_+ = [D']_+ + \sum_{r=0}^{\infty} (-)^r [(f \ h) : C_r \oplus E_r \longrightarrow D_r]_+ \in K_0(\mathbb{P}_Y(\mathbb{A})) .$$

Eliminating $[D']_+$ gives the sum formula in general.

□

Thus if $f : E \longrightarrow E'$ is a chain map of contractible finite chain complexes in $\mathbb{G}_{X,Y}(\mathbb{A})$ then f is a chain equivalence, and the algebraic mapping cone $C(f)$ is contractible, with end invariant

$$[C(f)] = [E']_+ - [E]_+ \in K_0(\mathbb{P}_Y(\mathbb{A})) .$$

PROPOSITION 3.9 *A finite chain complex C in $\mathbb{C}_X(\mathbb{A})$ is $\mathbb{C}_{X,Y}(\mathbb{A})$ -contractible if and only if it is $(\mathbb{C}_X(\mathbb{A}), \mathbb{N}_Y(\mathbb{A}))$ -finitely dominated.*

PROOF Exactly as for 3.1.

□

REMARK 3.10 The 1-dimensional $\mathbb{G}_{\mathbb{Z}^+, \{0\}}(\mathbb{A})$ -contractible chain complex C in $\mathbb{C}_{\mathbb{Z}^+}(\mathbb{A})$ associated in 3.3 to an object (M, p) in $\mathbb{P}_0(\mathbb{A})$ is $\mathbb{C}_{\mathbb{Z}^+, \{0\}}(\mathbb{A})$ -contractible if and only if $p = 0$. For $\mathbb{A} = \mathbb{B}^f(\mathbb{Z})$ and $(M, p) = (\mathbb{Z}, 1)$ this is the cellular chain complex $C = C(\mathbb{R}^+)$ of the tame end \mathbb{R}^+ regarded as a \mathbb{Z}^+ -graded CW complex, which is not bounded homotopy equivalent to a point.

□

EXAMPLE 3.11 As in 3.3 let $(X, Y) = (\mathbb{Z}^+, \{0\})$. For any object (M, p)

in $\mathbb{P}_0(\mathbb{A})$ the 1-dimensional chain complex C defined in $\mathbb{C}_X(\mathbb{A})$ by

$$d_C = \begin{pmatrix} 1-p & 0 & 0 & \dots \\ p & 1-p & 0 & \dots \\ 0 & p & 1-p & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$: C_1 = M[z] = \sum_{k=0}^{\infty} z^k M \longrightarrow C_0 = M[z] = \sum_{k=0}^{\infty} z^k M$$

is $\mathbb{C}_{X,Y}(\mathbb{A})$ -contractible, since d_C fits into the direct sum system in $\mathbb{P}_0(\mathbb{C}_X(\mathbb{A}))$

$$M[z] \begin{array}{c} \xrightarrow{d_C} \\ \xleftarrow{\Gamma} \end{array} M[z] \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{j} \end{array} (M, p)$$

with

$$\Gamma = \begin{pmatrix} 1-p & p & 0 & \dots \\ 0 & 1-p & p & \dots \\ 0 & 0 & 1-p & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} :$$

$$M[z] = \sum_{k=0}^{\infty} z^k M \longrightarrow M[z] = \sum_{k=0}^{\infty} z^k M ,$$

$$i = (p \ 0 \ 0 \ \dots) : M[z] = \sum_{k=0}^{\infty} z^k M \longrightarrow M ,$$

$$j = \begin{pmatrix} p \\ 0 \\ 0 \\ \vdots \end{pmatrix} : M \longrightarrow M[z] = \sum_{k=0}^{\infty} z^k M .$$

The direct sum system includes a $(\mathbb{C}_X(\mathbb{A}), \mathbb{C}_Y(\mathbb{A}))$ -domination (M, i, j, Γ) of C with $ij = p : M \longrightarrow M$, so that C has end invariant

$$[C]_+ = [M, p] \in K_0(\mathbb{P}_Y(\mathbb{A})) = K_0(\mathbb{P}_0(\mathbb{A})) .$$

□

PROPOSITION 3.12 *Let $f : L \longrightarrow M$, $f' : M \longrightarrow L$ be morphisms in $\mathbb{C}_X(\mathbb{A})$ such that that the germs $[f]$, $[f']$ are inverse isomorphisms in $\mathbb{G}_{X,Y}(\mathbb{A})$*

$$[f'f] = 1 : L \longrightarrow L , \quad [ff'] = 1 : M \longrightarrow M ,$$

i.e. such that there exist integers $b, c \geq 0$ with

$$(f'f - 1)(L) \subseteq L(\mathcal{N}_b(Y, X)) , \quad (ff' - 1)(M) \subseteq M(\mathcal{N}_c(Y, X)) .$$

The end invariant of f is the projective class

$$[f]_+ = [D, p] \in K_0(\mathbb{P}_Y(\mathbb{A}))$$

of the chain homotopy projection (D, p) defined in $\mathbb{C}_{\mathcal{N}_d(Y, X)}(\mathbb{A})$ for any such $b, c \geq 0$ with $f(L(\mathcal{N}_b(Y, X))) \subseteq M(\mathcal{N}_c(Y, X))$, with $d \geq \max(b, c)$ and

$$d_D = f| : D_1 = L(\mathcal{N}_b(Y, X)) \longrightarrow D_0 = M(\mathcal{N}_c(Y, X)) ,$$

$$p = \begin{cases} ff' - 1 : D_0 \longrightarrow D_0 \\ f'f - 1 : D_1 \longrightarrow D_0 . \end{cases}$$

PROOF Let C be the 1-dimensional $\mathbb{C}_{X, Y}(\mathbb{A})$ -contractible chain complex in $\mathbb{C}_X(\mathbb{A})$ defined by

$$d_C = f : C_1 = L \longrightarrow C_0 = M ,$$

The chain maps $i : C \longrightarrow D$, $j : D \longrightarrow C$ defined in $\mathbb{C}_X(\mathbb{A})$ by

$$i = \begin{cases} ff' - 1 : C_0 = M \longrightarrow D_0 = M(\mathcal{N}_c(Y, X)) \\ f'f - 1 : C_1 = L \longrightarrow D_1 = L(\mathcal{N}_b(Y, X)) , \end{cases}$$

$$j = \begin{cases} \text{inclusion} : D_0 = M(\mathcal{N}_c(Y, X)) \longrightarrow C_0 = M \\ \text{inclusion} : D_1 = L(\mathcal{N}_b(Y, X)) \longrightarrow C_1 = L \end{cases}$$

are such that

$$k : ji \simeq 1 : C \longrightarrow C , \quad ij = p : D \longrightarrow D$$

with the chain homotopy k given by

$$k = f' : C_0 = M \longrightarrow C_1 = L .$$

Thus (C, i, j) is a splitting in $\mathbb{C}_X(\mathbb{A})$ of (D, p) , and the projective class of (D, p) coincides with the end invariant of f

$$[D, p] = [C] = [f]_+ \in K_0(\mathbb{P}_Y(\mathbb{A})) .$$

□

§4. Excision and transversality in K -theory

The main result of §4 is a Mayer-Vietoris exact sequence (4.16) in the algebraic K -groups of the bounded categories associated to a union $X = X^+ \cup X^-$ of metric spaces

$$K'_1(\mathbb{C}_{X^+ \cap X^-}(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_{X^+}(\mathbb{A})) \oplus K_1(\mathbb{C}_{X^-}(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_X(\mathbb{A}))$$

$$\xrightarrow{\partial} K'_0(\mathbb{P}_{X^+ \cap X^-}(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_{X^+}(\mathbb{A})) \oplus K_0(\mathbb{P}_{X^-}(\mathbb{A})) ,$$

with the K' -groups certain generalizations of the algebraic K -groups $K_1(\mathbb{C}_{X^+ \cap X^-}(\mathbb{A}))$ and $K_0(\mathbb{P}_{X^+ \cap X^-}(\mathbb{A}))$. This exact sequence is obtained using algebraic excision and transversality properties of chain complexes in $\mathbb{C}_X(\mathbb{A})$. These are similar to the corresponding properties in geometric bordism and the algebraic K -theory localization exact sequence, which are now recalled.

The Mayer-Vietoris exact sequence of bordism groups for a union

$X = X^+ \cup X^-$ of reasonable spaces (e.g. polyhedra) with $X^+ \cap X^-$ closed and bicollared in X

$$\begin{aligned} \dots \longrightarrow \Omega_n(X^+ \cap X^-) &\longrightarrow \Omega_n(X^+) \oplus \Omega_n(X^-) \\ &\longrightarrow \Omega_n(X) \xrightarrow{\partial} \Omega_{n-1}(X^+ \cap X^-) \longrightarrow \dots \end{aligned}$$

is a direct consequence of the transversality of manifolds. Any map $f : M \longrightarrow X$ from an n -dimensional manifold M can be made transverse regular at $X^+ \cap X^- \subset X$, with

$$N = f^{-1}(X^+ \cap X^-) \subset M$$

a framed codimension 1 submanifold, giving the connecting map

$$\partial : \Omega_n(X) \longrightarrow \Omega_{n-1}(X^+ \cap X^-) ; (M, f) \longrightarrow (N, f|) .$$

The Mayer-Vietoris exact sequence can also be derived from the excision isomorphisms of relative bordism groups

$$\Omega_n(X, X^+) \cong \Omega_n(X^-, X^+ \cap X^-)$$

with the connecting maps given by

$$\begin{aligned} \partial : \Omega_n(X) &\longrightarrow \Omega_n(X, X^+) \cong \Omega_n(X^-, X^+ \cap X^-) \\ &\longrightarrow \Omega_{n-1}(X^+ \cap X^-) . \end{aligned}$$

The localization exact sequence of algebraic K -theory (Bass [7]) for a ring morphism $A \longrightarrow S^{-1}A$ inverting a multiplicative subset $S \subset A$

$$\begin{aligned} \dots \longrightarrow K_1(A) &\longrightarrow K_1(S^{-1}A) \longrightarrow K_1(A, S) \\ &\longrightarrow K_0(A) \longrightarrow K_0(S^{-1}A) \longrightarrow \dots \end{aligned}$$

identifies the relative torsion K -groups $K_1(A \longrightarrow S^{-1}A)$ with the class group $K_1(A, S) = K_0(\mathbb{H}(A, S))$ of the exact category $\mathbb{H}(A, S)$ of S -torsion A -modules of homological dimension 1.

The following result identifies the relative torsion group $K_1(\mathbb{P}_Y(\mathbb{A}) \longrightarrow \mathbb{P}_X(\mathbb{A}))$ for a pair of metric spaces $(X, Y \subseteq X)$ and a filtered additive category \mathbb{A} with the torsion group $K_1(\mathbb{C}_{X,Y}(\mathbb{A}))$ of the bounded germ category $\mathbb{C}_{X,Y}(\mathbb{A})$.

THEOREM 4.1 *For any pair of metric spaces $(X, Y \subseteq X)$ and any filtered additive category \mathbb{A} there is a natural excision isomorphism*

$$K_1(\mathbb{P}_Y(\mathbb{A}) \longrightarrow \mathbb{P}_X(\mathbb{A})) \cong K_1(\mathbb{C}_{X,Y}(\mathbb{A})) .$$

The connecting map in the exact sequence

$$\begin{aligned} K_1(\mathbb{C}_Y(\mathbb{A})) &\longrightarrow K_1(\mathbb{C}_X(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_{X,Y}(\mathbb{A})) \\ &\xrightarrow{\partial} K_0(\mathbb{P}_Y(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_X(\mathbb{A})) \end{aligned}$$

is given by

$$\partial : K_1(\mathbb{C}_{X,Y}(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_Y(\mathbb{A})) ; \tau([f] : M \longrightarrow M) \longrightarrow [f]_+ ,$$

sending the torsion $\tau([f])$ of an automorphism $[f] : M \longrightarrow M$ in $\mathbb{C}_{X,Y}(\mathbb{A})$ to the end invariant $[f]_+$.

PROOF The relative K -group $K_1(\mathbf{i})$ of the additive functor

$$i = \text{inclusion} : \mathbb{P}_Y(\mathbb{A}) \longrightarrow \mathbb{P}_X(\mathbb{A}) ; M \longrightarrow M$$

fits into an exact sequence

$$\begin{aligned} K_1(\mathbb{C}_Y(\mathbb{A})) &\longrightarrow K_1(\mathbb{C}_X(\mathbb{A})) \longrightarrow K_1(\mathbf{i}) \\ &\xrightarrow{\partial} K_0(\mathbb{P}_Y(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_X(\mathbb{A})) , \end{aligned}$$

identifying

$$K_1(\mathbb{P}_X(\mathbb{A})) = K_1(\mathbb{C}_X(\mathbb{A})) , \quad K_1(\mathbb{P}_Y(\mathbb{A})) = K_1(\mathbb{C}_Y(\mathbb{A})) .$$

An element in $K_1(\mathbf{i})$ is an equivalence class of triples (P, Q, f) consisting of objects P, Q in $\mathbb{P}_Y(\mathbb{A})$ and a stable isomorphism $[f] : P \longrightarrow Q$ in $\mathbb{P}_X(\mathbb{A})$, with

$$\partial : K_1(\mathbf{i}) \longrightarrow K_0(\mathbb{P}_Y(\mathbb{A})) ; (P, Q, f) \longrightarrow [P] - [Q] .$$

By definition, $(P, Q, f) = 0 \in K_1(\mathbf{i})$ if and only if there exists a stable isomorphism $[g] : P \longrightarrow Q$ in $\mathbb{P}_Y(\mathbb{A})$ such that

$$\tau([g]^{-1}[f] : P \longrightarrow P) = 0 \in K_1(\mathbb{P}_X(\mathbb{A})) .$$

Inverse isomorphisms

$$K_1(\mathbf{i}) \longrightarrow K_1(\mathbb{C}_{X,Y}(\mathbb{A})) ; (P, Q, f) \longrightarrow \tau([f_{RR}] : R \longrightarrow R) ,$$

$$K_1(\mathbb{C}_{X,Y}(\mathbb{A})) \longrightarrow K_1(\mathbf{i}) ; \tau([e] : M \longrightarrow M) \longrightarrow (P, Q, f)$$

will now be defined, as follows.

Given an element $(P, Q, f) \in K_1(\mathbf{i})$ represent the stable isomorphism $[f] : P \longrightarrow Q$ in $\mathbb{P}_X(\mathbb{A})$ by the isomorphism

$$f = \begin{pmatrix} f_{QP} & f_{QR} \\ f_{RP} & f_{RR} \end{pmatrix} : P \oplus R \longrightarrow Q \oplus R$$

in $\mathbb{P}_X(\mathbb{A})$, with R defined in $\mathbb{C}_X(\mathbb{A}) \subset \mathbb{P}_X(\mathbb{A})$. Write the inverse of f as

$$f^{-1} = g = \begin{pmatrix} g_{PQ} & g_{PR} \\ g_{RQ} & g_{RR} \end{pmatrix} : Q \oplus R \longrightarrow P \oplus R .$$

The composite morphism in $\mathbb{C}_X(\mathbb{A})$

$$f_{RP}g_{PR} : R \longrightarrow P \longrightarrow R$$

factors through an object in $\mathbb{C}_Y(\mathbb{A})$ (e.g. M if $P = (M, p)$), so that it has germ

$$[f_{RP}g_{PR}] = 0 : R \longrightarrow R .$$

Similarly, the composite $g_{RQ}f_{QR} : R \longrightarrow Q \longrightarrow R$ has germ

$$[g_{RQ}f_{QR}] = 0 : R \longrightarrow R .$$

Thus the germs $[f_{RR}], [g_{RR}] : R \longrightarrow R$ are inverse isomorphisms in $\mathbb{C}_{X,Y}(\mathbb{A})$, with

$$[f_{RR}][g_{RR}] = [f_{RR}g_{RR}] = [1 - f_{RP}g_{PR}] = 1 : R \longrightarrow R ,$$

$$[g_{RR}][f_{RR}] = [g_{RR}f_{RR}] = [1 - g_{RQ}f_{QR}] = 1 : R \longrightarrow R ,$$

and the morphism

$$K_1(i) \longrightarrow K_1(\mathbb{C}_{X,Y}(\mathbb{A})) ; (P, Q, f) \longrightarrow \tau([f_{RR}] : R \longrightarrow R)$$

is well-defined.

Given an automorphism $[e] : M \longrightarrow M$ in $\mathbb{C}_{X,Y}(\mathbb{A})$ let C be the 1-dimensional chain complex C defined in $\mathbb{C}_X(\mathbb{A})$ by

$$d_C = e : C_1 = M \longrightarrow C_0 = M ,$$

using any representative e of $[e]$. By 3.9 C is $\mathbb{C}_{N_b(Y,X)}(\mathbb{A})$ -finitely dominated in $\mathbb{C}_X(\mathbb{A})$ for some $b \geq 0$, so that there exists a 1-dimensional chain complex D in $\mathbb{P}_Y(\mathbb{A})$ with a chain equivalence $g : D \longrightarrow C$ in $\mathbb{P}_X(\mathbb{A})$. The algebraic mapping cone $C(g)$ is contractible, and for any chain contraction $\Gamma : 0 \simeq 1 : C(g) \longrightarrow C(g)$ there is defined an isomorphism in $\mathbb{P}_X(\mathbb{A})$

$$f = d_{C(g)} + \Gamma : C(g)_{\text{odd}} = D_0 \oplus C_1 \longrightarrow C(g)_{\text{even}} = D_1 \oplus C_0 .$$

The morphism

$$K_1(\mathbb{C}_{X,Y}(\mathbb{A})) \longrightarrow K_1(i) ; \tau([e]) \longrightarrow (D_0, D_1, [f])$$

is well-defined, and such that both the composites

$$K_1(\mathbb{C}_{X,Y}(\mathbb{A})) \longrightarrow K_1(i) \longrightarrow K_1(\mathbb{C}_{X,Y}(\mathbb{A}))$$

$$K_1(i) \longrightarrow K_1(\mathbb{C}_{X,Y}(\mathbb{A})) \longrightarrow K_1(i)$$

are identity maps. □

DEFINITION 4.2 For subsets $U, V \subseteq X$ of a metric space X and $b, c \geq 0$ write the intersection of the b -neighbourhood of U and the c -neighbourhood of V in X as

$$\begin{aligned} \mathcal{N}_{b,c}(U, V, X) &= \mathcal{N}_b(U, X) \cap \mathcal{N}_c(V, X) \\ &= \{x \in X \mid d(x, u) \leq b \text{ and } d(x, v) \leq c \text{ for some } u \in U, v \in V\} . \end{aligned}$$

For $b = c$ this is abbreviated

$$\mathcal{N}_{b,b}(U, V, X) = \mathcal{N}_b(U, V, X) .$$

□

In general, the inclusions

$$U \cap V \longrightarrow \mathcal{N}_{b,c}(U, V, X)$$

are not homotopy equivalences in the proper eventually Lipschitz category. This was first observed by Carlsson [16], using an example of the

type

$$\begin{aligned} X &= \{(x, |x|/(1 - |x|)) \mid -1 < x < 1\} \subset \mathbb{R}^2, \\ U &= \{(x, |x|/(1 - |x|)) \mid 0 \leq x < 1\} \subset X, \\ V &= \{(x, |x|/(1 - |x|)) \mid -1 < x \leq 0\} \subset X, \\ U \cup V &= X, \quad U \cap V = \{(0, 0)\} \subset X, \end{aligned}$$

for which

$$\mathcal{N}_2(U, V, X) = \mathcal{N}_2(U, X) = \mathcal{N}_2(V, X) = X.$$

Mayer-Vietoris presentations will be used in 4.8 to prove that

$$K_1(\mathbb{C}_{X,Y}(\mathbb{A})) = \varinjlim_b K_1(\mathbb{C}_{\mathcal{N}_b(X \setminus Y, X), \mathcal{N}_b(Y, X \setminus Y, X)}(\mathbb{A})),$$

showing that $K_*(\mathbb{C}_{X,Y}(\mathbb{A}))$ depends on the neighbourhoods in X of Y and the complement $X \setminus Y$ rather than just the quotient X/Y (with some metric).

Given subobjects $M^+, M^- \subseteq M$ in $\mathbb{G}_X(\mathbb{A})$ let

$$\begin{aligned} X^+ &= \text{supp}(M^+) = \{x \in X \mid M^+(x) \neq 0\} \subseteq X, \\ X^- &= \text{supp}(M^-) = \{x \in X \mid M^-(x) \neq 0\} \subseteq X \end{aligned}$$

so that

$$M^+ = M(X^+) \quad , \quad M^- = M(X^-).$$

Let $M^+ \cap M^-, M^+ + M^- \subseteq M$ be the subobjects defined by

$$M^+ \cap M^- = M(X^+ \cap X^-) \quad , \quad M^+ + M^- = M(X^+ \cup X^-).$$

DEFINITION 4.3 The *Mayer-Vietoris sequence* for an object M in $\mathbb{G}_X(\mathbb{A})$ and subobjects $M^+, M^- \subseteq M$ such that $M = M^+ + M^-$ is the exact sequence

$$\mathbb{M} : 0 \longrightarrow M^+ \cap M^- \xrightarrow{i} M^+ \oplus M^- \xrightarrow{j} M \longrightarrow 0$$

defined using the inclusions

$$i^\pm : M^+ \cap M^- \longrightarrow M^\pm \quad , \quad j^\pm : M^\pm \longrightarrow M,$$

with

$$\begin{aligned} i &= \begin{pmatrix} i^+ \\ -i^- \end{pmatrix} : M^+ \cap M^- \longrightarrow M^+ \oplus M^-, \\ j &= (j^+ \ j^-) : M^+ \oplus M^- \longrightarrow M. \end{aligned}$$

□

EXAMPLE 4.4 The Mayer-Vietoris presentation of an object M in $\mathbb{C}_X(\mathbb{A})$ with respect to an expression of X as a union

$$X = X^+ \cup X^-$$

is an exact sequence in $\mathbb{C}_X(\mathbb{A})$

$$\mathbb{M} : 0 \longrightarrow M(X^+ \cap X^-) \xrightarrow{i} M(X^+) \oplus M(X^-) \xrightarrow{j} M \longrightarrow 0$$

using the subobjects $M(X^+), M(X^-) \subseteq M$ such that

$$M(X^+) \cap M(X^-) = M(X^+ \cap X^-) \quad , \quad M(X^+) + M(X^-) = M \quad .$$

□

DEFINITION 4.5 A *Mayer-Vietoris presentation* with bound $b \geq 0$ of a chain complex E in $\mathbb{C}_X(\mathbb{A})$ with respect to a decomposition $X = X^+ \cup X^-$ is the Mayer-Vietoris exact sequence in $\mathbb{C}_X(\mathbb{A})$

$$\mathbb{E} : 0 \longrightarrow E^+ \cap E^- \xrightarrow{i} E^+ \oplus E^- \xrightarrow{j} E \longrightarrow 0$$

determined by subcomplexes $E^\pm \subseteq E$ such that

- (i) $E^+ + E^- = E$,
- (ii) E^\pm is defined in $\mathbb{C}_{\mathcal{N}_b(X^\pm, X)}(\mathbb{A})$, so that $E^+ \cap E^-$ is defined in $\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$,
- (iii) $E_r(X^\pm) \subseteq E_r^\pm$ ($r \geq 0$) .

□

The following result is an algebraic analogue of codimension 1 transversality of manifolds:

PROPOSITION 4.6 *Every finite chain complex E in $\mathbb{C}_X(\mathbb{A})$ admits a Mayer-Vietoris presentation \mathbb{E} , for any $X = X^+ \cup X^-$.*

PROOF An object L in $\mathbb{C}_X(\mathbb{A})$ can be regarded as a 0-dimensional chain complex. For any integer $b \geq 0$ define a Mayer-Vietoris presentation of L

$$\mathbb{L}\langle b \rangle : 0 \longrightarrow L^+\langle b \rangle \cap L^-\langle b \rangle \longrightarrow L^+\langle b \rangle \oplus L^-\langle b \rangle \longrightarrow L \longrightarrow 0$$

by the construction of 4.4, with

$$L^\pm\langle b \rangle = L(\mathcal{N}_b(X^\pm, X)) \quad , \quad L^+\langle b \rangle \cap L^-\langle b \rangle = L(\mathcal{N}_b(X^+, X^-, X)) \quad .$$

Given a morphism $f : L \longrightarrow M$ in $\mathbb{C}_X(\mathbb{A})$ and any integer $b \geq 0$ there exists an integer $c \geq 0$ such that

$$f(L^\pm\langle b \rangle) \subseteq M^\pm\langle c \rangle \quad ,$$

so that f can be resolved by a morphism of Mayer-Vietoris presentations

$$\begin{array}{ccccccc} \mathbb{L}\langle b \rangle : 0 & \longrightarrow & L^+\langle b \rangle \cap L^-\langle b \rangle & \longrightarrow & L^+\langle b \rangle \oplus L^-\langle b \rangle & \longrightarrow & L \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{M}\langle c \rangle : 0 & \longrightarrow & M^+\langle c \rangle \cap M^-\langle c \rangle & \longrightarrow & M^+\langle c \rangle \oplus M^-\langle c \rangle & \longrightarrow & M \longrightarrow 0 \end{array}$$

Let E be n -dimensional. There exists a sequence $b = (b_0, b_1, \dots, b_n)$ of integers ≥ 0 such that

$$d(E_r(\mathcal{N}_{b_r}(X^\pm, X))) \subseteq E_{r-1}(\mathcal{N}_{b_{r-1}}(X^\pm, X)) \quad (1 \leq r \leq n) .$$

For every such sequence b there is defined a Mayer-Vietoris presentation of E

$$\mathbb{E}\langle b \rangle : 0 \longrightarrow E^+ \langle b \rangle \cap E^- \langle b \rangle \longrightarrow E^+ \langle b \rangle \oplus E^- \langle b \rangle \longrightarrow E \longrightarrow 0 ,$$

with

$$E^\pm \langle b \rangle_r = E_r^\pm \langle b_r \rangle = E_r(\mathcal{N}_{b_r}(X^\pm, X)) \quad (0 \leq r \leq n) .$$

□

The Mayer-Vietoris presentations $\mathbb{E}\langle b \rangle$ constructed in 4.6 have the property that any Mayer-Vietoris presentation \mathbb{E} of E can be embedded as a subobject $\mathbb{E} \subseteq \mathbb{E}\langle b \rangle$ for some b .

PROPOSITION 4.7 *Let $X = X^+ \cup X^-$, and let E be a finite chain complex in $\mathbb{C}_X(\mathbb{A})$ with a Mayer-Vietoris presentation*

$$\mathbb{E} : 0 \longrightarrow E^+ \cap E^- \xrightarrow{i} E^+ \oplus E^- \xrightarrow{j} E \longrightarrow 0$$

with bound $b \geq 0$.

(i) *E is $\mathbb{G}_{X, X^+ \cap X^-}(\mathbb{A})$ -contractible if and only if E^+ is*

$\mathbb{G}_{\mathcal{N}_b(X^+, X), \mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ -contractible and E^- is

$\mathbb{G}_{\mathcal{N}_b(X^-, X), \mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ -contractible.

(ii) *E is $\mathbb{C}_{X, X^+ \cap X^-}(\mathbb{A})$ -contractible if and only if E^+ is*

$\mathbb{C}_{\mathcal{N}_b(X^+, X), \mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ -contractible and E^- is

$\mathbb{C}_{\mathcal{N}_b(X^-, X), \mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ -contractible.

PROOF (i) By 3.1 the chain complex $\begin{cases} E \\ E^+ \\ E^- \end{cases}$

$$\begin{cases} \mathbb{G}_{X, X^+ \cap X^-}(\mathbb{A}) \\ \mathbb{G}_{\mathcal{N}_b(X^+, X), \mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})\text{-contractible if and only if it is} \\ \mathbb{G}_{\mathcal{N}_b(X^-, X), \mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}) \\ (\mathbb{G}_X(\mathbb{A}), \mathbb{C}_{\mathcal{N}_c(X^+ \cap X^-, X)}(\mathbb{A})) \\ (\mathbb{G}_{\mathcal{N}_b(X^+, X)}(\mathbb{A}), \mathbb{C}_{\mathcal{N}_{b,d}(X^+, X^-, X)}(\mathbb{A}))\text{-finitely dominated for some} \\ (\mathbb{G}_{\mathcal{N}_b(X^-, X)}(\mathbb{A}), \mathbb{C}_{\mathcal{N}_{e,b}(X^+, X^-, X)}(\mathbb{A})) \\ c \\ d \geq 0. \text{ Since } E^+ \cap E^- \text{ is defined in } \mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}) \text{ the chain com-} \\ e \end{cases}$$

plex E is $\mathbb{G}_{X, X^+ \cap X^-}(\mathbb{A})$ -contractible if and only if E^+ is

$\mathbb{G}_{\mathcal{N}_b(X^+, X), \mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ -contractible and E^- is

$\mathbb{G}_{\mathcal{N}_b(X^-, X), \mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ -contractible.

(ii) As for (i), but using 3.9 instead of 3.1.

□

PROPOSITION 4.8 *For $X = X^+ \cup X^-$ there is a natural identification*

$$K_1(\mathbb{C}_{X, X^+}(\mathbb{A})) = \varinjlim_b K_1(\mathbb{C}_{\mathcal{N}_b(X^-, X), \mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) ,$$

with an exact sequence

$$\begin{aligned} \varinjlim_b K_1(\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) &\longrightarrow K_1(\mathbb{C}_{X^-}(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_{X, X^+}(\mathbb{A})) \\ &\xrightarrow{\partial} \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_{X^-}(\mathbb{A})) . \end{aligned}$$

PROOF Given a morphism $f : L \longrightarrow M$ in $\mathbb{C}_X(\mathbb{A})$ with bound b let C, D, E be the 1-dimensional chain complexes defined by

$$\begin{aligned} d_C &= f : C_1 = L \longrightarrow C_0 = M , \\ d_D &= f_{X^+} : D_1 = L(X^+) \longrightarrow D_0 = M(\mathcal{N}_b(X^+, X)) , \\ d_E &= f_{X^-} : E_1 = L(X^-) \longrightarrow E_0 = M(\mathcal{N}_b(X^-, X)) , \end{aligned}$$

with f_{X^+}, f_{X^-} the restrictions of f . The morphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & L(X^+ \cap X^-) & \longrightarrow & L(X^+) \oplus L(X^-) & \longrightarrow & L \longrightarrow 0 \\ & & \downarrow & & \downarrow f_{X^+} \oplus f_{X^-} & & \downarrow f \\ 0 & \rightarrow & M(\mathcal{N}_b(X^+, X^-, X)) & \rightarrow & M(\mathcal{N}_b(X^+, X)) \oplus M(\mathcal{N}_b(X^-, X)) & \rightarrow & M \rightarrow 0 . \end{array}$$

is a Mayer-Vietoris presentation of C . By 4.7 (ii) the morphism

$$\begin{cases} f : L \longrightarrow M \\ f_{X^+} : L(X^+) \longrightarrow M(\mathcal{N}_b(X^+, X)) \\ f_{X^-} : L(X^-) \longrightarrow M(\mathcal{N}_b(X^-, X)) \end{cases} \text{ is a } \begin{cases} \mathbb{C}_{X, X^+}(\mathbb{A}) \\ \mathbb{C}_{\mathcal{N}_b(X^+, X), \mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}) - \\ \mathbb{C}_{\mathcal{N}_b(X^-, X), \mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}) \end{cases}$$

isomorphism in $\begin{cases} \mathbb{C}_X(\mathbb{A}) \\ \mathbb{C}_{\mathcal{N}_b(X^+, X)}(\mathbb{A}) \\ \mathbb{C}_{\mathcal{N}_b(X^-, X)}(\mathbb{A}) \end{cases}$ if and only if the chain complex $\begin{cases} C \\ D \\ E \end{cases}$

is $\begin{cases} (\mathbb{C}_X(\mathbb{A}), \mathbb{C}_{\mathcal{N}_e(X^+, X)}(\mathbb{A})) \\ (\mathbb{C}_{\mathcal{N}_b(X^+, X)}(\mathbb{A}), \mathbb{C}_{\mathcal{N}_{b,d}(X^+, X^-, X)}(\mathbb{A})) \\ (\mathbb{C}_{\mathcal{N}_b(X^-, X)}(\mathbb{A}), \mathbb{C}_{\mathcal{N}_{e,b}(X^+, X^-, X)}(\mathbb{A})) \end{cases}$ -finitely dominated for some

$\begin{cases} c \\ d \geq 0 \\ e \end{cases}$. Now $M(\mathcal{N}_b(X^+, X^-, X))$ is defined in $\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$, so

that f is a $\mathbb{C}_{X,X^+}(\mathbb{A})$ -isomorphism if f_{X^+} is a $\mathbb{C}_{\mathcal{N}_b(X^+,X),\mathcal{N}_b(X^+,X^-,X)}(\mathbb{A})$ -isomorphism and f_{X^-} is a $\mathbb{C}_{\mathcal{N}_b(X^-,X),\mathcal{N}_b(X^+,X^-,X)}(\mathbb{A})$ -isomorphism. In particular, if $f : M \longrightarrow M$ is a $\mathbb{C}_{X,X^+}(\mathbb{A})$ -automorphism then

$$\begin{aligned} f_{X^-} \oplus 0 : M(\mathcal{N}_b(X^-,X)) &= M(X^-) \oplus M(\mathcal{N}_b(X^-,X) \setminus X^-) \\ &\longrightarrow M(\mathcal{N}_b(X^-,X)) \end{aligned}$$

is a $\mathbb{C}_{\mathcal{N}_b(X^-,X),\mathcal{N}_b(X^+,X^-,X)}(\mathbb{A})$ -automorphism. The morphism

$$K_1(\mathbb{C}_{X,X^+}(\mathbb{A})) \longrightarrow \varinjlim_b K_1(\mathbb{C}_{\mathcal{N}_b(X^-,X),\mathcal{N}_b(X^+,X^-,X)}(\mathbb{A})) ;$$

$$\tau([f] : M \longrightarrow M) \longrightarrow$$

$$\tau([f_{X^-} \oplus 0] : M(\mathcal{N}_b(X^-,X)) \longrightarrow M(\mathcal{N}_b(X^-,X)))$$

is an isomorphism inverse to the morphism

$$\varinjlim_b K_1(\mathbb{C}_{\mathcal{N}_b(X^-,X),\mathcal{N}_b(X^+,X^-,X)}(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_{X,X^+}(\mathbb{A}))$$

induced by the inclusions

$$\mathbb{C}_{\mathcal{N}_b(X^-,X),\mathcal{N}_b(X^+,X^-,X)}(\mathbb{A}) \longrightarrow \mathbb{C}_{X,X^+}(\mathbb{A}) \quad (b \geq 0) .$$

The exact sequences of 4.1

$$\begin{aligned} K_1(\mathbb{C}_{\mathcal{N}_b(X^+,X^-,X)}(\mathbb{A})) &\longrightarrow K_1(\mathbb{C}_{\mathcal{N}_b(X^-,X)}(\mathbb{A})) \\ &\longrightarrow K_1(\mathbb{C}_{\mathcal{N}_b(X^-,X),\mathcal{N}_b(X^+,X^-,X)}(\mathbb{A})) \\ &\xrightarrow{\partial} K_0(\mathbb{P}_{\mathcal{N}_b(X^+,X^-,X)}(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_{\mathcal{N}_b(X^-,X)}(\mathbb{A})) \end{aligned}$$

combined with the natural identifications

$$\begin{aligned} K_1(\mathbb{C}_{\mathcal{N}_b(X^-,X)}(\mathbb{A})) &= K_1(\mathbb{C}_{X^-}(\mathbb{A})) , \\ K_0(\mathbb{P}_{\mathcal{N}_b(X^-,X)}(\mathbb{A})) &= K_0(\mathbb{P}_{X^-}(\mathbb{A})) \end{aligned}$$

give the exact sequence of the statement, on passing to the direct limit as $b \rightarrow \infty$.

□

REMARK 4.9 Working on the level of permutative categories as in Pedersen and Weibel [53],[54] and Anderson, Connolly, Ferry and Pedersen [2] it is possible to extend 4.8 to all the algebraic K -groups of the idempotent completions, with natural excision isomorphisms

$$K_*(\mathbb{P}_{X,X^+}(\mathbb{A})) = \varinjlim_b K_*(\mathbb{P}_{\mathcal{N}_b(X^-,X),\mathcal{N}_b(X^+,X^-,X)}(\mathbb{A}))$$

leading to the Mayer-Vietoris exact sequence of Carlsson [16]

$$\begin{aligned}
\cdots &\longrightarrow \varinjlim_b K_n(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \longrightarrow K_n(\mathbb{P}_{X^+}(\mathbb{A})) \oplus K_n(\mathbb{P}_{X^-}(\mathbb{A})) \\
&\longrightarrow K_n(\mathbb{P}_X(\mathbb{A})) \xrightarrow{\partial} \varinjlim_b K_{n-1}(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \longrightarrow \cdots \\
&\hspace{15em} (n \in \mathbb{Z}) .
\end{aligned}$$

This sequence will be obtained in 4.16 below for $n = 1$ by direct chain complex methods. □

DEFINITION 4.10 Let $X = X^+ \cup X^-$.

(i) A *chain complex band* E is a finite chain complex in $\mathbb{C}_X(\mathbb{A})$ which is $\mathbb{G}_{X, X^+ \cap X^-}(\mathbb{A})$ -contractible, or equivalently (by 3.1)

$(\mathbb{G}_X(\mathbb{A}), \mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}))$ -finitely dominated for some $b \geq 0$.

(ii) The *positive end invariant* of a chain complex band E is the end invariant

$$[E]_+ = [E(X^+)] \in \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}))$$

of the $\mathbb{G}_{\mathcal{N}_b(X^+, X), \mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ -contractible chain complex $E(X^+)$ in $\mathbb{C}_{\mathcal{N}_b(X^+, X), \mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$.

(iii) The *negative end invariant* of a chain complex band E is the end invariant

$$[E]_- = [E(X^-)] \in \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}))$$

of the $\mathbb{G}_{\mathcal{N}_b(X^-, X), \mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ -contractible chain complex $E(X^-)$ in $\mathbb{C}_{\mathcal{N}_b(X^-, X), \mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$. □

EXAMPLE 4.11 Let

$$X = \mathbb{R} , \quad X^+ = \mathbb{R}^+ , \quad X^- = \mathbb{R}^- ,$$

so that

$$\mathcal{N}_b(X^+, X) = [-b, \infty) , \quad \mathcal{N}_b(X^-, X) = (-\infty, b] ,$$

$$\mathcal{N}_b(X^+, X^-, X) = [-b, b] ,$$

with

$$\varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) = K_0(\mathbb{P}_0(\mathbb{A})) .$$

Siebenmann [74] defined a band to be a finite CW complex W together with a finitely dominated infinite cyclic cover \overline{W} . (Such spaces arise in the obstruction theory of Farrell [21] and Siebenmann [76] for fibering

manifolds over the circle - see §20). A classifying map $c : W \longrightarrow S^1$ for \overline{W} lifts to a proper \mathbb{Z} -equivariant map $p = \bar{c} : \overline{W} \longrightarrow X = \mathbb{R}$, and

$$\overline{W} = \overline{W}^+ \cup \overline{W}^- = p^{-1}(X^+) \cup p^{-1}(X^-)$$

has two tame ends ϵ^+, ϵ^- such that

$$\pi_1(\epsilon^+) = \pi_1(\epsilon^-) = \pi_1(\overline{W})$$

with $\overline{W}^+ \cap \overline{W}^- = p^{-1}(X^+ \cap X^-)$ compact.

ϵ^-	\overline{W}^-	$\overline{W}^+ \cap \overline{W}^-$	\overline{W}^+	ϵ^+
\overline{W}				

The cellular chain complex of the universal cover \widetilde{W} of W is a chain complex band $E = C(\widetilde{W})$ in $\mathbb{C}_X(\mathbb{B}^f(\mathbb{Z}[\pi_1(\overline{W})]))$, such that the reduced positive and negative end invariants of E are the end invariants of ϵ^+, ϵ^-

$$[E]_{\pm} = [\epsilon^{\pm}] = [\overline{W}^{\pm}] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\overline{W})]) .$$

□

EXAMPLE 4.12 A contractible finite chain complex E in $\mathbb{C}_{X^+ \cup X^-}(\mathbb{A})$ is a chain complex band.

□

PROPOSITION 4.13 *For any Mayer-Vietoris presentation \mathbb{E} of an n -dimensional chain complex band E in $\mathbb{C}_{X^+ \cup X^-}(\mathbb{A})$*

$$\mathbb{E} : 0 \longrightarrow E^+ \cap E^- \longrightarrow E^+ \oplus E^- \longrightarrow E \longrightarrow 0$$

the subcomplexes $E^+, E^-, E^+ \cap E^- \subseteq E$ are also chain complex bands, with

$$[E^+] - [E]_+ = [E^+ / E(X^+)] ,$$

$$[E^-] - [E]_- = [E^- / E(X^-)] ,$$

$$[E]_+ + [E]_- - [E] = [E(X^+ \cap X^-)]$$

$$\in \text{im}(\varinjlim_b K_0(\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \longrightarrow \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}))) .$$

The reduced end invariants are such that

$$[E]_+ = [E^+] , [E]_- = [E^-] ,$$

$$[E]_+ + [E]_- = [E] \in \varinjlim_b \widetilde{K}_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) .$$

PROOF The subcomplexes $E^+, E^- \subseteq E$ are bands by 4.7 (i), and the intersection $E^+ \cap E^-$ is a band since it is defined in $\mathbb{C}_{\mathcal{N}_b(X, X^+, X^-)}(\mathbb{A})$. The various identities involving the end invariants

$$[E]_+ = [E(X^+)] , \quad [E]_- = [E(X^-)] \in \varinjlim_b \tilde{K}_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}))$$

are given by the projective class sum formula 1.1, using the Noether isomorphism

$$E^+ \cap E^- / E(X^+ \cap X^-) \cong E^+ / E(X^+) \oplus E^- / E(X^-) .$$

□

DEFINITION 4.14 The *end invariants* $[f]_{\pm}$ of an isomorphism $f : L \longrightarrow M$ in $\mathbb{C}_{X^+ \cup X^-}(\mathbb{A})$ are the end invariants $[E]_{\pm}$ of the contractible 1-dimensional chain complex E defined in $\mathbb{C}_{X^+ \cup X^-}(\mathbb{A})$ by

$$d_E = f : E_1 = L \longrightarrow E_0 = M ,$$

that is

$$[f]_{\pm} = [E]_{\pm} \in \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) .$$

□

The $\begin{cases} \text{positive} \\ \text{negative} \end{cases}$ end invariant defined in 4.14 is the image of the end invariant in the sense of 3.6 of the isomorphism

$$\begin{cases} [f(X^+, X^+)] : L(X^+) \longrightarrow M(X^+) \\ [f(X^-, X^-)] : L(X^-) \longrightarrow M(X^-) \end{cases}$$

$$\text{in } \begin{cases} \mathbb{C}_{\mathcal{N}_b(X^+, X), \mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}) \\ \mathbb{C}_{\mathcal{N}_b(X^-, X), \mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}) \end{cases} \begin{cases} [f]_+ = [f(X^+, X^+)]_+ \\ [f]_- = [f(X^-, X^-)]_+ \end{cases} \in \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) .$$

The sum formula of 4.13 gives

$$\begin{aligned} [f]_+ + [f]_- &= [M(X^+ \cap X^-)] - [L(X^+ \cap X^-)] \\ &\in \text{im}(K_0(\mathbb{C}_{X^+ \cap X^-}(\mathbb{A})) \longrightarrow \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}))) . \end{aligned}$$

For any isomorphism $f : L \longrightarrow M$ in $\mathbb{C}_{X^+ \cup X^-}(\mathbb{A})$ the construction of 3.12 gives chain homotopy projections (D^{\pm}, p^{\pm}) in $\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ such that

$$[f]_{\pm} = [D^{\pm}, p^{\pm}] \in \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) .$$

Explicit representatives in $\mathbb{P}_{\mathcal{N}_c(X^+, X^-, X)}(\mathbb{A})$ for $[f]_+$ and $[f]_-$ are now obtained, for some $c \geq 0$.

For any integers $c \geq b \geq 0$ such that

$$f(L(X^+)) \subseteq M(\mathcal{N}_b(X^+, X)) , \quad f^{-1}(M(\mathcal{N}_b(X^+, X))) \subseteq L(\mathcal{N}_c(X^+, X))$$

define an object in the idempotent completion $\mathbb{P}_{\mathcal{N}_c(X^+, X^-, X)}(\mathbb{A})$

$$P_b^+(f) = (L(\mathcal{N}_c(X^+, X) \setminus X^+), p_X - f^{-1} p_{\mathcal{N}_b(X^+, X)} f)$$

such that there is defined a direct sum system in $\mathbb{P}_{\mathcal{N}_c(X^+, X)}(\mathbb{A})$

$$(L(X^+), 1) \begin{array}{c} \xrightarrow{f^+} \\ \xleftarrow{p_X + f^{-1}} \end{array} (M(\mathcal{N}_b(X^+, X)), 1) \begin{array}{c} \xrightarrow{p_X - f^{-1}} \\ \xleftarrow{p_{\mathcal{N}_b(X^+, X)} f} \end{array} P_b^+(f) ,$$

with $f^+ : L(X^+) \rightarrow M(\mathcal{N}_b(X^+, X))$ the restriction of f . Similarly, for any integers $c \geq b \geq 0$ such that

$$f(L(X^-)) \subseteq M(\mathcal{N}_b(X^-, X)) , \quad f^{-1}(M(\mathcal{N}_b(X^-, X))) \subseteq L(\mathcal{N}_c(X^-, X))$$

define an object in the idempotent completion $\mathbb{P}_{\mathcal{N}_c(X^+, X^-, X)}(\mathbb{A})$

$$P_b^-(f) = (L(\mathcal{N}_c(X^-, X) \setminus X^-), p_X + f^{-1} p_{\mathcal{N}_b(X^-, X)} f)$$

such that there is defined a direct sum system in $\mathbb{P}_{\mathcal{N}_c(X^+, X^-, X)}(\mathbb{A})$

$$(L(X^-), 1) \begin{array}{c} \xrightarrow{f^-} \\ \xleftarrow{p_X - f^{-1}} \end{array} (M(\mathcal{N}_b(X^-, X)), 1) \begin{array}{c} \xrightarrow{p_X + f^{-1}} \\ \xleftarrow{p_{\mathcal{N}_b(X^-, X)} f} \end{array} P_b^-(f) ,$$

with $f^- : L(X^-) \rightarrow M(\mathcal{N}_b(X^-, X))$ the restriction of f . There is also defined a direct sum system in $\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$

$$(L(X^+ \cap X^-), 1) \begin{array}{c} \xrightarrow{f^+ \cap f^-} \\ \xleftarrow{p_{X^+ \cap X^-} f^{-1}} \end{array} (M(\mathcal{N}_b(X^+, X^-, X)), 1) \\ \begin{array}{c} \xrightarrow{p_X - f^{-1} \oplus p_{X^+} f^{-1}} \\ \xleftarrow{p_{\mathcal{N}_b(X^+, X)} f \oplus p_{\mathcal{N}_b(X^-, X)} f} \end{array} P_b^+(f) \oplus P_b^-(f) .$$

PROPOSITION 4.15 *The end invariants of an isomorphism $f : L \rightarrow M$ in $\mathbb{C}_{X^+ \cup X^-}(\mathbb{A})$ are given by*

$$[f]_+ = [P_b^+(f)] - [M(\mathcal{N}_b(X^+, X) \setminus X^+)] ,$$

$$[f]_- = [P_b^-(f)] - [M(\mathcal{N}_b(X^-, X) \setminus X^-)]$$

$$\in \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) .$$

PROOF As in 4.10 let E be the 1-dimensional contractible chain complex in $\mathbb{C}_{X^+ \cup X^-}(\mathbb{A})$ defined by

$$d_E = f : E_1 = L \rightarrow E_0 = M .$$

For any integer $b \geq 0$ such that

$$f(L(X^+)) \subseteq M(\mathcal{N}_b(X^+, X)) , \quad f(L(X^-)) \subseteq M(\mathcal{N}_b(X^-, X))$$

define a Mayer-Vietoris presentation $\mathbb{E}\langle b \rangle$ of E

$$\begin{array}{ccccccc}
0 & \longrightarrow & L(X^+ \cap X^-) & \longrightarrow & L(X^+) \oplus L(X^-) & \longrightarrow & L \longrightarrow 0 \\
& & \downarrow f^+ \cap f^- & & \downarrow f^+ \oplus f^- & & \downarrow f \\
0 & \longrightarrow & M(\mathcal{N}_b(X^+ \cap X^-)) & \longrightarrow & M(\mathcal{N}_b(X^+)) \oplus M(\mathcal{N}_b(X^-)) & \longrightarrow & M \longrightarrow 0
\end{array}$$

such that

$$[E^\pm \langle b \rangle] = [P_b^\pm(f)] \in \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) .$$

A direct application of 4.13 gives

$$\begin{aligned}
[f]_+ &= [E]_+ = [E^+ \langle b \rangle] - [E^+ \langle b \rangle / E(X^+)] \\
&= [P_b^+(f)] - [M(\mathcal{N}_b(X^+, X) \setminus X^+)] , \\
[f]_- &= [E]_- = [E^- \langle b \rangle] - [E^- \langle b \rangle / E(X^-)] \\
&= [P_b^-(f)] - [M(\mathcal{N}_b(X^-, X) \setminus X^-)] \\
&\in \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) .
\end{aligned}$$

□

The end invariants of an automorphism $f : M \longrightarrow M$ in $\mathbb{C}_{X^+ \cup X^-}(\mathbb{A})$ are such that

$$[f]_+ + [f]_- = 0 \in \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) .$$

THEOREM 4.16 *The torsion and projective class groups associated to $X = X^+ \cup X^-$ are related by a Mayer-Vietoris exact sequence*

$$\begin{aligned}
&\varinjlim_b K_1(\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_{X^+}(\mathbb{A})) \oplus K_1(\mathbb{C}_{X^-}(\mathbb{A})) \\
&\longrightarrow K_1(\mathbb{C}_X(\mathbb{A})) \xrightarrow{\partial} \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \\
&\longrightarrow K_0(\mathbb{P}_{X^+}(\mathbb{A})) \oplus K_0(\mathbb{P}_{X^-}(\mathbb{A})) ,
\end{aligned}$$

with the connecting map ∂ defined by either of the end invariants

$$\partial : K_1(\mathbb{C}_X(\mathbb{A})) \longrightarrow \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) ;$$

$$\tau(f : M \longrightarrow M) \longrightarrow [f]_+ = -[f]_- .$$

PROOF The relative K -group $K_1(\Delta_b)$ of the additive functor

$$\begin{aligned}
\Delta_b : \mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}) &\longrightarrow \mathbb{P}_{\mathcal{N}_b(X^+, X)}(\mathbb{A}) \times \mathbb{P}_{\mathcal{N}_b(X^-, X)}(\mathbb{A}) ; \\
M &\longrightarrow (M, M)
\end{aligned}$$

fits into an exact sequence

$$\begin{aligned} K_1(\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) &\longrightarrow K_1(\mathbb{C}_{\mathcal{N}_b(X^+, X)}(\mathbb{A})) \oplus K_1(\mathbb{C}_{\mathcal{N}_b(X^-, X)}(\mathbb{A})) \\ &\longrightarrow K_1(\Delta_b) \xrightarrow{\partial} K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \\ &\longrightarrow K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X)}(\mathbb{A})) \oplus K_0(\mathbb{P}_{\mathcal{N}_b(X^-, X)}(\mathbb{A})) . \end{aligned}$$

An element in $K_1(\Delta_b)$ is an equivalence class of quadruples (P, Q, e^+, e^-) consisting of objects P, Q in $\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ and stable isomorphisms $[e^\pm] : P \longrightarrow Q$ in $\mathbb{P}_{\mathcal{N}_b(X^\pm, X)}(\mathbb{A})$. By definition, $(P, Q, e^+, e^-) = 0 \in K_1(\Delta_b)$ if and only if there exists a stable isomorphism $[i] : P \longrightarrow Q$ in $\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ such that

$$\begin{aligned} \tau([e^+]^{-1}[i] : P \longrightarrow P) &= 0 \in K_1(\mathbb{P}_{\mathcal{N}_b(X^+, X)}(\mathbb{A})) , \\ \tau([e^-]^{-1}[i] : P \longrightarrow P) &= 0 \in K_1(\mathbb{P}_{\mathcal{N}_b(X^-, X)}(\mathbb{A})) . \end{aligned}$$

Given objects P^\pm, Q^\pm in $\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$, stable isomorphisms $[g^\pm] : P^\pm \longrightarrow Q^\pm$ in $\mathbb{P}_{\mathcal{N}_b(X^\pm, X)}(\mathbb{A})$ and a stable isomorphism $[h] : P^+ \oplus P^- \longrightarrow Q^+ \oplus Q^-$ in $\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ there is defined an element

$$(P^+, Q^+, g^+, (g^-)^{-1}h) = -(P^-, Q^-, (g^+)^{-1}h, g^-) \in K_1(\Delta_b)$$

with image

$$[P^+] - [Q^+] = [Q^-] - [P^-] \in K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) .$$

Define inverse isomorphisms

$$\begin{aligned} \varinjlim_b K_1(\Delta_b) &\longrightarrow K_1(\mathbb{C}_X(\mathbb{A})) ; (P, Q, e^+, e^-) \longrightarrow \\ &\tau((e^-)^{-1}e^+ : P \longrightarrow P) = -\tau(e^-(e^+)^{-1} : Q \longrightarrow Q) , \\ K_1(\mathbb{C}_X(\mathbb{A})) &\longrightarrow \varinjlim_b K_1(\Delta_b) ; \tau(f : M \longrightarrow M) \longrightarrow \end{aligned}$$

$$(P^+, Q^+, g^+, (g^-)^{-1}h) = -(P^-, Q^-, (g^+)^{-1}h, g^-) ,$$

with P^\pm, Q^\pm, g^\pm, h defined using the terminology of 4.15

$$P^+ = P_b^+(f) , P^- = P_b^-(f) ,$$

$$Q^+ = M(\mathcal{N}_b(X^+, X) \setminus X^+) , Q^- = M(\mathcal{N}_b(X^-, X) \setminus X^-) ,$$

$$g^\pm = (f^\pm p_{\mathcal{N}_b(X^\pm)}f) :$$

$$M(X^\pm) \oplus P^\pm \longrightarrow M(\mathcal{N}_b(X^\pm)) = Q^\pm \oplus M(X^\pm) ,$$

$$h = (f^+ \cap f^- p_{\mathcal{N}_b(X^+, X)}f p_{\mathcal{N}_b(X^-, X)}f) :$$

$$M(X^+ \cap X^-) \oplus P^+ \oplus P^- \longrightarrow$$

$$M(\mathcal{N}_b(X^+ \cap X^-)) = M(X^+ \cap X^-) \oplus Q^+ \oplus Q^- .$$

The composite

$$\varinjlim_b K_1(\Delta_b) \longrightarrow K_1(\mathbb{C}_X(\mathbb{A})) \longrightarrow \varinjlim_b K_1(\Delta_b)$$

is the identity, since for any $(P, Q, e^+, e^-) \in K_1(\Delta_b)$ the stable automorphism

$$\alpha = (e^-)^{-1}e^+ : P \longrightarrow P$$

has positive end invariant

$$[\alpha]_+ = [P] - [Q] \in K_0(\mathbb{P}_{X+\cap X^-}(\mathbb{A})) .$$

In order to verify that the composite

$$K_1(\mathbb{C}_X(\mathbb{A})) \longrightarrow \varinjlim_b K_1(\Delta_b) \longrightarrow K_1(\mathbb{C}_X(\mathbb{A}))$$

is the identity consider the evaluation on the torsion $\tau(f)$ of an automorphism $f : M \longrightarrow M$ in $\mathbb{C}_X(\mathbb{A})$. Let

$$M^+ = M(X^+) , \quad M^- = M(X^-) \subseteq M ,$$

$$i_M^\pm = \text{inclusion} : M^+ \cap M^- \longrightarrow M^\pm ,$$

$$j_M^\pm = \text{inclusion} : M^\pm \longrightarrow M ,$$

$$i_P^\pm = p_{X^\mp} f^{-1} p_{\mathcal{N}_b(X^\pm, X)} f : P^\pm \longrightarrow M^\mp ,$$

$$j_P^\pm = j_M^\mp i_P^\pm = f^{-1} p_{\mathcal{N}_b(X^\pm, X)} f : P^\pm \longrightarrow M ,$$

$$i_Q^\pm = \text{inclusion} : Q^\pm \longrightarrow M^\mp ,$$

$$j_Q^\pm = j_M^\mp i_Q^\pm = \text{inclusion} : Q^\pm \longrightarrow M ,$$

so that f is resolved by an isomorphism of exact sequences

$$\begin{array}{ccccccc} 0 \longrightarrow & M^+ \cap M^- \oplus P^+ \oplus P^- & \xrightarrow{u_P} & M^+ \oplus M^- \oplus P^+ \oplus P^- & \xrightarrow{v_P} & M & \longrightarrow 0 \\ & \downarrow h & & \downarrow g^+ \oplus g^- & & \downarrow f & \\ 0 \longrightarrow & M^+ \cap M^- \oplus Q^+ \oplus Q^- & \xrightarrow{u_Q} & M^+ \oplus M^- \oplus Q^+ \oplus Q^- & \xrightarrow{v_Q} & M & \longrightarrow 0 \end{array}$$

with

$$u_P = \begin{pmatrix} i_M^+ & 0 & -i_P^- \\ -i_M^- & -i_P^+ & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} :$$

$$M^+ \cap M^- \oplus P^+ \oplus P^- \longrightarrow M^+ \oplus M^- \oplus P^+ \oplus P^- ,$$

$$v_P = (j_M^+ \ j_M^- \ j_P^+ \ j_P^-) : M^+ \oplus M^- \oplus P^+ \oplus P^- \longrightarrow M ,$$

$$u_Q = \begin{pmatrix} i_M^+ & 0 & -i_Q^- \\ -i_M^- & -i_Q^+ & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} :$$

$$M^+ \cap M^- \oplus Q^+ \oplus Q^- \longrightarrow M^+ \oplus M^- \oplus Q^+ \oplus Q^- ,$$

$$v_Q = (j_M^+ \ j_M^- \ j_Q^+ \ j_Q^-) : M^+ \oplus M^- \oplus Q^+ \oplus Q^- \longrightarrow M .$$

The automorphisms

$$\begin{aligned} \beta_P &= \begin{pmatrix} 1 & 0 & 0 & i_P^- \\ 0 & 1 & i_P^+ & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &: M^+ \oplus M^- \oplus P^+ \oplus P^- \longrightarrow M^+ \oplus M^- \oplus P^+ \oplus P^- , \\ \beta_Q &= \begin{pmatrix} 1 & 0 & 0 & i_Q^- \\ 0 & 1 & i_Q^+ & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &: M^+ \oplus M^- \oplus Q^+ \oplus Q^- \longrightarrow M^+ \oplus M^- \oplus Q^+ \oplus Q^- \end{aligned}$$

have torsion

$$\tau(\beta_P) = \tau(\beta_Q) = 0 \in K_1(\mathbb{P}_X(\mathbb{A})) = K_1(\mathbb{C}_X(\mathbb{A})) ,$$

and are such that there is defined an isomorphism of exact sequences

$$\begin{array}{ccccccc} 0 \longrightarrow & M^+ \cap M^- \oplus P^+ \oplus P^- & \xrightarrow{i_M \oplus 1} & M^+ \oplus M^- \oplus P^+ \oplus P^- & \xrightarrow{j_M \oplus 1} & M & \longrightarrow 0 \\ & \downarrow h & & \downarrow g & & \downarrow f & \\ 0 \longrightarrow & M^+ \cap M^- \oplus Q^+ \oplus Q^- & \xrightarrow{i_M \oplus 1} & M^+ \oplus M^- \oplus Q^+ \oplus Q^- & \xrightarrow{j_M \oplus 1} & M & \longrightarrow 0 \end{array}$$

with

$$\begin{aligned} i_M &= \begin{pmatrix} i_M^+ \\ -i_M^- \end{pmatrix} : M^+ \cap M^- \longrightarrow M^+ \oplus M^- , \\ j_M &= (j_M^+ \ j_M^-) : M^+ \oplus M^- \longrightarrow M , \\ g &= (\beta_Q)^{-1}(g^+ \oplus g^-)\beta_P : M^+ \oplus M^- \oplus P^+ \oplus P^- \\ &\longrightarrow M^+ \oplus M^- \oplus Q^+ \oplus Q^- . \end{aligned}$$

Applying Lemma 1.4

$$\tau(f) = \tau([h]^{-1}[g]) = \tau([h]^{-1}[g^+ \oplus g^-]) \in K_1(\mathbb{C}_X(\mathbb{A})) ,$$

so that the composite

$$K_1(\mathbb{C}_X(\mathbb{A})) \longrightarrow \varinjlim_b K_1(\Delta_b) \longrightarrow K_1(\mathbb{C}_X(\mathbb{A}))$$

is the identity. This gives the exact sequence of Carlsson [16]

$$\begin{aligned}
& \varinjlim_b K_1(\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \longrightarrow \\
& \quad \varinjlim_b K_1(\mathbb{C}_{\mathcal{N}_b(X^+, X)}(\mathbb{A})) \oplus \varinjlim_b K_1(\mathbb{C}_{\mathcal{N}_b(X^-, X)}(\mathbb{A})) \\
& \longrightarrow K_1(\mathbb{C}_X(\mathbb{A})) \xrightarrow{\partial} \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \\
& \longrightarrow \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X)}(\mathbb{A})) \oplus \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^-, X)}(\mathbb{A})) .
\end{aligned}$$

The inclusions

$$\mathbb{C}_{X^\pm}(\mathbb{A}) \longrightarrow \mathbb{C}_{\mathcal{N}_b(X^\pm, X)}(\mathbb{A}) \quad (b \geq 0)$$

are homotopy equivalences in the proper eventually Lipschitz category, giving rise to identifications

$$\begin{aligned}
\varinjlim_b K_1(\mathbb{C}_{\mathcal{N}_b(X^\pm, X)}(\mathbb{A})) &= K_1(\mathbb{C}_{X^\pm}(\mathbb{A})) , \\
\varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^\pm, X)}(\mathbb{A})) &= K_0(\mathbb{P}_{X^\pm}(\mathbb{A})) ,
\end{aligned}$$

and hence to the exact sequence of the statement. □

The connecting map in 4.15 can be expressed as the composite

$$\begin{aligned}
& \partial : K_1(\mathbb{C}_X(\mathbb{A})) \longrightarrow \\
& K_1(\mathbb{C}_{X, X^+}(\mathbb{A})) = \varinjlim_b K_1(\mathbb{C}_{\mathcal{N}_b(X \setminus X^+, X), \mathcal{N}_b(X^+, X \setminus X^+, X)}(\mathbb{A})) \\
& \xrightarrow{\lim \partial_b} \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X \setminus X^+, X)}(\mathbb{A})) = \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) ,
\end{aligned}$$

with the connecting maps ∂_b given by 4.8.

REMARK 4.16 Working on the level of permutative categories as in Pedersen and Weibel [53], [54] and Carlsson [16] it is possible to extend the results of §4 to all the algebraic K -groups of the idempotent completions, with natural isomorphisms

$$\begin{aligned}
K_*(\mathbb{P}_Y(\mathbb{A}) \longrightarrow \mathbb{P}_X(\mathbb{A})) &\cong K_*(\mathbb{P}_{X, Y}(\mathbb{A})) \\
&\cong \varinjlim_b K_*(\mathbb{P}_{\mathcal{N}_b(X \setminus Y, X), \mathcal{N}_b(X \setminus Y, Y, X)}(\mathbb{A}))
\end{aligned}$$

with $\mathbb{P}_{X, Y}(\mathbb{A})$ the idempotent completion of $\mathbb{C}_{X, Y}(\mathbb{A})$, and a Mayer-

Vietoris exact sequence

$$\begin{aligned} \dots \longrightarrow \varinjlim_b K_n(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) &\longrightarrow K_n(\mathbb{P}_{X^+}(\mathbb{A})) \oplus K_n(\mathbb{P}_{X^-}(\mathbb{A})) \\ &\longrightarrow K_n(\mathbb{P}_X(\mathbb{A})) \xrightarrow{\partial} \varinjlim_b K_{n-1}(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \longrightarrow \dots \\ &\quad (n \in \mathbb{Z}) . \end{aligned}$$

□

§5. Isomorphism torsion

In general, it is not possible to define the torsion $\tau(f) \in K_1(\mathbb{A})$ of an isomorphism $f : L \longrightarrow M$ in \mathbb{A} , only for an automorphism $f : M \longrightarrow M$. This is necessary in order to define the torsion $\tau(C) \in K_1(\mathbb{A})$ for a contractible finite chain complex C in \mathbb{A} . The isomorphism torsion theory of Ranicki [65] is now applied to the bounded categories, for use in §§6,7.

The *isomorphism torsion group* $K_1^{iso}(\mathbb{A})$ was defined in [65] for any additive category \mathbb{A} to be the abelian group with one generator $\tau(f)$ for each isomorphism $f : L \longrightarrow M$ in \mathbb{A} , subject to the relations

$$\begin{aligned} \text{(i)} \quad \tau(gf : L \longrightarrow M \longrightarrow N) &= \tau(f : L \longrightarrow M) + \tau(g : M \longrightarrow N) , \\ \text{(ii)} \quad \tau(f \oplus f' : L \oplus L' \longrightarrow M \oplus M') \\ &= \tau(f : L \longrightarrow M) + \tau(f' : L' \longrightarrow M') . \end{aligned}$$

The *isomorphism torsion* of a contractible finite chain complex E in \mathbb{A} is defined by

$$\tau(E) = \tau(d + \Gamma : E_{odd} \longrightarrow E_{even}) \in K_1^{iso}(\mathbb{A}) ,$$

with

$$d + \Gamma = \begin{pmatrix} d & 0 & 0 & \dots \\ \Gamma & d & 0 & \dots \\ 0 & \Gamma & d & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} :$$

$$E_{odd} = E_1 \oplus E_3 \oplus E_5 \oplus \dots \longrightarrow E_{even} = E_0 \oplus E_2 \oplus E_4 \oplus \dots$$

the isomorphism in \mathbb{A} defined for any chain contraction

$$\Gamma : 0 \simeq 1 : E \longrightarrow E .$$

Given objects L, M in \mathbb{A} define the *sign*

$$\epsilon(L, M) = \tau\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : L \oplus M \longrightarrow M \oplus L\right) \in K_1^{iso}(\mathbb{A}) .$$

Also, given finite chain complexes C, D in \mathbb{A} define

$$\begin{aligned} \beta(C, D) &= \tau((C \oplus D)_{\text{even}} \longrightarrow C_{\text{even}} \oplus D_{\text{even}}) \\ &\quad - \tau((C \oplus D)_{\text{odd}} \longrightarrow C_{\text{odd}} \oplus D_{\text{odd}}) \\ &= \sum_{i>j} (\epsilon(C_{2i}, D_{2j}) - \epsilon(C_{2i+1}, D_{2j+1})) \in K_1^{\text{iso}}(\mathbb{A}) . \end{aligned}$$

The *isomorphism torsion* of a chain equivalence $f : C \longrightarrow D$ of round finite chain complexes in \mathbb{A} was defined in Ranicki [65] to be

$$\tau(f) = \tau(C(f)) - \beta(D, SC) \in K_1^{\text{iso}}(\mathbb{A}) ,$$

with $C(f)$ the algebraic mapping cone of f .

PROPOSITION 5.1 (i) *For any chain equivalences $f : C \longrightarrow D, g : D \longrightarrow E$ of round finite chain complexes in \mathbb{A}*

$$\tau(gf : C \longrightarrow E) = \tau(f : C \longrightarrow D) + \tau(g : D \longrightarrow E) \in K_1^{\text{iso}}(\mathbb{A}) .$$

(ii) *Let C be a contractible finite chain complex in \mathbb{A} , and let*

$$0 \longrightarrow C'' \xrightarrow{f} C' \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence in \mathbb{A} with C', C'' round finite. Then $f : C'' \longrightarrow C'$ is a chain equivalence with torsion

$$\tau(f) = \tau(C) + \sum_{r=0}^{\infty} (-)^r \tau((f h) : C''_r \oplus C_r \longrightarrow C'_r) + \beta(C'', C) \in K_1^{\text{iso}}(\mathbb{A})$$

for any splitting morphisms $h : C_r \longrightarrow C'_r$ of $g : C'_r \longrightarrow C_r$ ($r \geq 0$). If also C', C'' are contractible then

$$\tau(f) = \tau(C') - \tau(C'') \in K_1^{\text{iso}}(\mathbb{A}) .$$

PROOF See [65, 4.2].

□

The *reduced isomorphism torsion group* of an additive category \mathbb{A} is defined by

$$\tilde{K}_1^{\text{iso}}(\mathbb{A}) = \text{coker}(\epsilon : K_0(\mathbb{A}) \otimes K_0(\mathbb{A}) \longrightarrow K_1^{\text{iso}}(\mathbb{A})) ,$$

with ϵ the sign pairing. The *torsion* of a chain equivalence $f : C \longrightarrow D$ of finite chain complexes in \mathbb{A} is the reduced torsion of the algebraic mapping cone

$$\tau(f) = \tau(C(f)) \in \tilde{K}_1^{\text{iso}}(\mathbb{A}) .$$

The *reduced automorphism torsion group* of an additive category \mathbb{A} is defined by

$$\tilde{K}_1(\mathbb{A}) = \text{coker}(\epsilon : K_0(\mathbb{A}) \otimes K_0(\mathbb{A}) \longrightarrow K_1(\mathbb{A})) ,$$

with ϵ the sign pairing.

DEFINITION 5.2 A *stable canonical structure* $[\phi]$ on an additive category \mathbb{A} is a collection of stable isomorphisms $[\phi_{M,N}] : M \longrightarrow N$, one for each ordered pair (M, N) of stably isomorphic objects in \mathbb{A} , such that

- (i) $[\phi_{M,M}] = [1_M] : M \longrightarrow M$,
- (ii) $[\phi_{M,P}] = [\phi_{N,P}][\phi_{M,N}] : M \longrightarrow N \longrightarrow P$,
- (iii) $[\phi_{M \oplus M', N \oplus N'}] = [\phi_{M,N}] \oplus [\phi_{M',N'}] : M \oplus M' \longrightarrow N \oplus N'$.

□

A stable canonical structure $[\phi]$ on \mathbb{A} splits the natural map

$$K_1(\mathbb{A}) \longrightarrow K_1^{iso}(\mathbb{A}) ; \tau(f : M \longrightarrow M) \longrightarrow \tau(f)$$

by

$$K_1^{iso}(\mathbb{A}) \longrightarrow K_1(\mathbb{A}) ;$$

$$\tau(f : M \longrightarrow N) \longrightarrow \tau(f) = \tau([\phi_{N,M}]f : M \longrightarrow M) .$$

The automorphism torsion of a contractible finite chain complex E in \mathbb{A} is then defined to be the image of the isomorphism torsion

$$\tau(E) = \tau(d + \Gamma : E_{odd} \longrightarrow E_{even}) \in K_1(\mathbb{A}) .$$

A stable isomorphism $[f] : M \longrightarrow N$ has an isomorphism torsion $\tau([f]) \in K_1^{iso}(\mathbb{A})$ and hence also an automorphism torsion $\tau([f]) \in K_1(\mathbb{A})$. Similarly for reduced torsion.

For a ring A the automorphism torsion groups of the additive category $\mathbb{B}^f(A)$ of based f.g. free A -modules are the usual torsion groups

$$K_1(\mathbb{B}^f(A)) = K_1(A) ,$$

$$\tilde{K}_1(\mathbb{B}^f(A)) = \tilde{K}_1(A) = \text{coker}(K_1(\mathbb{Z}) \longrightarrow K_1(A)) .$$

If A is such that the rank of based f.g. free A -module is well-defined then $\mathbb{B}^f(A)$ has the canonical (un)stable structure $[\phi]$ with $\phi_{M,N} : M \longrightarrow N$ the isomorphism sending the base of M to the base of N .

A stable canonical structure (5.2) on an additive category \mathbb{A} allows the definition of torsion $\tau(f) \in K_1(\mathbb{A})$ for an isomorphism $f : L \longrightarrow M$ in \mathbb{A} , and hence the definition of torsion $\tau(C) \in K_1(\mathbb{A})$ for a contractible finite chain complex C in \mathbb{A} . A ‘flasque structure’ on an additive category \mathbb{A} determines a stable canonical structure; the bounded graded categories over open cones have flasque structures, allowing torsion to be defined for bounded homotopy equivalences over open cones.

DEFINITION 5.3 A *flasque structure* $\{\Sigma, \sigma, \rho\}$ on an additive category \mathbb{A} consists of

- (i) an object ΣM for each object M of \mathbb{A} ,
- (ii) an isomorphism $\sigma_M : M \oplus \Sigma M \longrightarrow \Sigma M$ for each object M of \mathbb{A} ,

- (iii) an isomorphism $\rho_{M,N} : \Sigma(M \oplus N) \longrightarrow \Sigma M \oplus \Sigma N$ for each pair of objects M, N in \mathbb{A} , such that

$$\begin{aligned} \sigma_{M \oplus N} &= (\rho_{M,N})^{-1}(\sigma_M \oplus \sigma_N)(1_{M \oplus N} \oplus \rho_{M,N}) : \\ M \oplus N \oplus \Sigma(M \oplus N) &\longrightarrow \Sigma(M \oplus N) . \end{aligned}$$

□

LEMMA 5.4 *A flasque structure $\{\Sigma, \sigma, \rho\}$ on \mathbb{A} determines a stable canonical structure $[\phi]$ on \mathbb{A} , with*

$$\phi_{M,N} = (\sigma_N \oplus 1)^{-1}(\sigma_M \oplus 1) : M \oplus \Sigma M \oplus \Sigma N \longrightarrow N \oplus \Sigma M \oplus \Sigma N ,$$

and hence a splitting map $K_1^{iso}(\mathbb{A}) \longrightarrow K_1(\mathbb{A})$ allowing the definition of torsion $\tau(C) \in K_1(\mathbb{A})$ for any contractible finite chain complex C on \mathbb{A} .

□

In particular, if \mathbb{A} admits a flasque structure every object M in \mathbb{A} is stably isomorphic to 0 (via σ_M), so that $K_0(\mathbb{A}) = 0$.

DEFINITION 5.5 A flasque structure $\{\Sigma, \sigma, \rho\}$ on \mathbb{A} is *natural* if the assignment $M \longrightarrow \Sigma M$ on objects of \mathbb{A} can be extended to morphisms

$$(f : M \longrightarrow N) \longrightarrow (\Sigma f : \Sigma M \longrightarrow \Sigma N) ,$$

defining an additive functor $\Sigma : \mathbb{A} \longrightarrow \mathbb{A}$ such that the isomorphisms $\sigma_M : M \oplus \Sigma M \longrightarrow \Sigma M$ define a natural equivalence of functors

$$\sigma : 1_{\mathbb{A}} \oplus \Sigma \longrightarrow \Sigma : \mathbb{A} \longrightarrow \mathbb{A} .$$

□

If \mathbb{A} admits a natural flasque structure then $K_*(\mathbb{A}) = 0$, and in particular $K_1(\mathbb{A}) = 0$.

EXAMPLE 5.6 An additive category \mathbb{A} with countable direct sums has a natural flasque structure $\{\Sigma, \sigma, \rho\}$ by the Eilenberg swindle, with

$$\Sigma M = \sum_1^{\infty} M ,$$

$$\sigma_M : M \oplus \Sigma M \longrightarrow \Sigma M ; (x_0, (x_1, x_2, \dots)) \longrightarrow (x_0, x_1, x_2, \dots) .$$

□

PROPOSITION 5.7 (i) $\mathbb{C}_{\mathbb{R}}(\mathbb{A})$ has a flasque structure.

(ii) $\mathbb{C}_{\mathbb{R}^+}(\mathbb{A})$ has a natural flasque structure.

PROOF (i) For any object M in $\mathbb{C}_{\mathbb{R}}(\mathbb{A})$ let TM be the object defined by

$$(TM)(t) = \begin{cases} M(t-1) & \text{if } t \geq 1 \\ M(t+1) & \text{if } t < -1 \\ 0 & \text{otherwise} , \end{cases}$$

and let $T : M \longrightarrow TM$ be the isomorphism bounded by 1 defined by

$$T(v, u) : M(u) \longrightarrow TM(v) ; a \longrightarrow \begin{cases} a & \text{if } u \geq 0 \text{ and } v = u + 1 \\ a & \text{if } u < 0 \text{ and } v = u - 1 \\ 0 & \text{otherwise .} \end{cases}$$

Define a flasque structure $\{\Sigma, \sigma, \rho\}$ on $\mathbb{C}_{\mathbb{R}}(\mathbb{A})$ by

$$\Sigma M = \sum_{i=1}^{\infty} T^i M ,$$

$$\sigma_M : M \oplus \Sigma M \longrightarrow \Sigma M ;$$

$$(a_0, (a_1, a_2, \dots)) \longrightarrow (Ta_0, Ta_1, Ta_2, \dots) ,$$

$$\rho_{M,N} : \Sigma(M \oplus N) \longrightarrow \Sigma M \oplus \Sigma N ; (a, b) \longrightarrow (a, b) .$$

(ii) The flasque structure on $\mathbb{C}_{\mathbb{R}}(\mathbb{A})$ defined in (i) restricts to a natural flasque structure on $\mathbb{C}_{\mathbb{R}^+}(\mathbb{A})$.

□

LEMMA 5.8 *If $F : \mathbb{A} \longrightarrow \mathbb{A}'$ is a functor of additive categories such that every object M' of \mathbb{A}' is the image $M' = F(M)$ of an object M in \mathbb{A} then a flasque structure $\{\Sigma, \sigma, \rho\}$ on \mathbb{A} determines a flasque structure $\{\Sigma', \sigma', \rho'\}$ on \mathbb{A}' , with*

$$\Sigma' M' = F(\Sigma M) ,$$

$$\begin{aligned} \sigma'_{M'} = F(\sigma_M) : M' \oplus \Sigma' M' &= F(M \oplus \Sigma M) \\ &\longrightarrow \Sigma' M' = F(\Sigma M) , \end{aligned}$$

$$\begin{aligned} \rho_{M', N'} = F(\rho_{M, N}) : \Sigma'(M' \oplus N') &= F(\Sigma(M \oplus N)) \\ &\longrightarrow \Sigma' M' \oplus \Sigma' N' = F(\Sigma M \oplus \Sigma N) . \end{aligned}$$

□

In particular, 5.8 applies to the functor $F : \mathbb{C}_X(\mathbb{A}) \longrightarrow \mathbb{C}_{X,Y}(\mathbb{A})$ induced by the inclusion.

PROPOSITION 5.9 *Let $(X, Y \subseteq X)$ be a pair of metric spaces, and let \mathbb{A} be any filtered additive category.*

(i) *The end invariant defines a morphism*

$$\partial^{iso} : K_1^{iso}(\mathbb{C}_{X,Y}(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_Y(\mathbb{A})) ; \tau(E) \longrightarrow [E]_+$$

such that the connecting map ∂ in the exact sequence of 4.1

$$\begin{aligned} K_1(\mathbb{C}_Y(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_X(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_{X,Y}(\mathbb{A})) \\ \xrightarrow{\partial} K_0(\mathbb{P}_Y(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_X(\mathbb{A})) \end{aligned}$$

is the composite

$$\partial : K_1(\mathbb{C}_{X,Y}(\mathbb{A})) \longrightarrow K_1^{iso}(\mathbb{C}_{X,Y}(\mathbb{A})) \xrightarrow{\partial^{iso}} K_0(\mathbb{P}_Y(\mathbb{A})) ,$$

and

$$\mathrm{im}(\partial) \subseteq \mathrm{im}(\partial^{iso}) .$$

(ii) *The sequence*

$$K_1^{iso}(\mathbb{C}_{X,Y}(\mathbb{A})) \xrightarrow{\tilde{\partial}^{iso}} \tilde{K}_0(\mathbb{P}_Y(\mathbb{A})) \longrightarrow \tilde{K}_0(\mathbb{P}_X(\mathbb{A}))$$

is exact.

(iii) *If $K_0(\mathbb{C}_X(\mathbb{A})) = \{0\}$ then*

$$\mathrm{im}(\partial) = \mathrm{im}(\partial^{iso}) .$$

(iv) *If $\mathbb{C}_X(\mathbb{A})$ admits a flasque structure $\{\Sigma, \sigma, \rho\}$ and $\mathbb{C}_{X,Y}(\mathbb{A})$ has the induced flasque structure then*

$$\partial^{iso} : K_1^{iso}(\mathbb{C}_{X,Y}(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_{X,Y}(\mathbb{A})) \xrightarrow{\partial} K_0(\mathbb{P}_Y(\mathbb{A})) .$$

The automorphism torsion $\tau(E) \in K_1(\mathbb{C}_{X,Y}(\mathbb{A}))$ of any contractible finite chain complex E is such that $\partial\tau(E) = [E]_+$.

PROOF (i) The end invariant of an isomorphism $f : M \longrightarrow N$ in $\mathbb{C}_{X,Y}(\mathbb{A})$ is the element $[f]_+ \in K_0(\mathbb{P}_Y(\mathbb{A}))$ defined in 3.6. It is necessary to prove that for any contractible finite chain complex E in $\mathbb{C}_{X,Y}(\mathbb{A})$ the element $\partial\tau(E) = [E]_+ \in K_0(\mathbb{P}_Y(\mathbb{A}))$ is the end invariant $[d + \Gamma]_+$ of the isomorphism $d + \Gamma : E_{\text{odd}} \longrightarrow E_{\text{even}}$ in $\mathbb{C}_{X,Y}(\mathbb{A})$ used to define $\tau(E)$, for any chain contraction $\Gamma : 0 \simeq 1 : E \longrightarrow E$. Consider first the 1-dimensional case, with

$$d_E = f : E_1 = L \longrightarrow E_0 = M$$

an isomorphism in $\mathbb{C}_{X,Y}(\mathbb{A})$, so that $\partial\tau([f]) \in K_0(\mathbb{P}_Y(\mathbb{A}))$ is the end invariant $[f]_+$ of f . The n -dimensional case is reduced to the 1-dimensional case by the “folding” process of Whitehead [88], as follows.

Let then E be n -dimensional

$$E : \dots \longrightarrow 0 \longrightarrow E_n \xrightarrow{d} E_{n-1} \longrightarrow \dots \longrightarrow E_0 .$$

As E is contractible there exists a morphism $\Gamma : E_{n-1} \longrightarrow E_n$ in $\mathbb{C}_{X,Y}(\mathbb{A})$ which splits $d : E_n \longrightarrow E_{n-1}$ with

$$\Gamma d = 1 : E_n \longrightarrow E_n .$$

Let C be the contractible $(n-1)$ -dimensional chain complex in $\mathbb{C}_{X,Y}(\mathbb{A})$ defined by

$$C : E_{n-1} \xrightarrow{\begin{pmatrix} d \\ \Gamma \end{pmatrix}} E_{n-2} \oplus E_n \xrightarrow{(d \ 0)} E_{n-3} \longrightarrow \dots \longrightarrow E_0 ,$$

and let D be the $\mathbb{C}_{X,Y}(\mathbb{A})$ -contractible n -dimensional chain complex in

$\mathbb{C}_X(\mathbb{A})$ defined by

$$\begin{aligned} D : E_n &\xrightarrow{\begin{pmatrix} d \\ -1 \end{pmatrix}} E_{n-1} \oplus E_n \xrightarrow{\begin{pmatrix} d & 0 \\ \Gamma & 1 \end{pmatrix}} \\ &E_{n-2} \oplus E_n \xrightarrow{\begin{pmatrix} d & 0 \end{pmatrix}} E_{n-3} \longrightarrow \dots \longrightarrow E_0 . \end{aligned}$$

There are defined exact sequences of contractible chain complexes in $\mathbb{C}_{X,Y}(\mathbb{A})$

$$\begin{aligned} 0 &\longrightarrow C' \longrightarrow D \longrightarrow C \longrightarrow 0 , \\ 0 &\longrightarrow E' \longrightarrow D \longrightarrow E \longrightarrow 0 , \end{aligned}$$

with C', E' the elementary n -dimensional chain complexes

$$\begin{aligned} C' : E_n &\xrightarrow{1} E_n \longrightarrow 0 \longrightarrow \dots \longrightarrow 0 , \\ E' : 0 &\longrightarrow E_n \xrightarrow{1} E_n \longrightarrow 0 \longrightarrow \dots \longrightarrow 0 . \end{aligned}$$

Now

$$\tau(C) = \tau(D) = \tau(E) \in K_1(\mathbb{C}_{X,Y}(\mathbb{A})) ,$$

and by the sum formula 3.8

$$[C]_+ = [D]_+ = [E]_+ \in K_0(\mathbb{P}_Y(\mathbb{A})) .$$

As C is $(n-1)$ -dimensional this gives the inductive step in the proof that the connecting map ∂ of 4.1 sends the torsion $\tau(E)$ to the positive end invariant $[d + \Gamma : E_{\text{odd}} \longrightarrow E_{\text{even}}]_+$.

(ii) The objects P, Q in $\mathbb{P}_Y(\mathbb{A})$ are such that $[P] = [Q] \in \tilde{K}_0(\mathbb{P}_X(\mathbb{A}))$ if and only if there exists an isomorphism in $\mathbb{P}_X(\mathbb{A})$

$$f = \begin{pmatrix} f_{QP} & f_{QR} \\ f_{SP} & f_{SR} \end{pmatrix} : P \oplus R \longrightarrow Q \oplus S$$

for some objects R, S in $\mathbb{C}_X(\mathbb{A})$, in which case $[f_{SR}] : R \longrightarrow S$ is an isomorphism in $\mathbb{C}_{X,Y}(\mathbb{A})$ with isomorphism torsion

$$\tau([f_{SR}]) \in K_1^{\text{iso}}(\mathbb{C}_{X,Y}(\mathbb{A}))$$

such that

$$\tilde{\partial}^{\text{iso}} \tau([f_{SR}]) = [Q] - [P] \in \tilde{K}_0(\mathbb{P}_Y(\mathbb{A})) .$$

(iii) The objects of $\mathbb{C}_{X,Y}(\mathbb{A})$ are the objects of $\mathbb{C}_X(\mathbb{A})$, so for any isomorphism $f : M \longrightarrow N$ in $\mathbb{C}_{X,Y}(\mathbb{A})$ the element $\partial^{\text{iso}} \tau(f) \in K_0(\mathbb{P}_Y(\mathbb{A}))$ has image

$$\partial^{\text{iso}} \tau(f) = [N] - [M] \in \text{im}(K_0(\mathbb{C}_X(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_Y(\mathbb{A}))) = \{0\} .$$

(iv) By 3.7 (iii) the end invariants of the isomorphisms $\sigma_M : M \oplus \Sigma M \longrightarrow \Sigma M$ in $\mathbb{C}_X(\mathbb{A})$ have end invariants

$$[\sigma_M]_+ = 0 \in K_0(\mathbb{P}_Y(\mathbb{A})) .$$

The torsion of an isomorphism $f : L \longrightarrow M$ in $\mathbb{C}_{X,Y}(\mathbb{A})$ is defined by

$$\tau([f]) = \tau([\sigma_M][f][\sigma_L]^{-1}) \in K_1(\mathbb{C}_{X,Y}(\mathbb{A})) ,$$

so $\partial\tau([f]) \in K_0(\mathbb{P}_Y(\mathbb{A}))$ is the end invariant of the stable automorphism $[\sigma_M][f][\sigma_L]^{-1} : 0 \longrightarrow 0$ in $\mathbb{C}_{X,Y}(\mathbb{A})$. By the sum formula 3.7 (ii)

$$\begin{aligned} \partial(\tau[f]) &= [[\sigma_M][f][\sigma_L]^{-1}]_+ \\ &= [\sigma_M]_+ + [f]_+ - [\sigma_L]_+ = [f]_+ \in K_0(\mathbb{P}_Y(\mathbb{A})) , \end{aligned}$$

verifying the 1-dimensional case of $\partial\tau(E) = [E]_+$. The n -dimensional case for $n \geq 2$ is reduced to the case $n = 1$ by folding as in (i). □

§6. Open cones

The *open cone* of a subspace $X \subseteq S^k$ is the metric space

$$O(X) = \{tx \in \mathbb{R}^{k+1} \mid t \in \mathbb{R}^+, x \in X\} .$$

In particular,

$$O(S^k) = \mathbb{R}^{k+1} .$$

For the empty set \emptyset it is understood that

$$O(\emptyset) = \{0\} , \quad \mathbb{C}_{O(\emptyset)}(\mathbb{A}) = \mathbb{A} .$$

For any $Y \subseteq X \subseteq S^k$ and any $b \geq 0$ let

$$\begin{aligned} O_b(Y, X) &= \mathcal{N}_b(O(Y), O(X)) \\ &= \{x \in O(X) \mid d(x, y) \leq b \text{ for some } y \in O(Y)\} . \end{aligned}$$

Also, for any $Y, Z \subseteq X \subseteq S^k$ and any $b \geq 0$ let

$$\begin{aligned} O_b(Y, Z, X) &= \mathcal{N}_b(O(Y), O(Z), O(X)) \\ &= \{x \in O(X) \mid d(x, y) \leq b \text{ for some } y \in O(Y) \\ &\quad \text{and } d(x, z) \leq b \text{ for some } z \in O(Z)\} . \end{aligned}$$

LEMMA 6.1 *Let $X \subseteq S^k$ be a compact polyhedron which is expressed as a union $X = X^+ \cup X^-$ of compact polyhedra. For any $b \geq 0$ the inclusion*

$$O(X^+ \cap X^-) \longrightarrow O_b(X^+, X^-, X)$$

is a homotopy equivalence in the proper eventually Lipschitz category.

PROOF Following a suggestion of Steve Ferry, replace X^+, X^- by Lipschitz homotopy equivalent cubical subcomplexes of S^k , for which there is an identity

$$O_b(X^+, X^-, X) = O_b(X^+ \cap X^-, X) .$$

□

The results of §1-§5 will now be specialized to the $O(X)$ -bounded categories $\mathbb{C}_{O(X)}(\mathbb{A})$.

PROPOSITION 6.2 (i) *Let $Y \subseteq X \subseteq S^k$. The torsion and class groups of the bounded categories over the open cones are related by an excision isomorphism*

$$K_1(\mathbb{P}_{O(Y)}(\mathbb{A})) \longrightarrow \mathbb{P}_{O(X)}(\mathbb{A}) \cong K_1(\mathbb{C}_{O(X), O(Y)}(\mathbb{A}))$$

and an exact sequence

$$\begin{aligned} K_1(\mathbb{C}_{O(Y)}(\mathbb{A})) &\longrightarrow K_1(\mathbb{C}_{O(X)}(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_{O(X), O(Y)}(\mathbb{A})) \\ &\xrightarrow{\partial} K_0(\mathbb{P}_{O(Y)}(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_{O(X)}(\mathbb{A})) . \end{aligned}$$

(ii) *For a compact polyhedron $X = X^+ \cup X^- \subseteq S^k$ there is defined a Mayer-Vietoris exact sequence*

$$\begin{aligned} K_1(\mathbb{C}_{O(X^+ \cap X^-)}(\mathbb{A})) &\longrightarrow K_1(\mathbb{C}_{O(X^+)}(\mathbb{A})) \oplus K_1(\mathbb{C}_{O(X^-)}(\mathbb{A})) \\ &\longrightarrow K_1(\mathbb{C}_{O(X^+ \cup X^-)}(\mathbb{A})) \xrightarrow{\partial} K_0(\mathbb{P}_{O(X^+ \cap X^-)}(\mathbb{A})) \\ &\longrightarrow K_0(\mathbb{P}_{O(X^+)}(\mathbb{A})) \oplus K_0(\mathbb{P}_{O(X^-)}(\mathbb{A})) . \end{aligned}$$

PROOF (i) A special case of 4.1.

(ii) A special case of 4.16, identifying

$$O(X^+ \cup X^-) = O(X^+) \cup O(X^-) , \quad O(X^+ \cap X^-) = O(X^+) \cap O(X^-) ,$$

and using 6.1 to identify

$$\begin{aligned} K_1(\mathbb{C}_{O_b(X^+, X^-, X)}(\mathbb{A})) &= K_1(\mathbb{C}_{O(X^+, X) \cap O(X^-, X)}(\mathbb{A})) , \\ K_0(\mathbb{P}_{O_b(X^+, X^-, X)}(\mathbb{A})) &= K_0(\mathbb{P}_{O(X^+, X) \cap O(X^-, X)}(\mathbb{A})) . \end{aligned}$$

□

Pedersen and Weibel [54] identified the algebraic K -theory of $\mathbb{P}_{O(X)}(\mathbb{A})$ for a compact polyhedron $X \subseteq S^k$ with the reduced generalized homology groups of X with coefficients in a non-connective de-looping of the algebraic K -theory of the idempotent completion $\mathbb{P}_0(\mathbb{A})$

$$K_*(\mathbb{P}_{O(X)}(\mathbb{A})) = H_*^{lf}(O(X); \mathbb{K}(\mathbb{P}_0(\mathbb{A}))) = \tilde{H}_{*-1}(X; \mathbb{K}(\mathbb{P}_0(\mathbb{A}))) .$$

The excision isomorphisms and the Mayer-Vietoris exact sequence of 6.2 are particular consequences of this identification. However, the elementary techniques used here in the cases $* \leq 1$ give an explicit formula for the connecting map relating torsion and projective class, and are better suited for the generalization to L -theory obtained in §14 below.

Next, it is shown that $\mathbb{C}_{O(X)}(\mathbb{A})$ admits a flasque structure for non-empty X , so that the isomorphism torsion theory of §5 applies.

Given a non-empty subspace $X \subseteq S^k$, and a base point $x \in X$ define for any object M in $\mathbb{C}_{O(X)}(\mathbb{A})$ an object TM in $\mathbb{C}_{O(X)}(\mathbb{A})$ by

$$(TM)(ty) = \begin{cases} M((t-1)y) & \text{if } t > 1 \\ M(0) & \text{if } t = 1 \text{ and } y = x \\ 0 & \text{otherwise} . \end{cases}$$

Let $T : M \longrightarrow TM$ be the isomorphism in $\mathbb{C}_{O(X)}(\mathbb{A})$ defined by

$$T(uz, ty) : M(ty) \longrightarrow TM(uz) ;$$

$$a \longrightarrow \begin{cases} a & \text{if } u = t + 1 \text{ and } z = y \\ a & \text{if } t = 0 \text{ and } u = 1 \text{ and } z = x \\ 0 & \text{otherwise .} \end{cases}$$

DEFINITION 6.3 The x -based flasque structure $\{\Sigma, \sigma, \rho\}$ on $\mathbb{C}_{O(X)}(\mathbb{A})$ is defined by

$$\Sigma M = \sum_{i=1}^{\infty} T^i M ,$$

$$\sigma_M : M \oplus \Sigma M \longrightarrow \Sigma M ;$$

$$(a_0, (a_1, a_2, \dots)) \longrightarrow (Ta_0, Ta_1, Ta_2, \dots) ,$$

$$\rho_{M,N} : \Sigma(M \oplus N) \longrightarrow \Sigma M \oplus \Sigma N ; (a, b) \longrightarrow (a, b) .$$

The x -based torsion of an isomorphism $f : M \longrightarrow N$ in $\mathbb{C}_{O(X)}(\mathbb{A})$ is the torsion of f with respect to the x -based flasque structure

$$\tau_x(f) = \tau([\sigma_N][f][\sigma_M]^{-1} : 0 \longrightarrow M \longrightarrow N \longrightarrow 0) \in K_1(\mathbb{C}_{O(X)}(\mathbb{A})) .$$

□

In particular, the natural flasque structure defined on $\mathbb{C}_{\mathbb{R}^+}(\mathbb{A})$ in 5.6 is the 1-based flasque structure $\{\Sigma, \sigma, \rho\}$ of 6.3.

Applying Lemma 5.4, it follows that $\mathbb{C}_{O(X)}(\mathbb{A})$ is equipped with a stable canonical structure, and that $K_0(\mathbb{C}_{O(X)}(\mathbb{A})) = 0$ for non-empty X . The x -based torsion is defined for any contractible finite chain complex C in $\mathbb{C}_{O(X)}(\mathbb{A})$ by

$$\tau_x(C) = \tau_x(d + \Gamma : C_{\text{odd}} \longrightarrow C_{\text{even}}) \in K_1(\mathbb{C}_{O(X)}(\mathbb{A})) ,$$

with $\Gamma : 0 \simeq 1 : C \longrightarrow C$ any chain contraction.

The filtered structure on \mathbb{A} is used to identify

$$\mathbb{C}_{X_1 \times X_2}(\mathbb{A}) = \mathbb{C}_{X_1}(\mathbb{C}_{X_2}(\mathbb{A})) ,$$

for any metric spaces X_1, X_2 with the max metric on the product $X_1 \times X_2$

$$d_{X_1 \times X_2}((x_1, x_2), (y_1, y_2)) = \max(d_{X_1}(x_1, y_1), d_{X_2}(x_2, y_2)) .$$

The cone and suspension of X are defined as usual by

$$CX = X \times [0, 1] / X \times \{1\} , \quad \Sigma X = C^+ X \cup_X C^- X .$$

For $X \subseteq S^k$ identify

$$O(CX) = O(X) \times \mathbb{R}^+ , \quad O(\Sigma X) = O(X) \times \mathbb{R} .$$

PROPOSITION 6.4 (i) For non-empty $X \subseteq S^k$ and any $x \in X$ the x -based flasque structure on $\mathbb{C}_{O(CX)}(\mathbb{A})$ is natural, so that

$$K_*(\mathbb{C}_{O(CX)}(\mathbb{A})) = K_*(\mathbb{P}_{O(CX)}(\mathbb{A})) = 0 .$$

(ii) *The bounded $Y \times \mathbb{R}^+$ -graded category $\mathbb{C}_{Y \times \mathbb{R}^+}(\mathbb{A})$ has a natural flasque structure, for any metric space Y .*

PROOF (i) The x -based flasque structure on $\mathbb{C}_{O(\{x\})}(\mathbb{A}) = \mathbb{C}_{\mathbb{R}^+}(\mathbb{A})$ is natural, with $\Sigma f : \Sigma M \longrightarrow \Sigma N$ defined for any morphism $f : M \longrightarrow N$ by

$$\Sigma f : \Sigma M \longrightarrow \Sigma N ; (a_1, a_2, \dots) \longrightarrow (f(a_1), f(a_2), \dots) .$$

It follows that for any non-empty $X \subseteq S^k$ and any $x \in X$ the x -based flasque structure on

$$\mathbb{C}_{O(CX)}(\mathbb{A}) = \mathbb{C}_{\mathbb{R}^+}(\mathbb{C}_{O(X)}(\mathbb{A}))$$

is natural.

(ii) Immediate from (i) and the identifications

$$\mathbb{C}_{Y \times \mathbb{R}^+}(\mathbb{A}) = \mathbb{C}_{\mathbb{R}^+}(\mathbb{C}_Y(\mathbb{A})) , \quad \mathbb{R}^+ = O(\{\text{pt.}\}) .$$

□

PROPOSITION 6.5 *The functor*

$$\mathbb{C}_{X \cup Y \times \mathbb{R}^+}(\mathbb{A}) \longrightarrow \mathbb{C}_{X,Y}(\mathbb{A}) ; M \longrightarrow M(X) = \sum_{x \in X} M(x)$$

induces an isomorphism

$$K_1(\mathbb{C}_{X \cup Y \times \mathbb{R}^+}(\mathbb{A})) \cong K_1(\mathbb{C}_{X,Y}(\mathbb{A})) ,$$

and there is defined an exact sequence

$$\begin{aligned} K_1(\mathbb{C}_Y(\mathbb{A})) &\longrightarrow K_1(\mathbb{C}_X(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_{X \cup Y \times \mathbb{R}^+}(\mathbb{A})) \\ &\xrightarrow{\partial} K_0(\mathbb{P}_Y(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_X(\mathbb{A})) \end{aligned}$$

with the connecting map

$$\begin{aligned} \partial : K_1(\mathbb{C}_{X \cup Y \times \mathbb{R}^+}(\mathbb{A})) &\longrightarrow K_0(\mathbb{P}_Y(\mathbb{A})) ; \\ \tau(f : M \longrightarrow M) &\longrightarrow [f(X, X)]_+ \end{aligned}$$

sending the torsion $\tau(f)$ of an automorphism $f : M \longrightarrow M$ in $\mathbb{C}_{X \cup Y \times \mathbb{R}^+}(\mathbb{A})$ to the end invariant $[f(X, X)]_+$ of the automorphism $[f(X, X)] : M(X) \longrightarrow M(X)$ in $\mathbb{C}_{X,Y}(\mathbb{A})$ defined by the restriction of f to the subobject $M(X) \subseteq M$.

PROOF Theorem 4.1 gives an exact sequence

$$\begin{aligned} K_1(\mathbb{C}_{Y \times \mathbb{R}^+}(\mathbb{A})) &\longrightarrow K_1(\mathbb{C}_{X \cup Y \times \mathbb{R}^+}(\mathbb{A})) \\ &\longrightarrow K_1(\mathbb{C}_{X \cup Y \times \mathbb{R}^+, Y \times \mathbb{R}^+}(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_{Y \times \mathbb{R}^+}(\mathbb{A})) . \end{aligned}$$

By 6.4 (ii) $\mathbb{C}_{Y \times \mathbb{R}^+}(\mathbb{A})$ has a natural flasque structure, so that the idempotent completion $\mathbb{P}_{Y \times \mathbb{R}^+}(\mathbb{A})$ also has a natural flasque structure and

$$K_1(\mathbb{C}_{Y \times \mathbb{R}^+}(\mathbb{A})) = K_0(\mathbb{P}_{Y \times \mathbb{R}^+}(\mathbb{A})) = 0 .$$

Thus there is a natural identification

$$K_1(\mathbb{C}_{X \cup Y \times \mathbb{R}^+}(\mathbb{A})) = K_1(\mathbb{C}_{X \cup Y \times \mathbb{R}^+, Y \times \mathbb{R}^+}(\mathbb{A})) .$$

Every object M in $\mathbb{C}_{X \cup Y \times \mathbb{R}^+}(\mathbb{A})$ is stably isomorphic to the object $M(X)$ in $\mathbb{C}_X(\mathbb{A})$. The germs away from Y of morphisms in $\mathbb{C}_X(\mathbb{A})$ are the germs away from $Y \times \mathbb{R}^+$ of morphisms in $\mathbb{C}_{X \cup Y \times \mathbb{R}^+}(\mathbb{A})$. It follows that the functor

$$\mathbb{C}_{X \cup Y \times \mathbb{R}^+, Y \times \mathbb{R}^+}(\mathbb{A}) \longrightarrow \mathbb{C}_{X, Y}(\mathbb{A}) ; M \longrightarrow M(X)$$

induces isomorphisms in algebraic K -theory, allowing the natural identification

$$K_1(\mathbb{C}_{X \cup Y \times \mathbb{R}^+, Y \times \mathbb{R}^+}(\mathbb{A})) = K_1(\mathbb{C}_{X, Y}(\mathbb{A})) .$$

□

In fact, 6.5 is a special case of a general result: the functor

$$\mathbb{C}_{X \cup Y \times \mathbb{R}^+}(\mathbb{A}) \longrightarrow \mathbb{C}_{X, Y}(\mathbb{A})$$

induces isomorphisms in all the algebraic K -groups

$$K_n(\mathbb{P}_{X \cup Y \times \mathbb{R}^+}(\mathbb{A})) \cong K_n(\mathbb{C}_{X, Y}(\mathbb{A})) \quad (n \in \mathbb{Z}) ,$$

with a consequent long exact sequence

$$\begin{aligned} \dots \longrightarrow K_n(\mathbb{P}_Y(\mathbb{A})) \longrightarrow K_n(\mathbb{P}_X(\mathbb{A})) \longrightarrow K_n(\mathbb{C}_{X \cup Y \times \mathbb{R}^+}(\mathbb{A})) \\ \xrightarrow{\partial} K_{n-1}(\mathbb{P}_Y(\mathbb{A})) \longrightarrow K_{n-1}(\mathbb{P}_X(\mathbb{A})) \longrightarrow \dots \end{aligned}$$

(Ferry and Pedersen [28], Hambleton and Pedersen [31]).

DEFINITION 6.6 (i) Given a map $\omega : S^0 \longrightarrow X$ let

$$\begin{aligned} \bar{B}_0(\omega) : K_0(\mathbb{A}) \longrightarrow K_0(\mathbb{P}_0(\mathbb{A})) \xrightarrow{\bar{B}} \\ K_1(\mathbb{C}_{\mathbb{R}}(\mathbb{A})) = K_1(\mathbb{C}_{O(S^0)}(\mathbb{A})) \xrightarrow{\omega_*} K_1(\mathbb{C}_{O(X)}(\mathbb{A})) . \end{aligned}$$

(ii) The *Whitehead group* of $\mathbb{C}_{O(X)}(\mathbb{A})$ is defined by

$$\begin{aligned} Wh(\mathbb{C}_{O(X)}(\mathbb{A})) \\ = \begin{cases} \text{coker}(\sum_{\omega} \bar{B}_0(\omega) : \sum_{\omega} K_0(\mathbb{A}) \longrightarrow K_1(\mathbb{C}_{O(X)}(\mathbb{A}))) & \text{if } X \neq \emptyset \\ \tilde{K}_1(\mathbb{A}) & \text{if } X = \emptyset \end{cases} \end{aligned}$$

with the sum taken over all the maps $\omega : S^0 \longrightarrow X$.

□

The *class at 0* of a finite chain complex C in $\mathbb{C}_{O(X)}(\mathbb{A})$ is

$$[C(0)] = \sum_{r=0}^{\infty} (-)^r [C_r(0)] \in K_0(\mathbb{A}) .$$

PROPOSITION 6.7 (i) If $\omega : S^0 \longrightarrow X$ extends to a map $D^1 \longrightarrow X$ then

$$\bar{B}_0(\omega) = 0 : K_0(\mathbb{A}) \longrightarrow K_1(\mathbb{C}_{O(X)}(\mathbb{A})) .$$

(ii) If X is non-empty and connected

$$Wh(\mathbb{C}_{O(X)}(\mathbb{A})) = K_1(\mathbb{C}_{O(X)}(\mathbb{A})) .$$

If X has n components X_1, X_2, \dots, X_n with $n \geq 2$ then

$$Wh(\mathbb{C}_{O(X)}(\mathbb{A})) = \text{coker}\left(\sum_{i=1}^{n-1} \bar{B}_0(\omega_i) : \sum_{i=1}^{n-1} K_0(\mathbb{A}) \longrightarrow K_1(\mathbb{C}_{O(X)}(\mathbb{A}))\right)$$

for any maps $\omega_i : S^0 \longrightarrow X$ such that $\omega_i(+1) \in X_1$, $\omega_i(-1) \in X_{i+1}$.

(iii) For any $x, x' \in X$ let $\{\Sigma, \sigma, \rho\}$, $\{\Sigma', \sigma', \rho'\}$ be the x - and x' -based flasque structures on $\mathbb{C}_{O(X)}(\mathbb{A})$. The difference of the x - and x' -based torsions of a contractible finite chain complex C in $\mathbb{C}_{O(X)}(\mathbb{A})$ is

$$\tau_x(C) - \tau_{x'}(C) = \bar{B}_0(\omega)[C(0)] \in K_1(\mathbb{C}_{O(X)}(\mathbb{A})) ,$$

with

$$\omega : S^0 \longrightarrow X ; +1 \longrightarrow x , -1 \longrightarrow x' .$$

(iv) The Whitehead torsion of a contractible finite chain complex C in $\mathbb{C}_{O(X)}(\mathbb{A})$

$$\tau(C) = \tau_x(C) \in Wh(\mathbb{C}_{O(X)}(\mathbb{A}))$$

is independent of the choice of base point $x \in X$ used to define the torsion $\tau_x(C) \in K_1(\mathbb{C}_{O(X)}(\mathbb{A}))$.

PROOF (i) If ω extends to a map $D^1 \longrightarrow X$ there is a factorization

$$\begin{aligned} \bar{B}_0(\omega) : K_0(\mathbb{A}) &\longrightarrow K_1(\mathbb{C}_{O(S^0)}(\mathbb{A})) \\ &\longrightarrow K_1(\mathbb{C}_{O(D^1)}(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_{O(X)}(\mathbb{A})) \end{aligned}$$

and $K_1(\mathbb{C}_{O(D^1)}(\mathbb{A})) = 0$ (by 6.4 (i)).

(ii) Immediate from (i).

(iii) The automorphism defined for any object M in $\mathbb{C}_{O(X)}(\mathbb{A})$ by

$$\alpha_M = (\sigma'_M \oplus 1_{\Sigma M})^{-1} (\sigma_M \oplus 1_{\Sigma' M}) : M \oplus \Sigma M \oplus \Sigma' M \longrightarrow M \oplus \Sigma M \oplus \Sigma' M$$

has torsion

$$\tau(\alpha_M) = \bar{B}_0(\omega)[M(0)] \in K_1(\mathbb{C}_{O(X)}(\mathbb{A})) .$$

(iv) Immediate from (iii). □

REMARK 6.8 In view of 6.7 (ii) and Anderson [1] the Whitehead group $Wh(\mathbb{C}_{O(X)}(\mathbb{A}))$ is a special case of the Whitehead groups defined by Anderson and Munkholm [4, p.146]. In particular, for $X = S^0$

$$\begin{aligned} Wh(\mathbb{C}_{O(S^0)}(\mathbb{A})) &= Wh(\mathbb{C}_{\mathbb{R}}(\mathbb{A})) \\ &= \text{coker}(\bar{B}_0 : K_0(\mathbb{A}) \longrightarrow K_1(\mathbb{C}_{\mathbb{R}}(\mathbb{A}))) . \end{aligned}$$

□

For any object M in $\mathbb{C}_{O(X+\cap X^-)}(\mathbb{A})$

$$\begin{aligned} [M] &= [M(0)] \in \text{im}(K_0(\mathbb{C}_{O(X+\cap X^-)}(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_{O(X+\cap X^-)}(\mathbb{A}))) \\ &= \text{im}(K_0(\mathbb{A}) \longrightarrow K_0(\mathbb{P}_{O(X+\cap X^-)}(\mathbb{A}))) \\ &= \begin{cases} \{0\} & \text{if } X^+ \cap X^- \neq \emptyset \\ \text{im}(K_0(\mathbb{A}) \longrightarrow K_0(\mathbb{P}_0(\mathbb{A}))) & \text{if } X^+ \cap X^- = \emptyset. \end{cases} \end{aligned}$$

Given a map

$$\omega : S^0 \longrightarrow X^+ \cup X^- ; +1 \longrightarrow x^+ , -1 \longrightarrow x^- ,$$

such that $x^\pm \in X^\pm$ define the abelian group morphism

$$\begin{aligned} \bar{B}_0(\omega) : K_0(\mathbb{C}_{O(X+\cap X^-)}(\mathbb{A})) &\longrightarrow K_1(\mathbb{C}_{O(X+\cup X^-)}(\mathbb{A})) ; \\ [M] &\longrightarrow \tau([\sigma_M^-]^{-1}[\sigma_M^+] : M \longrightarrow M) \end{aligned}$$

with $\{\Sigma^\pm, \sigma^\pm, \rho^\pm\}$ the x^\pm -based flasque structure on $\mathbb{C}_{O(X^\pm)}(\mathbb{A})$ given by 6.3. The morphism $\bar{B}_0(\omega)$ of 6.7 (i) is the composite

$$\begin{aligned} \bar{B}_0(\omega) : K_0(\mathbb{A}) &\longrightarrow K_0(\mathbb{C}_{O(X+\cap X^-)}(\mathbb{A})) \\ &\xrightarrow{\bar{B}_0(\omega)} K_1(\mathbb{C}_{O(X+\cup X^-)}(\mathbb{A})) ; \\ [M] &\longrightarrow \omega_* \tau(z : M[z, z^{-1}] \longrightarrow M[z, z^{-1}]) . \end{aligned}$$

By 6.7 (iii) the x^+ -based and x^- -based torsions of a contractible finite chain complex E in $\mathbb{C}_{O(X+\cup X^-)}(\mathbb{A})$ differ by

$$\begin{aligned} \tau_{x^+}(E) - \tau_{x^-}(E) &= \bar{B}_0(\omega)[E(0)] \\ &\in \text{im}(\bar{B}_0(\omega) : K_0(\mathbb{A}) \longrightarrow K_1(\mathbb{C}_{O(X+\cup X^-)}(\mathbb{A}))) . \end{aligned}$$

The composite

$$\begin{aligned} K_0(\mathbb{C}_{O(X+\cap X^-)}(\mathbb{A})) &\xrightarrow{\bar{B}_0(\omega)} K_1(\mathbb{C}_{O(X+\cup X^-)}(\mathbb{A})) \\ &\xrightarrow{\partial} K_0(\mathbb{P}_{O(X+\cap X^-)}(\mathbb{A})) \end{aligned}$$

is independent of ω , being the natural map

$$\begin{aligned} \partial \bar{B}_0(\omega) : K_0(\mathbb{C}_{O(X+\cap X^-)}(\mathbb{A})) &\longrightarrow K_0(\mathbb{P}_{O(X+\cap X^-)}(\mathbb{A})) ; \\ [M] &\longrightarrow [M] = [M(0)] . \end{aligned}$$

PROPOSITION 6.9 *The connecting map*

$$\partial : K_1(\mathbb{C}_{O(X+\cup X^-)}(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_{O(X+\cap X^-)}(\mathbb{A}))$$

sends the x -based torsion $\tau_x(E) \in K_1(\mathbb{C}_{O(X+\cup X^-)}(\mathbb{A}))$ of a contractible

finite chain complex E in $\mathbb{C}_{O(X+\cup X^-)}(\mathbb{A})$ to the projective class

$$\begin{aligned} \partial\tau_x(E) &= [E]_+ + \sum_{r=0}^{\infty} (-)^r [\sigma_{E_r} : E_r \oplus \Sigma E_r \longrightarrow \Sigma E_r]_+ \\ &= \begin{cases} [E]_+ & \text{if } x \in X^+ \\ [E]_+ - [E(0)] & \text{if } x \notin X^+ \end{cases} \in K_0(\mathbb{P}_{O(X+\cap X^-)}(\mathbb{A})) . \end{aligned}$$

PROOF By construction, ∂ is the composite

$$\begin{aligned} \partial : K_1(\mathbb{C}_{O(X+\cup X^-)}(\mathbb{A})) &\longrightarrow \\ &K_1(\mathbb{C}_{O(X+\cup X^-), O(X^-)}(\mathbb{A})) \cong K_1(\mathbb{C}_{O(X^+), O(X+\cap X^-)}(\mathbb{A})) \\ &\xrightarrow{\partial} K_0(\mathbb{P}_{O(X^+), O(X+\cap X^-)}(\mathbb{A})) . \end{aligned}$$

For $x \in X^+$ the image of $\tau_x(E)$ in $K_1(\mathbb{C}_{O(X^+), O(X+\cap X^-)}(\mathbb{A}))$ is the x -based torsion $\tau_x(E(O(X^+)))$ of the contractible chain complex $E(O(X^+))$ in $\mathbb{C}_{O(X^+), O(X+\cap X^-)}(\mathbb{A})$, and the identity

$$\partial\tau_x(E) = [E]_+ \in K_0(\mathbb{P}_{O(X+\cap X^-)}(\mathbb{A}))$$

is immediate from 5.8. For $x \notin X^+$ choose a base point $x' \in X^+$, and define $\omega : S^0 \longrightarrow X$ by $\omega(+1) = x'$, $\omega(-1) = x$, so that

$$\tau_x(E) = \tau_{x'}(E) - \bar{B}_0(\omega)[E(0)] \in K_1(\mathbb{C}_{O(X+\cup X^-)}(\mathbb{A}))$$

and

$$\begin{aligned} \partial\tau_x(E) &= \partial\tau_{x'}(E) - \partial\bar{B}_0(\omega)[E(0)] \\ &= [E]_+ - [E(0)] \in K_0(\mathbb{P}_{O(X+\cup X^-)}(\mathbb{A})) . \end{aligned}$$

□

If $X^+ \cap X^-$ is non-empty then $K_0(\mathbb{C}_{O(X+\cap X^-)}(\mathbb{A})) = 0$, since there exists a flasque structure on $\mathbb{C}_{O(X+\cap X^-)}(\mathbb{A})$ (by 6.3). Thus if E is round at 0 or if $X^+ \cap X^-$ is non-empty then

$$\partial\tau(E) = [E]_+ \in K_0(\mathbb{P}_{O(X+\cap X^-)}(\mathbb{A}))$$

for all base points $x \in X^+ \cup X^-$, in agreement with 6.7 (iii).

PROPOSITION 6.10 *The image under the connecting map ∂ of the x -based torsion $\tau_x(f) \in K_1(\mathbb{C}_{O(X+\cup X^-)}(\mathbb{A}))$ of a chain equivalence $f : C \longrightarrow D$ of band complexes in $\mathbb{C}_{O(X+\cup X^-)}(\mathbb{A})$ is given by*

$$\partial\tau(f) = \begin{cases} [D]_+ - [C]_+ & \text{if } x \in X^+ \\ [D]_+ - [C]_+ - ([D(0)] - [C(0)]) & \text{if } x \notin X^+ . \end{cases}$$

PROOF The x -based torsion of f is the x -based torsion of the algebraic mapping cone $C(f)$

$$\tau_x(f) = \tau_x(C(f)) \in K_1(\mathbb{C}_{O(X+\cup X^-)}(\mathbb{A})) .$$

(The sign term β in the definition of $\tau(f)$ in Ranicki [65, p.223] is 0, since $X^+ \cup X^-$ is non-empty and $K_0(\mathbb{C}_{O(X+\cup X^-)}(\mathbb{A})) = 0$.) The result

follows from 6.9 and the sum formula of 3.12 for the end invariants of the band complexes in the exact sequence

$$0 \longrightarrow D \longrightarrow C(f) \longrightarrow SC \longrightarrow 0 .$$

□

For any $X = X^+ \cup X^- \subseteq S^k$ and $b \geq 0$ write

$$\begin{aligned} O_b(X^+) &= \mathcal{N}_b(O(X^+), O(X)) \\ &= O(X^+, X) \cup \mathcal{N}_b(O(X^+ \cap X^-), O(X^-)) , \\ O_b(X^-) &= \mathcal{N}_b(O(X^-), O(X)) \\ &= O(X^-, X) \cup \mathcal{N}_b(O(X^+ \cap X^-), O(X^+)) . \end{aligned}$$

PROPOSITION 6.11 (i) *The Whitehead torsion and reduced projective class groups associated to $X = X^+ \cup X^- \subseteq S^k$ are related by a Mayer-Vietoris exact sequence*

$$\begin{aligned} Wh(\mathbb{C}_{O(X^+ \cap X^-)}(\mathbb{A})) &\longrightarrow Wh(\mathbb{C}_{O(X^+)}(\mathbb{A})) \oplus Wh(\mathbb{C}_{O(X^-)}(\mathbb{A})) \\ &\longrightarrow Wh(\mathbb{C}_{O(X^+ \cup X^-)}(\mathbb{A})) \xrightarrow{\tilde{\partial}} \tilde{K}_0(\mathbb{P}_{O(X^+ \cap X^-)}(\mathbb{A})) \\ &\longrightarrow \tilde{K}_0(\mathbb{P}_{O(X^+)}(\mathbb{A})) \oplus \tilde{K}_0(\mathbb{P}_{O(X^-)}(\mathbb{A})) . \end{aligned}$$

(ii) *If E is a contractible finite chain complex in $\mathbb{C}_{O_b(X^\pm)}(\mathbb{A})$ for some $b \geq 0$ then*

$$\tau(E) \in \text{im}(Wh(\mathbb{C}_{O(X^\pm)}(\mathbb{A})) \longrightarrow Wh(\mathbb{C}_{O(X^+ \cup X^-)}(\mathbb{A}))) .$$

(iii) *If E is a contractible finite chain complex in $\mathbb{C}_{O(X^+ \cup X^-)}(\mathbb{A})$ such that*

$$\tilde{\partial}\tau(E) = 0 \in \tilde{K}_0(\mathbb{P}_{O(X^+ \cap X^-)}(\mathbb{A}))$$

then for any Mayer-Vietoris presentation of E

$$E : 0 \longrightarrow E^+ \cap E^- \xrightarrow{i} E^+ \oplus E^- \xrightarrow{j} E \longrightarrow 0$$

there exist finite chain complexes F^+, F^- in $\mathbb{C}_{O(X^+ \cap X^-)}(\mathbb{A})$ and chain equivalences $\phi^\pm : F^\pm \longrightarrow E^\pm$ in $\mathbb{C}_{O_b(X^\pm)}(\mathbb{A})$ for some $b \geq 0$, with

$$\begin{aligned} \tau(E) &= \tau(\phi^+) + \tau(\phi^-) + \tau((\phi^+ \oplus \phi^-)^{-1}i : E^+ \cap E^- \longrightarrow F^+ \oplus F^-) \\ &\in \text{im}(Wh(\mathbb{C}_{O(X^+)}(\mathbb{A})) \oplus Wh(\mathbb{C}_{O(X^-)}(\mathbb{A})) \longrightarrow Wh(\mathbb{C}_{O(X^+ \cup X^-)}(\mathbb{A}))) . \end{aligned}$$

PROOF (i) By 6.7 there is no loss of generality in assuming that X^+ and

X^- are connected, so that

$$\begin{aligned}
Wh(\mathbb{C}_{O(X^\pm)}(\mathbb{A})) &= K_1(\mathbb{C}_{O(X^\pm)}(\mathbb{A})) , \\
\tilde{K}_0(\mathbb{P}_{O(X^\pm)}(\mathbb{A})) &= K_0(\mathbb{P}_{O(X^\pm)}(\mathbb{A})) , \\
\tilde{K}_0(\mathbb{P}_{O(X^+ \cap X^-)}(\mathbb{A})) \\
&= \left\{ \begin{array}{l} K_0(\mathbb{P}_{O(X^+ \cap X^-)}(\mathbb{A})) \\ \text{coker}(K_0(\mathbb{A}) \longrightarrow K_0(\mathbb{P}_{O(X^+ \cap X^-)}(\mathbb{A}))) \end{array} \right. \\
Wh(\mathbb{C}_{O(X^+ \cup X^-)}(\mathbb{A})) \\
&= \left\{ \begin{array}{l} K_1(\mathbb{C}_{O(X^+ \cup X^-)}(\mathbb{A})) \\ \text{coker}(\bar{B}_0(\omega) : K_0(\mathbb{A}) \longrightarrow K_1(\mathbb{C}_{O(X^+ \cup X^-)}(\mathbb{A}))) \end{array} \right. \\
Wh(\mathbb{C}_{O(X^+ \cap X^-)}(\mathbb{A})) \\
&= \left\{ \begin{array}{l} \text{coker}(\sum_{\eta} \bar{B}_0(\eta) : \sum_{\eta} K_0(\mathbb{A}) \longrightarrow K_1(\mathbb{C}_{O(X^+ \cap X^-)}(\mathbb{A}))) \\ K_1(\mathbb{A}) \end{array} \right. \\
&\quad \text{if } \left\{ \begin{array}{l} X^+ \cap X^- \neq \emptyset \\ X^+ \cap X^- = \emptyset \end{array} \right.
\end{aligned}$$

with the sums taken over all the maps

$$\omega : S^0 \longrightarrow X^+ \cup X^- , \quad \eta : S^0 \longrightarrow X^+ \cap X^-$$

with $\omega(\pm 1) = x^\pm \in X^\pm$. For any such η the composites

$$K_0(\mathbb{A}) \xrightarrow{\bar{B}_0(\eta)} K_1(\mathbb{C}_{O(X^+ \cap X^-)}(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_{O(X^\pm)}(\mathbb{A}))$$

are 0, since X^+ and X^- are connected. Thus the reduced K -theory Mayer-Vietoris exact sequence is a quotient of the absolute K -theory Mayer-Vietoris exact sequence of 6.2 (ii).

(ii) For any object M in $\mathbb{C}_{O_b(X^+)}(\mathbb{A})$ let M' be the object defined in $\mathbb{C}_{O(X^+)}(\mathbb{A})$ by

$$M'(x') = \begin{cases} M(x') & \text{if } x' \in O(X^+) \setminus \{0\} \\ M(0) \oplus M(O_b(X^+) \setminus O(X^+)) & \text{if } x' = 0 . \end{cases}$$

For any base point $x^- \in X^-$ the evident regrading map $\theta_M : M \longrightarrow M'$ is an isomorphism in $\mathbb{C}_{O_b(X^+)}(\mathbb{A})$ with x^- -based torsion

$$\tau_{x^-}(\theta_M) = \tau_{x^-}(M(O_b(X^+) \cap O(X^-)) \longrightarrow M'(O_b(X^+) \cap O(X^-)))$$

$$\in \text{im}(K_1(\mathbb{C}_{O_b(X^+) \cap O(X^-)}(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_{O(X^+ \cup X^-)}(\mathbb{A}))) = \{0\} ,$$

so that the Whitehead torsion is

$$\tau(\theta_M) = 0 \in Wh(\mathbb{C}_{O(X^+ \cup X^-)}(\mathbb{A})) .$$

Thus for any isomorphism $f : M \longrightarrow N$ in $\mathbb{C}_{O_b(X^+)}(\mathbb{A})$ there is defined an isomorphism in $\mathbb{C}_{O(X^+)}(\mathbb{A})$

$$f' = \theta_N f(\theta_M)^{-1} : M' \longrightarrow M \longrightarrow N \longrightarrow N'$$

such that

$$\tau_{x^-}(f') = \tau_{x^-}(f) + \tau_{x^-}(\theta_N) - \tau_{x^-}(\theta_M) \in K_1(\mathbb{C}_{O(X+\cup X^-)}(\mathbb{A})) ,$$

and the Whitehead torsion of f is

$$\tau(f) = \tau(f') \in \text{im} (Wh(\mathbb{C}_{O(X+)}(\mathbb{A})) \longrightarrow Wh(\mathbb{C}_{O(X+\cup X^-)}(\mathbb{A}))) .$$

Similarly for chain complexes.

(iii) The reduced end invariant $[E]_{\pm}$ is the $\mathbb{C}_{O(X+\cap X^-)}(\mathbb{A})$ -finiteness obstruction of E^{\pm}

$$[E]_{\pm} = [E^{\pm}] \in \tilde{K}_0(\mathbb{P}_{O(X+\cap X^-)}(\mathbb{A})) .$$

By the sum formula and (ii) the Whitehead torsion of E is

$$\begin{aligned} \tau(E) &= \tau(i : E^+ \cap E^- \longrightarrow E^+ \oplus E^-) \\ &= \tau(\phi^+) + \tau(\phi^-) + \tau((\phi^+ \oplus \phi^-)^{-1} i : E^+ \cap E^- \longrightarrow F^+ \oplus F^-) \\ &\in \text{im} (Wh(\mathbb{C}_{O(X+)}(\mathbb{A})) \oplus Wh(\mathbb{C}_{O(X-)}(\mathbb{A})) \longrightarrow Wh(\mathbb{C}_{O(X+\cup X^-)}(\mathbb{A}))) . \end{aligned}$$

□

§7. K -theory of $\mathbb{C}_1(\mathbb{A})$

We now consider the K -theory of the bounded \mathbb{Z} -graded category

$$\mathbb{C}_1(\mathbb{A}) = \mathbb{C}_{\mathbb{Z}}(\mathbb{A}) ,$$

using resolutions in the \mathbb{Z} -graded category $\mathbb{G}_1(\mathbb{A}) = \mathbb{G}_{\mathbb{Z}}(\mathbb{A})$. The bounded \mathbb{R} -graded category $\mathbb{C}_{\mathbb{R}}(\mathbb{A})$ equivalent to $\mathbb{C}_1(\mathbb{A})$ is used to obtain a chain complex interpretation of the isomorphism

$$K_1(\mathbb{C}_{\mathbb{R}}(\mathbb{A})) \cong K_0(\mathbb{P}_0(\mathbb{A}))$$

originally obtained by Pedersen [49] (for $\mathbb{A} = \mathbb{B}^f(A)$) and by Pedersen and Weibel [53] (for any \mathbb{A}).

In order to apply the K -theory exact sequences of §4 to $\mathbb{C}_{\mathbb{R}}(\mathbb{A})$ use the expression of $X = \mathbb{R}$ as a union $X = X^+ \cup X^-$ with

$$X^+ = \mathbb{R}^+ = [0, \infty) , \quad X^- = \mathbb{R}^- = (-\infty, 0] , \quad X^+ \cap X^- = \{0\} ,$$

or the equivalent subsets of \mathbb{Z} . The b -neighbourhoods for $b \geq 0$ are the intervals

$$\mathcal{N}_b(X^+, X) = [-b, \infty) , \quad \mathcal{N}_b(X^-, X) = (-\infty, b] ,$$

$$\mathcal{N}_b(X^+, X^-, X) = [-b, b] \subset \mathbb{R} .$$

The various graded categories of a filtered additive category \mathbb{A} associated to \mathbb{Z} , $\mathbb{Z}^+ = \{n \in \mathbb{Z} \mid n \geq 0\}$ and $\mathbb{Z}^- = \{n \in \mathbb{Z} \mid n \leq 0\}$ are denoted by

$$\mathbb{B}_{\mathbb{Z}}(\mathbb{A}) = \mathbb{B}_1(\mathbb{A}) ,$$

$$\mathbb{B}_{\mathbb{Z}^+}(\mathbb{A}) = \mathbb{B}_+(\mathbb{A}) , \quad \mathbb{B}_{\mathbb{Z}^-}(\mathbb{A}) = \mathbb{B}_-(\mathbb{A}) ,$$

$$\mathbb{B}_{\mathbb{Z}^+, \{0\}}(\mathbb{A}) = \mathbb{B}_{+,0}(\mathbb{A}) , \quad \mathbb{B}_{\mathbb{Z}^-, \{0\}}(\mathbb{A}) = \mathbb{B}_{-,0}(\mathbb{A}) .$$

with $\mathbb{B} = \mathbb{C}$ or \mathbb{G} . The inclusion $\mathbb{Z} \longrightarrow \mathbb{R}$ is a homotopy equivalence in the proper eventually Lipschitz category, with homotopy inverse

$$\mathbb{R} \longrightarrow \mathbb{Z} ; x \longrightarrow [x] ,$$

and similarly for $\mathbb{Z}^\pm \longrightarrow \mathbb{R}^\pm$. Thus $\mathbb{C}_1(\mathbb{A})$ is equivalent to $\mathbb{C}_{\mathbb{R}}(\mathbb{A})$ and

$$K_*(\mathbb{C}_1(\mathbb{A})) = K_*(\mathbb{C}_{\mathbb{R}}(\mathbb{A})) .$$

The flasque structure defined on $\mathbb{C}_{\mathbb{R}}(\mathbb{A})$ in 5.7 (i) restricts to a flasque structure on $\mathbb{C}_1(\mathbb{A})$, and this will be used to define torsions in $K_1(\mathbb{C}_1(\mathbb{A}))$ for isomorphisms in $\mathbb{C}_1(\mathbb{A})$. The flasque structure on $\mathbb{C}_1(\mathbb{A})$ restricts to natural flasque structures on $\mathbb{C}_{\pm}(\mathbb{A})$, so that

$$K_*(\mathbb{C}_{\pm}(\mathbb{A})) = K_*(\mathbb{C}_{\mathbb{R}^\pm}(\mathbb{A})) = 0 .$$

The induced flasque structure on $\mathbb{C}_{\pm, \{0\}}(\mathbb{A})$ is not natural, in general. The inclusions

$$\mathbb{C}_{\pm, 0}(\mathbb{A}) \longrightarrow \mathbb{C}_{\mathbb{R}^\pm, \{0\}}(\mathbb{A})$$

are also equivalences of additive categories, so that

$$K_*(\mathbb{C}_{\pm, 0}(\mathbb{A})) = K_*(\mathbb{C}_{\mathbb{R}^\pm, \{0\}}(\mathbb{A})) .$$

Given an object M in \mathbb{A} let $M[z]$ be the object of $\mathbb{C}_+(\mathbb{A})$ defined by

$$M[z](k) = z^k M \quad (k \geq 0) ,$$

with $z^k M$ a copy of M . Given a collection of morphisms in \mathbb{A}

$$\{f_j \in \text{Hom}_{\mathbb{A}}(L, M) \mid j \geq 0\}$$

such that $\{j \in \mathbb{Z} \mid f_j \neq 0\}$ is finite let

$$f = \sum_{j=0}^{\infty} z^j f_j : L[z, z^{-1}] \longrightarrow M[z, z^{-1}]$$

be the morphism in $\mathbb{C}_+(\mathbb{A})$ defined by

$$f(k, j) = \begin{cases} f_{k-j} & \text{if } k \geq j \\ 0 & \text{if } k < j \end{cases} : L[z](j) = z^j L \longrightarrow M[z](k) = z^k M .$$

The *polynomial extension category* $\mathbb{A}[z]$ of \mathbb{A} is the subcategory of $\mathbb{C}_+(\mathbb{A})$ with objects $M[z]$ and morphisms $\sum_j z^j f_j$. Similarly for $\mathbb{A}[z^{-1}]$.

The end invariant $[E]_+ \in K_0(\mathbb{P}_0(\mathbb{A}))$ of a $\mathbb{G}_{+, 0}(\mathbb{A})$ -contractible finite chain complex E in $\mathbb{C}_{+, 0}(\mathbb{A})$ is defined as in §3.

PROPOSITION 7.1 *The end invariant defines an isomorphism*

$$\partial : K_1(\mathbb{C}_{+, 0}(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_0(\mathbb{A})) ; \tau(E) \longrightarrow [E]_+ ,$$

with inverse

$$\bar{B} : K_0(\mathbb{P}_0(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_{+, 0}(\mathbb{A})) ;$$

$$[M, p] \longrightarrow \tau(1 - p + zp : M[z] \longrightarrow M[z]) .$$

PROOF ∂ is the connecting map in the exact sequence of 4.1 for $(X, Y) = (\mathbb{Z}^+, \{0\})$

$$\begin{aligned} K_1(\mathbb{A}) &\longrightarrow K_1(\mathbb{C}_+(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_{+,0}(\mathbb{A})) \\ &\xrightarrow{\partial} K_0(\mathbb{P}_0(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_+(\mathbb{A})) \end{aligned}$$

with $\mathbb{P}_+(\mathbb{A}) = \mathbb{P}_0(\mathbb{C}_+(\mathbb{A}))$. It follows from the natural flasque structure on $\mathbb{C}_+(\mathbb{A})$ given by 5.7 (ii) that

$$K_1(\mathbb{C}_+(\mathbb{A})) = K_0(\mathbb{P}_+(\mathbb{A})) = 0 ,$$

so ∂ is an isomorphism. The identification $\partial\tau(E) = [E]_+$ is given by 5.9. In order to verify that \bar{B} is the inverse of ∂ it is necessary to prove that for any object (M, p) in $\mathbb{P}_0(\mathbb{A})$ the automorphism in $\mathbb{C}_{+,0}(\mathbb{A})$

$$1 - p + zp : M[z] \longrightarrow M[z]$$

has positive end invariant

$$[1 - p + zp]_+ = [M, p] \in K_0(\mathbb{P}_0(\mathbb{A})) .$$

This has already been done in 3.11. □

Given an object M in \mathbb{A} let $M[z, z^{-1}]$ be the object of $\mathbb{C}_1(\mathbb{A})$ defined by

$$M[z, z^{-1}](k) = z^k M \quad (k \in \mathbb{Z}) ,$$

with $z^k M$ a copy of M . Given a collection of morphisms in \mathbb{A}

$$\{f_j \in \text{Hom}_{\mathbb{A}}(L, M) \mid j \in \mathbb{Z}\}$$

such that $\{j \in \mathbb{Z} \mid f_j \neq 0\}$ is finite let

$$f = \sum_{j=-\infty}^{\infty} z^j f_j : L[z, z^{-1}] \longrightarrow M[z, z^{-1}]$$

be the morphism in $\mathbb{C}_1(\mathbb{A})$ defined by

$$f(k, j) = f_{k-j} : L[z, z^{-1}](j) = z^j L \longrightarrow M[z, z^{-1}](k) = z^k M .$$

The Laurent polynomial extension $\mathbb{A}[z, z^{-1}]$ of \mathbb{A} is the subcategory of $\mathbb{C}_1(\mathbb{A})$ with objects $M[z, z^{-1}]$ and morphisms $\sum_j z^j f_j$.

The end invariant of a $\mathbb{G}_{-,0}(\mathbb{A})$ -contractible finite chain complex E in $\mathbb{C}_{-,0}(\mathbb{A})$ is denoted by $[E]_- \in K_0(\mathbb{P}_0(\mathbb{A}))$.

Let $\mathbb{N}_0(\mathbb{A})$ be the full subcategory of $\mathbb{C}_1(\mathbb{A})$ with objects M of finite support (2.8 (i)), corresponding to the neighbourhood $\mathbb{N}_{\{0\}}(\mathbb{A})$ of $\mathbb{C}_{\{0\}}(\mathbb{A})$ in $\mathbb{C}_{\mathbb{R}}(\mathbb{A})$ (which is the full subcategory with objects of compact support). Similarly, let $\mathbb{N}_{\pm,0}(\mathbb{A})$ be the neighbourhood of $\mathbb{C}_{\{0\}}(\mathbb{A})$ in $\mathbb{C}_{\pm}(\mathbb{A})$, the full subcategory with objects of finite support. The inclusions

$$\mathbb{A} = \mathbb{C}_{\{0\}}(\mathbb{A}) \longrightarrow \mathbb{N}_0(\mathbb{A}) , \quad \mathbb{A} \longrightarrow \mathbb{N}_{\pm,0}(\mathbb{A})$$

are equivalences of additive categories.

Recall from $\left\{ \begin{smallmatrix} 3.1 \\ 3.7 \end{smallmatrix} \right.$ that a finite chain complex C in $\mathbb{C}_\pm(\mathbb{A})$ is $\left\{ \begin{smallmatrix} \mathbb{G}_{\pm,0}(\mathbb{A}) \\ \mathbb{C}_{\pm,0}(\mathbb{A}) \end{smallmatrix} \right.$ -contractible if and only if it is $\left\{ \begin{smallmatrix} (\mathbb{G}_\pm(\mathbb{A}), \mathbb{N}_{\pm,0}(\mathbb{A})) \\ (\mathbb{C}_\pm(\mathbb{A}), \mathbb{N}_{\pm,0}(\mathbb{A})) \end{smallmatrix} \right.$ -finitely dominated.

A *chain complex band* E in $\mathbb{C}_1(\mathbb{A})$ is a finite chain complex which is $(\mathbb{G}_1(\mathbb{A}), \mathbb{N}_0(\mathbb{A}))$ -finitely dominated, so that E is a chain complex band in $\mathbb{C}_\mathbb{R}(\mathbb{A})$ in the sense of 4.10 and the end invariants $[E]_\pm \in K_0(\mathbb{P}_0(\mathbb{A}))$ are defined for $X = \mathbb{Z} = \mathbb{Z}^+ \cup \mathbb{Z}^-$, with sum

$$[E]_+ + [E]_- = [E] + [E(0)] \in K_0(\mathbb{P}_0(\mathbb{A})) .$$

The end invariants of an automorphism $f : L \longrightarrow L$ in $\mathbb{C}_1(\mathbb{A})$ are defined as in 4.14 to be the end invariants of the 1-dimensional (contractible) chain complex band E given by

$$d_E = f : E_1 = L \longrightarrow E_0 = L ,$$

that is

$$[f]_\pm = [E]_\pm \in K_0(\mathbb{P}_0(\mathbb{A}))$$

with $[f]_+ + [f]_- = 0 \in K_0(\mathbb{P}_0(\mathbb{A}))$.

PROPOSITION 7.2 (i) *The end invariants of automorphisms in $\mathbb{C}_1(\mathbb{A})$ define an isomorphism*

$\partial : K_1(\mathbb{C}_1(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_0(\mathbb{A})) ; \tau(f : L \longrightarrow L) \longrightarrow [f]_+ = -[f]_-$
with inverse

$$\bar{B} : K_0(\mathbb{P}_0(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_1(\mathbb{A})) ;$$

$$[M, p] \longrightarrow \tau(1 - p + zp : M[z, z^{-1}] \longrightarrow M[z, z^{-1}]) .$$

(ii) *The isomorphism ∂ of (i) sends the torsion $\tau(f)$ of a chain equivalence $f : E \longrightarrow E'$ of bands in $\mathbb{C}_1(\mathbb{A})$ to the difference of the positive end invariants*

$$\partial\tau(f) = [E']_+ - [E]_+ \in K_0(\mathbb{P}_0(\mathbb{A})) .$$

PROOF (i) ∂ is the connecting map in the Mayer-Vietoris exact sequence of 4.15 for $X = X^+ \cup X^-$ given by $\mathbb{Z} = \mathbb{Z}^+ \cup \mathbb{Z}^-$. The categories $\mathbb{C}_{X^\pm}(\mathbb{A}) = \mathbb{C}_{\mathbb{Z}^\pm}(\mathbb{A})$ have natural flasque structures (by 5.7 (ii)), so that

$$K_*(\mathbb{C}_{\mathbb{Z}^\pm}(\mathbb{A})) = K_*(\mathbb{P}_{\mathbb{Z}^\pm}(\mathbb{A})) = 0 .$$

The exact sequence of 4.16 includes

$$0 \longrightarrow K_1(\mathbb{C}_1(\mathbb{A})) \xrightarrow{\partial} K_0(\mathbb{P}_0(\mathbb{A})) \longrightarrow 0 ,$$

so that ∂ is an isomorphism. In order to verify that \bar{B} is the inverse of ∂ note that by 3.10 (again) for any object (M, p) in $\mathbb{P}_0(\mathbb{A})$ the automorphism in $\mathbb{C}_1(\mathbb{A})$

$$1 - p + zp : M[z, z^{-1}] \longrightarrow M[z, z^{-1}]$$

has positive end invariant

$$[1 - p + zp]_+ = [M, p] \in K_0(\mathbb{P}_0(\mathbb{A})) .$$

(ii) It is immediate from 5.9 that ∂ sends the torsion $\tau(C) \in K_1(\mathbb{C}_1(\mathbb{A}))$ of a contractible finite chain complex C in $\mathbb{C}_1(\mathbb{A})$ to the positive end invariant

$$\partial\tau(C) = [C]_+ \in K_0(\mathbb{P}_0(\mathbb{A})) .$$

The algebraic mapping cone $C(f)$ is contractible, and fits into an exact sequence of band in $\mathbb{C}_1(\mathbb{A})$

$$0 \longrightarrow E' \longrightarrow C(f) \longrightarrow SE \longrightarrow 0 .$$

Applying the sum formula of 3.8

$$\begin{aligned} \partial\tau(f) &= \partial\tau(C(f)) \\ &= [C(f)]_+ = [E']_+ + [SE]_+ = [E']_+ - [E]_+ \in K_0(\mathbb{P}_0(\mathbb{A})) . \end{aligned}$$

□

DEFINITION 7.3 The *Whitehead group* of $\mathbb{C}_1(\mathbb{A})$ is

$$Wh(\mathbb{C}_1(\mathbb{A})) = \text{coker}(\bar{B}_0 : K_0(\mathbb{A}) \longrightarrow K_1(\mathbb{C}_1(\mathbb{A})))$$

with

$$\begin{aligned} \bar{B}_0 : K_0(\mathbb{A}) &\longrightarrow K_0(\mathbb{P}_0(\mathbb{A})) \xrightarrow{\bar{B}} K_1(\mathbb{C}_1(\mathbb{A})) ; \\ [M] &\longrightarrow \tau(z : M[z, z^{-1}] \longrightarrow M[z, z^{-1}]) . \end{aligned}$$

□

PROPOSITION 7.4 (i) *The reduced end invariants of isomorphisms in $\mathbb{C}_1(\mathbb{A})$ define an isomorphism*

$$\partial : Wh(\mathbb{C}_1(\mathbb{A})) \longrightarrow \tilde{K}_0(\mathbb{P}_0(\mathbb{A})) ; \tau(f : L \longrightarrow M) \longrightarrow [f]_+ = -[f]_-$$

with inverse

$$\bar{B} : \tilde{K}_0(\mathbb{P}_0(\mathbb{A})) \longrightarrow Wh(\mathbb{C}_1(\mathbb{A})) ;$$

$$[M, p] \longrightarrow \tau(1 - p + zp : M[z, z^{-1}] \longrightarrow M[z, z^{-1}]) .$$

(ii) *The Whitehead torsion $\tau(f) \in Wh(\mathbb{C}_1(\mathbb{A}))$ of a chain equivalence $f : E \longrightarrow E'$ of bands in $\mathbb{C}_1(\mathbb{A})$ is sent by the isomorphism ∂ to the difference of the reduced positive end invariants*

$$\partial\tau(f) = [E']_+ - [E]_+ \in K_0(\mathbb{P}_0(\mathbb{A})) .$$

PROOF (i) Immediate from 7.2 (i).

(ii) Immediate from (i) and 7.2 (ii).

□

REMARK 7.5 (i) Browder [9] studied the problem of homotoping a homeomorphism $W \longrightarrow M \times \mathbb{R}$ to a *PL* homeomorphism, with W an open n -dimensional *PL* manifold and M a closed $(n-1)$ -dimensional *PL* manifold, proving that there is no obstruction in the case $n \geq 6$, $\pi_1(W) = \{1\}$.

(ii) If W is an open n -dimensional manifold with an \mathbb{R} -bounded homotopy equivalence $h : W \longrightarrow X \times \mathbb{R}$ (e.g. a homeomorphism) for some finite $(n - 1)$ -dimensional geometric Poincaré complex X then W is a band with two tame ends. The cellular chain complex $C(\widetilde{W})$ of the universal cover \widetilde{W} is a band in $\mathbb{C}_{\mathbb{R}}(\mathbb{B}^f(\mathbb{Z}[\pi]))$, and

$$[W]_{\pm} = [C(\widetilde{W})]_{\pm} \in K_0(\mathbb{Z}[\pi])$$

with $\pi = \pi_1(W) = \pi_1(X)$. The end invariants are such that

$$[W]_+ + [W]_- = [W] = [X] = 0 \in \widetilde{K}_0(\mathbb{Z}[\pi]) .$$

By the main result of Siebenmann [73] (cf. Pacheco and Bryant [13, §4]) for $n \geq 6$ the following conditions are equivalent:

- (a) $[W]_+ = 0 \in \widetilde{K}_0(\mathbb{Z}[\pi])$,
- (b) the \mathbb{R} -bounded homotopy equivalence $h : W \longrightarrow X \times \mathbb{R}$ can be made transverse regular at $X \times \{0\} \subset X \times \mathbb{R}$ with the restriction

$$f = h| : M = h^{-1}(X \times \{0\}) \longrightarrow X$$

a homotopy equivalence,

- (c) W is homeomorphic to $M \times \mathbb{R}$ for a compact $(n - 1)$ -manifold M homotopy equivalent to X .

See 15.4 below for the surgery-theoretic interpretation.

(iii) For any filtered additive category \mathbb{A} a band chain complex E in $\mathbb{C}_{\mathbb{R}}(\mathbb{A})$ is simple chain equivalent to $C(1 - z : F[z, z^{-1}] \longrightarrow F[z, z^{-1}])$ for a finite chain complex F in \mathbb{A} if and only if $[E]_+ = [E]_- = 0 \in \widetilde{K}_0(\mathbb{P}_0(\mathbb{A}))$, where simple means $\tau = 0 \in Wh(\mathbb{C}_{\mathbb{R}}(\mathbb{A})) = Wh(\mathbb{C}_1(\mathbb{A}))$.

□

Let $\mathbb{C}_{\mathbb{R}}(\mathbb{A})^{\mathbb{Z}}$ be the subcategory of $\mathbb{C}_{\mathbb{R}}(\mathbb{A})$ with objects L which are invariant under the \mathbb{Z} -action on \mathbb{R}

$$L(x) = L(x + 1) \quad (x \in \mathbb{R}) ,$$

with the \mathbb{Z} -equivariant morphisms. Let $\mathbb{A}[z, z^{-1}]$ be the subcategory of $\mathbb{C}_1(\mathbb{A})$ with one object $L[z, z^{-1}]$ for each object L in \mathbb{A} , and morphisms the \mathbb{Z} -equivariant morphisms in $\mathbb{C}_1(\mathbb{A})$

$$\sum_{j=-\infty}^{\infty} z^j f_j : L[z, z^{-1}] \longrightarrow M[z, z^{-1}]$$

defined by collections $\{f_j \in \text{Hom} \mid j \in \mathbb{Z}\}$ with $\{j \mid f_j \neq 0\}$ finite. The functor

$$\mathbb{A}[z, z^{-1}] \longrightarrow \mathbb{C}_{\mathbb{R}}(\mathbb{A})^{\mathbb{Z}} ; \quad M[z, z^{-1}] \longrightarrow M[z, z^{-1}]$$

is an equivalence of additive categories.

PROPOSITION 7.6 *For any filtered additive category \mathbb{A} the map*

$$\begin{aligned} \bar{B} : K_0(\mathbb{P}_0(\mathbb{A})) &\longrightarrow K_1(\mathbb{C}_{\mathbb{R}}(\mathbb{A})^{\mathbb{Z}}) ; \\ [M, p] &\longrightarrow \tau(zp + 1 - p : M[z, z^{-1}] \longrightarrow M[z, z^{-1}]) . \end{aligned}$$

is a split injection.

PROOF The composite

$$\bar{B} : K_0(\mathbb{P}_0(\mathbb{A})) \xrightarrow{\bar{B}} K_1(\mathbb{C}_{\mathbb{R}}(\mathbb{A})^{\mathbb{Z}}) \longrightarrow K_1(\mathbb{C}_{\mathbb{R}}(\mathbb{A}))$$

is an isomorphism, by 7.2 (i). □

For a ring A and the additive category $\mathbb{A} = \mathbb{B}^f(A)$ of based f.g. free A -modules there is an equivalence of additive categories

$$\mathbb{B}^f(A[z, z^{-1}]) \longrightarrow \mathbb{A}[z, z^{-1}] ; M[z, z^{-1}] \longrightarrow M[z, z^{-1}] ,$$

with $A[z, z^{-1}]$ the Laurent polynomial extension ring of A . The map \bar{B} is the original injection due to Bass, Heller and Swan [8] of the projective class group of a ring A in the torsion group of the Laurent polynomial extension $A[z, z^{-1}]$

$$\bar{B} : K_0(A) \longrightarrow K_1(A[z, z^{-1}]) ; [P] \longrightarrow \tau(z : P[z, z^{-1}] \longrightarrow P[z, z^{-1}]) .$$

The K -theory of the polynomial extension category $\mathbb{A}[z, z^{-1}]$ will be examined in greater detail in §10 below.

§8. The Laurent polynomial extension category $\mathbb{A}[z, z^{-1}]$

The finite Laurent polynomial extension category $\mathbb{A}[z, z^{-1}]$ of a filtered additive category \mathbb{A} was defined in §7 to be the subcategory of $\mathbb{C}_1(\mathbb{A})$ with objects the polynomial extensions of objects L in \mathbb{A}

$$L[z, z^{-1}] = \sum_{j=-\infty}^{\infty} z^j L$$

and with \mathbb{Z} -equivariant morphisms. The infinite Laurent polynomial extension category $\mathbb{G}_1(\mathbb{A})[[z, z^{-1}]]$ is defined to be a subcategory of $\mathbb{G}_{\mathbb{Z}^2}(\mathbb{A})$ containing $\mathbb{A}[z, z^{-1}]$ as a subcategory. The \mathbb{Z} -equivariant version of the algebraic transversality of §4 is used to construct ‘finite Mayer-Vietoris presentations’ in $\mathbb{A}[z, z^{-1}]$ for finite chain complexes in $\mathbb{A}[z, z^{-1}]$, as subobjects of canonical ‘infinite Mayer-Vietoris presentations’ in $\mathbb{G}_1(\mathbb{A})[[z, z^{-1}]]$. This algebraic transversality is an abstract version of the existence of fundamental domains for free cocompact actions of the infinite cyclic group \mathbb{Z} on manifolds. Algebraic transversality will be used in §10 to express the torsion groups of $\mathbb{A}[z, z^{-1}]$ in terms of the algebraic K -groups of \mathbb{A} .

DEFINITION 8.1 (i) The *finite Laurent extension* of an additive group A is the additive group $A[z, z^{-1}]$ of formal power series $\sum_{k=-\infty}^{\infty} a_k z^k$ with coefficients $a_k \in A$ such that $\{k \in \mathbb{Z} \mid a_k \neq 0 \in A\}$ is finite.

(ii) The *infinite Laurent extension* of a countable product of additive groups $A = \prod_{j=-\infty}^{\infty} A(j)$ is the additive group $A[[z, z^{-1}]]$ of formal power series $\sum_{k=-\infty}^{\infty} a_k z^k$ with coefficients

$$a_k = \prod_{j=-\infty}^{\infty} a_k(j) \in A = \prod_{j=-\infty}^{\infty} A(j)$$

such that $\{k \in \mathbb{Z} \mid a_k(j) \neq 0 \in A(j)\}$ is finite for each $j \in \mathbb{Z}$.

□

In particular, if A is the additive group of a ring then $A[z, z^{-1}]$ is also a ring, and if M is an A -module then $M[z, z^{-1}]$ is an $A[z, z^{-1}]$ -module. For a f.g. free A -module M and any A -module N

$$\mathrm{Hom}_{A[z, z^{-1}]}(M[z, z^{-1}], N[z, z^{-1}]) = \mathrm{Hom}_A(M, N)[z, z^{-1}] .$$

For a countable direct sum $M = \sum_{j=-\infty}^{\infty} M(j)$ of f.g. free A -modules

$$\mathrm{Hom}_A(M, N) = \prod_{j=-\infty}^{\infty} \mathrm{Hom}_A(M(j), N) ,$$

$$\mathrm{Hom}_{A[z, z^{-1}]}(M[z, z^{-1}], N[z, z^{-1}]) = \mathrm{Hom}_A(M, N)[[z, z^{-1}]] .$$

Given an additive category \mathbb{A} let $\mathbb{G}_1(\mathbb{A}) = \mathbb{G}_{\mathbb{Z}}(\mathbb{A})$ be the \mathbb{Z} -graded category.

EXAMPLE 8.2 For any ring A let

$$\mathbb{B}^f(A) = \{\text{based f.g. free } A\text{-modules}\} ,$$

$$\mathbb{B}(A) = \{\text{countably based free } A\text{-modules}\} .$$

The forgetful functor

$$\mathbb{G}_1(\mathbb{B}^f(A)) \longrightarrow \mathbb{B}(A) ; M \longrightarrow \sum_{j=-\infty}^{\infty} M(j)$$

is an equivalence of additive categories.

□

For any object L in \mathbb{A} let $L[z, z^{-1}]$ be the object in $\mathbb{G}_1(\mathbb{A})$ defined by

$$L[z, z^{-1}](j) = z^j L \quad (j \in \mathbb{Z})$$

with $z^j L$ a copy of L . A morphism $f : L[z, z^{-1}] \longrightarrow L'[z, z^{-1}]$ in $\mathbb{G}_1(\mathbb{A})$ is *homogeneous of degree k* if

(i) for all $j, j' \in \mathbb{Z}$ with $j' - j \neq k$

$$f(j', j) = 0 : z^j L \longrightarrow z^{j'} L' ,$$

(ii) for all $i, j \in \mathbb{Z}$

$$f(i + k, i) = f(j + k, j) = f_k :$$

$$z^i L = z^j L = L \longrightarrow z^{i+k} L' = z^{j+k} L' = L'$$

for some morphism $f_k : L \longrightarrow L'$ in \mathbb{A} ,

in which case f is written as

$$f = z^k f_k : L[z, z^{-1}] \longrightarrow L'[z, z^{-1}] .$$

DEFINITION 8.3 The *finite Laurent extension* $\mathbb{A}[z, z^{-1}]$ of an additive category \mathbb{A} is the subcategory of $\mathbb{G}_1(\mathbb{A})$ with one object $L[z, z^{-1}]$ for each object L in \mathbb{A} , with morphisms finite sums of homogeneous morphisms

$$f = \sum_{k=-\infty}^{\infty} z^k f_k : L[z, z^{-1}] \longrightarrow L'[z, z^{-1}] ,$$

so that

$$\mathrm{Hom}_{\mathbb{A}[z, z^{-1}]}(L[z, z^{-1}], L'[z, z^{-1}]) = \mathrm{Hom}_{\mathbb{A}}(L, L')[z, z^{-1}] .$$

□

For each object $M = L[z, z^{-1}]$ in $\mathbb{A}[z, z^{-1}]$ define a homogeneous degree 1 automorphism

$$\zeta(M) = z : M \longrightarrow M$$

by

$$\zeta(M)_1 = \text{identity} : L \longrightarrow L .$$

The morphisms in $\mathbb{A}[z, z^{-1}]$ are the morphisms $f : M \longrightarrow M'$ in $\mathbb{G}_1(\mathbb{A})$ which are ζ -equivariant that is

$$f\zeta(M) = \zeta(M')f : M \longrightarrow M' .$$

The finite Laurent extension $\mathbb{A}[z, z^{-1}]$ of a filtered additive category \mathbb{A} is a subcategory of the bounded \mathbb{Z} -graded category $\mathbb{C}_1(\mathbb{A})$, namely the subcategory

$$\mathbb{A}[z, z^{-1}] = \mathbb{C}_1(\mathbb{A})^{\mathbb{Z}}$$

with objects $L[z, z^{-1}]$ and the ζ -equivariant morphisms.

EXAMPLE 8.4 For any ring A the finite Laurent extension of the additive category $\mathbb{B}^f(A)$ of based f.g. free A -modules is the additive category of based f.g. free $A[z, z^{-1}]$ -modules

$$\mathbb{B}^f(A)[z, z^{-1}] = \mathbb{B}^f(A[z, z^{-1}]) .$$

□

For any ring A the inclusion defines a morphism of rings

$$i : A \longrightarrow A[z, z^{-1}] ; a \longrightarrow a = \sum_{j=-\infty}^{\infty} a_j z^j , \quad a_j = \begin{cases} a & \text{if } j = 0 \\ 0 & \text{if } j \neq 0. \end{cases}$$

Induction and restriction define functors

$$i_! : (A\text{-modules}) \longrightarrow (A[z, z^{-1}]\text{-modules}) ,$$

$$i^! : (A[z, z^{-1}]\text{-modules}) \longrightarrow (A\text{-modules}) .$$

An A -module L induces an $A[z, z^{-1}]$ -module

$$i_! L = A[z, z^{-1}] \otimes_A L = L[z, z^{-1}] = \sum_{j=-\infty}^{\infty} z^j L .$$

The restriction $i^! M$ of an $A[z, z^{-1}]$ -module M is the A -module defined by the additive group of M with A acting by the restriction of $A[z, z^{-1}]$ -action to $A \subset A[z, z^{-1}]$. An $A[z, z^{-1}]$ -module morphism $f : L[z, z^{-1}] \longrightarrow L'[z, z^{-1}]$ can be expressed as a polynomial

$$f = \sum_{j=-\infty}^{\infty} z^j f_j : L[z, z^{-1}] \longrightarrow L'[z, z^{-1}] ;$$

$$\sum_{k=-\infty}^{\infty} z^k x_k \longrightarrow \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} z^{j+k} f_j(x_k)$$

with coefficients $f_j \in \text{Hom}_A(L, L')$ ($j \in \mathbb{Z}$) such that for each $x \in L$ the set $\{j \in \mathbb{Z} \mid f_j(x) \neq 0\}$ is finite. For f.g. L this condition is equivalent to the set $\{j \in \mathbb{Z} \mid f_j \neq 0\}$ being finite, but this need not be the case for infinitely generated L . In particular, if $L = i^! i_! K$ for an A -module K then the $A[z, z^{-1}]$ -module morphism

$$p = \sum_{j=-\infty}^{\infty} z^j p_j : i_! L = i_! i^! i_! K \longrightarrow i_! K ; a \otimes (b \otimes x) \longrightarrow ab \otimes x$$

$$(a, b \in A[z, z^{-1}] , \quad x \in K)$$

has each coefficient $p_j : L \longrightarrow K$ non-zero, namely

$$p_j : L = i^! i_! K = \sum_{k=-\infty}^{\infty} z^k K \longrightarrow K ; \quad \sum_{k=-\infty}^{\infty} z^k x_k \longrightarrow x_j .$$

For each $k \in \mathbb{Z}$ there is only a finite number of $j \in \mathbb{Z}$ with $p_j(z^k K) \neq \{0\}$ (only $j = k$), so that

$$p = \sum_{j=-\infty}^{\infty} p_j z^j \in \text{Hom}_A(L, K)[[z, z^{-1}]] ,$$

with the infinite Laurent extension defined as in 8.1 (ii) with respect to

the countable product structure

$$\mathrm{Hom}_A(L, K) = \prod_{k=-\infty}^{\infty} \mathrm{Hom}_A(z^k K, K) .$$

DEFINITION 8.5 For any additive category \mathbb{A} the *induction* and *restriction* functors are given by

$$i_! : \mathbb{A} \longrightarrow \mathbb{A}[z, z^{-1}] ; L \longrightarrow i_! L = L[z, z^{-1}] = \sum_{j=-\infty}^{\infty} z^j L ,$$

$$i^! : \mathbb{A}[z, z^{-1}] \longrightarrow \mathbb{G}_1(\mathbb{A}) ;$$

$$M = L[z, z^{-1}] \longrightarrow i^! M , (i^! M)(j) = z^j L .$$

The object $i^!(L[z, z^{-1}])$ will usually be written $L[z, z^{-1}]$. A morphism $f : L \longrightarrow L'$ in \mathbb{A} induces a homogeneous degree 0 morphism $i_! f : i_! L \longrightarrow i_! L'$ in $\mathbb{A}[z, z^{-1}]$ with $(i_! f)_0 = f : L \longrightarrow L'$. A morphism $f : L[z, z^{-1}] \longrightarrow L'[z, z^{-1}]$ in $\mathbb{A}[z, z^{-1}]$ restricts to a morphism in $\mathbb{G}_1(\mathbb{A})$

$$i^! f : L[z, z^{-1}] \longrightarrow L'[z, z^{-1}]$$

with

$$(i^! f)(j, k) = f_{j-k} : L[z, z^{-1}](k) = z^k L \longrightarrow L'[z, z^{-1}](j) = z^j L' .$$

□

EXAMPLE 8.6 For a ring A and the additive category $\mathbb{A} = \mathbb{B}^f(A)$ of based f.g. free A -modules the induction and restriction functors of 8.5 are the induction and restriction associated to the inclusion $i : A \longrightarrow A[z, z^{-1}]$

$$i_! : \mathbb{A} = \mathbb{B}^f(A) \longrightarrow \mathbb{A}[z, z^{-1}] = \mathbb{B}^f(A[z, z^{-1}]) ,$$

$$i^! : \mathbb{A}[z, z^{-1}] = \mathbb{B}^f(A[z, z^{-1}]) \longrightarrow \mathbb{G}_1(\mathbb{A}) \approx \mathbb{B}(A) .$$

□

Let $\mathbb{G}_2(\mathbb{A}) = \mathbb{G}_{\mathbb{Z}^2}(\mathbb{A})$ be the \mathbb{Z}^2 -graded category. Given an object L in $\mathbb{G}_1(\mathbb{A})$ let $L[z, z^{-1}]$ be the object in $\mathbb{G}_2(\mathbb{A})$ defined by

$$L[z, z^{-1}](j, k) = z^k L(j) \quad (j, k \in \mathbb{Z}) .$$

For any objects L, L' in $\mathbb{G}_1(\mathbb{A})$ use the countable direct product structure

$$\mathrm{Hom}_{\mathbb{G}_1(\mathbb{A})}(L, L') = \prod_{j=-\infty}^{\infty} \left(\sum_{j'=-\infty}^{\infty} \mathrm{Hom}_{\mathbb{A}}(L(j), L'(j')) \right)$$

to identify the infinite Laurent polynomial extension

$$\mathrm{Hom}_{\mathbb{G}_1(\mathbb{A})}(L, L')[[z, z^{-1}]]$$

with the image of the injection

$$\begin{aligned} \mathrm{Hom}_{\mathbb{G}_1(\mathbb{A})}(L, L')[[z, z^{-1}]] &\longrightarrow \mathrm{Hom}_{\mathbb{G}_2(\mathbb{A})}(L[z, z^{-1}], L'[z, z^{-1}]) ; \\ \sum_{k=-\infty}^{\infty} f_k z^k &\longrightarrow f \end{aligned}$$

defined by

$$\begin{aligned} f((j', k'), (j, k)) &= f_{k'-k}(j', j) : \\ z^k L(j) &= L(j) \longrightarrow z^{k'} L'(j') = L'(j') . \end{aligned}$$

DEFINITION 8.7 The *infinite Laurent extension* $\mathbb{G}_1(\mathbb{A})[[z, z^{-1}]]$ of $\mathbb{G}_1(\mathbb{A})$ is the subcategory of $\mathbb{G}_2(\mathbb{A})$ with one object $L[z, z^{-1}]$ for each object L in $\mathbb{G}_1(\mathbb{A})$, and morphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbb{G}_1(\mathbb{A})}([z, z^{-1}]](L[z, z^{-1}], L'[z, z^{-1}]) &= \mathrm{Hom}_{\mathbb{G}_1(\mathbb{A})}(L, L')[[z, z^{-1}]] \\ &\subseteq \mathrm{Hom}_{\mathbb{G}_2(\mathbb{A})}(L[z, z^{-1}], L'[z, z^{-1}]) . \end{aligned}$$

□

EXAMPLE 8.8 The additive category $\mathbb{B}(A[z, z^{-1}])$ of countably based free A -modules is such that the forgetful functor

$$\mathbb{G}_1(\mathbb{B}^f(A))[[z, z^{-1}]] \longrightarrow \mathbb{B}(A[z, z^{-1}])$$

is an equivalence.

□

A morphism $f = \sum_{j=-\infty}^{\infty} z^j f_j : M \longrightarrow M'$ in $\mathbb{G}_1(\mathbb{A})[[z, z^{-1}]]$ is *homogeneous of degree k* if $f_j = 0$ for $j \neq k$. A morphism $f : M \longrightarrow M'$ in $\mathbb{G}_1(\mathbb{A})[[z, z^{-1}]]$ is *linear* if $f_j = 0$ for $j \neq 0, 1$, so that

$$f = f_0 + z f_1 : M \longrightarrow M' .$$

In the applications it is convenient to introduce a change of sign, writing linear morphisms as

$$f = f_+ - z f_- : M \longrightarrow M'$$

with $f_+ = f_0$, $f_- = -f_1$.

The Laurent polynomial extension category $\mathbb{A}[z, z^{-1}]$ will be identified with the full subcategory of $\mathbb{G}_1(\mathbb{A})[[z, z^{-1}]]$ with objects $L[z, z^{-1}]$ such that $L(j) = 0$ for $j \neq 0$. Also, $\mathbb{C}_0(\mathbb{A})[z, z^{-1}]$ is identified with the full subcategory of $\mathbb{G}_1(\mathbb{A})[[z, z^{-1}]]$ with objects $L[z, z^{-1}]$ such that $\{j \in \mathbb{Z} \mid L(j) \neq 0\}$ is finite.

DEFINITION 8.9 A *Mayer-Vietoris presentation* of an object M in $\mathbb{A}[z, z^{-1}]$ is an exact sequence in $\mathbb{G}_1(\mathbb{A})[[z, z^{-1}]]$

$$\mathrm{MV} : 0 \longrightarrow L''[z, z^{-1}] \xrightarrow{f} L'[z, z^{-1}] \xrightarrow{g} M \longrightarrow 0$$

with $f = f_+ - zf_-$ linear. The Mayer-Vietoris presentation is *finite* if \mathbb{M} is an exact sequence in $\mathbb{C}_0(\mathbb{A})[z, z^{-1}]$, in which case it is equivalent to an exact sequence in $\mathbb{A}[z, z^{-1}]$, and the *torsion* of \mathbb{M} is defined by

$$\tau(\mathbb{M}) = \tau((f \ h) : L''[z, z^{-1}] \oplus M \longrightarrow L'[z, z^{-1}]) \in K_1^{iso}(\mathbb{A}[z, z^{-1}])$$

for any morphism $h : M \longrightarrow L'[z, z^{-1}]$ splitting g .

□

For a ring A and $\mathbb{A} = \mathbb{B}^f(\mathbb{A})$ the finite Mayer-Vietoris presentations of objects in $\mathbb{A}[z, z^{-1}] = \mathbb{B}^f(A)[z, z^{-1}]$ are ‘Mayer-Vietoris presentations’ in the sense of Waldhausen [82].

The restriction of an object $M = i_!L$ in $\mathbb{A}[z, z^{-1}]$ is an object $i^!M$ in $\mathbb{G}_1(\mathbb{A})$. The induced object $i_!i^!M$ in $\mathbb{G}_1(\mathbb{A})[[z, z^{-1}]]$ is equipped with commuting automorphisms

$$\zeta_1(M) = \zeta(i_!i^!M) \ , \ \zeta_2(M) = i_!i^!\zeta(M) : i_!i^!M \longrightarrow i_!i^!M$$

such that

$$i_!i^!M = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \zeta_1(M)^j \zeta_2(M)^k L \ .$$

Define a morphism in $\mathbb{G}_1(\mathbb{A})[[z, z^{-1}]]$

$$p(M) = \sum_{j=-\infty}^{\infty} z^j p(M)_j : i_!i^!M \longrightarrow M = i_!L ;$$

$$\zeta_1(M)^j \zeta_2(M)^k x \longrightarrow \zeta(M)^{j+k} x$$

by

$$p(M)_j = j\text{th projection} : i^!M = \sum_{k=-\infty}^{\infty} \zeta(M)^k L \longrightarrow L ;$$

$$\sum_{k=-\infty}^{\infty} \zeta(M)^k x_k \longrightarrow x_j \ .$$

DEFINITION 8.10 The exact sequence

$$\mathbb{M}(\infty) : 0 \longrightarrow i_!i^!M \xrightarrow{1 - z\zeta(M)^{-1}} i_!i^!M \xrightarrow{p(M)} M \longrightarrow 0$$

is the *universal Mayer-Vietoris presentation* of M in $\mathbb{G}_1(\mathbb{A})[[z, z^{-1}]]$.

□

Let $\mathbb{Z}^+ = \{r \in \mathbb{Z} \mid r \geq 0\} \cup \{\infty\}$, and let \mathbb{N} be the lattice of pairs $(N^+, N^-) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, with

$$(N \cup N')^\pm = \max(N^\pm, N'^\pm) \ , \ (N \cap N')^\pm = \min(N^\pm, N'^\pm) \ .$$

\mathbb{N} has the maximum element $\infty = (\infty; \infty)$. An element $N = (N^+, N^-)$

$\in \mathbb{N}$ is *finite* if $N^+ \neq \infty$ and $N^- \neq \infty$. Let $\mathbb{N}^f \subset \mathbb{N}$ be the sublattice of finite elements.

PROPOSITION 8.11 (i) *For every object $M = i_!L$ in $\mathbb{A}[z, z^{-1}]$ and every $N = (N^+, N^-) \in \mathbb{N}$ there is defined a Mayer-Vietoris presentation of M in $\mathbb{G}_1(\mathbb{A})[[z, z^{-1}]]$*

$$\mathbb{M}\langle N \rangle : 0 \longrightarrow M''\langle N \rangle[z, z^{-1}] \xrightarrow{f\langle N \rangle} M'\langle N \rangle[z, z^{-1}] \xrightarrow{g\langle N \rangle} M \longrightarrow 0$$

with

$$\begin{aligned} M'\langle N \rangle &= \sum_{k=-N^+}^{N^-} z^k L, \quad M''\langle N \rangle = \sum_{k=-N^++1}^{N^-} z^k L, \\ f\langle N \rangle &= f_+\langle N \rangle - z f_-\langle N \rangle : M''\langle N \rangle[z, z^{-1}] \longrightarrow M'\langle N \rangle[z, z^{-1}], \\ f_+\langle N \rangle &: M''\langle N \rangle \longrightarrow M'\langle N \rangle; \quad \sum_{k=-N^++1}^{N^-} z^k x_k \longrightarrow \sum_{k=-N^++1}^{N^-} z^k x_k, \\ f_-\langle N \rangle &: M''\langle N \rangle \longrightarrow M'\langle N \rangle; \quad \sum_{k=-N^++1}^{N^-} z^k x_k \longrightarrow \sum_{k=-N^++1}^{N^-} z^{k-1} x_k, \\ g\langle N \rangle_j &: M'\langle N \rangle \longrightarrow L; \quad \sum_{k=-N^+}^{N^-} z^k x_k \longrightarrow x_j. \end{aligned}$$

(ii) *If $N \in \mathbb{N}$ is finite then $\mathbb{M}\langle N \rangle$ is a finite linear presentation of M in $\mathbb{C}_0(\mathbb{A})[z, z^{-1}]$.*

(iii) *If $N, N' \in \mathbb{N}$ are such that $N \leq N'$ then inclusion defines a morphism of Mayer-Vietoris presentations $\mathbb{M}\langle N \rangle \longrightarrow \mathbb{M}\langle N' \rangle$ resolving $1 : M \longrightarrow M$.*

PROOF $\mathbb{M}\langle N \rangle$ is a subobject of the universal Mayer-Vietoris presentation $\mathbb{M}\langle \infty \rangle$ of M .

□

DEFINITION 8.12 A *Mayer-Vietoris presentation* of a finite chain complex E in $\mathbb{A}[z, z^{-1}]$ is an exact sequence of finite chain complexes in $\mathbb{G}_1(\mathbb{A})[[z, z^{-1}]]$

$$\mathbb{E} : 0 \longrightarrow E''[z, z^{-1}] \xrightarrow{f} E'[z, z^{-1}] \xrightarrow{g} E \longrightarrow 0$$

with E', E'' finite chain complexes in $\mathbb{G}_1(\mathbb{A})$ and f linear. \mathbb{E} is *finite* if it is defined in $\mathbb{C}_0(\mathbb{A})[z, z^{-1}]$, in which case it is equivalent to an exact sequence of chain complexes in $\mathbb{A}[z, z^{-1}]$ and the *torsion* of \mathbb{E} is defined

by

$$\begin{aligned}\tau(\mathbb{E}) &= \tau((g \ 0) : C(f_+ - zf_- : E'' \longrightarrow E') \longrightarrow E) \\ &= \sum_{r=0}^{\infty} (-)^r \tau(\mathbb{E}_r) \in K_1(\mathbb{A}[z, z^{-1}]) .\end{aligned}$$

□

Given an n -dimensional chain complex E in $\mathbb{A}[z, z^{-1}]$ with $E_r = i_! F_r$ ($0 \leq r \leq n$) let $\mathbb{N}(E)$ be the lattice of $(2n+2)$ -tuples of elements of \mathbb{Z}^+

$$N = (N_0^+, N_1^+, \dots, N_n^+; N_0^-, N_1^-, \dots, N_n^-)$$

such that

$$d_E\left(\sum_{j=-N_r^+}^{N_r^-} z^j F_r\right) \subseteq \sum_{k=-N_{r-1}^+}^{N_{r-1}^-} z^k F_{r-1} \quad (1 \leq r \leq n) .$$

$\mathbb{N}(E)$ has the maximum element

$$\infty = (\infty, \dots, \infty; \infty, \dots, \infty) \in \mathbb{N}(E) .$$

An element $N \in \mathbb{N}(E)$ is *finite* if

$$N_r^\pm \neq \infty \quad (0 \leq r \leq n) .$$

Let $\mathbb{N}^f(E) \subset \mathbb{N}(E)$ be the sublattice of finite elements.

PROPOSITION 8.13 (i) *For every n -dimensional chain complex E in $\mathbb{A}[z, z^{-1}]$ with $E_r = i_! F_r$ ($0 \leq r \leq n$) and every $N \in \mathbb{N}(E)$ there is defined a Mayer-Vietoris presentation of E in $\mathbb{G}_1(\mathbb{A})[[z, z^{-1}]]$*

$$\begin{aligned}\mathbb{E}\langle N \rangle : 0 \longrightarrow E''\langle N \rangle[z, z^{-1}] &\xrightarrow{f_+\langle N \rangle - zf_-\langle N \rangle} E'\langle N \rangle[z, z^{-1}] \\ &\xrightarrow{g\langle N \rangle} E \longrightarrow 0 .\end{aligned}$$

(ii) $\mathbb{N}^f(E)$ is non-empty, and $\mathbb{E}\langle N \rangle$ is a finite Mayer-Vietoris presentation of E for any $N \in \mathbb{N}^f(E)$.

(iii) If $N, N' \in \mathbb{N}(E)$ are such that $N \leq N'$ then inclusion defines a morphism of Mayer-Vietoris presentations $\mathbb{E}\langle N \rangle \longrightarrow \mathbb{E}\langle N' \rangle$ resolving $1 : E \longrightarrow E$.

PROOF Set

$$\mathbb{E}\langle N \rangle_r = \mathbb{E}_r\langle N_r \rangle \quad (0 \leq r \leq n)$$

the Mayer-Vietoris presentation of E_r associated to $N_r = (N_r^+, N_r^-) \in \mathbb{N}$.

□

The *universal Mayer-Vietoris presentation* of a finite chain complex E in $\mathbb{A}[z, z^{-1}]$ is the Mayer-Vietoris presentation of E in $\mathbb{G}_1(\mathbb{A})[[z, z^{-1}]]$

$$\mathbb{E}\langle \infty \rangle : 0 \longrightarrow i_! i^! E \xrightarrow{1 - z\zeta(E)^{-1}} i_! i^! E \xrightarrow{p(E)} E \longrightarrow 0$$

associated by 8.12 to the maximal element $\infty \in \mathbb{N}(E)$. The universal property is that for any chain map $f : E' \longrightarrow E$ of finite chain complexes in $\mathbb{A}[z, z^{-1}]$ and any Mayer-Vietoris presentation \mathbb{E}' of E' there is defined a unique morphism $f : \mathbb{E}' \longrightarrow \mathbb{E}(\infty)$ resolving f .

DEFINITION 8.14 (i) For any filtered additive category \mathbb{A} let

$$\begin{aligned} \bar{B} : K_0(\mathbb{P}_0(\mathbb{A})) &\longrightarrow K_1(\mathbb{A}[z, z^{-1}]) ; \\ [M, p] &\longrightarrow \tau(zp + 1 - p : M[z, z^{-1}] \longrightarrow M[z, z^{-1}]) , \\ \bar{B}_0 : K_0(\mathbb{A}) &\longrightarrow K_0(\mathbb{P}_0(\mathbb{A})) \xrightarrow{\bar{B}} K_1(\mathbb{A}[z, z^{-1}]) ; \\ [M] &\longrightarrow \tau(z : M[z, z^{-1}] \longrightarrow M[z, z^{-1}]) . \end{aligned}$$

(ii) The *Whitehead group* of $\mathbb{A}[z, z^{-1}]$ is defined by

$$Wh(\mathbb{A}[z, z^{-1}]) = \text{coker}(\bar{B}_0 : K_0(\mathbb{A}) \longrightarrow \tilde{K}_1(\mathbb{A}[z, z^{-1}])) .$$

□

For a ring A and the additive category $\mathbb{A} = \mathbb{B}^f(A)$ of based f.g. free A -modules this is the original injection due to Bass, Heller and Swan [8] of the projective class group of a ring A in the torsion group of the Laurent polynomial extension $A[z, z^{-1}]$

$$\bar{B} : K_0(A) \longrightarrow K_1(A[z, z^{-1}]) ; [P] \longrightarrow \tau(z : P[z, z^{-1}] \longrightarrow P[z, z^{-1}]) .$$

The Whitehead group of the polynomial extension category

$$\mathbb{A}[z, z^{-1}] = \mathbb{B}^f(A)[z, z^{-1}] = \mathbb{B}^f(A[z, z^{-1}])$$

is

$$\begin{aligned} Wh(\mathbb{A}[z, z^{-1}]) &= \text{coker}(\bar{B}_0 : K_0(\mathbb{A}) \longrightarrow \tilde{K}_1(\mathbb{A}[z, z^{-1}])) \\ &= K_1(A[z, z^{-1}]) / \{\tau(\pm z : A[z, z^{-1}] \longrightarrow A[z, z^{-1}])\} , \end{aligned}$$

since $K_0(\mathbb{A})$ is generated by $[A]$. (If A is such that the rank of f.g. free A -modules is well-defined then $K_0(\mathbb{A}) = \mathbb{Z}$). The Laurent polynomial extension of a group ring $A = \mathbb{Z}[\pi]$ is the group ring

$$A[z, z^{-1}] = \mathbb{Z}[\pi \times \mathbb{Z}] ,$$

and the actual Whitehead group of $\pi \times \mathbb{Z}$ is

$$\begin{aligned} Wh(\pi \times \mathbb{Z}) &= K_1(\mathbb{Z}[\pi \times \mathbb{Z}]) / \{\tau(\pm z^j g) \mid g \in \pi, j \in \mathbb{Z}\} \\ &= Wh(\mathbb{B}^f(\mathbb{Z}[\pi])[z, z^{-1}]) / \{\tau(g) \mid g \in \pi\} . \end{aligned}$$

A *chain complex band* E in $\mathbb{A}[z, z^{-1}]$ is a finite chain complex such that the restriction $i^!E$ is $\mathbb{C}_0(\mathbb{A})$ -finitely dominated in $\mathbb{G}_1(\mathbb{A})$, so that E is a chain complex band in $\mathbb{G}_1(\mathbb{A})$ and the end invariants $[E]_{\pm} \in K_0(\mathbb{P}_0(\mathbb{A}))$ are defined as in 4.10.

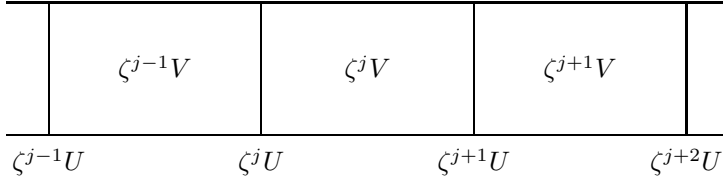
EXAMPLE 8.15 Let X be a finite n -dimensional CW complex with fundamental group $\pi_1(X) = \pi \times \mathbb{Z}$, so that $\mathbb{Z}[\pi_1(X)] = \mathbb{Z}[\pi][z, z^{-1}]$. The

cellular chain complex of the universal cover \tilde{X} is a finite n -dimensional chain complex $C(\tilde{X})$ in $\mathbb{B}^f(\mathbb{Z}[\pi_1(X)])$. The Mayer-Vietoris presentations of $C(\tilde{X})$ constructed in 8.12 are obtained by an algebraic transversality which mimics the geometric codimension 1 transversality for maps $p : X \rightarrow S^1$ inducing

$$p_* = \text{projection} : \pi_1(X) = \pi \times \mathbb{Z} \rightarrow \pi_1(S^1) = \mathbb{Z} .$$

Let $\bar{X} = \tilde{X}/\pi$ be the infinite cyclic cover of X classified by p , and let $\zeta : \bar{X} \rightarrow \bar{X}$ be a generating covering translation. If X is a compact n -dimensional manifold it is possible to choose $p : X \rightarrow S^1$ transverse regular at a point $* \in X$, so that $V = p^{-1}(\{*\})$ is a codimension 1 framed submanifold of X , and cutting X along V defines a compact cobordism $(V; U, \zeta U)$ between U and a copy ζU of U such that

$$\bar{X} = \bigcup_{j=-\infty}^{\infty} \zeta^j(V; U, \zeta U) .$$



Thus codimension 1 manifold transversality determines a finite Mayer-Vietoris presentation of $C(\bar{X})$

$$0 \rightarrow C(\tilde{U})[z, z^{-1}] \xrightarrow{f_+ - zf_-} C(\tilde{V})[z, z^{-1}] \rightarrow C(\bar{X}) \rightarrow 0 ,$$

with \tilde{U}, \tilde{V} the covers of U, V induced from the universal cover \tilde{X} of \bar{X} . More generally, for any finite CW complex X it is possible to develop a codimension 1 CW transversality theory using the mapping torus of ζ

$$T(\zeta) = \bar{X} \times [0, 1] / \{(x, 0) = (\zeta(x), 1) | x \in \bar{X}\} ,$$

which has an infinite cyclic cover

$$\bar{T}(\zeta) = \bar{X} \times \mathbb{R}$$

with generating covering translation

$$\bar{T}(\zeta) \rightarrow \bar{T}(\zeta) ; (x, t) \rightarrow (\zeta(x), t + 1) .$$

Define a *subfundamental domain* to be a subcomplex $Y \subseteq \bar{X}$ such that

$$\bigcup_{j=-\infty}^{\infty} \zeta^j Y = \bar{X} .$$

A *fundamental domain* is a subfundamental domain such that

$$Y \cap \zeta^j Y = \emptyset \text{ for } j \neq -1, 0, 1 .$$

For any subfundamental domain Y (e.g. \bar{X}) define

$$(V; U, \zeta U) = (Y \times [0, 1]; Y \cap \zeta^{-1}Y \times 0, \zeta Y \cap Y \times 1) \\ \subseteq \bar{T}(\zeta) = \bar{X} \times \mathbb{R} ,$$

$T_Y(\zeta) = Y \times [0, 1] / \{(y, 0) = (\zeta(y), 1) \mid y \in Y \cap \zeta^{-1}Y\} \subseteq T(\zeta)$,
so that $T_Y(\zeta)$ has infinite cyclic cover with fundamental domain V

$$\bar{T}_Y(\zeta) = \bigcup_{j=-\infty}^{\infty} \zeta^j(V; U, \zeta U) \subseteq \bar{T}(\zeta) .$$

The projection

$$p_Y : T_Y(\zeta) \longrightarrow X ; (x, t) \longrightarrow p(x)$$

has contractible point inverses, and is thus a homotopy equivalence, corresponding to the Mayer-Vietoris presentation

$$0 \longrightarrow C(U)[z, z^{-1}] \xrightarrow{f_+ - zf_-} C(V)[z, z^{-1}] \longrightarrow C(\tilde{X}) \longrightarrow 0 .$$

In particular, for $Y = \bar{X}$

$$V = \bar{X} \times [0, 1] , \quad U = \bar{X} , \quad T_Y(\zeta) = T(\zeta) ,$$

corresponding to the universal Mayer-Vietoris presentation

$$0 \longrightarrow C(\tilde{X})[z, z^{-1}] \xrightarrow{1 - z\tilde{\zeta}^{-1}} C(\tilde{X})[z, z^{-1}] \longrightarrow C(\tilde{X}) \longrightarrow 0$$

with $\tilde{\zeta} : \tilde{X} \longrightarrow \tilde{X}$ a $\pi_1(X)$ -equivariant lift of $\zeta : X \longrightarrow X$. Subfundamental domains $Y \subseteq \bar{X}$ can be constructed in exactly the same way as the subcomplexes $E'\langle N \rangle \subseteq i^!E$ used to define $\mathbb{E}\langle N \rangle$ in 8.12. Let I_r ($0 \leq r \leq n$) be an indexing set for the r -cells $e^r \subseteq X$, so that

$$X = \bigcup_{I_0} e^0 \cup \bigcup_{I_1} e^1 \cup \dots \cup \bigcup_{I_n} e^n ,$$

and choose a lift $\bar{e}^r \subseteq \bar{X}$ for each cell $e^r \subseteq X$ so that

$$\bar{X} = \bigcup_{j=-\infty}^{\infty} (\bigcup_{I_0} \zeta^j \bar{e}^0 \cup \bigcup_{I_1} \zeta^j \bar{e}^1 \cup \dots \cup \bigcup_{I_n} \zeta^j \bar{e}^n) .$$

For any element

$$N = (N_0^+, N_1^+, \dots, N_n^+; N_0^-, N_1^-, \dots, N_n^-) \in \mathbb{N}(C(\tilde{X}))$$

such that

$$\bigcup_{j=-N_r^+}^{N_r^-} \bigcup_{I_r} \zeta^j \partial \bar{e}^r \subseteq \bar{X}^{(r-2)} \cup \bigcup_{k=-N_{r-1}^+}^{N_{r-1}^-} \bigcup_{I_{r-1}} \zeta^k \bar{e}^{r-1} \quad (1 \leq r \leq n)$$

there is defined a subfundamental domain

$$\bar{Y}\langle N \rangle = \bigcup_{j=-N_0^+}^{N_0^-} \bigcup_{I_0} \zeta^j \bar{e}^0 \cup \bigcup_{j=-N_1^+}^{N_1^-} \bigcup_{I_1} \zeta^j \bar{e}^1 \cup \dots \cup \bigcup_{j=-N_n^+}^{N_n^-} \bigcup_{I_n} \zeta^j \bar{e}^n \subseteq \bar{X} ,$$

corresponding to the Mayer-Vietoris presentation $\mathbb{E}\langle N \rangle$ of $E = C(\tilde{X})$. N is finite if and only if $\tilde{Y}\langle N \rangle$ is compact.

□

§9. Nilpotent class

The construction of the nilpotent class groups $\text{Nil}_0(A)$, $\widetilde{\text{Nil}}_0(A)$ for a ring A of Bass [7] are now extended to define the analogues $\text{Nil}_0(\mathbb{A})$, $\widetilde{\text{Nil}}_0(\mathbb{A})$ for an additive category \mathbb{A} . The nilpotent class groups are required for the splitting theorem of §10 for the torsion groups $K_1(\mathbb{A}[z, z^{-1}])$, $K_1^{\text{iso}}(\mathbb{A}[z, z^{-1}])$ of the Laurent polynomial extension category $\mathbb{A}[z, z^{-1}]$.

An endomorphism $\nu : M \longrightarrow M$ is *nilpotent* if for some integer $N \geq 0$

$$\nu^N = 0 : M \longrightarrow M .$$

Let $\text{Nil}(\mathbb{A})$ be the additive category with objects pairs

$$(M = \text{object of } \mathbb{A} , \nu : M \longrightarrow M \text{ nilpotent}) .$$

A morphism in $\text{Nil}(\mathbb{A})$

$$f : (M, \nu) \longrightarrow (M', \nu')$$

is defined by a morphism $f : M \longrightarrow M'$ in \mathbb{A} such that

$$\nu' f = f \nu : M \longrightarrow M' .$$

$\text{Nil}(\mathbb{A})$ is given the (non-split) exact structure in which a sequence

$$0 \longrightarrow (M'', \nu'') \longrightarrow (M', \nu') \longrightarrow (M, \nu) \longrightarrow 0$$

is exact if the underlying sequence $0 \longrightarrow M'' \longrightarrow M' \longrightarrow M \longrightarrow 0$ is exact in \mathbb{A} .

The *nilpotent class group* $\text{Nil}_0(\mathbb{A})$ of an additive category \mathbb{A} is the class group of $\text{Nil}(\mathbb{A})$

$$\text{Nil}_0(\mathbb{A}) = K_0(\text{Nil}(\mathbb{A})) .$$

The inclusion

$$i : \mathbb{A} \longrightarrow \text{Nil}(\mathbb{A}) ; M \longrightarrow (M, 0)$$

is split by

$$j : \text{Nil}(\mathbb{A}) \longrightarrow \mathbb{A} ; (M, \nu) \longrightarrow M .$$

The *reduced nilpotent class group* $\widetilde{\text{Nil}}_0(\mathbb{A})$ is defined by

$$\widetilde{\text{Nil}}_0(\mathbb{A}) = \text{coker}(i : K_0(\mathbb{A}) \longrightarrow \text{Nil}_0(\mathbb{A})) ,$$

and is such that

$$\text{Nil}_0(\mathbb{A}) = K_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) .$$

EXAMPLE 9.1 The Nil-groups of a ring A are the Nil-groups of the additive category $\mathbb{B}^f(A)$ of based f.g. free A -modules

$$\text{Nil}_0(A) = \text{Nil}_0(\mathbb{B}^f(A)) \quad , \quad \widetilde{\text{Nil}}_0(A) = \widetilde{\text{Nil}}_0(\mathbb{B}^f(A)) \quad .$$

□

A finite chain complex C in \mathbb{A} and a chain homotopy nilpotent chain map $\nu : C \longrightarrow C$ have an invariant $[C, \nu] \in \text{Nil}_0(\mathbb{A})$, which will now be defined. In fact, $\text{Nil}_0(\mathbb{A})$ will be expressed in terms of such pairs (C, ν) , by analogy with the following expression of $K_0(\mathbb{A})$ in terms of finite chain complexes C in \mathbb{A} .

Let $K_0^{(\infty)}(\mathbb{A})$ be the abelian group with one generator $[C]$ for each finite chain complex C in \mathbb{A} , subject to the relations

- (i) $[C] = [C']$ if C is chain equivalent to C' ,
- (ii) $[C] - [C'] + [C''] = 0$ if there is defined an exact sequence in \mathbb{A}

$$0 \longrightarrow C'' \longrightarrow C' \longrightarrow C \longrightarrow 0 \quad .$$

For any finite chain complex C in \mathbb{A} there is defined an exact sequence of finite chain complexes in \mathbb{A}

$$0 \longrightarrow C \longrightarrow C(1 : C \longrightarrow C) \longrightarrow SC \longrightarrow 0$$

with $C(1 : C \longrightarrow C)$ contractible, so that

$$[SC] = -[C] \in K_0^{(\infty)}(\mathbb{A}) \quad .$$

PROPOSITION 9.2 *The natural maps define inverse isomorphisms*

$$K_0(\mathbb{A}) \longrightarrow K_0^{(\infty)}(\mathbb{A}) ; [M] \longrightarrow [M] \quad ,$$

$$K_0^{(\infty)}(\mathbb{A}) \longrightarrow K_0(\mathbb{A}) ; [C] \longrightarrow [C] \quad .$$

The morphisms are defined by regarding objects M in \mathbb{A} as 0-dimensional chain complexes, and by sending finite chain complexes in \mathbb{A} to their class.

PROOF Let $K_0^{(n)}(\mathbb{A})$ ($n \geq 0$) be the abelian group defined exactly as $K_0^{(\infty)}(\mathbb{A})$ but using only n -dimensional finite chain complexes in \mathbb{A} . In particular, $K_0^{(0)}(\mathbb{A}) = K_0(\mathbb{A})$.

An n -dimensional chain complex C is also $(n+1)$ -dimensional, so there are defined natural maps

$$K_0^{(n)}(\mathbb{A}) \longrightarrow K_0^{(n+1)}(\mathbb{A}) ; [C] \longrightarrow [C] \quad (n \geq 0) \quad .$$

Given an $(n+1)$ -dimensional chain complex C let C' be the n -dimensional chain complex

$$C' : C_{n+1} \xrightarrow{d} C_n \longrightarrow \dots \longrightarrow C_2 \xrightarrow{d} C_1 \quad ,$$

and let C'' be the $(n+1)$ -dimensional chain complex

$$C'' : C_{n+1} \xrightarrow{d} C_n \longrightarrow \dots \longrightarrow C_2 \xrightarrow{\begin{pmatrix} d \\ 0 \end{pmatrix}} C_1 \oplus C_0 \xrightarrow{(d \ 1)} C_0 .$$

Now C'' is chain equivalent to SC' , and there is defined an exact sequence

$$0 \longrightarrow C \longrightarrow C'' \longrightarrow SC_0 \longrightarrow 0 ,$$

so that

$$[C] = [C''] - [SC_0] = [SC'] - [SC_0] = [C_0] - [C'] \in K_0^{(n+1)}(\mathbb{A}) .$$

Thus the natural map is an isomorphism $K_0^{(n)}(\mathbb{A}) \longrightarrow K_0^{(n+1)}(\mathbb{A})$, with inverse

$$K_0^{(n+1)}(\mathbb{A}) \longrightarrow K_0^{(n)}(\mathbb{A}) ; [C] \longrightarrow [C_0] - [C'] \quad (n \geq 0) .$$

□

Define $\text{Nil}_0^{(\infty)}(\mathbb{A})$ to be the abelian group with one generator $[C, \nu]$ for each finite chain complex C in \mathbb{A} with a chain homotopy nilpotent self chain map $\nu : C \longrightarrow C$, subject to the relations

- (i) $[C, \nu] = [C', \nu']$ if there exists a chain equivalence $f : C \longrightarrow C'$ such that

$$f\nu \simeq \nu'f : C \longrightarrow C' ,$$

- (ii) $[C, \nu] - [C', \nu'] + [C'', \nu''] = 0$ if there is defined an exact sequence

$$0 \longrightarrow (C'', \nu'') \longrightarrow (C', \nu') \longrightarrow (C, \nu) \longrightarrow 0 .$$

The suspension $S(C, \nu) = (SC, \nu)$ is such that there is defined an exact sequence

$$0 \longrightarrow (C, \nu) \longrightarrow (C(1 : C \longrightarrow C), \nu \oplus \nu) \longrightarrow S(C, \nu) \longrightarrow 0$$

with $C(1)$ contractible, so that

$$[S(C, \nu)] = -[C, \nu] \in \text{Nil}_0^{(\infty)}(\mathbb{A}) .$$

The reduced Nil-group defined by

$$\widetilde{\text{Nil}}_0^{(\infty)}(\mathbb{A}) = \text{coker}(i : K_0^{(\infty)}(\mathbb{A}) \longrightarrow \text{Nil}_0^{(\infty)}(\mathbb{A}))$$

is such that

$$\text{Nil}_0^{(\infty)}(\mathbb{A}) = K_0^{(\infty)}(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0^{(\infty)}(\mathbb{A}) .$$

PROPOSITION 9.3 *The natural maps define isomorphisms*

$$\text{Nil}_0(\mathbb{A}) \longrightarrow \text{Nil}_0^{(\infty)}(\mathbb{A}) ; [M, \nu] \longrightarrow [M, \nu] ,$$

$$\widetilde{\text{Nil}}_0(\mathbb{A}) \longrightarrow \widetilde{\text{Nil}}_0^{(\infty)}(\mathbb{A}) ; [M, \nu] \longrightarrow [M, \nu] .$$

PROOF In view of 9.2 it is sufficient to consider the absolute Nil-groups.

Let $\text{Nil}_0^{(n)}(\mathbb{A})$ ($n \geq 0$) be the abelian group defined exactly as $\text{Nil}_0^{(\infty)}(\mathbb{A})$ but using only pairs (C, ν) with C an n -dimensional chain complex in \mathbb{A} . In particular,

$$\text{Nil}_0^{(0)}(\mathbb{A}) = \text{Nil}_0(\mathbb{A}) .$$

The proof that the natural maps $\text{Nil}_0^{(n)}(\mathbb{A}) \longrightarrow \text{Nil}_0^{(n+1)}(\mathbb{A})$ ($n \geq 0$) are isomorphisms proceeds by analogy with the proof of 9.2.

Given an $(n+1)$ -dimensional pair (C, ν) let $N \geq 0$ be so large that there exists a chain homotopy $\eta : \nu^N \simeq 0 : C \longrightarrow C$, so that $\eta : C_0 \longrightarrow C_1$ is such that

$$\nu^N = d\eta : C_0 \longrightarrow C_0 .$$

Let (M, μ) be the object of $\text{Nil}(\mathbb{A})$ defined by

$$\begin{aligned} \mu = \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix} : M = \sum_{j=1}^{N-1} C_0 = C_0 \oplus C_0 \oplus \dots \oplus C_0 \\ \longrightarrow M = \sum_{j=1}^{N-1} C_0 = C_0 \oplus C_0 \oplus \dots \oplus C_0 . \end{aligned}$$

Also, let e, f be the morphisms in \mathbb{A} defined by

$$e = (\nu^{N-1} \ \nu^{N-2} \ \dots \ \nu) : M = C_0 \oplus C_0 \oplus \dots \oplus C_0 \longrightarrow C_0 ,$$

$$f = (\eta \ 0 \ \dots \ 0) : M = C_0 \oplus C_0 \oplus \dots \oplus C_0 \longrightarrow C_1 ,$$

and let (C', ν') be the $(n+1)$ -dimensional pair defined by

$$\begin{aligned} C' : C_{n+1} &\xrightarrow{d} C_n \longrightarrow \dots \longrightarrow C_2 \xrightarrow{\begin{pmatrix} d \\ 0 \end{pmatrix}} C_1 \oplus M \xrightarrow{(d \ e)} C_0 , \\ \nu' = \nu : C'_r &= C_r \longrightarrow C'_r = C_r \ (r \neq 1) , \\ \nu' = \begin{pmatrix} \nu & f \\ 0 & \mu \end{pmatrix} : C'_1 &= C_1 \oplus M \longrightarrow C'_1 = C_1 \oplus M . \end{aligned}$$

The N th power of $\nu' : C'_1 \longrightarrow C'_1$ has an upper triangular matrix of the type

$$\nu'^N = \begin{pmatrix} \nu^N & f'' \\ 0 & \mu^N \end{pmatrix} : C'_1 = C_1 \oplus M \longrightarrow C'_1 = C_1 \oplus M$$

with $\mu^N = 0$. The chain map $\nu'' : C' \longrightarrow C'$ defined by

$$\nu'' = \begin{cases} \begin{pmatrix} 0 & f'' - \eta e \\ 0 & 0 \end{pmatrix} : C'_1 = C_1 \oplus M \longrightarrow C'_1 = C_1 \oplus M \\ 0 : C'_r \longrightarrow C'_r \text{ if } r \neq 1 \end{cases}$$

is nilpotent, with $\nu'^2 = 0$, and there is defined a chain homotopy

$$\eta \oplus 0 : \nu'^N \simeq \nu'' : C' \longrightarrow C' .$$

Powers of chain homotopic self chain maps are chain homotopic, so that $\nu'^{2N} = (\nu'^N)^2$ is chain homotopic to $\nu''^2 = 0$, and $\nu' : C' \longrightarrow C'$ is chain homotopy nilpotent. The morphism

$$\eta' = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} : C'_0 = C_0 \longrightarrow C'_1 = C_1 \oplus C_0 \oplus \dots \oplus C_0 = C_1 \oplus M$$

is such that

$$d'\eta' = \nu' (= \nu) : C'_0 \longrightarrow C'_0 .$$

The chain homotopy nilpotent self chain map

$$\hat{\nu}' = \nu' - (d'\eta' + \eta'd') : C' \longrightarrow C'$$

is chain homotopic to $\nu' : C' \longrightarrow C'$, and such that $\hat{\nu}' = 0 : C'_0 \longrightarrow C'_0$. Thus

$$[C', \nu'] = [C', \hat{\nu}'] \in \text{Nil}_0^{(n+1)}(\mathbb{A}) ,$$

and there is defined an exact sequence

$$0 \longrightarrow (C_0, 0) \longrightarrow (C', \hat{\nu}') \longrightarrow S(C'', \hat{\nu}'') \longrightarrow 0$$

with $(C'', \hat{\nu}'')$ the n -dimensional pair defined by

$$C''_r = C'_{r+1} \quad (0 \leq r \leq n) \quad , \quad \hat{\nu}'' = \hat{\nu}' .$$

There is also defined an exact sequence

$$0 \longrightarrow (C, \nu) \longrightarrow (C', \nu') \longrightarrow S(M, \mu) \longrightarrow 0 .$$

It follows that

$$\begin{aligned} [C, \nu] &= [C', \nu'] + [M, \mu] = [C_0, 0] + [M, \mu] - [C'', \hat{\nu}''] \\ &\in \text{im}(\text{Nil}_0^{(n)}(\mathbb{A}) \longrightarrow \text{Nil}_0^{(n+1)}(\mathbb{A})) . \end{aligned}$$

□

The inclusion $\mathbb{A} \longrightarrow \mathbb{P}_0(\mathbb{A}); M \longrightarrow (M, 1)$ induces an isomorphism of reduced nilpotent class groups

$$\widetilde{\text{Nil}}_0(\mathbb{A}) \longrightarrow \widetilde{\text{Nil}}_0(\mathbb{P}_0(\mathbb{A})) ; [M, \nu] \longrightarrow [(M, 1), \nu]$$

with inverse

$$\widetilde{\text{Nil}}_0(\mathbb{P}_0(\mathbb{A})) \longrightarrow \widetilde{\text{Nil}}_0(\mathbb{A}) ; [(M, p), \nu] \longrightarrow [M, \nu] .$$

These isomorphisms will be used to identify

$$\widetilde{\text{Nil}}_0(\mathbb{P}_0(\mathbb{A})) = \widetilde{\text{Nil}}_0(\mathbb{A}) .$$

Given a pair (C, ν) with C a $\mathbb{C}_0(\mathbb{A})$ -finitely dominated chain complex

in $\mathbb{G}_1(\mathbb{A})$ and $\nu : C \longrightarrow C$ chain homotopy nilpotent it follows from 9.3 that there is defined a reduced nilpotent class invariant

$$[C, \nu] \in \widetilde{\text{Nil}}_0(\mathbb{P}_0(\mathbb{A})) = \widetilde{\text{Nil}}_0(\mathbb{A}) .$$

This will be used in §10 below.

§10. K -theory of $\mathbb{A}[z, z^{-1}]$

The splitting theorem of Bass, Heller and Swan [8] and Bass [7] for the torsion group of the Laurent polynomial extension $A[z, z^{-1}]$ of a ring A

$$K_1(A[z, z^{-1}]) = K_1(A) \oplus K_0(A) \oplus \widetilde{\text{Nil}}_0(A) \oplus \widetilde{\text{Nil}}_0(A)$$

will now be generalized to the torsion group of the finite Laurent extension $\mathbb{A}[z, z^{-1}]$ of a filtered additive category \mathbb{A}

$$K_1(\mathbb{A}[z, z^{-1}]) = K_1(\mathbb{A}) \oplus K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) .$$

The proof makes use the Mayer-Vietoris presentations of §8 and the nilpotent objects of §9 to obtain a split exact sequence

$$\begin{aligned} 0 \longrightarrow K_1^{iso}(\mathbb{A}) \xrightarrow{i_!} K_1^{iso}(\mathbb{A}[z, z^{-1}]) \xrightarrow{B \oplus N_+ \oplus N_-} \\ K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \longrightarrow 0 \end{aligned}$$

and the analogue with K_1^{iso} replace by K_1

$$\begin{aligned} 0 \longrightarrow K_1(\mathbb{A}) \xrightarrow{i_!} K_1(\mathbb{A}[z, z^{-1}]) \xrightarrow{B \oplus N_+ \oplus N_-} \\ K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \longrightarrow 0 , \end{aligned}$$

as well as a version for the Whitehead torsion and reduced class groups

$$\begin{aligned} 0 \longrightarrow Wh(\mathbb{A}) \xrightarrow{i_!} Wh(\mathbb{A}[z, z^{-1}]) \xrightarrow{B \oplus N_+ \oplus N_-} \\ \widetilde{K}_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \longrightarrow 0 . \end{aligned}$$

Given an object L in \mathbb{A} and $j \in \mathbb{Z}$ define objects in $\mathbb{G}_1(\mathbb{A})$

$$\begin{aligned} \zeta^j L^+ &= \zeta^j(L[z, z^{-1}])^+ = \sum_{k=j}^{\infty} z^k L , \\ \zeta^j L^- &= \zeta^j(L[z, z^{-1}])^- = \sum_{k=-\infty}^{j-1} z^k L . \end{aligned}$$

The projections onto $\zeta^j L^+$ and $\zeta^j L^-$ define morphisms in $\mathbb{G}_1(\mathbb{A})$

$$\begin{aligned} p_{\zeta^j L^+} &= 1 \oplus 0 : L[z, z^{-1}] = \zeta^j L^+ \oplus \zeta^j L^- \\ &\longrightarrow L[z, z^{-1}] = \zeta^j L^+ \oplus \zeta^j L^- , \\ p_{\zeta^j L^-} &= 0 \oplus 1 : L[z, z^{-1}] = \zeta^j L^+ \oplus \zeta^j L^- \\ &\longrightarrow L[z, z^{-1}] = \zeta^j L^+ \oplus \zeta^j L^- . \end{aligned}$$

DEFINITION 10.1 The split surjection

$$B \oplus N_+ \oplus N_- :$$

$$\begin{aligned} K_1^{iso}(\mathbb{A}[z, z^{-1}]) &\longrightarrow K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) ; \\ \tau(f) &\longrightarrow ([f]_+, [P_+, \nu_+], [P_-, \nu_-]) \end{aligned}$$

sends the torsion of an isomorphism in $\mathbb{A}[z, z^{-1}]$

$$f = \sum_{j=-s}^t z^j f_j : L[z, z^{-1}] \longrightarrow M[z, z^{-1}]$$

with inverse

$$f^{-1} = \sum_{j=-s_1}^{t_1} z^j (f^{-1})_j : M[z, z^{-1}] \longrightarrow L[z, z^{-1}]$$

to the end invariant

$$\begin{aligned} B\tau(f) &= [f]_+ = -[f]_- \\ &= [P_+] - \left[\sum_{j=-s}^{-1} z^j M \right] = \left[\sum_{j=0}^{t-1} z^j M \right] - [P_-] \in K_0(\mathbb{P}_0(\mathbb{A})) , \end{aligned}$$

and the reduced nilpotent classes

$$N_{\pm} \tau(f) = [P_{\pm}, \nu_{\pm}] \in \widetilde{\text{Nil}}_0(\mathbb{A})$$

of the objects (P_{\pm}, ν_{\pm}) in $\text{Nil}(\mathbb{P}_0(\mathbb{A}))$ given by

$$P_+ = L^- \cap f^{-1}(\zeta^{-s} M^+) = \left(\sum_{j=-s-s_1}^{-1} z^j L, p_{L^-} f^{-1} p_{\zeta^{-s} M^+} f \right) ,$$

$$P_- = L^+ \cap f^{-1}(\zeta^t M^-) = \left(\sum_{j=0}^{t+t_1-1} z^j L, p_{L^+} f^{-1} p_{\zeta^t M^-} f \right) ,$$

$$\nu_+ = p_{L^-} f^{-1} \zeta p_{\zeta^{-s} M^+} f : P_+ \longrightarrow P_+ ,$$

$$\nu_- = p_{\zeta L^+} f^{-1} \zeta^{-1} p_{\zeta^t M^-} f : P_- \longrightarrow P_- .$$

□

There are two distinct ways of splitting $K_1^{iso}(\mathbb{A}[z, z^{-1}])$, as the *algebraically significant* direct sum system

$$\begin{array}{ccc} & i_! & \\ K_1^{iso}(\mathbb{A}) & \xrightarrow{\quad} & K_1^{iso}(\mathbb{A}[z, z^{-1}]) \\ & j_! & \\ & B \oplus N_+ \oplus N_- & \\ & \xrightarrow{\quad} & K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \\ & \xleftarrow{\quad} & \\ & \bar{B} \oplus \bar{N}_+ \oplus \bar{N}_- & \end{array}$$

and the *geometrically significant* direct sum system

$$\begin{array}{ccc}
 & i_! & \\
 K_1^{iso}(\mathbb{A}) & \xrightarrow{\quad} & K_1^{iso}(\mathbb{A}[z, z^{-1}]) \\
 & \xleftarrow{\quad} & \\
 & j'_! & \\
 & B \oplus N_+ \oplus N_- & \\
 & \xrightarrow{\quad} & K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) . \\
 & \xleftarrow{\quad} & \\
 & \bar{B}' \oplus \bar{N}'_+ \oplus \bar{N}'_- &
 \end{array}$$

Similarly for K_1 instead of K_1^{iso} . The algebraically significant splitting $j_!$ of $i_!$ is induced by the functor

$$\begin{aligned}
 j : \mathbb{A}[z, z^{-1}] &\longrightarrow \mathbb{A} ; M = L[z, z^{-1}] \longrightarrow j_! M = L , \\
 (f = \sum_{k=-\infty}^{\infty} z^k f_k : L[z, z^{-1}] &\longrightarrow L'[z, z^{-1}]) \\
 &\longrightarrow j_! f = \sum_{k=-\infty}^{\infty} f_k : L \longrightarrow L' .
 \end{aligned}$$

DEFINITION 10.2 (i) The *algebraically significant* injection

$$\bar{B} \oplus \bar{N}_+ \oplus \bar{N}_- : K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \longrightarrow K_1^{iso}(\mathbb{A}[z, z^{-1}])$$

is the splitting of $B \oplus N_+ \oplus N_-$ with components

$$\begin{aligned}
 \bar{B}([P]) &= \tau(z : P[z, z^{-1}] \longrightarrow P[z, z^{-1}]) , \\
 \bar{N}_+[P, \nu] &= \tau(1 - (z^{-1} - 1)\nu : P[z, z^{-1}] \longrightarrow P[z, z^{-1}]) , \\
 \bar{N}_-[P, \nu] &= \tau(1 - (z - 1)\nu : P[z, z^{-1}] \longrightarrow P[z, z^{-1}]) .
 \end{aligned}$$

(ii) The *geometrically significant* injection splitting of $B \oplus N_+ \oplus N_-$

$$\bar{B}' \oplus \bar{N}'_+ \oplus \bar{N}'_- : K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \longrightarrow K_1^{iso}(\mathbb{A}[z, z^{-1}])$$

has components

$$\begin{aligned}
 \bar{B}'([P]) &= \tau(-z : P[z, z^{-1}] \longrightarrow P[z, z^{-1}]) , \\
 \bar{N}'_+[P, \nu] &= \tau(1 - z^{-1}\nu : P[z, z^{-1}] \longrightarrow P[z, z^{-1}]) , \\
 \bar{N}'_-[P, \nu] &= \tau(1 - z\nu : P[z, z^{-1}] \longrightarrow P[z, z^{-1}]) .
 \end{aligned}$$

□

REMARK 10.3 For a ring A and the additive category $\mathbb{A} = \mathbb{B}^f(A)$ of based f.g. free A -modules the algebraically significant direct sum decomposition of the automorphism torsion group $K_1(\mathbb{A}[z, z^{-1}]) =$

$$K_1(A[z, z^{-1}])$$

$$\begin{array}{ccc} & i_! & \\ K_1(A) & \xrightarrow{\quad} & K_1(A[z, z^{-1}]) \\ & j_! & \\ & B \oplus N_+ \oplus N_- & \\ & \xrightarrow{\quad} & K_0(A) \oplus \widetilde{\text{Nil}}_0(A) \oplus \widetilde{\text{Nil}}_0(A) \\ & \xleftarrow{\quad} & \\ & \bar{B} \oplus \bar{N}_+ \oplus \bar{N}_- & \end{array}$$

is the original decomposition of Chapter XII of Bass [7], with $j_! : K_1(A[z, z^{-1}]) \longrightarrow K_1(A)$ induced by the surjection of rings

$$j : A[z, z^{-1}] \longrightarrow A ; \quad \sum_{k=-\infty}^{\infty} a_k z^k \longrightarrow \sum_{k=-\infty}^{\infty} a_k$$

splitting the inclusion $i : A \longrightarrow A[z, z^{-1}]$. The relative merits of algebraic and geometric significance have already been discussed in Ranicki [66]. The geometrically significant direct sum decomposition of the Whitehead group of a product $\pi \times \mathbb{Z}$

$$\begin{array}{ccc} & i_! & \\ Wh(\pi) & \xrightarrow{\quad} & Wh(\pi \times \mathbb{Z}) \\ & j'_! & \\ & B \oplus N_+ \oplus N_- & \\ & \xrightarrow{\quad} & \tilde{K}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\text{Nil}}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\text{Nil}}_0(\mathbb{Z}[\pi]) \\ & \xleftarrow{\quad} & \\ & \bar{B}' \oplus \bar{N}'_+ \oplus \bar{N}'_- & \end{array}$$

includes the split injection

$$\bar{B}' : \tilde{K}_0(\mathbb{Z}[\pi]) \longrightarrow Wh(\pi \times \mathbb{Z}) ; [P] \longrightarrow \tau(-z : P[z, z^{-1}] \longrightarrow P[z, z^{-1}])$$

which was identified in [66] with the split injection defined geometrically by Ferry [27], sending the finiteness obstruction $[X] \in \tilde{K}_0(\mathbb{Z}[\pi])$ of a finitely dominated CW complex X with $\pi_1(X) = \pi$ to the Whitehead torsion

$$\bar{B}'([X]) = \tau(1 \times -1 : X \times S^1 \longrightarrow X \times S^1) \in Wh(\pi \times \mathbb{Z}) .$$

It is also geometrically significant that the image of $\bar{B}' : \tilde{K}_0(\mathbb{Z}[\pi]) \longrightarrow Wh(\pi \times \mathbb{Z})$ is the subgroup of the transfer invariant elements, as will be explicitly verified in §12 below.

□

The objects P_{\pm} of 10.1 fit into direct sum systems in $\mathbb{P}_0(\mathbb{G}_1(\mathbb{A}))$

$$\begin{array}{ccccc}
 (L^+, 1) & \xrightleftharpoons[p_{L^+} f^{-1}]{f} & (\zeta^{-s} M^+, 1) & \xrightleftharpoons[p_{\zeta^{-s} M^+} f]{p_{L^+} f^{-1}} & P_+ , \\
 (L^-, 1) & \xrightleftharpoons[p_{L^-} f^{-1}]{f} & (\zeta^t M^-, 1) & \xrightleftharpoons[p_{\zeta^t M^-} f]{p_{L^+} f^{-1}} & P_- .
 \end{array}$$

The nilpotent endomorphisms $\nu_{\pm} : P_{\pm} \longrightarrow P_{\pm}$ of 10.1 fit into endomorphisms of exact sequences in $\mathbb{P}_0(\mathbb{G}_1(\mathbb{A}))$

$$\begin{array}{ccccccc}
 0 & \longrightarrow & L^+ & \xrightarrow{f} & \zeta^{-s} M^+ & \longrightarrow & P_+ \longrightarrow 0 \\
 & & \downarrow \zeta & & \downarrow \zeta & & \downarrow \nu_+ \\
 0 & \longrightarrow & L^+ & \xrightarrow{f} & \zeta^{-s} M^+ & \longrightarrow & P_+ \longrightarrow 0 \\
 0 & \longrightarrow & L^- & \xrightarrow{f} & \zeta^t M^- & \longrightarrow & P_- \longrightarrow 0 \\
 & & \downarrow \zeta^{-1} & & \downarrow \zeta^{-1} & & \downarrow \nu_- \\
 0 & \longrightarrow & L^- & \xrightarrow{f} & \zeta^t M^- & \longrightarrow & P_- \longrightarrow 0
 \end{array}$$

and are such that

$$\begin{aligned}
 (\nu_+)^{s+s_1} &= 0 : P_+ \longrightarrow P_+ , \\
 (\nu_-)^{t+t_1} &= 0 : P_- \longrightarrow P_- .
 \end{aligned}$$

PROPOSITION 10.4 *For any object $M = L[z, z^{-1}]$ in $\mathbb{A}[z, z^{-1}]$ and any $N = (N^+, N^-) \in \mathbb{N}^f$ the torsion $\tau(\mathbb{M}\langle N \rangle) \in K_1^{iso}(\mathbb{A}[z, z^{-1}])$ of the finite Mayer-Vietoris presentation $\mathbb{M}\langle N \rangle$ of M is such that*

$$\begin{aligned}
 (B \oplus N_+ \oplus N_-) \tau(\mathbb{M}\langle N \rangle) &= ([\sum_{j=-N^+}^{-1} z^j L], 0, 0) \\
 &\in K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) .
 \end{aligned}$$

PROOF The map $g\langle N \rangle$ in the exact sequence

$$\begin{array}{ccccc}
 \mathbb{M}\langle N \rangle : 0 & \longrightarrow & M''\langle N \rangle[z, z^{-1}] & \xrightarrow{f\langle N \rangle} & M'\langle N \rangle[z, z^{-1}] \\
 & & & & \downarrow g\langle N \rangle \\
 & & & & M \longrightarrow 0
 \end{array}$$

is split by the homogeneous degree 0 map

$$h\langle N \rangle : M = L[z, z^{-1}] \longrightarrow M'\langle N \rangle[z, z^{-1}]$$

defined by

$$h\langle N \rangle_0 = \text{inclusion} : L \longrightarrow M'\langle N \rangle = \sum_{k=-N^+}^{N^-} z^k L ,$$

The isomorphism

$$f = (f\langle N \rangle \ h\langle N \rangle) : M''\langle N \rangle[z, z^{-1}] \oplus M \longrightarrow M'\langle N \rangle[z, z^{-1}]$$

is such that

$$\tau(\mathbb{M}\langle N \rangle) = \tau(f) \in K_1^{iso}(\mathbb{A}[z, z^{-1}]) .$$

The nilpotent objects (P_{\pm}, ν_{\pm}) associated to f in 10.1 are given up to isomorphism by

$$\begin{aligned} \nu_+ : P_+ &= \sum_{j=-N^+}^{-1} z^j L \longrightarrow P_+ ; \quad \sum_{j=-N^+}^{-1} z^j x_j \longrightarrow \sum_{j=-N^+}^{-2} z^{j+1} x_j \\ \nu_- : P_- &= \sum_{j=0}^{N^-} z^j L \longrightarrow P_- ; \quad \sum_{j=0}^{N^-} z^j x_j L \longrightarrow \sum_{j=1}^{N^-} z^{j-1} x_j . \end{aligned}$$

P_+ and P_- fit into exact sequences in $\mathbb{G}_1(\mathbb{A})$

$$\begin{aligned} 0 \longrightarrow M''\langle N \rangle^+ \oplus L^+ &\xrightarrow{f} M'\langle N \rangle^+ \xrightarrow{p_+} P_+ \longrightarrow 0 \\ 0 \longrightarrow M''\langle N \rangle^- \oplus L^- &\xrightarrow{f} \zeta M'\langle N \rangle^- \xrightarrow{p_-} P_- \longrightarrow 0 \end{aligned}$$

with p_+ , p_- the projections. The matrices of ν_+ and ν_- are upper triangular, so that

$$[P_+, \nu_+] = [P_-, \nu_-] = 0 \in \widetilde{\text{Nil}}_0(\mathbb{A}) .$$

Thus the components of $(B \oplus N_+ \oplus N_-)\tau(\mathbb{M}\langle N \rangle)$ are given by

$$\begin{aligned} B\tau(\mathbb{M}\langle N \rangle) &= B\tau(f) = [f]_+ = [P_+] \in K_0(\mathbb{P}_0(\mathbb{A})) , \\ N_{\pm}\tau(\mathbb{M}\langle N \rangle) &= N_{\pm}\tau(f) = [P_{\pm}, \nu_{\pm}] = 0 \in \widetilde{\text{Nil}}_0(\mathbb{A}) . \end{aligned}$$

□

REMARK 10.5 The algebraic K -theory splitting theorem of Bass, Heller and Swan [8] used the linearization trick of Higman [34] to represent every element of $K_1(\mathbb{A}[z, z^{-1}])$ as the difference of the torsions of linear automorphisms. Atiyah [6, 2.2.4] proves the Bott periodicity theorem in topological K -theory by applying the linearization trick to polynomial clutching functions of bundles. See Bass [7, IV] and Karoubi [39, III.1] for the connection between the algebraic K -theory of polynomial extensions and Bott periodicity.

□

For any additive category \mathbb{A} every element of $K_1^{iso}(\mathbb{A}[z, z^{-1}])$ is the difference of the torsions of linear isomorphisms:

PROPOSITION 10.6 *An isomorphism in $\mathbb{A}[z, z^{-1}]$*

$$f = \sum_{j=-s}^t z^j f_j : L[z, z^{-1}] \longrightarrow M[z, z^{-1}]$$

determines a commutative diagram of isomorphisms in $\mathbb{A}[z, z^{-1}]$

$$\begin{array}{ccc} (L \oplus M'')[z, z^{-1}] & \xrightarrow{\begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix}} & (L \oplus M'')[z, z^{-1}] \\ \left(\begin{array}{cc} f & 0 \\ 0 & 1 \end{array} \right) \downarrow & & \downarrow f' = (e' \ i) \\ (M \oplus M'')[z, z^{-1}] & \xrightarrow{f'' = (h \ i)} & M'[z, z^{-1}] \end{array}$$

with f' and f'' linear, so that

$$\tau(f) = \tau(f') - \tau(f'') \in K_1^{\text{iso}}(\mathbb{A}[z, z^{-1}]) .$$

The nilpotent objects in $\mathbb{P}_0(\mathbb{A})$ associated to f , f' , f'' are such that

$$(P_{\pm}, \nu_{\pm}) = (P'_{\pm}, \nu'_{\pm}) ,$$

$$\nu''_+ : P''_+ = \sum_{k=-s}^{-1} z^k M \longrightarrow P''_+ ; \quad \sum_{k=-s}^{-1} z^k x_k \longrightarrow \sum_{k=-s+1}^{-1} z^k x_{k-1} ,$$

$$\nu''_- : P''_- = \sum_{k=0}^t z^k M \longrightarrow P''_- ; \quad \sum_{k=0}^t z^k x_k \longrightarrow \sum_{k=0}^{t-1} z^k x_{k+1} ,$$

and

$$B\tau(f') = [P_+] , \quad B\tau(f'') = [P''_+] \in K_0(\mathbb{P}_0(\mathbb{A})) ,$$

$$N_{\pm}\tau(f') = [P_{\pm}, \nu_{\pm}] , \quad N_{\pm}\tau(f'') = 0 \in \widetilde{\text{Nil}}_0(\mathbb{A}) .$$

PROOF Define a contractible 1-dimensional chain complex E in $\mathbb{A}[z, z^{-1}]$ by

$$d_E = f : E_1 = L[z, z^{-1}] \longrightarrow E_0 = M[z, z^{-1}] .$$

Let $N = (s, 0; t, 0) \in \mathbb{N}^f(E)$, so that by 8.8 there is defined a finite Mayer-Vietoris presentation of E in $\mathbb{C}_0(\mathbb{A})[z, z^{-1}]$

$$\begin{array}{ccc} \mathbb{E}\langle N \rangle : 0 \longrightarrow E''\langle N \rangle[z, z^{-1}] & \xrightarrow{f_+\langle N \rangle - z f_-\langle N \rangle} & E'\langle N \rangle[z, z^{-1}] \\ & & \xrightarrow{g\langle N \rangle} E \longrightarrow 0 \end{array}$$

which is written as

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & L[z, z^{-1}] & \xrightarrow{1} & L[z, z^{-1}] \longrightarrow 0 \\
& & \downarrow & & \downarrow e' & & \downarrow f \\
0 & \longrightarrow & M''[z, z^{-1}] & \xrightarrow{i = i_+ - zi_-} & M'[z, z^{-1}] & \xrightarrow{i'} & M[z, z^{-1}] \longrightarrow 0 .
\end{array}$$

Passing from $\mathbb{C}_0(\mathbb{A})$ to \mathbb{A} note that $E'\langle N \rangle$ and $E''\langle N \rangle$ are 1-dimensional chain complexes in \mathbb{A} with

$$E''\langle N \rangle_1 = 0 \quad , \quad E'\langle N \rangle_1 = L \quad ,$$

$$f_+\langle N \rangle = i_+ = \text{inclusion} \quad :$$

$$E''\langle N \rangle_0 = M'' = \sum_{k=-s+1}^t z^k M \longrightarrow E'\langle N \rangle_0 = M' = \sum_{k=-s}^t z^k M \quad ,$$

$$f_-\langle N \rangle = i_- = \zeta^{-1}(\text{inclusion}) \quad :$$

$$E''\langle N \rangle_0 = M'' = \sum_{k=-s+1}^t z^k M \longrightarrow E'\langle N \rangle_0 = M' = \sum_{k=-s}^t z^k M \quad ,$$

$$g\langle N \rangle_j = i'_j = j\text{th projection} \quad :$$

$$E'\langle N \rangle_0 = M' = \sum_{k=-s}^t z^k M \longrightarrow E_0 = M \quad ; \quad \sum_{k=-s}^t z^k x_k \longrightarrow x_j \quad ,$$

$$d_{E'\langle N \rangle} = e' = \sum_{k=-s}^t z^k f_k \quad :$$

$$E'\langle N \rangle_1 = L \longrightarrow E'\langle N \rangle_0 = \sum_{k=-s}^t z^k M \quad .$$

The homogeneous degree 0 morphism $h : M[z, z^{-1}] \longrightarrow M'[z, z^{-1}]$ defined by

$$h_0 = \text{inclusion} : M \longrightarrow M' = \sum_{k=-s}^t z^k M$$

splits $i' : M'[z, z^{-1}] \longrightarrow M[z, z^{-1}]$. The morphism defined in $\mathbb{A}[z, z^{-1}]$

$$e = \sum_{k=-s}^t z^k e_k : L[z, z^{-1}] \longrightarrow M''[z, z^{-1}]$$

by

$$e_k = \begin{cases} \sum_{j=0}^{t-k-1} -z^j f_{j+k+1} : L \longrightarrow M'' & \text{if } k \geq 0 \\ \sum_{j=-s-k}^0 -z^j f_{j+k} : L \longrightarrow M'' & \text{if } k \leq -1 \end{cases}$$

is such that

$$ie = hf - e' : L[z, z^{-1}] \longrightarrow M'[z, z^{-1}] .$$

□

THEOREM 10.7 *The isomorphism torsion group $K_1^{iso}(\mathbb{A}[z, z^{-1}])$ fits into the split exact sequence*

$$0 \longrightarrow K_1^{iso}(\mathbb{A}) \xrightarrow{i_!} K_1^{iso}(\mathbb{A}[z, z^{-1}]) \xrightarrow{B \oplus N_+ \oplus N_-} K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \longrightarrow 0 ,$$

with B the composite

$$B : K_1^{iso}(\mathbb{A}[z, z^{-1}]) \longrightarrow K_1^{iso}(\mathbb{C}_1(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_1(\mathbb{A})) \xrightarrow{B} K_0(\mathbb{P}_0(\mathbb{A}))$$

induced by the inclusion $\mathbb{A}[z, z^{-1}] \subset \mathbb{C}_1(\mathbb{A})$. Similarly, the automorphism torsion group $K_1(\mathbb{A}[z, z^{-1}])$ fits into the split exact sequence

$$0 \longrightarrow K_1(\mathbb{A}) \xrightarrow{i_!} K_1(\mathbb{A}[z, z^{-1}]) \xrightarrow{B \oplus N_+ \oplus N_-} K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \longrightarrow 0 ,$$

and the Whitehead group $Wh(\mathbb{A}[z, z^{-1}])$ fits into the split exact sequence

$$0 \longrightarrow K_1(\mathbb{A}) \xrightarrow{i_!} Wh(\mathbb{A}[z, z^{-1}]) \xrightarrow{B \oplus N_+ \oplus N_-} \widetilde{K}_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \longrightarrow 0 .$$

PROOF For any linear isomorphism in $\mathbb{A}[z, z^{-1}]$

$$f = f_+ - zf_- : L[z, z^{-1}] \longrightarrow M[z, z^{-1}]$$

there is defined a commutative diagram of isomorphisms in $\mathbb{P}_0(\mathbb{A})[z, z^{-1}]$

$$\begin{array}{ccccc}
& & P_+[z, z^{-1}] \oplus P_-[z, z^{-1}] & & \\
& \nearrow \begin{pmatrix} -z & 0 \\ 0 & 1 \end{pmatrix} & & \nwarrow \begin{pmatrix} 1 - z^{-1}\nu_+ & 0 \\ 0 & 1 - z\nu_- \end{pmatrix} & \\
P_+[z, z^{-1}] \oplus P_-[z, z^{-1}] & & & & P_+[z, z^{-1}] \oplus P_-[z, z^{-1}] \\
& \nwarrow h'' = \begin{pmatrix} h''_+ \\ h''_- \end{pmatrix} & & \nearrow h' = \begin{pmatrix} h'_+ \\ h'_- \end{pmatrix} & \\
& L[z, z^{-1}] & \xrightarrow{f} & M[z, z^{-1}] &
\end{array}$$

with

$$\begin{aligned}
h' &= \begin{pmatrix} h'_+ \\ h'_- \end{pmatrix} : M \longrightarrow P_+ \oplus P_- , \\
h'' &= \begin{pmatrix} h''_+ \\ h''_- \end{pmatrix} : L \longrightarrow P_+ \oplus P_-
\end{aligned}$$

the isomorphisms in $\mathbb{P}_0(\mathbb{A})$ defined by

$$\begin{aligned}
h'_+ &= p_{L-} f^{-1} : M \longrightarrow P_+ , \quad h'_- = p_{M+} f^{-1} : M \longrightarrow P_- , \\
h''_+ &= \zeta^{-1}(\text{inclusion}) : L \longrightarrow P_+ , \quad h''_- = \text{inclusion} : L \longrightarrow P_- .
\end{aligned}$$

The torsion of f can thus be expressed as

$$\begin{aligned}
\tau(f) &= i_! \tau(h : L \longrightarrow M) + \tau(-z : P_+[z, z^{-1}] \longrightarrow P_+[z, z^{-1}]) \\
&\quad + \tau(1 - z^{-1}\nu_+ : P_+[z, z^{-1}] \longrightarrow P_+[z, z^{-1}]) \\
&\quad + \tau(1 - z\nu_- : P_-[z, z^{-1}] \longrightarrow P_-[z, z^{-1}]) \\
&= i_! \tau(h) + \bar{B}'([P]) + \bar{N}'_+[P_+, \nu_+] + \bar{N}'_-[P_-, \nu_-] \in K_1^{\text{iso}}(\mathbb{A}[z, z^{-1}])
\end{aligned}$$

with (P_{\pm}, ν_{\pm}) the nilpotent objects associated to f in 10.1 and $h = h'^{-1}h'' : L \longrightarrow M$ an isomorphism in \mathbb{A} . By 10.6 every element of $K_1^{\text{iso}}(\mathbb{A}[z, z^{-1}])$ can be expressed as a difference of the torsions of linear isomorphisms in $\mathbb{A}[z, z^{-1}]$, thus establishing the geometrically significant

direct sum system

$$\begin{array}{ccc}
 K_1^{iso}(\mathbb{A}) & \xrightleftharpoons{i_!} & K_1^{iso}(\mathbb{A}[z, z^{-1}]) \\
 & j'_! & \\
 & B \oplus N_+ \oplus N_- & \\
 & \xrightleftharpoons{\quad} & K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) . \\
 & \bar{B}' \oplus \bar{N}'_+ \oplus \bar{N}'_- &
 \end{array}$$

The geometrically significant surjection $j'_!$ splitting $i_!$ is thus given for linear isomorphism $f : L[z, z^{-1}] \longrightarrow M[z, z^{-1}]$ by

$$j'_! : K_1^{iso}(\mathbb{A}[z, z^{-1}]) \longrightarrow K_1^{iso}(\mathbb{A}) ; \tau(f) \longrightarrow \tau(h : L \longrightarrow M) .$$

Define abelian group morphisms

$$\omega : K_0(\mathbb{P}_0(\mathbb{A})) \longrightarrow K_1^{iso}(\mathbb{P}_0(\mathbb{A})) = K_1^{iso}(\mathbb{A}) ;$$

$$[M, p] \longrightarrow \tau(-1 : (M, p) \longrightarrow (M, p)) = \tau(1 - 2p : M \longrightarrow M) ,$$

$$\mu : \widetilde{\text{Nil}}_0(\mathbb{A}) \longrightarrow K_1^{iso}(\mathbb{A}) ; [M, \nu] \longrightarrow \tau(1 - \nu : M \longrightarrow M) .$$

The splitting maps in the algebraically significant direct sum system

$$\begin{array}{ccc}
 K_1^{iso}(\mathbb{A}) & \xrightleftharpoons{i_!} & K_1^{iso}(\mathbb{A}[z, z^{-1}]) \\
 & j_! & \\
 & B \oplus N_+ \oplus N_- & \\
 & \xrightleftharpoons{\quad} & K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \\
 & \bar{B} \oplus \bar{N}_+ \oplus \bar{N}_- &
 \end{array}$$

are related to the geometrically significant splitting maps by

$$\bar{B}' = \bar{B} + \omega , \quad \bar{N}'_{\pm} = \bar{N}_{\pm} + \mu ,$$

$$j'_! = j_! + B\omega + \mu N_+ + \mu N_- ,$$

so that the algebraically significant direct sum system has also been established.

For the automorphism torsion groups note that for an automorphism in $\mathbb{A}[z, z^{-1}]$

$$f : M[z, z^{-1}] \longrightarrow M[z, z^{-1}]$$

the objects M' , M'' in the commutative diagram of 10.5

$$\begin{array}{ccc}
 (L \oplus M'')[z, z^{-1}] & \xrightarrow{\begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix}} & (L \oplus M'')[z, z^{-1}] \\
 \left(\begin{array}{cc} f & 0 \\ 0 & 1 \end{array} \right) \downarrow & & \downarrow f' = (e' \ i) \\
 (M \oplus M'')[z, z^{-1}] & \xrightarrow{f'' = (h \ i)} & M'[z, z^{-1}]
 \end{array}$$

are defined by

$$M' = \sum_{j=-s}^t z^j M, \quad M'' = \sum_{j=-s+1}^t z^j M$$

and there exists a homogeneous degree 0 isomorphism in $\mathbb{A}[z, z^{-1}]$

$$\phi : M'[z, z^{-1}] \longrightarrow (M \oplus M'')[z, z^{-1}].$$

Thus $\phi f'$ and $\phi f''$ are linear automorphisms such that

$$\tau(f) = \tau(\phi f') - \tau(\phi f'') \in K_1(\mathbb{A}[z, z^{-1}]),$$

and every element of $K_1(\mathbb{A}[z, z^{-1}])$ is the difference of the torsions of linear automorphisms in $\mathbb{A}[z, z^{-1}]$. The verification of split exactness now proceeds as for the isomorphism torsion groups. □

PROPOSITION 10.8 *The projection*

$B \oplus N_+ \oplus N_- : K_1^{iso}(\mathbb{A}[z, z^{-1}]) \longrightarrow K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A})$
sends the torsion $\tau(E) \in K_1^{iso}(\mathbb{A}[z, z^{-1}])$ of a contractible finite chain complex E in $\mathbb{A}[z, z^{-1}]$ with $E_r = i_r F_r$ to

$$(B \oplus N_+ \oplus N_-)\tau(E) = ([E]_+, [\zeta^{-N^+} E^+, \nu_+], [\zeta^{N^-} E^-, \nu_-]) \\ \in K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}),$$

with ν_+, ν_- the chain homotopy nilpotent self chain maps of the $\mathbb{C}_0(\mathbb{A})$ -finitely dominated chain complexes $\zeta^{-N^+} E^+, \zeta^{N^-} E^-$ in $\mathbb{G}_1(\mathbb{A})$ defined for any $N \in \mathbb{N}^f(E)$ by

$$\begin{aligned} \nu_+ : (\zeta^{-N^+} E^+)_r &= \sum_{j=-N_r^+}^{\infty} z^j F_r \longrightarrow (\zeta^{-N^+} E^+)_r ; \\ &\sum_{j=-N_r^+}^{\infty} z^j x_j \longrightarrow \sum_{j=-N_r^+}^{\infty} z^{j+1} x_j, \\ \nu_- : (\zeta^{N^-} E^-)_r &= \sum_{j=-\infty}^{N_r^-} z^j F_r \longrightarrow (\zeta^{N^-} E^-)_r ; \\ &\sum_{j=-\infty}^{N_r^-} z^j x_j \longrightarrow \sum_{j=-\infty}^{N_r^-} z^{j-1} x_j, \end{aligned}$$

and with

$$[E]_+ = [\zeta^{-N^+} E^+] - \sum_{r=0}^n (-)^r \left[\sum_{j=-N_r^+}^{-1} z^j F_r \right] \in K_0(\mathbb{P}_0(\mathbb{A}))$$

the positive end invariant (4.10) of E regarded as a chain complex in $\mathbb{C}_1(\mathbb{A})$.

PROOF Applying the torsion sum formula of 5.1 (ii) to the exact sequence of finite chain complexes in $\mathbb{A}[z, z^{-1}]$

$$\begin{array}{c} \mathbb{E}\langle N \rangle : 0 \longrightarrow E''\langle N \rangle[z, z^{-1}] \xrightarrow{f = f_+\langle N \rangle - zf_-\langle N \rangle} E'\langle N \rangle[z, z^{-1}] \\ \xrightarrow{g\langle N \rangle} E \longrightarrow 0 \end{array}$$

gives

$$\begin{aligned} \tau(E) &= \tau(f) - \sum_{r=0}^{\infty} (-)^r \tau(\mathbb{E}_r\langle N_r \rangle) - \beta(E''\langle N \rangle[z, z^{-1}], E) \\ &\in K_1^{iso}(\mathbb{A}[z, z^{-1}]) . \end{aligned}$$

The sign term β does not contribute to $(B \oplus N_+ \oplus N_-)\tau(E)$, since

$$\begin{aligned} \beta(E''\langle N \rangle[z, z^{-1}], E) &\in \text{im}(i_! : K_1^{iso}(\mathbb{A}) \longrightarrow K_1^{iso}(\mathbb{A}[z, z^{-1}])) \\ &= \ker(B \oplus N_+ \oplus N_- : K_1^{iso}(\mathbb{A}[z, z^{-1}]) \\ &\longrightarrow K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A})) . \end{aligned}$$

From 10.4

$$\begin{aligned} (B \oplus N_+ \oplus N_-)\tau(\mathbb{E}_r\langle N_r \rangle) &= ([\sum_{j=-N_r^+}^{-1} z^j F_r], 0, 0) \\ &\in K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) . \end{aligned}$$

The linear chain equivalence f fits into a commutative diagram of chain equivalences in $\mathbb{G}_1(\mathbb{A})[z, z^{-1}]$

$$\begin{array}{ccc}
& \zeta^{-N^+} E^+[z, z^{-1}] \oplus \zeta^{N^-} E^-[z, z^{-1}] & \\
\begin{pmatrix} -z & 0 \\ 0 & 1 \end{pmatrix} \nearrow & & \searrow \begin{pmatrix} 1 - z^{-1}\nu_+ & 0 \\ 0 & 1 - z\nu_- \end{pmatrix} \\
\zeta^{-N^+} E^+[z, z^{-1}] \oplus \zeta^{N^-} E^-[z, z^{-1}] & & \zeta^{-N^+} E^+[z, z^{-1}] \oplus \zeta^{N^-} E^-[z, z^{-1}] \\
\begin{matrix} \nearrow \\ h'' = \begin{pmatrix} h''_+ \\ h''_- \end{pmatrix} \end{matrix} & & \begin{matrix} \nwarrow \\ h' = \begin{pmatrix} h'_+ \\ h'_- \end{pmatrix} \end{matrix} \\
E''\langle N \rangle[z, z^{-1}] & \xrightarrow{f} & E'\langle N \rangle[z, z^{-1}]
\end{array}$$

with h' , h'' the homogeneous degree 0 chain equivalences with components

$$h'_+ = z^{-1}(\text{inclusion}) : E'\langle N \rangle \longrightarrow \zeta^{-1}E'\langle N^+ \rangle = \zeta^{-N^+}E^+ ,$$

$$h'_- = \text{inclusion} : E'\langle N \rangle \longrightarrow E'\langle N^- \rangle = \zeta^{N^-}E^- ,$$

$$h''_+ = \text{inclusion} : E''\langle N \rangle \longrightarrow E''\langle N^+ \rangle = \zeta^{-N^+}E^+ ,$$

$$h''_- = \text{inclusion} : E''\langle N \rangle \longrightarrow E''\langle N^- \rangle = \zeta^{N^-}E^- ,$$

so that

$$\begin{aligned}
(B \oplus N_+ \oplus N_-)\tau(f) &= ([\zeta^{-N^+}E^+]_+, [\zeta^{-N^+}E^+, \nu_+], [\zeta^{N^-}E^-, \nu_-]) \\
&\in K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) .
\end{aligned}$$

□

REMARK 10.9 The following conditions on a contractible finite chain complex E in $\mathbb{A}[z, z^{-1}]$ are equivalent :

- (i) $\tau(E) \in \text{im}(i_! : Wh(\mathbb{A}) \longrightarrow Wh(\mathbb{A}[z, z^{-1}]))$,
- (ii) $(B \oplus N_+ \oplus N_-)\tau(E) = 0 \in \widetilde{K}_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A})$,
- (iii) there exists a finite Mayer-Vietoris presentation of E

$$\mathbb{E} : 0 \longrightarrow E''[z, z^{-1}] \longrightarrow E'[z, z^{-1}] \longrightarrow E \longrightarrow 0$$

with E' and E'' contractible in \mathbb{A} and $\tau(\mathbb{E}) = 0 \in Wh(\mathbb{A}[z, z^{-1}])$,

so that

$$\tau(E) = i_!(\tau(E') - \tau(E'')) \in \text{im}(i_! : Wh(\mathbb{A}) \longrightarrow Wh(\mathbb{A}[z, z^{-1}])) .$$

This is the abstract version of the codimension 1 splitting theorem of

Farrell and Hsiang [24], which in its untwisted form states that a homotopy equivalence $f : M \longrightarrow X \times S^1$ of compact n -dimensional manifolds is such that

$$\begin{aligned} \tau(f) &\in \text{im}(i_! : Wh(\pi) \longrightarrow Wh(\pi \times \mathbb{Z})) \\ &= \ker(B \oplus N_+ \oplus N_- : Wh(\pi \times \mathbb{Z}) \\ &\quad \longrightarrow \widetilde{K}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\text{Nil}}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\text{Nil}}_0(\mathbb{Z}[\pi])) \\ &\quad (\pi = \pi_1(X)) \end{aligned}$$

if (and for $n \geq 6$ only if) f splits, i.e. is homotopic to a map (also denoted by f) which is transverse regular at $X \times \{*\} \subset X \times S^1$ and such that the restriction

$$f| : f^{-1}(X \times \{*\}) \longrightarrow X$$

is a homotopy equivalence of $(n-1)$ -dimensional manifolds. The components of the splitting obstruction $(B \oplus N_+ \oplus N_-)\tau(f)$ are given by

$$\begin{aligned} B\tau(f) &= [\bar{M}^-] = [i^!C(\widetilde{M})/\zeta^{-N^+}C(\widetilde{M})^+] \in \widetilde{K}_0(\mathbb{Z}[\pi]) , \\ N_+\tau(f) &= [i^!C(\widetilde{M})/\zeta^{-N^+}C(\widetilde{M})^+, \zeta] , \\ N_-\tau(f) &= [i^!C(\widetilde{M})/\zeta^{N^-}C(\widetilde{M})^-, \zeta^{-1}] \in \widetilde{\text{Nil}}_0(\mathbb{Z}[\pi]) \end{aligned}$$

for any $N = (N^+, N^-) \in \mathbb{N}^f(C(\widetilde{M}))$, with $\bar{M} = f^*(X \times \mathbb{R})$ the pullback infinite cyclic cover of M . By duality

$$\tau(f) + (-)^n \tau(f)^* = \tau(M) - \tau(X \times S^1) = 0 \in Wh(\pi \times \mathbb{Z}) ,$$

so that

$$\begin{aligned} B\tau(f) &= (-)^n (B\tau(f))^* \in \widetilde{K}_0(\mathbb{Z}[\pi]) , \\ N_+\tau(f) &= (-)^{n-1} (N_-\tau(f))^* \in \widetilde{\text{Nil}}_0(\mathbb{Z}[\pi]) . \end{aligned}$$

The splitting obstruction $(B \oplus N_+ \oplus N_-)\tau(f)$ vanishes for a simple homotopy equivalence $f : M \longrightarrow X \times S^1$, in which case there is defined a Wh_2 -invariant

$$\tau_2(f) \in \widehat{H}^{n+1}(\mathbb{Z}_2; Wh_2(\pi \times \mathbb{Z})) .$$

Let $B_2 : Wh_2(\pi \times \mathbb{Z}) \longrightarrow Wh(\pi)$ be the Wh_2 -analogue of $B : Wh(\pi \times \mathbb{Z}) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\pi])$. The image $B_2\tau_2(f) \in \widehat{H}^n(\mathbb{Z}_2; Wh(\pi))$ is the obstruction of Šlosman [78] and Wall [84, 12B], such that $B_2\tau_2(f) = 0$ if and only if the restriction $f| : f^{-1}(X \times \{*\}) \longrightarrow X$ can be chosen to be a simple homotopy equivalence. See §20 for the generalization to the fibering obstruction.

□

§11. Lower K -theory

The lower K -groups $K_{-m}(A)$ ($m \geq 1$) of a ring A were defined by Bass [7] using the polynomial extension rings $A[z]$, $A[z^{-1}]$, $A[z, z^{-1}]$, to fit into split exact sequences

$$\begin{aligned} 0 \longrightarrow K_{1-m}(A) &\longrightarrow K_{1-m}(A[z]) \oplus K_{1-m}(A[z^{-1}]) \\ &\longrightarrow K_{1-m}(A[z, z^{-1}]) \longrightarrow K_{-m}(A) \longrightarrow 0 . \end{aligned}$$

The lower K -groups of an additive category \mathbb{A} were defined by Karoubi [38] to be

$$K_{-m}(\mathbb{A}) = K_1(S^{m+1}\mathbb{A}) \quad (m \geq 1)$$

with $S^{m+1}\mathbb{A}$ the $(m+1)$ -fold “suspension” of \mathbb{A} , and are such that

$$K_{-m}(\mathbb{A}) = K_{-m}(\mathbb{P}_0(\mathbb{A})) \quad (m \geq 1) .$$

For the additive category $\mathbb{A} = \mathbb{B}^f(A)$ of based f.g. free A -modules

$$K_{-m}(\mathbb{B}^f(A)) = K_{-m}(A) \quad (m \geq 1) .$$

Define a metric on \mathbb{Z}^m by

$$d(J, K) = \max\{|j_i - k_i| \mid 1 \leq i \leq m\} \geq 0$$

for $J = (j_1, j_2, \dots, j_m)$, $K = (k_1, k_2, \dots, k_m) \in \mathbb{Z}^m$. Write the bounded \mathbb{Z}^m -graded category of a filtered additive category \mathbb{A} and its idempotent completion as

$$\mathbb{C}_{\mathbb{Z}^m}(\mathbb{A}) = \mathbb{C}_m(\mathbb{A}) \quad , \quad \mathbb{P}_0(\mathbb{C}_m(\mathbb{A})) = \mathbb{P}_m(\mathbb{A}) \quad (m \geq 1) .$$

Since $\mathbb{Z}^{m+1} = \mathbb{Z}^m \times \mathbb{Z}$ the bounded \mathbb{Z}^{m+1} -graded category can be expressed as

$$\mathbb{C}_{m+1}(\mathbb{A}) = \mathbb{C}_1(\mathbb{C}_m(\mathbb{A})) ,$$

and by 6.2 there is an identification

$$K_1(\mathbb{C}_{m+1}(\mathbb{A})) = K_0(\mathbb{P}_m(\mathbb{A})) \quad (m \geq 0) .$$

Pedersen [49] (for $\mathbb{A} = \mathbb{B}^f(A)$) and Pedersen and Weibel [53] expressed the lower K -groups of a filtered additive category \mathbb{A} as

$$K_{-m}(\mathbb{A}) = K_1(\mathbb{C}_{m+1}(\mathbb{A})) = K_0(\mathbb{P}_m(\mathbb{A})) \quad (m \geq 1) .$$

The lower K -groups $K_{-m}(\mathbb{A})$ ($m \geq 1$) of any filtered additive category \mathbb{A} will now be shown to fit into split exact sequences

$$\begin{aligned} 0 \longrightarrow K_{1-m}(\mathbb{P}_0(\mathbb{A})) &\longrightarrow K_{1-m}(\mathbb{P}_0(\mathbb{A}[z])) \oplus K_{1-m}(\mathbb{P}_0(\mathbb{A}[z^{-1}])) \\ &\longrightarrow K_{1-m}(\mathbb{P}_0(\mathbb{A}[z, z^{-1}])) \longrightarrow K_{-m}(\mathbb{A}) \longrightarrow 0 , \end{aligned}$$

exactly as in the original case $\mathbb{A} = \mathbb{B}^f(A)$.

As in §7 define the polynomial extension $\mathbb{A}[z]$ of an additive category

\mathbb{A} to be the additive category with one object

$$L[z] = \sum_{k=0}^{\infty} z^k L$$

for each object L in \mathbb{A} , and one morphism

$$f = \sum_{k=0}^{\infty} z^k f_k : L[z] \longrightarrow L'[z]$$

for each collection $\{f_k \in \text{Hom}_{\mathbb{A}}(L, L') \mid k \geq 0\}$ of morphisms in \mathbb{A} with $\{k \geq 0 \mid f_k \neq 0\}$ finite. Regard $\mathbb{A}[z]$ as a subcategory of $\mathbb{C}_1(\mathbb{A})$ with objects M such that

$$M(j) = \begin{cases} z^j M(0) & \text{if } j \geq 0 \\ 0 & \text{if } j < 0 . \end{cases}$$

The functor

$$j_+ : \mathbb{A}[z] \longrightarrow \mathbb{A}[z, z^{-1}] ; L[z] \longrightarrow L[z, z^{-1}]$$

defines an inclusion of $\mathbb{A}[z]$ as a subcategory of $\mathbb{A}[z, z^{-1}]$. The polynomial extension $\mathbb{A}[z^{-1}]$ is defined similarly, with one object

$$L[z^{-1}] = \sum_{k=-\infty}^0 z^k L$$

for each object L in \mathbb{A} , with morphisms

$$f = \sum_{k=-\infty}^0 z^k f_k : L[z^{-1}] \longrightarrow L'[z^{-1}]$$

and with an inclusion

$$j_- : \mathbb{A}[z^{-1}] \longrightarrow \mathbb{A}[z, z^{-1}] ; L[z^{-1}] \longrightarrow L[z, z^{-1}] .$$

The inclusions define a commutative square of additive functors

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{i_+} & \mathbb{A}[z] \\ i_- \downarrow & & \downarrow j_+ \\ \mathbb{A}[z^{-1}] & \xrightarrow{j_-} & \mathbb{A}[z, z^{-1}] . \end{array}$$

Given a functor

$$F : \{\text{additive categories}\} \longrightarrow \{\text{abelian groups}\}$$

define the functor

$$LF : \{\text{additive categories}\} \longrightarrow \{\text{abelian groups}\} ; \mathbb{A} \longrightarrow LF(\mathbb{A})$$

by

$$LF(\mathbb{A}) = \text{coker}((j_+ \ j_-) : F(\mathbb{A}[z]) \oplus F(\mathbb{A}[z^{-1}]) \longrightarrow F(\mathbb{A}[z, z^{-1}])) .$$

DEFINITION 11.1 The *lower K-groups* of an additive category \mathbb{A} are defined by

$$K_{-m}(\mathbb{A}) = L^m K_0(\mathbb{P}_0(\mathbb{A})) \quad (m \geq 1) .$$

□

Following Bass [7, p.659] define a functor

$$F : \{\text{additive categories}\} \longrightarrow \{\text{abelian groups}\}$$

to be *contracted* if the chain complex

$$\begin{aligned} 0 \longrightarrow F(\mathbb{A}) &\xrightarrow{\begin{pmatrix} i_+ \\ -i_- \end{pmatrix}} F(\mathbb{A}[z]) \oplus F(\mathbb{A}[z^{-1}]) \\ &\xrightarrow{(j_+ \ j_-)} F(\mathbb{A}[z, z^{-1}]) \xrightarrow{B} LF(\mathbb{A}) \longrightarrow 0 \end{aligned}$$

has a natural chain contraction, with B the natural projection.

PROPOSITION 11.2 *The functors*

$$\begin{aligned} L^m K_1 : \{\text{additive categories}\} &\longrightarrow \{\text{abelian groups}\} ; \mathbb{A} \longrightarrow L^m K_1(\mathbb{A}) \\ &(m \geq 0, L^0 K_1 = K_1) \end{aligned}$$

are contracted. For $m \geq 1$ there are natural identifications

$$L^m K_1(\mathbb{A}) = K_{1-m}(\mathbb{P}_0(\mathbb{A})) = K_1(\mathbb{C}_m(\mathbb{A})) = K_0(\mathbb{P}_{m-1}(\mathbb{A}))$$

and for $m \geq 2$

$$L^m K_1(\mathbb{A}) = K_{1-m}(\mathbb{A}) .$$

PROOF Consider first the case $m = 0$. Working as in the proof of 10.7 there are defined naturally split exact sequences

$$0 \longrightarrow K_1(\mathbb{A}) \xrightarrow{i_+} K_1(\mathbb{A}[z]) \xrightarrow{N_- j_+} \widetilde{\text{Nil}}_0(\mathbb{A}) \longrightarrow 0 ,$$

$$0 \longrightarrow K_1(\mathbb{A}) \xrightarrow{i_-} K_1(\mathbb{A}[z^{-1}]) \xrightarrow{N_+ j_-} \widetilde{\text{Nil}}_0(\mathbb{A}) \longrightarrow 0$$

and hence a naturally contracted chain complex

$$\begin{aligned} 0 \longrightarrow K_1(\mathbb{A}) &\xrightarrow{\begin{pmatrix} i_+ \\ -i_- \end{pmatrix}} K_1(\mathbb{A}[z]) \oplus K_1(\mathbb{A}[z^{-1}]) \\ &\xrightarrow{(j_+ \ j_-)} K_1(\mathbb{A}[z, z^{-1}]) \xrightarrow{B} K_0(\mathbb{P}_0(\mathbb{A})) \longrightarrow 0 \end{aligned}$$

so that

$$LK_1(\mathbb{A}) = K_0(\mathbb{P}_0(\mathbb{A})) .$$

For $m \geq 1$ replace \mathbb{A} by $\mathbb{C}_m(\mathbb{A})$ and use the identifications of filtered additive categories

$$\mathbb{C}_m(\mathbb{A}[z^{\pm 1}]) = \mathbb{C}_m(\mathbb{A})[z^{\pm 1}] , \quad \mathbb{C}_m(\mathbb{A}[z, z^{-1}]) = \mathbb{C}_m(\mathbb{A})[z, z^{-1}]$$

to obtain a naturally contracted chain complex

$$\begin{aligned} 0 \longrightarrow K_1(\mathbb{C}_m(\mathbb{A})) &\xrightarrow{\begin{pmatrix} i_+ \\ -i_- \end{pmatrix}} K_1(\mathbb{C}_m(\mathbb{A}[z])) \oplus K_1(\mathbb{C}_m(\mathbb{A}[z^{-1}])) \\ &\xrightarrow{(j_+ \ j_-)} K_1(\mathbb{C}_m(\mathbb{A}[z, z^{-1}])) \xrightarrow{B} K_0(\mathbb{P}_m(\mathbb{A})) \longrightarrow 0 \end{aligned}$$

which can be written as

$$\begin{aligned} 0 \longrightarrow K_0(\mathbb{P}_{m-1}(\mathbb{A})) &\longrightarrow K_0(\mathbb{P}_{m-1}(\mathbb{A}[z])) \oplus K_0(\mathbb{P}_{m-1}(\mathbb{A}[z^{-1}])) \\ &\longrightarrow K_0(\mathbb{P}_{m-1}(\mathbb{A}[z, z^{-1}])) \longrightarrow K_0(\mathbb{P}_m(\mathbb{A})) \longrightarrow 0 . \end{aligned}$$

Thus the functor $F : \mathbb{A} \longrightarrow K_0(\mathbb{P}_{m-1}(\mathbb{A}))$ is contracted with

$$LF(\mathbb{A}) = K_0(\mathbb{P}_m(\mathbb{A})) = K_{-m}(\mathbb{A}) .$$

□

EXAMPLE 11.3 Given a ring A let $A[z]$ (resp. $A[z^{-1}]$) be the subring of $A[z, z^{-1}]$ consisting of the polynomials $\sum_{k=0}^{\infty} a_k z^k$ (resp. $\sum_{k=-\infty}^0 a_k z^k$) such that $a_k = 0$ for $k < 0$ (resp. $k > 0$). For the additive category $\mathbb{A} = \mathbb{B}^f(A)$ of based f.g. free A -modules

$$\mathbb{A}[z] = \mathbb{B}^f(A[z]) , \quad \mathbb{A}[z^{-1}] = \mathbb{B}^f(A[z^{-1}]) ,$$

and the functors

$$i_{\pm} : \mathbb{A} \longrightarrow \mathbb{A}[z^{\pm}] , \quad j_{\pm} : \mathbb{A}[z^{\pm}] \longrightarrow \mathbb{A}[z, z^{-1}]$$

are induced by the inclusions of rings

$$i_{\pm} : A \longrightarrow A[z^{\pm}] , \quad j_{\pm} : A[z^{\pm}] \longrightarrow A[z, z^{-1}] .$$

Bass [7] defines a functor

$$F : \{\text{rings}\} \longrightarrow \{\text{abelian groups}\}$$

to be contracted if the functor

$$LF : \{\text{rings}\} \longrightarrow \{\text{abelian groups}\} ; A \longrightarrow LF(A)$$

defined by

$$LF(A) = \text{coker}((j_+ \ j_-) : F(A[z]) \oplus F(A[z^{-1}]) \longrightarrow F(A[z, z^{-1}]))$$

is such that the chain complex

$$\begin{aligned} 0 \longrightarrow F(A) &\xrightarrow{\begin{pmatrix} i_+ \\ -i_- \end{pmatrix}} F(A[z]) \oplus F(A[z^{-1}]) \\ &\xrightarrow{(j_+ \ j_-)} F(A[z, z^{-1}]) \xrightarrow{B} LF(A) \longrightarrow 0 \end{aligned}$$

has a natural chain contraction, with B the natural projection. The “fundamental theorem of algebraic K -theory” ([7]) is that the functors

$$K_{1-m} : \{\text{rings}\} \longrightarrow \{\text{abelian groups}\} ; A \longrightarrow K_{1-m}(A) \quad (m \geq 0)$$

are contracted, with natural identifications

$$LK_{1-m}(A) = K_{-m}(A) .$$

This is the special case of 11.2 with $\mathbb{A} = \mathbb{B}^f(A)$.

□

For $m \geq 1$ define the m -fold *Laurent polynomial extension* of an additive category \mathbb{A} inductively to be the additive category

$$\mathbb{A}[\mathbb{Z}^m] = \mathbb{A}[\mathbb{Z}^{m-1}][z_m, z_m^{-1}] , \quad \mathbb{A}[\mathbb{Z}] = \mathbb{A}[z_1, z_1^{-1}] ,$$

and write

$$\mathbb{A}[\mathbb{Z}^m] = \mathbb{A}[z_1, z_1^{-1}, z_2, z_2^{-1}, \dots, z_m, z_m^{-1}] .$$

Alternatively, $\mathbb{A}[\mathbb{Z}^m]$ can be viewed as the subcategory of the bounded \mathbb{Z}^m -graded category $\mathbb{C}_m(\mathbb{A}) = \mathbb{C}_{\mathbb{Z}^m}(\mathbb{A})$ with one object $M[\mathbb{Z}^m]$ for each object M in \mathbb{A} , graded by

$$M[\mathbb{Z}^m](j_1, j_2, \dots, j_m) = z_1^{j_1} z_2^{j_2} \dots z_m^{j_m} M ,$$

with the \mathbb{Z}^m -equivariant morphisms.

THEOREM 11.4 *The torsion group of the m -fold Laurent polynomial extension of \mathbb{A} is such that up to natural isomorphism*

$$K_1(\mathbb{A}[\mathbb{Z}^m]) = \left(\sum_{i=0}^m \binom{m}{i} K_{1-i}(\mathbb{P}_0(\mathbb{A})) \right) \oplus 2 \left(\sum_{i=1}^m \binom{m}{i} \widetilde{\text{Nil}}_{1-i}(\mathbb{A}) \right)$$

with

$$\widetilde{\text{Nil}}_*(\mathbb{A}) = \text{coker}(K_*(\mathbb{A}) \longrightarrow \text{Nil}_*(\mathbb{A})) , \quad \text{Nil}_*(\mathbb{A}) = K_*(\text{Nil}(\mathbb{A})) .$$

PROOF Iterate one of the splittings of §10

$$K_1(\mathbb{A}[z, z^{-1}]) = K_1(\mathbb{A}) \oplus K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) .$$

(As usual, there are algebraically and geometrically significant splittings.)

□

For a ring A and $\mathbb{A} = \mathbb{B}^f(A)$ there are evident identifications

$\mathbb{A}[\mathbb{Z}^m] = \mathbb{B}^f(A[\mathbb{Z}^m])$, $\text{Nil}_*(\mathbb{A}) = \text{Nil}_*(A)$, $\widetilde{\text{Nil}}_*(\mathbb{A}) = \widetilde{\text{Nil}}_*(A)$,
with $A[\mathbb{Z}^m] = A[z_1, z_1^{-1}, z_2, z_2^{-1}, \dots, z_m, z_m^{-1}]$ the m -fold Laurent polynomial extension of A .

The K_0 - and K_1 -groups of \mathbb{Z} are

$$K_0(\mathbb{Z}) = \mathbb{Z} , \quad K_1(\mathbb{Z}) = \mathbb{Z}_2 .$$

EXAMPLE 11.5 The lower K -groups and the lower Nil-groups of \mathbb{Z} are

$$K_{-m}(\mathbb{Z}) = \text{Nil}_{-m}(\mathbb{Z}) = \widetilde{\text{Nil}}_{-m}(\mathbb{Z}) = 0 \quad (m \geq 1) ,$$

by virtue of the computation $Wh(\mathbb{Z}^m) = 0$ of Bass, Heller and Swan [8].

□

§12. Transfer in K -theory

The lower K -group $K_{1-m}(\mathbb{P}_0(\mathbb{A}))$ ($= K_{1-m}(\mathbb{A})$ for $m \geq 2$) of an additive category \mathbb{A} will now be identified with the subgroup $K_1(\mathbb{A}[\mathbb{Z}^m])^{INV}$ of the T^m -transfer invariant elements in $K_1(\mathbb{A}[\mathbb{Z}^m])$. This is significant because the reduced lower K -group $\tilde{K}_{1-m}(\mathbb{Z}[\pi])$ ($= K_{1-m}(\mathbb{Z}[\pi])$ for $m \geq 2$) arises geometrically as the T^m -transfer invariant subgroup of the Whitehead group $Wh(\pi \times \mathbb{Z}^m)$, in connection with the ‘wrapping up’ procedure for passing from \mathbb{R}^m -bounded open n -dimensional manifolds with fundamental group π to closed $(m+n)$ -dimensional manifolds with fundamental group $\pi \times \mathbb{Z}^m$.

In the first instance consider the case $m = 1$.

For each integer $q \geq 1$ let

$$q : S^1 \longrightarrow S^1 ; z = [t] \longrightarrow z^q = [qt]$$

be the canonical q -fold cover of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ by itself. Given a map $c : X \longrightarrow S^1$ let

$$X^! = \{(x, [t]) \in X \times S^1 \mid c(x) = [qt] \in S^1\}$$

be the pullback q -fold cover of X

$$\begin{array}{ccc} X^! & \xrightarrow{c^!} & S^1 \\ \downarrow & & \downarrow q \\ X & \xrightarrow{c} & S^1 . \end{array}$$

The infinite cyclic cover of X classified by c

$$\bar{X} = \{(x, t) \in X \times \mathbb{R} \mid c(x) = [t] \in S^1\}$$

has covering projection

$$\bar{X} \longrightarrow X ; (x, t) \longrightarrow x$$

and generating covering translation

$$\zeta : \bar{X} \longrightarrow \bar{X} ; (x, t) \longrightarrow (x, t + 1) .$$

Let $\bar{X}^!$ denote the space \bar{X} regarded as the infinite cyclic cover of $X^!$ classified by

$$c^! : X^! \longrightarrow S^1 ; (x, [t]) \longrightarrow [t] ,$$

with covering projection

$$\bar{X}^! = \bar{X} \longrightarrow X^! ; (x, t) \longrightarrow (x, [t/q])$$

and generating covering translation

$$\zeta^! = \zeta^q : \bar{X}^! = \bar{X} \longrightarrow \bar{X}^! = \bar{X} ; (x, t) \longrightarrow (x, t + q) ,$$

so that there is defined a pullback square

$$\begin{array}{ccc} \bar{X}^! & \xrightarrow{\bar{c}^!} & \mathbb{R} \\ \downarrow & & \downarrow \bar{q} \\ \bar{X} & \xrightarrow{\bar{c}} & \mathbb{R} \end{array}$$

with

$$\bar{q} : \mathbb{R} \longrightarrow \mathbb{R} ; x \longrightarrow x + q .$$

If X is a compact manifold and $(V; U, \zeta U)$ is a fundamental domain for \bar{X} then

$$(V^!; U, \zeta^q U) = \bigcup_{k=0}^{q-1} (\zeta^k V; \zeta^{k-1} U, \zeta^k U)$$

is a fundamental domain for $\bar{X}^!$. More generally, the q -fold transfer of an \mathbb{R} -graded CW complex $(K, \rho_K : K \longrightarrow \mathbb{R})$ (not necessarily of the type $K = \bar{X}$) may be defined to be the \mathbb{R} -graded CW complex

$$(K^!, \rho_{K^!}) = (K, \rho_K / q) ,$$

such that

$$(\rho_{K^!})^{-1}[j, j+1] = \bigcup_{k=0}^{q-1} (\rho_K)^{-1}[qj+k, qj+k+1] \quad (j \in \mathbb{Z}) .$$

Define the q -fold transfer functor

$$q^! : \mathbb{C}_1(\mathbb{A}) \longrightarrow \mathbb{C}_1(\mathbb{A}) ; M \longrightarrow q^! M , \quad q^! M(j) = \sum_{k=0}^{q-1} M(qj+k) .$$

The q -fold transfer of a morphism $f : M \longrightarrow N$ in $\mathbb{C}_1(\mathbb{A})$ is the morphism $q^! f : q^! M \longrightarrow q^! N$ defined by

$$\begin{aligned} q^! f(k, j) &= \begin{pmatrix} f(qk, qj) & f(qk, qj+1) & \dots & f(qk, qj+q-1) \\ f(qk+1, qj) & f(qk+1, qj+1) & \dots & f(qk+1, qj+q-1) \\ \vdots & \vdots & \ddots & \vdots \\ f(qk+q-1, qj) & f(qk+q-1, qj+1) & \dots & f(qk+q-1, qj+q-1) \end{pmatrix} \\ &: q^! M(j) = M(qj) \oplus M(qj+1) \oplus \dots \oplus M(qj+q-1) \\ &\longrightarrow q^! N(k) = N(qk) \oplus N(qk+1) \oplus \dots \oplus N(qk+q-1) . \end{aligned}$$

The q -fold transfer of a linear morphism $f = f_+ - zf_- : M \longrightarrow N$ in $\mathbb{C}_1(\mathbb{A})$ is the linear morphism

$$q^! f = q^! f_+ - zq^! f_- : q^! M \longrightarrow q^! N$$

with

$$\begin{aligned}
 q^! f_+(j) &= \begin{pmatrix} f_+(qj) & 0 & 0 & \dots & 0 \\ -f_-(qj) & f_+(qj+1) & 0 & \dots & 0 \\ 0 & -f_-(qj+1) & f_+(qj+2) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & f_+(qj+q-1) \end{pmatrix} \\
 : q^! M(j) &= M(qj) \oplus M(qj+1) \oplus \dots \oplus M(qj+q-1) \\
 &\longrightarrow q^! N(j) = N(qj) \oplus N(qj+1) \oplus \dots \oplus N(qj+q-1),
 \end{aligned}$$

$$\begin{aligned}
 q^! f_-(j) &= \begin{pmatrix} 0 & 0 & \dots & f_-(qj+q-1) \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \\
 : q^! M(j) &= M(qj) \oplus M(qj+1) \oplus \dots \oplus M(qj+q-1) \\
 \longrightarrow q^! N(j+1) &= N(qj+q) \oplus N(qj+q+1) \oplus \dots \oplus N(qj+2q-1).
 \end{aligned}$$

The q -fold transfer of a ζ -equivariant morphism is ζ -equivariant, so that there is also defined a functor $q^! : \mathbb{A}[z, z^{-1}] \longrightarrow \mathbb{A}[z, z^{-1}]$.

The q -fold transfer functors

$$q^! : \mathbb{C}_1(\mathbb{A}) \longrightarrow \mathbb{C}_1(\mathbb{A}) \quad , \quad q^! : \mathbb{A}[z, z^{-1}] \longrightarrow \mathbb{A}[z, z^{-1}]$$

induce the q -fold transfer maps in the algebraic K -groups

$$\begin{aligned}
 q^! &: K_1(\mathbb{C}_1(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_1(\mathbb{A})) \quad , \\
 q^! &: K_1(\mathbb{A}[z, z^{-1}]) \longrightarrow K_1(\mathbb{A}[z, z^{-1}]) \quad .
 \end{aligned}$$

DEFINITION 12.1 The S^1 -transfer invariant subgroup of $K_1(\mathbb{A}[z, z^{-1}])$ is

$$K_1(\mathbb{A}[z, z^{-1}])^{INV} = \{ \tau \in K_1(\mathbb{A}[z, z^{-1}]) \mid q^!(\tau) = \tau \text{ for every } q \geq 2 \} .$$

□

For any group π there are analogously defined q -fold transfer maps in the Whitehead group of $\pi \times \mathbb{Z}$

$$q^! : Wh(\pi \times \mathbb{Z}) \longrightarrow Wh(\pi \times \mathbb{Z})$$

and an S^1 -transfer invariant subgroup $Wh(\pi \times \mathbb{Z})^{INV} \subseteq Wh(\pi \times \mathbb{Z})$.

If X is a finitely dominated CW complex then $X \times S^1$ is equipped with a homotopy equivalence

$$\phi : X \times S^1 \longrightarrow K$$

to a finite CW complex K (Mather [46]) with infinite cyclic cover $\bar{K} \simeq X$. The map

$$c : K \xrightarrow{\phi^{-1}} X \times S^1 \xrightarrow{\text{proj.}} S^1$$

has the property that for every cover $q : S^1 \rightarrow S^1$ the pullback cover $K^!$ is simple homotopy equivalent to K . The map representing $-1 \in \pi_1(S^1) = S^1$

$$-1 : S^1 \rightarrow S^1 ; t \mapsto 1 - t$$

covers itself

$$\begin{array}{ccc} S^1 & \xrightarrow{-1} & S^1 \\ q \downarrow & & \downarrow q \\ S^1 & \xrightarrow{-1} & S^1 \end{array} .$$

Thus the self-homotopy equivalence

$$f = \phi(1 \times -1)\phi^{-1} : K \xrightarrow{\phi^{-1}} X \times S^1 \xrightarrow{1 \times -1} X \times S^1 \xrightarrow{\phi} K$$

is such that

$$\begin{aligned} q^! \tau(f : K \rightarrow K) &= \tau(f^! : K^! \rightarrow K^!) = \tau(f : K \rightarrow K) \\ &\in Wh(\pi \times \mathbb{Z})^{INV} \subseteq Wh(\pi \times \mathbb{Z}) \quad (\pi = \pi_1(X)) . \end{aligned}$$

In fact, the image of the split injection of Ferry [27]

$$\bar{B}'_{geo} : \tilde{K}_0(\mathbb{Z}[\pi]) \rightarrow Wh(\pi \times \mathbb{Z}) ; [X] \mapsto \tau(f)$$

is precisely the S^1 -transfer invariant subgroup $Wh(\pi \times \mathbb{Z})^{INV}$. See Ranicki [66] for the identification of \bar{B}'_{geo} with the geometrically significant split injection

$$\bar{B}' : \tilde{K}_0(\mathbb{Z}[\pi]) \rightarrow Wh(\pi \times \mathbb{Z}) ; [P] \mapsto \tau(-z : P[z, z^{-1}] \rightarrow P[z, z^{-1}]) .$$

PROPOSITION 12.2 (i) *The effect of $q^!$ on the geometrically significant direct sum decomposition of $K_1(\mathbb{A}[z, z^{-1}])$ is given by the commutative diagram*

$$\begin{array}{ccc} K_1(\mathbb{A}) \oplus K_0(\mathbb{P}_0(\mathbb{A})) \oplus \tilde{\text{Nil}}_0(\mathbb{A}) \oplus \tilde{\text{Nil}}_0(\mathbb{A}) & \xrightarrow{i_1 \oplus \bar{B}' \oplus N'_+ \oplus N'_-} & K_1(\mathbb{A}[z, z^{-1}]) \\ \downarrow q \oplus 1 \oplus q^! \oplus q^! & & \downarrow q^! \\ K_1(\mathbb{A}) \oplus K_0(\mathbb{P}_0(\mathbb{A})) \oplus \tilde{\text{Nil}}_0(\mathbb{A}) \oplus \tilde{\text{Nil}}_0(\mathbb{A}) & \xrightarrow{i_1 \oplus \bar{B}' \oplus N'_+ \oplus N'_-} & K_1(\mathbb{A}[z, z^{-1}]) \end{array}$$

with

$$q^! : \widetilde{\text{Nil}}_0(\mathbb{A}) \longrightarrow \widetilde{\text{Nil}}_0(\mathbb{A}) ; [M, \nu] \longrightarrow [M, \nu^q] .$$

(ii) The image of the geometrically significant split injection

$$\bar{B}' : K_0(\mathbb{P}_0(\mathbb{A})) \longrightarrow K_1(\mathbb{A}[z, z^{-1}]) ;$$

$$[M, p] \longrightarrow \tau(1 - p - zp : M[z, z^{-1}] \longrightarrow M[z, z^{-1}])$$

is the subgroup of the S^1 -transfer invariant elements

$$\text{im}(\bar{B}') = K_1(\mathbb{A}[z, z^{-1}])^{INV} .$$

(iii) The effect of $q^!$ on $K_1(\mathbb{C}_1(\mathbb{A}))$ is

$$q^! = 1 : K_1(\mathbb{C}_1(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_1(\mathbb{A})) .$$

PROOF (i) For a linear automorphism in $\mathbb{A}[z, z^{-1}]$

$$f = f_+ - zf_- : M[z, z^{-1}] \longrightarrow M[z, z^{-1}]$$

the q -fold transfer is the linear automorphism

$$q^! f = \begin{pmatrix} f_+ & 0 & 0 & \dots & -zf_- \\ -f_- & f_+ & 0 & \dots & 0 \\ 0 & -f_- & f_+ & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & f_+ \end{pmatrix}$$

$$: \left(\sum_{j=0}^{q-1} M \right) [z, z^{-1}] = (M \oplus M \oplus \dots \oplus M) [z, z^{-1}]$$

$$\longrightarrow (M \oplus M \oplus \dots \oplus M) [z, z^{-1}] .$$

Define an automorphism in \mathbb{A}

$$g = j_! f = f_+ - f_- : M \longrightarrow M .$$

The linear automorphism in $\mathbb{A}[z, z^{-1}]$

$$f' = g^{-1} f = f'_+ - zf'_- = g^{-1} f_+ - zg^{-1} f_- : M[z, z^{-1}] \longrightarrow M[z, z^{-1}]$$

is such that

$$f'_+ f'_- = f'_- f'_+ , \quad f'_+ - f'_- = 1 : M \longrightarrow M .$$

Elementary row and column operations show that

$$\tau(q^! f') = \tau((f'_+)^q - z(f'_-)^q : M[z, z^{-1}] \longrightarrow M[z, z^{-1}]) \in K_1(\mathbb{A}[z, z^{-1}]) .$$

It follows that

$$\begin{aligned} \tau(q^! f) &= \tau(q^! (gf')) = \tau(q^! f') + \tau(q^! (i_! g)) \\ &= \tau((f'_+)^q - z(f'_-)^q : M[z, z^{-1}] \longrightarrow M[z, z^{-1}]) \\ &\quad + qi_! \tau(g : M \longrightarrow M) \in K_1(\mathbb{A}[z, z^{-1}]) . \end{aligned}$$

In particular, if $f_+f_- = f_-f_+$ then

$$\tau(q^!f) = \tau((f_+)^q - z(f_-)^q : M[z, z^{-1}] \longrightarrow M[z, z^{-1}]) \in K_1(\mathbb{A}[z, z^{-1}]) .$$

This is the case for each of the components of the geometrically significant decomposition of $K_1(\mathbb{A}[z, z^{-1}])$:

(I) if $f_- = 0$ then

$$\begin{aligned} \tau(f) &= i_! \tau(f_+) , \quad q^! \tau(f) = i_! \tau((f^+)^q) = q \tau(f) \\ &\in \text{im}(i_! : K_1(\mathbb{A}) \longrightarrow K_1(\mathbb{A}[z, z^{-1}])) , \end{aligned}$$

(II) if $f_+ = 1 - p$, $f_- = p$ for a projection $p^2 = p : M \longrightarrow M$ then

$$\begin{aligned} \bar{B}'([M, p]) &= \tau(f) , \quad q^! \bar{B}'([M, p]) = \tau((1 - p)^q - zp^q) = \tau(f) \\ &\in \text{im}(\bar{B}' : K_0(\mathbb{P}_0(\mathbb{A})) \longrightarrow K_1(\mathbb{A}[z, z^{-1}])) , \end{aligned}$$

(III) if $f_+ = \nu$, $f_- = 1$ for a nilpotent endomorphism $\nu : M \longrightarrow M$ then

$$\begin{aligned} \bar{N}'_+[M, \nu] &= \tau(1 - z^{-1}\nu) = \tau(f) - \bar{B}'([M]) , \\ q^! \bar{N}'_+[M, \nu] &= \tau(\nu^q - z) - \bar{B}'[M] = \tau(1 - z^{-1}\nu^q) = \bar{N}'_+(q^![M, \nu]) \\ &\in \text{im}(\bar{N}'_+ : \widetilde{\text{Nil}}_0(\mathbb{A}) \longrightarrow K_1(\mathbb{A}[z, z^{-1}])) , \end{aligned}$$

(IV) if $f_+ = 1$, $f_- = \nu$ for a nilpotent endomorphism $\nu : M \longrightarrow M$ then

$$\begin{aligned} \bar{N}'_-[M, \nu] &= \tau(f) , \quad q^! \bar{N}'_-[M, \nu] = \tau(1 - z\nu^q) = \bar{N}'_-(q^![M, \nu]) \\ &\in \text{im}(\bar{N}'_- : \widetilde{\text{Nil}}_0(\mathbb{A}) \longrightarrow K_1(\mathbb{A}[z, z^{-1}])) . \end{aligned}$$

(ii) Immediate from (i).

(iii) Immediate from (ii), since there is defined a commutative diagram

$$\begin{array}{ccccc} K_0(\mathbb{P}_0(\mathbb{A})) & \xrightarrow{\bar{B}'} & K_1(\mathbb{A}[z, z^{-1}]) & \longrightarrow & K_1(\mathbb{C}_1(\mathbb{A})) \\ \downarrow 1 & & \downarrow q^! & & \downarrow q^! \\ K_0(\mathbb{P}_0(\mathbb{A})) & \xrightarrow{\bar{B}'} & K_1(\mathbb{A}[z, z^{-1}]) & \longrightarrow & K_1(\mathbb{C}_1(\mathbb{A})) \end{array}$$

an isomorphism. □

Given $m \geq 1$ let $T^m = S^1 \times S^1 \times \dots \times S^1$ be the m -fold torus. For each m -tuple $Q = (q_1, q_2, \dots, q_m)$ of integers $q_j \geq 1$ let

$$Q : T^m \longrightarrow T^m ; (z_1, z_2, \dots, z_m) \longrightarrow ((z_1)^{q_1}, (z_2)^{q_2}, \dots, (z_m)^{q_m})$$

be the canonical finite covering of T^m by itself. For any such Q define the Q -fold transfer functors

$$Q^! : \mathbb{C}_m(\mathbb{A}) \longrightarrow \mathbb{C}_m(\mathbb{A}) , \quad Q^! : \mathbb{A}[\mathbb{Z}^m] \longrightarrow \mathbb{A}[\mathbb{Z}^m]$$

by sending the object M to the object $Q^!M$ with components

$$\begin{aligned} & Q^!M(j_1, j_2, \dots, j_m) \\ &= \sum_{k_1=0}^{q_1-1} \sum_{k_2=0}^{q_2-1} \dots \sum_{k_m=0}^{q_m-1} M(q_1j_1 + k_1, q_2j_2 + k_2, \dots, q_mj_m + k_m) , \end{aligned}$$

and similarly for morphisms.

By analogy with 12.1 and 12.2:

DEFINITION 12.3 The T^m -transfer invariant subgroup of $K_1(\mathbb{A}[\mathbb{Z}^m])$ is

$$K_1(\mathbb{A}[\mathbb{Z}^m])^{INV} = \{ \tau \in K_1(\mathbb{A}[\mathbb{Z}^m]) \mid Q^!(\tau) = \tau \text{ for every } Q \} .$$

□

PROPOSITION 12.4 For any $m \geq 1$ the geometrically significant split injection

$$\bar{B}'_1 \bar{B}'_2 \dots \bar{B}'_m : K_{1-m}(\mathbb{P}_0(\mathbb{A})) \longrightarrow K_1(\mathbb{A}[\mathbb{Z}^m])$$

maps the lower K -group $K_{1-m}(\mathbb{P}_0(\mathbb{A}))$ ($= K_{1-m}(\mathbb{A})$ for $m \geq 2$) to the subgroup of the T^m -transfer invariant elements

$$\text{im}(\bar{B}'_1 \bar{B}'_2 \dots \bar{B}'_m) = K_1(\mathbb{A}[\mathbb{Z}^m])^{INV} .$$

PROOF This follows from the case $m = 1$ dealt with in 12.1 and the factorization of the Q -fold transfer for $Q = (q_1, q_2, \dots, q_m)$ as the m -fold composition of the single transfers

$$Q^! = (q_1)^!(q_2)^! \dots (q_m)^! : \mathbb{A}[\mathbb{Z}^m] \longrightarrow \mathbb{A}[\mathbb{Z}^m]$$

with $(q_j)^!$ acting on the j th coordinate ($1 \leq j \leq m$).

□

§13. Quadratic L -theory

This section is a brief recollection from Ranicki [68, §3] of the quadratic L -theory of an additive category with involution \mathbb{A} .

An *involution* on an additive category \mathbb{A} is a contravariant functor

$$* : \mathbb{A} \longrightarrow \mathbb{A} ; M \longrightarrow M^*$$

together with a natural equivalence

$$e : 1 \longrightarrow ** : \mathbb{A} \longrightarrow \mathbb{A} ; M \longrightarrow (e(M) : M \longrightarrow M^{**})$$

such that for every object M in \mathbb{A}

$$e(M)^* = e(M^*)^{-1} : M^* \longrightarrow M^* .$$

Use $e(M)$ to identify

$$M^{**} = M$$

for every object M in \mathbb{A} . Given objects M, N in \mathbb{A} define

$$M \otimes_{\mathbb{A}} N = \text{Hom}_{\mathbb{A}}(M^*, N) .$$

The *transposition* isomorphism is defined by

$$T(M, N) : M \otimes_{\mathbb{A}} N \longrightarrow N \otimes_{\mathbb{A}} M ; f \longrightarrow f^* ,$$

and is such that

$$T(N, M)T(M, N) = 1 : M \otimes_{\mathbb{A}} N \longrightarrow M \otimes_{\mathbb{A}} N .$$

Given chain complexes C, D in \mathbb{A} let $C \otimes_{\mathbb{A}} D$ be the abelian group chain complex defined by

$$(C \otimes_{\mathbb{A}} D)_r = \sum_{p+q=r} C_p \otimes_{\mathbb{A}} D_q ,$$

$$d_{C \otimes_{\mathbb{A}} D} : (C \otimes_{\mathbb{A}} D)_r \longrightarrow (C \otimes_{\mathbb{A}} D)_{r-1} ; f \longrightarrow d_D f + (-)^q f(d_C)^* .$$

An n -cycle $\phi \in (C \otimes_{\mathbb{A}} D)_n$ is a chain map

$$\phi : C^{n-*} \longrightarrow D ,$$

with C^{n-*} the n -dual chain complex defined by

$$d_{C^{n-*}} = (-)^r d_C^* : (C^{n-*})_r = C^{n-r} = (C_{n-r})^* \longrightarrow (C^{n-*})_{r-1} .$$

The transposition isomorphism

$$T : C \otimes_{\mathbb{A}} D \longrightarrow D \otimes_{\mathbb{A}} C$$

is defined by

$$T = (-)^{pq} T(C_p, D_q) : C_p \otimes_{\mathbb{A}} D_q \longrightarrow D_q \otimes_{\mathbb{A}} C_p .$$

Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$ -module resolution of \mathbb{Z}

$$W : \dots \longrightarrow \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] .$$

Given a chain complex C in \mathbb{A} define the $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complex

$$W_{\%} C = W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes_{\mathbb{A}} C) ,$$

with $T \in \mathbb{Z}_2$ acting by the transposition involution $T : C \otimes_{\mathbb{A}} C \longrightarrow C \otimes_{\mathbb{A}} C$.

The boundary of the n -chain

$$\psi = \{\psi_s \in (C \otimes_{\mathbb{A}} C)_{n-s} \mid s \geq 0\} \in (W_{\%} C)_n$$

is the $(n-1)$ -chain $d(\psi) \in (W_{\%} C)_{n-1}$ with

$$\begin{aligned} d(\psi)_s &= d_{C \otimes_{\mathbb{A}} C}(\psi_s) + (-)^{n+s+1}(\psi_{s+1} + (-)^s T \psi_{s+1}) \\ &\in (C \otimes_{\mathbb{A}} C)_{n-s-1} \quad (s \geq 0) . \end{aligned}$$

The *quadratic Q -groups* of C are defined by

$$Q_n(C) = H_n(W_{\%} C) .$$

An n -dimensional quadratic (Poincaré) complex (C, ψ) in \mathbb{A} is an n -dimensional chain complex C in \mathbb{A} together with an element $\psi \in Q_n(C)$ (such that the chain map $(1+T)\psi_0 : C^{n-*} \longrightarrow C$ is a chain equivalence).

See Ranicki [68] for further details of the definition of *quadratic (Poincaré) pairs* the construction of the n -dimensional quadratic L -group $L_n(\mathbb{A})$ ($n \geq 0$) as the cobordism group of n -dimensional quadratic Poincaré complexes $(C, \psi \in Q_n(C))$ in \mathbb{A} , and for the proof of 4-periodicity

$$L_n(\mathbb{A}) = L_{n+4}(\mathbb{A}) .$$

$L_{2i}(\mathbb{A})$ (resp. $L_{2i+1}(\mathbb{A})$) is the Witt group of nonsingular $(-)^i$ -quadratic forms (resp. formations) in \mathbb{A} . The definition of $L_n(\mathbb{A})$ is extended to the range $n \leq -1$ by the 4-periodicity, that is $L_n(\mathbb{A}) = L_{n+4k}(\mathbb{A})$ for any k such that $n + 4k \geq 0$. The surgery obstruction groups $L_*(A)$ of Wall [84] are the quadratic L -groups of the additive category with involution $\mathbb{A} = \mathbb{B}^f(A)$ of based f.g. free A -modules for a ring with involution A .

A chain map $f : C \longrightarrow D$ in \mathbb{A} induces a \mathbb{Z} -module chain map

$$f_{\%} : W_{\%}C \longrightarrow W_{\%}D ; \psi = \{\psi_s | s \geq 0\} \longrightarrow f_{\%}\psi = \{f\psi_s f^* | s \geq 0\} .$$

Given an n -dimensional quadratic complex (C, ψ) and a subcomplex $B \subseteq C$ there is defined a quotient n -dimensional quadratic complex

$$(C, \psi)/B = (C/B, f_{\%}\psi) ,$$

with $f : C \longrightarrow C/B$ the projection. In dealing with subcomplexes it is always required that the sequences

$$0 \longrightarrow B_r \longrightarrow C_r \xrightarrow{f} (C/B)_r \longrightarrow 0$$

be split exact, so that there is defined a connecting chain map $g : C/B \longrightarrow SB$ such that $gf = 0$.

PROPOSITION 13.1 *There is a natural one-one correspondence between the homotopy equivalence classes of n -dimensional quadratic Poincaré pairs in \mathbb{A} and the homotopy equivalence classes of n -dimensional quadratic complexes in \mathbb{A} .*

PROOF This is just the categorical version of the one-one correspondence of Ranicki [61, 3.4] for the additive category with duality involution

$$\mathbb{P}(A) = \{ \text{f.g. projective } A\text{-modules} \} ,$$

and proceeds as follows.

The *boundary* of an n -dimensional quadratic complex (C, ψ) in \mathbb{A} is the $(n - 1)$ -dimensional quadratic Poincaré complex in \mathbb{A}

$$\partial(C, \psi) = (\partial C, \partial\psi)$$

with

$$\begin{aligned} d_{\partial C} &= \begin{pmatrix} d_C & (-)^r(1+T)\psi_0 \\ 0 & (-)^r d_C^* \end{pmatrix} : \\ \partial C_r &= C_{r+1} \oplus C^{m-r} \longrightarrow \partial C_{r-1} = C_r \oplus C^{m-r+1} , \end{aligned}$$

$$\begin{aligned} \partial \psi_0 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : \\ \partial C^{m-r-1} &= C^{m-r} \oplus C_{r+1} \longrightarrow \partial C_r = C_{r+1} \oplus C^{m-r} , \end{aligned}$$

$$\begin{aligned} \partial \psi_s &= \begin{pmatrix} (-)^{n-r-s-1} T \psi_{s-1} & 0 \\ 0 & 0 \end{pmatrix} : \\ \partial C^{m-r-s-1} &= C^{m-r-s} \oplus C_{r+s+1} \longrightarrow \partial C_r = C_{r+1} \oplus C^{m-r} \\ &\quad (s \geq 1) . \end{aligned}$$

Given an n -dimensional quadratic complex (C, ψ) in \mathbb{A} define the thickening n -dimensional quadratic Poincaré pair in \mathbb{A}

$$\delta \partial(C, \psi) = (p_C = \text{projection} : \partial C \longrightarrow C^{m-*} , (0, \partial \psi)) .$$

Conversely, given an n -dimensional quadratic Poincaré pair in \mathbb{A}

$$X = (f : C \longrightarrow D, (\delta \psi, \psi))$$

apply the algebraic Thom construction to obtain an n -dimensional quadratic complex

$$X/\partial X = (D, \delta \psi)/C = (C(f), \delta \psi/\psi)$$

with

$$\begin{aligned} (\delta \psi/\psi)_s &= \begin{pmatrix} \delta \psi_s & 0 \\ (-)^{n-r-1} \psi_s f^* & (-)^{n-r-s} T \psi_{s+1} \end{pmatrix} : \\ C(f)^{n-r-s+1} &= D^{n-r-s+1} \oplus C^{m-r-s} \longrightarrow C(f)_r = D_r \oplus C_{r-1} \\ &\quad (s \geq 0) , \end{aligned}$$

which is homotopy equivalent to $\delta \partial(X/\partial X)$.

□

The boundary construction of 13.1 generalizes to quadratic pairs. An n -dimensional quadratic pair $(f : C \longrightarrow D, (\delta \psi, \psi))$ determines an $(n-1)$ -dimensional quadratic Poincaré pair

$$(\partial f : \partial C \longrightarrow \partial D, (\delta \delta \psi, \partial \psi))$$

with

$$\begin{aligned} \partial C &= S^{-1}C((1+T)\psi_0 : C^{m-1-*} \longrightarrow C) , \\ \partial D &= S^{-1}C((1+T)\delta \psi_0 : C(f)^{n-*} \longrightarrow D) . \end{aligned}$$

Here, $(1 + T)\delta\psi_0$ is an abbreviation for the chain map $C(f)^{n-*} \longrightarrow D$ defined by

$$((1 + T)\delta\psi_0 \ f(1 + T)\psi_0) : C(f)^{n-r} = D^{n-r} \oplus C^{m-r-1} \longrightarrow D_r .$$

The quadratic complex (C, ψ) is Poincaré if and only if ∂C is contractible. Similarly, the quadratic pair $(f : C \longrightarrow D, (\delta\psi, \psi))$ is Poincaré if and only if ∂D is contractible, in which case ∂C is also contractible.

The n -dimensional quadratic pairs in \mathbb{A}

$$V^+ = (f^+ : C^+ \longrightarrow D^+, (\delta\psi^+, \psi^+)) ,$$

$$V^- = (f^- : C^- \longrightarrow D^-, (\delta\psi^-, \psi^-))$$

are *adjoining* if

$$C^+ = C^- , \quad \psi^+ = -\psi^- .$$

See Ranicki [61, §3] for the definition of the *union* of adjoining n -dimensional quadratic pairs

$$V^+ = (f^+ : C \longrightarrow D^+, (\delta\psi^+, \psi)) ,$$

$$V^- = (f^- : C \longrightarrow D^-, (\delta\psi^-, -\psi))$$

with common boundary

$$\partial V^+ = -\partial V^- = (C, \psi) .$$

The union is an n -dimensional quadratic complex in \mathbb{A}

$$V^+ \cup V^- = (D, \delta\psi) = (D^+ \cup D^-, \delta\psi^+ \cup \delta\psi^-) ,$$

with

$$D^+ \cup D^- = C\left(\begin{pmatrix} f^+ \\ f^- \end{pmatrix} : C \longrightarrow D^+ \oplus D^-\right) .$$

The duality chain maps fit into a map of homotopy exact sequences of chain complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & D^{n-*} & \longrightarrow & (D^+)^{n-*} \oplus (D^-)^{n-*} & \longrightarrow & C^{m-*} \longrightarrow 0 \\ (1+T)\delta\psi_0 \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & D & \longrightarrow & C(f^+) \oplus C(f^-) & \longrightarrow & SC \longrightarrow 0 \end{array}$$

so that there is defined an exact sequence of algebraic mapping cones

$$0 \longrightarrow \partial D \longrightarrow \partial D^+ \oplus \partial D^- \longrightarrow S\partial C \longrightarrow 0 .$$

If ∂D^+ and ∂D^- are contractible then so are ∂C and ∂D , i.e. the union of quadratic Poincaré pairs is a quadratic Poincaré complex.

DEFINITION 13.2 A *splitting* (V^+, V^-, h) of an n -dimensional quadratic complex (E, θ) in \mathbb{A} consists of adjoining n -dimensional quadratic pairs

V^+ , V^- , together with a homotopy equivalence

$$h : V^+ \cup V^- \longrightarrow (E, \theta) .$$

The splitting is *Poincaré* if the quadratic pairs V^+ , V^- are Poincaré, in which case the complex (E, θ) is Poincaré.

□

See Ranicki [63, 7.5], [69, §23] for the applications of Poincaré splittings to the algebraic theory of codimension 1 surgery, and Yamasaki [89] for an application to controlled surgery.

DEFINITION 13.3 Given an n -dimensional quadratic complex (E, θ) and a subcomplex $D \subseteq E$ in \mathbb{A} let (V^+, V^-, h) be the splitting of (E, θ) given by

$$V^+ = (f^+ : \partial(E/D) \longrightarrow D, (0, \partial p_{\%} \theta)) ,$$

$$V^- = \delta \partial((E, \theta)/D) = (f^- : \partial(E/D) \longrightarrow (E/D)^{n-*}, (0, \partial p_{\%} \theta)) ,$$

$$f^+ : \partial(E/D)_r = (E/D)_{r+1} \oplus (E/D)^{n-r} \longrightarrow (E/D)_{r+1} \longrightarrow D_r ,$$

$$f^- = \text{projection} :$$

$$\partial(E/D)_r = (E/D)_{r+1} \oplus (E/D)^{n-r} \longrightarrow (E/D)^{n-r} ,$$

$$h : C \begin{pmatrix} f^+ \\ f^- \end{pmatrix}_r = (E/D)^{n-r} \oplus \partial(E/D)_{r-1} \oplus D_r \longrightarrow D_r \longrightarrow E_r .$$

If (E, θ) is a Poincaré complex then (V^+, V^-, h) is a Poincaré splitting.

□

In fact, every splitting of a quadratic complex is homotopy equivalent to one constructed as in 13.3.

An involution on \mathbb{A} determines an involution on the projective class and torsion groups of \mathbb{A}

$$* : K_0(\mathbb{A}) \longrightarrow K_0(\mathbb{A}) ; [M] \longrightarrow [M^*] ,$$

$$* : K_1(\mathbb{A}) \longrightarrow K_1(\mathbb{A}) ; \tau(f : L \longrightarrow L) \longrightarrow \tau(f^* : L^* \longrightarrow L^*) .$$

In dealing with torsion it is required that the additive category \mathbb{A} be equipped with both an involution and a compatible stable canonical structure (see Ranicki [68, §7] for details).

A quadratic complex (C, ψ) in \mathbb{A} is *round* if the underlying chain complex C is round, i.e. such that $[C] = 0 \in K_0(\mathbb{A})$. The torsion of an n -dimensional quadratic Poincaré complex (C, ψ) in \mathbb{A} is defined by

$$\tau(C, \psi) = \tau((1 + T)\psi_0 : C^{n-*} \longrightarrow C) \in \begin{cases} K_1(\mathbb{A}) & \text{for round } (C, \psi) \\ \tilde{K}_1(\mathbb{A}) & \text{for any } (C, \psi) , \end{cases}$$

and satisfies

$$\tau(C, \psi)^* = (-)^n \tau(C, \psi) .$$

There is a corresponding notion of torsion for an $(n+1)$ -dimensional quadratic Poincaré pair $V = (f : C \longrightarrow D, (\delta\psi, \psi))$

$$\tau(V) = \tau((1+T)\delta\psi_0 : C(f)^{n+1-*} \longrightarrow D) ,$$

satisfying

$$\tau(C, \psi) = \tau(V) + (-)^n \tau(V)^* .$$

If (V^+, V^-, h) is a Poincaré splitting of an n -dimensional quadratic Poincaré complex (E, θ) in \mathbb{A} the torsions satisfy

$$\tau(E, \theta) - \tau(V^+ \cup V^-) = \tau(h) + (-)^n \tau(h)^* ,$$

$$\tau(V^+ \cup V^-) = \tau(V^+) + \tau(V^-) - \tau(V^+ \cap V^-) .$$

The *intermediate quadratic L -groups* $L_*^S(\mathbb{A})$ are defined for a $*$ -invariant subgroup $\begin{cases} S \subseteq K_0(\mathbb{A}) \\ S \subseteq K_1(\mathbb{A}) \\ S \subseteq \tilde{K}_1(\mathbb{A}) \end{cases}$ to be the cobordism groups of quadratic

Poincaré complexes (C, ψ) in \mathbb{A} with $\begin{cases} [C] \in S \\ [C] = 0, \tau(C, \psi) \in S \\ \tau(C, \psi) \in S \end{cases}$. The intermediate L -groups associated to $*$ -invariant subgroups $S \subseteq S'$ are related by the usual Rothenberg exact sequence

$$\dots \longrightarrow L_n^S(\mathbb{A}) \longrightarrow L_n^{S'}(\mathbb{A}) \longrightarrow \hat{H}^n(\mathbb{Z}_2; S'/S) \longrightarrow L_{n-1}^S(\mathbb{A}) \longrightarrow \dots ,$$

with the Tate \mathbb{Z}_2 -cohomology groups defined by

$$\hat{H}^n(\mathbb{Z}_2; S'/S) = \{x \in S'/S \mid x^* = (-)^n x\} / \{y + (-)^n y^* \mid y \in S'/S\} .$$

The quadratic L -groups of \mathbb{A} are the special cases

$$L_*(\mathbb{A}) = L_*^{\tilde{K}_1(\mathbb{A})}(\mathbb{A}) = L_*^{K_0(\mathbb{A})}(\mathbb{A}) .$$

The *round quadratic L -groups* of \mathbb{A} are the special cases

$$L_*^r(\mathbb{A}) = L_*^{K_1(\mathbb{A})}(\mathbb{A}) = L_*^{\{0\} \subseteq K_0(\mathbb{A})}(\mathbb{A}) ,$$

For a ring with involution A and $\mathbb{A} = \mathbb{B}^f(A)$ these are the round quadratic L -groups $L_*^r(A)$ of Hambleton, Ranicki and Taylor [32].

Given an additive category with involution \mathbb{A} and a set X it is not possible in general to define an involution on the X -graded category $\mathbb{G}_X(\mathbb{A})$. Define the *locally finite X -graded category* $\mathbb{F}_X(\mathbb{A})$ to be the subcategory of $\mathbb{G}_X(\mathbb{A})$ with the same objects, in which a morphism

$$f = \{f(y, x)\} : L = \sum_{x \in X} L(x) \longrightarrow M = \sum_{y \in X} M(y)$$

is required to be such that for each $y \in X$ the set $\{x \in X \mid f(y, x) \neq 0 : L(x) \longrightarrow M(y)\}$ is finite. The involution on \mathbb{A} extends to an involution

on $\mathbb{F}_X(\mathbb{A})$ by

$$* : \mathbb{F}_X(\mathbb{A}) \longrightarrow \mathbb{F}_X(\mathbb{A}) ;$$

$$M = \sum_{x \in X} M(x) \longrightarrow M^* = \sum_{x \in X} M^*(x) , \quad M^*(x) = M(x)^* .$$

The dual of a morphism $f : L \longrightarrow M$ in $\mathbb{F}_X(\mathbb{A})$ with components $f(y, x) : L(x) \longrightarrow M(y)$ is the morphism $f^* : M^* \longrightarrow L^*$ with components

$$f^*(x, y) = f(y, x)^* :$$

$$M^*(y) = M(y)^* \longrightarrow L^*(x) = L(x)^* \quad (x, y \in X) .$$

An involution $* : \mathbb{A} \longrightarrow \mathbb{A}$ on a filtered additive category \mathbb{A} is required to preserve the filtration degree δ , i.e. for every morphism $f : L \longrightarrow M$ in \mathbb{A}

$$\delta(f^* : M^* \longrightarrow L^*) = \delta(f : L \longrightarrow M) .$$

For any metric space X the bounded X -graded category $\mathbb{C}_X(\mathbb{A})$ is a subcategory of $\mathbb{F}_X(\mathbb{A})$ which is invariant under the involution, so that it is also a filtered additive category with involution. Similarly, if $(X, Y \subseteq X)$ is a pair of metric spaces the bounded (X, Y) -graded category $\mathbb{C}_{X,Y}(\mathbb{A})$ of 2.6 is a filtered additive category with involution.

REMARK 13.4 Given a metric space X and an n -dimensional normal map $(f, b) : M \longrightarrow N$ from an n -dimensional X -bounded manifold M to an n -dimensional X -bounded geometric Poincaré complex N with constant bounded fundamental group $\pi_1(N) = \pi$ Ferry and Pedersen [28] associate an X -bounded surgery obstruction

$$\sigma_*^b(f, b) \in L_n(\mathbb{C}_X(\mathbb{Z}[\pi])) .$$

By the main result of [28] for $n \geq 5$ this is the only obstruction to making (f, b) X -bounded normal bordant to an X -bounded homotopy equivalence, provided that X is a reasonable metric space (such as the open cone $O(K)$ on a compact polyhedron $K \subset S^k$). For compact X this is just the main result of Wall [84], with $L_n(\mathbb{C}_X(\mathbb{Z}[\pi])) = L_n(\mathbb{Z}[\pi])$. \square

§14. Excision and transversality in L -theory

The localization exact sequence of algebraic L -theory (Ranicki [63, §3]) for a morphism of rings with involution $A \longrightarrow S^{-1}A$ inverting a multiplicative subset $S \subset A$

$$\dots \longrightarrow L_n^I(A) \longrightarrow L_n(S^{-1}A) \longrightarrow L_n(A, S) \longrightarrow L_{n-1}^I(A) \longrightarrow \dots$$

relates the free L -groups $L_n(S^{-1}A)$, the intermediate projective L -groups $L_n^I(A)$ with

$$I = \ker(\tilde{K}_0(A) \longrightarrow \tilde{K}_0(S^{-1}A))$$

and the L -groups $L_n(A, S)$ of cobordism classes of $(n-1)$ -dimensional quadratic Poincaré complexes (C, ψ) over A such that C is an $S^{-1}A$ -contractible f.g. projective A -module chain complex, corresponding to an excision isomorphism

$$L_n^I(A \longrightarrow S^{-1}A) \cong L_n(A, S) .$$

A similar exact sequence

$$\begin{aligned} \dots \longrightarrow L_n^J(\mathbb{P}_Y(\mathbb{A})) \longrightarrow L_n(\mathbb{C}_X(\mathbb{A})) \longrightarrow L_n(\mathbb{C}_{X,Y}(\mathbb{A})) \\ \xrightarrow{\partial} L_{n-1}^J(\mathbb{P}_Y(\mathbb{A})) \longrightarrow \dots \end{aligned}$$

will now be obtained for any pair of metric spaces $(X, Y \subseteq X)$ and filtered additive category with involution \mathbb{A} , with

$$J = \ker(\tilde{K}_0(\mathbb{P}_Y(\mathbb{A})) \longrightarrow \tilde{K}_0(\mathbb{P}_X(\mathbb{A}))) ,$$

by analogy with the K -theory exact sequence of 4.1

$$\begin{aligned} K_1(\mathbb{C}_Y(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_X(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_{X,Y}(\mathbb{A})) \\ \xrightarrow{\partial} K_0(\mathbb{P}_Y(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_X(\mathbb{A})) . \end{aligned}$$

The exact sequence is obtained from quadratic L -theory excision isomorphisms

$$L_n^J(\mathbb{P}_Y(\mathbb{A})) \longrightarrow \mathbb{C}_X(\mathbb{A}) \cong L_n(\mathbb{C}_{X,Y}(\mathbb{A}))$$

analogous to the K -theory excision isomorphism of 4.1

$$K_1(\mathbb{P}_Y(\mathbb{A})) \longrightarrow \mathbb{P}_X(\mathbb{A}) \cong K_1(\mathbb{C}_{X,Y}(\mathbb{A})) .$$

This requires the following L -theory analogue of the result of 2.7 that every finite chain complex in the bounded germ category $\mathbb{C}_{X,Y}(\mathbb{A})$ is of the type $C = [D]$ for a finite chain complex D in $\mathbb{C}_X(\mathbb{A})$.

LEMMA 14.1 (i) For every n -dimensional quadratic complex (C, ψ) in $\mathbb{C}_{X,Y}(\mathbb{A})$ there exists an n -dimensional quadratic complex (D, θ) in $\mathbb{C}_X(\mathbb{A})$ such that

$$(C, \psi) = [D, \theta] .$$

(ii) $L_n(\mathbb{C}_{X,Y}(\mathbb{A}))$ is naturally isomorphic to the cobordism group of n -dimensional quadratic $\mathbb{C}_{X,Y}(\mathbb{A})$ -Poincaré complexes in $\mathbb{C}_X(\mathbb{A})$.

PROOF (i) As in the proof of 2.7 for any representatives in $\mathbb{C}_X(\mathbb{A})$

$$d_C : C_r \longrightarrow C_{r-1} \quad (1 \leq r \leq n)$$

there exists a sequence $b = (b_0, b_1, \dots, b_n)$ of integers $b_r \geq 0$ such that

- (a) $d_C(C_r(\mathcal{N}_{b_r}(Y, X))) \subseteq C_{r-1}(\mathcal{N}_{b_{r-1}}(Y, X))$,
- (b) $(d_C)^2(C_r) \subseteq C_{r-2}(\mathcal{N}_{b_{r-2}}(Y, X))$,
- (c) the n -cycle

$$\psi \in (W_{\%}C)_n = \sum_{p+q+r=n} W_p \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C_q \otimes_{\mathbb{C}_{X,Y}(\mathbb{A})} C_r)$$

is represented by an n -chain

$$\psi' \in (W_{\%}C)'_n = \sum_{p+q+r=n} W_p \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C_q \otimes_{\mathbb{C}_X(\mathbb{A})} C_r)$$

with boundary

$$d_{(W_{\%}C)'}(\psi') \in \ker((W_{\%}C)'_{n-1} \longrightarrow (W_{\%}C)_{n-1}) .$$

For any such sequence b the n -dimensional quadratic complex (D, θ) in $\mathbb{C}_X(\mathbb{A})$ defined by

$$\begin{aligned} d_D &= \begin{pmatrix} 0 & 0 \\ 0 & d_C| \end{pmatrix} : \\ D_r &= C_r = C_r(\mathcal{N}_{b_r}(Y, X)) \oplus C_r(X \setminus \mathcal{N}_{b_r}(Y, X)) \longrightarrow \\ D_{r-1} &= C_{r-1} = C_{r-1}(\mathcal{N}_{b_{r-1}}(Y, X)) \oplus C_{r-1}(X \setminus \mathcal{N}_{b_{r-1}}(Y, X)) , \\ \theta_s &= \begin{pmatrix} 0 & 0 \\ 0 & [\psi_s] \end{pmatrix} : D^{n-r-s} = C^{n-r-s} \\ &= C^{n-r-s}(\mathcal{N}_{b_{n-r-s}}(Y, X)) \oplus C^{n-r-s}(X \setminus \mathcal{N}_{b_{n-r-s}}(Y, X)) \\ &\longrightarrow D_r = C_r = C_r(\mathcal{N}_{b_r}(Y, X)) \oplus C_r(X \setminus \mathcal{N}_{b_r}(Y, X)) \end{aligned}$$

is such that $[D, \theta] = (C, \psi)$.

(ii) Immediate from (i) and its analogue for quadratic pairs.

□

Let

$$\begin{aligned} J &= \text{im}(\tilde{\partial}^{iso} : K_1^{iso}(\mathbb{C}_{X,Y}(\mathbb{A})) \longrightarrow \tilde{K}_0(\mathbb{P}_Y(\mathbb{A}))) \\ &= \ker(\tilde{K}_0(\mathbb{P}_Y(\mathbb{A})) \longrightarrow \tilde{K}_0(\mathbb{P}_X(\mathbb{A}))) \subseteq \tilde{K}_0(\mathbb{P}_Y(\mathbb{A})) \end{aligned}$$

and let $L_*^J(\mathbb{P}_Y(\mathbb{A})) \longrightarrow \mathbb{C}_X(\mathbb{A})$ be the relative L -quadratic groups in the exact sequence

$$\begin{aligned} L_n^J(\mathbb{P}_Y(\mathbb{A})) &\longrightarrow L_n(\mathbb{C}_X(\mathbb{A})) \longrightarrow L_n^J(\mathbb{P}_Y(\mathbb{A})) \longrightarrow \mathbb{C}_X(\mathbb{A}) \\ &\xrightarrow{\partial} L_{n-1}^J(\mathbb{P}_Y(\mathbb{A})) \longrightarrow L_{n-1}(\mathbb{C}_X(\mathbb{A})) . \end{aligned}$$

THEOREM 14.2 *For any \mathbb{A} , $(X, Y \subseteq X)$ there are natural excision isomorphisms*

$$L_n^J(\mathbb{P}_Y(\mathbb{A})) \longrightarrow \mathbb{C}_X(\mathbb{A}) \cong L_n(\mathbb{C}_{X,Y}(\mathbb{A})) ,$$

with the connecting map in the exact sequence

$$\begin{aligned} L_n^J(\mathbb{P}_Y(\mathbb{A})) &\longrightarrow L_n(\mathbb{C}_X(\mathbb{A})) \longrightarrow L_n(\mathbb{C}_{X,Y}(\mathbb{A})) \\ &\xrightarrow{\partial} L_{n-1}^J(\mathbb{P}_Y(\mathbb{A})) \longrightarrow L_{n-1}(\mathbb{C}_X(\mathbb{A})) \end{aligned}$$

given by

$$\partial : L_n(\mathbb{C}_{X,Y}(\mathbb{A})) \longrightarrow L_{n-1}^J(\mathbb{C}_Y(\mathbb{A})) ; (C, \psi) \longrightarrow \partial(C, \psi)$$

with (C, ψ) any n -dimensional quadratic $\mathbb{C}_{X,Y}(\mathbb{A})$ -Poincaré complex in $\mathbb{C}_X(\mathbb{A})$.

PROOF The relative L -group $L_n^J(\mathbb{P}_Y(\mathbb{A}) \longrightarrow \mathbb{C}_X(\mathbb{A}))$ is the cobordism group of n -dimensional quadratic Poincaré pairs $(f : C \longrightarrow D, (\delta\psi, \psi))$ in $\mathbb{P}_X(\mathbb{A})$ such that (C, ψ) is defined in $\mathbb{P}_Y(\mathbb{A})$ and D is defined in $\mathbb{C}_X(\mathbb{A})$. The algebraic Thom construction (13.1) on such a pair gives an n -dimensional quadratic Poincaré complex $(C(f), \delta\psi/\psi)$ in $\mathbb{P}_{X,Y}(\mathbb{A})$ with reduced projective class

$$[C(f)] = [D] = 0 \in \tilde{K}_0(\mathbb{P}_{X,Y}(\mathbb{A})) .$$

The complex is homotopy $\mathbb{C}_{X,Y}(\mathbb{A})$ -finite, and represents an element

$$(C(f), \delta\psi/\psi) \in L_n(\mathbb{C}_{X,Y}(\mathbb{A})) .$$

The boundary of an n -dimensional quadratic $\mathbb{C}_{X,Y}(\mathbb{A})$ -Poincaré complex (C, ψ) in $\mathbb{C}_X(\mathbb{A})$ is an $(n-1)$ -dimensional quadratic Poincaré complex $\partial(C, \psi)$ in $\mathbb{C}_X(\mathbb{A})$ with $\partial C = S^{-1}C((1+T)\psi_0 : C^{n-*} \longrightarrow C)$ a $\mathbb{C}_{X,Y}(\mathbb{A})$ -contractible chain complex. By 3.9 there exists a bound $b \geq 0$ such that ∂C is $(\mathbb{C}_X(\mathbb{A}), \mathcal{C}_{\mathcal{N}_b(Y,X)}(\mathbb{A}))$ -finitely dominated, with reduced projective class

$$\begin{aligned} [\partial C] &= -\tilde{\partial}^{iso} \tau(C, \psi) \\ &\in J = \text{im}(\tilde{\partial}^{iso} : K_1^{iso}(\mathbb{C}_{X,Y}(\mathbb{A})) \longrightarrow \tilde{K}_0(\mathbb{P}_Y(\mathbb{A}))) . \end{aligned}$$

The n -dimensional quadratic Poincaré pair $(\partial C \longrightarrow C^{n-*}, (0, \partial\psi))$ obtained by thickening (13.1) represents an element in the relative L -group $L_n^J(\mathbb{P}_Y(\mathbb{A}) \longrightarrow \mathbb{C}_X(\mathbb{A}))$.

The algebraic Thom construction and thickening define inverse isomorphisms

$$\begin{aligned} L_n^J(\mathbb{P}_Y(\mathbb{A}) \longrightarrow \mathbb{C}_X(\mathbb{A})) &\longrightarrow L_n(\mathbb{C}_{X,Y}(\mathbb{A})) ; \\ (f : C \longrightarrow D, (\delta\psi, \psi)) &\longrightarrow (C(f), \delta\psi/\psi) , \\ L_n(\mathbb{C}_{X,Y}(\mathbb{A})) &\longrightarrow L_n^J(\mathbb{P}_Y(\mathbb{A}) \longrightarrow \mathbb{C}_X(\mathbb{A})) ; \\ (C, \psi) &\longrightarrow (\partial C \longrightarrow C^{n-*}, (0, \partial\psi)) . \end{aligned}$$

□

As in §4 write the complement of Y in X as

$$Z = X \setminus Y .$$

The L -theory analogues of the excision isomorphism of 4.8

$$K_1(\mathbb{C}_{X,Y}(\mathbb{A})) = \varinjlim_b K_1(\mathbb{C}_{\mathcal{N}_b(Z,X), \mathcal{N}_b(Y,Z,X)}(\mathbb{A}))$$

will now be obtained, along with the exact sequence

$$\begin{aligned} \varinjlim_b K_1(\mathbb{C}_{\mathcal{N}_b(Y,Z,X)}(\mathbb{A})) &\longrightarrow K_1(\mathbb{C}_Z(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_{X,Y}(\mathbb{A})) \\ &\xrightarrow{\partial} \varinjlim_b K_0(\mathbb{P}_{\mathcal{N}_b(Y,Z,X)}(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_Z(\mathbb{A})) . \end{aligned}$$

COROLLARY 14.3 *There is a natural identification*

$$L_n(\mathbb{C}_{X,Y}(\mathbb{A})) = \varinjlim_b L_n(\mathbb{C}_{\mathcal{N}_b(Z,X),\mathcal{N}_b(Y,Z,X)}(\mathbb{A})) ,$$

with an exact sequence

$$\begin{aligned} \dots &\longrightarrow \varinjlim_b L_n^{J_b}(\mathbb{P}_{\mathcal{N}_b(Y,Z,X)}(\mathbb{A})) \longrightarrow L_n(\mathbb{C}_Z(\mathbb{A})) \longrightarrow L_n(\mathbb{C}_{X,Y}(\mathbb{A})) \\ &\xrightarrow{\partial} \varinjlim_b L_{n-1}^{J_b}(\mathbb{P}_{\mathcal{N}_b(Y,Z,X)}(\mathbb{A})) \longrightarrow L_{n-1}(\mathbb{C}_Z(\mathbb{A})) \longrightarrow \dots , \end{aligned}$$

where

$$J_b = \ker(\tilde{K}_0(\mathbb{P}_{\mathcal{N}_b(Y,Z,X)}(\mathbb{A})) \longrightarrow \tilde{K}_0(\mathbb{P}_Z(\mathbb{A}))) \subseteq \tilde{K}_0(\mathbb{P}_{\mathcal{N}_b(Y,Z,X)}(\mathbb{A})) .$$

PROOF Given an n -dimensional quadratic Poincaré complex (C, ψ) in $\mathbb{C}_{X,Y}(\mathbb{A})$ there exists a subcomplex $D \subseteq C$ such that D is defined in $\mathbb{C}_{\mathcal{N}_b(Y,Z,X)}(\mathbb{A})$ for some $b \geq 0$ and the quotient C/D is defined in $\mathbb{C}_{\mathcal{N}_b(Z,X),\mathcal{N}_b(Y,Z,X)}(\mathbb{A})$. The morphism

$$\begin{aligned} L_n(\mathbb{C}_{X,Y}(\mathbb{A})) &\longrightarrow \varinjlim_b L_n(\mathbb{C}_{\mathcal{N}_b(Z,X),\mathcal{N}_b(Y,Z,X)}(\mathbb{A})) ; \\ (C, \psi) &\longrightarrow (C, \psi)/D \end{aligned}$$

is an isomorphism inverse to the morphism

$$\varinjlim_b L_n^{J_b}(\mathbb{C}_{\mathcal{N}_b(Z,X),\mathcal{N}_b(Y,Z,X)}(\mathbb{A})) \longrightarrow L_n(\mathbb{C}_{X,Y}(\mathbb{A}))$$

induced by the inclusions

$$\mathbb{C}_{\mathcal{N}_b(Z,X),\mathcal{N}_b(Y,Z,X)}(\mathbb{A}) \longrightarrow \mathbb{C}_{X,Y}(\mathbb{A}) \quad (b \geq 0) .$$

The exact sequences of 14.2

$$\begin{aligned} \dots &\longrightarrow L_n(\mathbb{P}_{\mathcal{N}_b(Y,Z,X)}(\mathbb{A})) \longrightarrow L_n(\mathbb{C}_{\mathcal{N}_b(Z,X)}(\mathbb{A})) \\ &\longrightarrow L_n(\mathbb{C}_{\mathcal{N}_b(Z,X),\mathcal{N}_b(Y,Z,X)}(\mathbb{A})) \\ &\xrightarrow{\partial} L_{n-1}^{J_b}(\mathbb{P}_{\mathcal{N}_b(Y,Z,X)}(\mathbb{A})) \longrightarrow L_{n-1}(\mathbb{C}_{\mathcal{N}_b(Z,X)}(\mathbb{A})) \longrightarrow \dots \end{aligned}$$

combined with the natural identifications

$$L_*(\mathbb{C}_{\mathcal{N}_b(Z,X)}(\mathbb{A})) = L_*(\mathbb{C}_Z(\mathbb{A}))$$

give the exact sequence of the statement, on passing to the limit as $b \rightarrow \infty$.

□

THEOREM 14.4 *For a metric space X expressed as a union $X = X^+ \cup X^-$ of subspaces $X^+, X^- \subseteq X$ the quadratic L -groups of the bounded X -graded category $\mathbb{C}_X(\mathbb{A})$ of a filtered additive category with involution \mathbb{A} fit into a Mayer-Vietoris exact sequence*

$$\begin{aligned} \dots \longrightarrow \varinjlim_b L_n^{J_b}(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) &\longrightarrow L_n(\mathbb{C}_{X^+}(\mathbb{A})) \oplus L_n(\mathbb{C}_{X^-}(\mathbb{A})) \\ &\longrightarrow L_n(\mathbb{C}_X(\mathbb{A})) \xrightarrow{\partial} \varinjlim_b L_{n-1}^{J_b}(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \longrightarrow \dots, \end{aligned}$$

with

$$J_b = \ker(\tilde{K}_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \longrightarrow \tilde{K}_0(\mathbb{P}_{X^+}(\mathbb{A})) \oplus \tilde{K}_0(\mathbb{P}_{X^-}(\mathbb{A}))) .$$

The connecting map is defined by

$$\begin{aligned} \partial : L_n(\mathbb{C}_X(\mathbb{A})) &\longrightarrow \varinjlim_b L_{n-1}^{J_b}(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) ; \\ &(E, \theta) \longrightarrow \partial((E, \theta)/E^+) \end{aligned}$$

with $E^+ \subseteq E$ any subcomplex appearing in a Mayer-Vietoris presentation of E

$$\mathbb{E} : 0 \longrightarrow E^+ \cap E^- \xrightarrow{i} E^+ \oplus E^- \xrightarrow{j} E \longrightarrow 0 .$$

PROOF For each $b \geq 0$ let $L_n^{J_b}(\Delta_b)$ be the cobordism group of adjoining pairs (V^+, V^-) of n -dimensional quadratic Poincaré pairs in $\mathbb{C}_X(\mathbb{A})$

$$V^+ = (f^+ : C \longrightarrow D^+, (\delta\psi^+, \psi)) ,$$

$$V^- = (f^- : C \longrightarrow D^-, (\delta\psi^-, -\psi))$$

such that C is $\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ -finitely dominated in $\mathbb{C}_X(\mathbb{A})$ with projective class

$$[C] \in J_b \subseteq \tilde{K}_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) ,$$

and such that D^\pm is homotopy $\mathbb{C}_{X^\pm}(\mathbb{A})$ -finite. The common boundary of V^+ and V^- is a $\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ -finitely dominated $(n-1)$ -dimensional quadratic Poincaré complex in $\mathbb{C}_X(\mathbb{A})$

$$V^+ \cap V^- = \partial V^+ = -\partial V^- = (C, \psi) .$$

By construction, $L_n^{J_b}(\Delta_b)$ is the relative L -group in the exact sequence

$$\begin{aligned} \dots \longrightarrow L_n^{J_b}(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \\ \longrightarrow L_n(\mathbb{C}_{\mathcal{N}_b(X^+, X)}(\mathbb{A})) \oplus L_n(\mathbb{C}_{\mathcal{N}_b(X^-, X)}(\mathbb{A})) \longrightarrow L_n^{J_b}(\Delta_b) \\ \xrightarrow{\partial} L_{n-1}^{J_b}(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \longrightarrow \dots , \end{aligned}$$

with

$$\partial : L_n^{J_b}(\Delta_b) \longrightarrow L_{n-1}^{J_b}(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) ; (V^+, V^-) \longrightarrow V^+ \cap V^- .$$

The abelian group morphisms defined by union

$$U : \varinjlim_b L_n^{J_b}(\Delta_b) \longrightarrow L_n(\mathbb{C}_X(\mathbb{A})) ; (V^+, V^-) \longrightarrow V^+ \cup V^-$$

will now be shown to be isomorphisms by exhibiting explicit inverses.

Define an abelian group morphism

$$V : L_n(\mathbb{C}_X(\mathbb{A})) \longrightarrow \varinjlim_b L_n^{J_b}(\Delta_b) ; (E, \theta) \longrightarrow (V^+, V^-)$$

with the adjoining n -dimensional quadratic Poincaré pairs

$$V^+ = (f^+ : C \longrightarrow D^+, (\delta\psi^+, \psi)) ,$$

$$V^- = (f^- : C \longrightarrow D^-, (\delta\psi^-, -\psi))$$

given by the construction in 13.3 of a splitting (V^+, V^-) for (E, θ) using the subcomplex $D = E^+ \subseteq E$, so that

$$(C, \psi) = \partial((E, \theta)/E^-) , D^+ = E^+ , D^- = (E/E^+)^{n-*} .$$

It will be proved that $V = U^{-1}$. The chain complexes D^\pm are defined in $\mathbb{C}_{\mathcal{N}_b(X^\pm, X)}(\mathbb{A})$, and hence homotopy $\mathbb{C}_{X^\pm}(\mathbb{A})$ -finite. In order to verify that the chain complex $C = \partial(E/E^+)$ is $\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ -finitely dominated in $\mathbb{C}_X(\mathbb{A})$ it is convenient to embed the given Mayer-Vietoris presentation of E as a subobject $\mathbb{E} \subseteq \mathbb{E}\langle b \rangle$ in one of the Mayer-Vietoris presentations constructed in 4.6

$$\mathbb{E}\langle b \rangle : 0 \longrightarrow E^+\langle b \rangle \cap E^-\langle b \rangle \longrightarrow E^+\langle b \rangle \oplus E^-\langle b \rangle \longrightarrow E \longrightarrow 0 ,$$

which is possible for some sufficiently large bound $b \geq 0$. The $(n-1)$ -quadratic Poincaré complex $\partial((E, \theta)/E^+\langle b \rangle)$ is obtained up to homotopy equivalence from $\partial((E, \theta)/E^+)$ by algebraic surgery (Ranicki [61, §4]) on the n -dimensional quadratic pair

$$(\partial(E/E^+) \longrightarrow (E^+\langle b \rangle/E^+)^{n-*}, (0, \partial[\theta]))$$

with $(E^+\langle b \rangle/E^+)^{n-*}$ defined in $\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$. Thus $\partial(E/E^+)$ is $\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ -finitely dominated if and only if $\partial(E/E^+\langle b \rangle)$ is $\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ -finitely dominated, and there is no loss of generality in assuming that $\mathbb{E} = \mathbb{E}\langle b \rangle$. The same argument also shows that it suffices to verify that $\partial(E/E^+\langle b \rangle)$ is $\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ -finitely dominated for any bound $b \geq 0$. It will now be shown that for sufficiently large b the Mayer-Vietoris presentation $\mathbb{E}\langle b \rangle$ is compatible with the quadratic structure $\theta \in Q_n(E)$, and proceed with the verification for such b .

Define the quadratic Q -groups of a Mayer-Vietoris presentation of a chain complex E in $\mathbb{C}_X(\mathbb{A})$

$$\mathbb{E} : 0 \longrightarrow E^+ \cap E^- \longrightarrow E^+ \oplus E^- \longrightarrow E \longrightarrow 0$$

by

$$Q_n(\mathbb{E}) = H_n(W_\%(E^+ \cap E^-) \longrightarrow (W_\%E^+) \oplus (W_\%E^-)) ,$$

to fit into an exact sequence

$$\begin{aligned} \dots \longrightarrow Q_n(E^+ \cap E^-) &\longrightarrow Q_n(E^+) \oplus Q_n(E^-) \longrightarrow Q_n(\mathbb{E}) \\ &\longrightarrow Q_{n-1}(E^+ \cap E^-) \longrightarrow \dots \end{aligned}$$

The long exact sequence of $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complexes

$$\begin{aligned} 0 \longrightarrow (E^+ \cap E^-) \otimes_{\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})} (E^+ \cap E^-) &\longrightarrow \\ (E^+ \otimes_{\mathbb{C}_{\mathcal{N}_b(X^+, X)}(\mathbb{A})} E^+) \oplus (E^- \otimes_{\mathbb{C}_{\mathcal{N}_b(X^-, X)}(\mathbb{A})} E^-) &\longrightarrow E \otimes_{\mathbb{C}_X(\mathbb{A})} E \\ \longrightarrow (E/E^+ \otimes_{\mathbb{C}_X(\mathbb{A})} E/E^-) \oplus (E/E^- \otimes_{\mathbb{C}_X(\mathbb{A})} E/E^+) &\longrightarrow 0 \end{aligned}$$

(with the free $\mathbb{Z}[\mathbb{Z}_2]$ -action on $(E/E^+ \otimes E/E^-) \oplus (E/E^- \otimes E/E^+)$) induces an exact sequence of homology \mathbb{Z} -modules

$$\begin{aligned} \dots \longrightarrow H_{n+1}(E/E^+ \otimes E/E^-) &\longrightarrow Q_n(\mathbb{E}) \longrightarrow Q_n(E) \\ &\longrightarrow H_n(E/E^+ \otimes E/E^-) \longrightarrow \dots, \end{aligned}$$

with $Q_n(\mathbb{E}) \longrightarrow Q_n(E)$ the union map. If the bound b in the construction of the Mayer-Vietoris presentation $\mathbb{E} = \mathbb{E}\langle b \rangle$ is chosen so large that the composite chain map

$$(E/E^+)^{n-*} \longrightarrow E^{n-*} \xrightarrow{(1+T)\theta_0} E \longrightarrow E/E^-$$

is zero then

$$\theta \in \ker(Q_n(E) \longrightarrow H_n(E/E^+ \otimes E/E^-)) = \text{im}(Q_n(\mathbb{E}) \longrightarrow Q_n(E)).$$

Choosing a lift of $\theta \in Q_n(E)$ to an element $(\theta^+, \theta^-, \theta^+ \cap \theta^-) \in Q_n(\mathbb{E})$ there is defined a splitting (U^+, U^-, g) of (E, θ) with

$$U^+ = (E^+ \cap E^- \longrightarrow E^+, (\theta^+, \theta^+ \cap \theta^-)),$$

$$U^- = (E^+ \cap E^- \longrightarrow E^-, (\theta^-, -\theta^+ \cap \theta^-)),$$

$$g : E_r^+ \oplus (E^+ \cap E^-)_{r-1} \oplus E_r^- \longrightarrow E_r ; (x^+, y, x^-) \longrightarrow x^+ + x^-.$$

(The splitting (U^+, U^-, g) is not Poincaré in general). The chain complexes $\partial E^+, \partial E^-$ fit into an exact sequence of chain complexes in $\mathbb{C}_X(\mathbb{A})$

$$0 \longrightarrow \partial(E^+ \cap E^-) \longrightarrow \partial E^+ \oplus \partial E^- \longrightarrow \partial E' \longrightarrow 0$$

with

$$\partial E' = S^{-1}C(C(E^+ \cap E^- \longrightarrow E^+ \oplus E^-)^{n-*} \longrightarrow E).$$

Now $C(E^+ \cap E^- \longrightarrow E^+ \oplus E^-)$ is chain equivalent to E , so that $\partial E'$ is chain equivalent to ∂E , which is contractible. Thus ∂E^+ and ∂E^- are complementary homotopy direct summands of $\partial(E^+ \cap E^-)$, and are both $\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ -finitely dominated in $\mathbb{C}_X(\mathbb{A})$. The quadratic complex $(E, \theta)/E^+$ is obtained from the quadratic Poincaré pair U^- by the algebraic Thom construction collapsing the boundary, so there is defined a chain equivalence

$$\partial(E/E^+) \simeq C(S^{-1}(E^+ \cap E^-) \longrightarrow \partial E^-),$$

and $\partial(E/E^+)$ is also $\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ -finitely dominated in $\mathbb{C}_X(\mathbb{A})$. The reduced end invariant of the isomorphism torsion of (E, θ)

$$\tau(E, \theta) = \tau((1+T)\theta_0 : E^{n-*} \longrightarrow E) = -\tau(\partial E) \in K_1^{iso}(\mathbb{C}_X(\mathbb{A}))$$

is given by

$$\partial\tau(E, \theta) = [\partial E^-] = [\partial(E/E^+)] = [C]$$

$$\in J_b = \text{im}(\tilde{\partial}^{iso} : K_1^{iso}(\mathbb{C}_X(\mathbb{A})) \longrightarrow \tilde{K}_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}))) .$$

Since $[M] \in J_b$ for every object M in $\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$ (assuming X^+ and X^- are non-empty) it follows that

$$[V^+ \cap V^-] = [C] \in J_b \subseteq \tilde{K}_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) ,$$

as required for $V(E, \theta) = (V^+, V^-)$ to represent an element in $L_n^{J_b}(\Delta_b)$.

For any n -dimensional quadratic Poincaré complex (E, θ) in $\mathbb{C}_X(\mathbb{A})$ let (V^+, V^-) be the pair constructed above to represent $V(E, \theta) \in L_n^{J_b}(\Delta_b)$. The union $V^+ \cup V^-$ is homotopy equivalent to (E, θ) . Homotopy equivalent quadratic Poincaré complexes are cobordant, so the composite

$$\begin{aligned} UV : L_n(\mathbb{C}_X(\mathbb{A})) &\longrightarrow \varinjlim_b L_n^{J_b}(\Delta_b) \longrightarrow L_n(\mathbb{C}_X(\mathbb{A})) ; \\ (E, \theta) &\longrightarrow V^+ \cup V^- \end{aligned}$$

is the identity.

For any adjoining pair of n -dimensional quadratic Poincaré pairs in $\mathbb{C}_X(\mathbb{A})$

$$V^+ = (f^+ : C \longrightarrow D^+, (\delta\psi^+, \psi)) ,$$

$$V^- = (f^- : C \longrightarrow D^-, (\delta\psi^-, -\psi))$$

representing an element $(V^+, V^-) \in L_n^{J_b}(\Delta_b)$ let $V^+ \cup V^- = (E, \theta)$ be the union n -dimensional quadratic Poincaré complex in $\mathbb{C}_X(\mathbb{A})$. Let (V'^+, V'^-, h') be the splitting of (E, θ) obtained by the construction of 13.3 using the subcomplex

$$D = D^+ \subseteq E = C \left(\begin{pmatrix} f^+ \\ f^- \end{pmatrix} : C \longrightarrow D^+ \oplus D^- \right) .$$

The pair (V'^+, V'^-) is homotopy equivalent to (V^+, V^-) . Homotopy equivalent pairs are cobordant, so the composite

$$\begin{aligned} VU : \varinjlim_b L_n^{J_b}(\Delta_b) &\longrightarrow L_n(\mathbb{C}_X(\mathbb{A})) \longrightarrow \varinjlim_b L_n^{J_b}(\Delta_b) ; \\ (V^+, V^-) &\longrightarrow (V'^+, V'^-) \end{aligned}$$

is the identity.

Thus U, V are inverse isomorphisms, and the L -theory Mayer-Vietoris sequence is exact.

□

REMARK 14.5 The algebraic Poincaré transversality used in the proof of 14.4 can be developed further, giving a purely algebraic proof of Proposition 7.4.1 of Ranicki [63], and hence of the Mayer-Vietoris exact sequence of Cappell [15] for the L -groups of an amalgamated product of rings with involution $A = A^+ *_B A^-$

$$\begin{aligned} \dots &\longrightarrow L_n^J(B) \oplus \text{UNil}_{n+1} \longrightarrow L_n(A^+) \oplus L_n(A^-) \\ &\longrightarrow L_n(A^+ *_B A^-) \xrightarrow{\partial} L_{n-1}^J(B) \oplus \text{UNil}_n \longrightarrow \dots \end{aligned}$$

with $J = \ker(\tilde{K}_0(B) \longrightarrow \tilde{K}_0(A^+) \oplus \tilde{K}_0(A^-))$, and the analogue for an HNN polynomial extension $A = A^+ *_B [z, z^{-1}]$

$$\begin{aligned} \dots &\longrightarrow L_n^J(B) \oplus \text{UNil}_{n+1} \longrightarrow L_n(A^+) \longrightarrow L_n(A^+ *_B [z, z^{-1}]) \\ &\xrightarrow{\partial} L_{n-1}^J(B) \oplus \text{UNil}_n \longrightarrow \dots \end{aligned}$$

with $J = \ker(\tilde{K}_0(B) \longrightarrow \tilde{K}_0(A^+))$. See §16 below for the L -theory of the Laurent polynomial extension $A = B[z, z^{-1}]$. See Connolly and Koźniewski [20] for an expression of the UNil -groups as the L -groups of an additive category with involution.

□

COROLLARY 14.6 *For a compact polyhedron $Y \subseteq S^k$ expressed as a union $Y = Y^+ \cup Y^-$ of compact polyhedra $Y^+, Y^- \subseteq Y$ the quadratic L -groups of the bounded $O(Y)$ -graded category $\mathbb{C}_{O(Y)}(\mathbb{A})$ of a filtered additive category with involution \mathbb{A} fit into a Mayer-Vietoris exact sequence*

$$\begin{aligned} \dots &\longrightarrow L_n^J(\mathbb{P}_{O(Y^+ \cap Y^-)}(\mathbb{A})) \longrightarrow L_n(\mathbb{C}_{O(Y^+)}(\mathbb{A})) \oplus L_n(\mathbb{C}_{O(Y^-)}(\mathbb{A})) \\ &\longrightarrow L_n(\mathbb{C}_{O(Y)}(\mathbb{A})) \xrightarrow{\partial} L_{n-1}^J(\mathbb{P}_{O(Y^+ \cap Y^-)}(\mathbb{A})) \longrightarrow \dots, \end{aligned}$$

with

$$\begin{aligned} J &= \ker(\tilde{K}_0(\mathbb{P}_{O(Y^+ \cap Y^-)}(\mathbb{A})) \longrightarrow \tilde{K}_0(\mathbb{P}_{O(Y^+)}(\mathbb{A})) \oplus \tilde{K}_0(\mathbb{P}_{O(Y^-)}(\mathbb{A}))) \\ &= \text{im}(\tilde{\partial} : \text{Wh}_{\mathbb{C}_{O(Y)}}(\mathbb{A}) \longrightarrow \tilde{K}_0(\mathbb{P}_{O(Y^+ \cap Y^-)}(\mathbb{A}))) \\ &\subseteq \tilde{K}_0(\mathbb{P}_{O(Y^+ \cap Y^-)}(\mathbb{A})). \end{aligned}$$

PROOF Apply 14.4 to the expression of the open cone $X = O(Y)$ as a union $X = X^+ \cup X^-$ of the open cones of subpolyhedra $Y^+, Y^- \subseteq Y$, with $X^\pm = O(Y^\pm)$. By 7.1 the inclusions

$$X^+ \cap X^- = O(Y^+ \cap Y^-) \longrightarrow \mathcal{N}_b(X^+, X^-, X) \quad (b \geq 0)$$

are homotopy equivalences in the proper eventually Lipschitz category. The induced isomorphisms in the projective class groups

$$\tilde{K}_0(\mathbb{P}_{X^+ \cap X^-}(\mathbb{A})) \longrightarrow \tilde{K}_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \quad (b \geq 0)$$

send J to J_b , so that there are also induced isomorphisms in the L -groups

$$L_*^J(\mathbb{P}_{X^+ \cap X^-}(\mathbb{A})) \longrightarrow L_*^{J_b}(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \quad (b \geq 0),$$

and the direct limits in the exact sequence of 14.4 are given by

$$\lim_{\substack{\longrightarrow \\ b}} L_*^{J_b}(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) = L_*^J(\mathbb{P}_{X^+ \cap X^-}(\mathbb{A})).$$

□

In order to formulate the intermediate versions of 14.4 it is convenient to assume that the bounded categories $\mathbb{C}_W(\mathbb{A})$ for $W = X, X^+, X^-$ have flasque structures (as is the case for the open cones $W = O(Y), O(Y^+), O(Y^-)$ of 14.6) so that the intermediate L -groups $L_*^T(\mathbb{C}_W(\mathbb{A}))$ are defined for any $*$ -invariant subgroup $T \subseteq K_1(\mathbb{C}_W(\mathbb{A}))$.

COROLLARY 14.7 *For $X = X^+ \cup X^-$ and any $*$ -invariant subgroup $T \subseteq K_1(\mathbb{C}_X(\mathbb{A}))$ such that*

$$\ker(\partial : K_1(\mathbb{C}_X(\mathbb{A})) \longrightarrow \lim_{\substack{\longrightarrow \\ b}} \widetilde{K}_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}))) \subseteq T$$

there is defined a Mayer-Vietoris exact sequence of intermediate L -groups

$$\begin{aligned} \dots &\longrightarrow \lim_{\substack{\longrightarrow \\ b}} L_n^{\partial T}(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \\ &\longrightarrow L_n(\mathbb{C}_{X^+}(\mathbb{A})) \oplus L_n(\mathbb{C}_{X^-}(\mathbb{A})) \\ &\longrightarrow L_n^T(\mathbb{C}_X(\mathbb{A})) \xrightarrow{\partial} \lim_{\substack{\longrightarrow \\ b}} L_{n-1}^{\partial T}(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \longrightarrow \dots \end{aligned}$$

PROOF Exactly as for 14.4, the special case $T = K_1(\mathbb{C}_X(\mathbb{A}))$ with $\partial T = \lim_{\substack{\longrightarrow \\ b}} J_b$.

□

Given $*$ -invariant subgroups

$$S^+ \subseteq K_1(\mathbb{C}_{X^+}(\mathbb{A})), \quad S^- \subseteq K_1(\mathbb{C}_{X^-}(\mathbb{A}))$$

define the $*$ -invariant subgroups

$$\begin{aligned} R_0 &= \ker(\lim_{\substack{\longrightarrow \\ b}} K_1(\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}))) \\ &\longrightarrow K_1(\mathbb{C}_{X^+}(\mathbb{A})) \oplus K_1(\mathbb{C}_{X^-}(\mathbb{A})), \\ R &= \ker(\lim_{\substack{\longrightarrow \\ b}} K_1(\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}))) \\ &\longrightarrow K_1(\mathbb{C}_{X^+}(\mathbb{A}))/S^+ \oplus K_1(\mathbb{C}_{X^-}(\mathbb{A}))/S^- \\ &\subseteq \lim_{\substack{\longrightarrow \\ b}} K_1(\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})), \\ T &= \text{im}(S^+ \oplus S^- \longrightarrow K_1(\mathbb{C}_X(\mathbb{A}))) \subseteq K_1(\mathbb{C}_X(\mathbb{A})) \end{aligned}$$

such that there is defined an exact sequence

$$0 \longrightarrow R/R_0 \longrightarrow S^+ \oplus S^- \longrightarrow T \longrightarrow 0 .$$

COROLLARY 14.8 *The intermediate quadratic L -groups fit into a Mayer-Vietoris exact sequence*

$$\begin{aligned} \dots \longrightarrow \varinjlim_b L_n^R(\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) &\longrightarrow L_n^{S^+}(\mathbb{C}_{X^+}(\mathbb{A})) \oplus L_n^{S^-}(\mathbb{C}_{X^-}(\mathbb{A})) \\ &\longrightarrow L_n^T(\mathbb{C}_X(\mathbb{A})) \xrightarrow{\partial} \varinjlim_b L_{n-1}^R(\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \longrightarrow \dots \end{aligned}$$

PROOF The proof of 14.4 shows that for any n -dimensional quadratic Poincaré complex (E, θ) in $\mathbb{C}_X(\mathbb{A})$ such that the torsion

$$\tau(E, \theta) \in K_1(\mathbb{C}_X(\mathbb{A}))$$

has reduced end invariant

$$\tilde{\partial}\tau(E, \theta) = 0 \in \varinjlim_b \tilde{K}_0(\mathbb{P}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A}))$$

there exists a Poincaré splitting (V^+, V^-, h) with V^+, V^- defined in $\mathbb{C}_{\mathcal{N}_b(X^\pm, X)}(\mathbb{A})$ for some $b \geq 0$, and $V^+ \cap V^-$ consequently defined in $\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$. (Specifically, replace $\partial(E/E^+)$ by any chain equivalent finite chain complex in $\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})$, for some $b \geq 0$.) The torsions

$$\begin{aligned} \tau(V^\pm) &\in K_1(\mathbb{C}_{X^\pm}(\mathbb{A})) , \quad \tau(V^+ \cap V^-) \in K_1(\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) , \\ \tau(h) , \quad \tau(V^+ \cup V^-) &\in K_1(\mathbb{C}_X(\mathbb{A})) \end{aligned}$$

are related by

$$\begin{aligned} \tau(E, \theta) - \tau(V^+ \cup V^-) &= \tau(h) + (-)^n \tau(h)^* , \\ \tau(V^+ \cup V^-) &= \tau(V^+) + (-)^n \tau(V^-)^* \\ &= \tau(V^-) + (-)^n \tau(V^+)^* \in K_1(\mathbb{C}_X(\mathbb{A})) , \\ \tau(V^+ \cap V^-) &= \tau(V^\pm) + (-)^{n-1} \tau(V^\pm)^* \in K_1(\mathbb{C}_{X^\pm}(\mathbb{A})) . \end{aligned}$$

If (E, θ) is such that $\tau(E, \theta) \in T$ then $\tilde{\partial}\tau(E, \theta) = 0$ and there exists such a Poincaré splitting (V^+, V^-, h) with

$$\begin{aligned} \tau(h) &\in T \subseteq K_1(\mathbb{C}_X(\mathbb{A})) , \\ \tau(V^\pm) &\in S^\pm \subseteq K_1(\mathbb{C}_{X^\pm}(\mathbb{A})) , \\ \tau(V^+ \cap V^-) &\in R \subseteq K_1(\mathbb{C}_{X^+ \cap X^-}(\mathbb{A})) . \end{aligned}$$

Working as in the proof of 14.4 define the cobordism groups $L_*^{S^+, S^-}(\Delta_b)$ of adjoining cobordisms (V^+, V^-) with $\tau(V^\pm) \in S^\pm$ to fit into the exact

sequence

$$\begin{aligned} \dots &\longrightarrow L_n^R(\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \\ &\longrightarrow L_n^{S^+}(\mathbb{C}_{X^+}(\mathbb{A})) \oplus L_n^{S^-}(\mathbb{C}_{X^-}(\mathbb{A})) \longrightarrow L_n^{S^+, S^-}(\Delta_b) \\ &\xrightarrow{\partial} L_{n-1}^R(\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \longrightarrow \dots, \end{aligned}$$

pass to the limits as $b \rightarrow \infty$ to obtain an exact sequence

$$\begin{aligned} \dots &\longrightarrow \varinjlim_b L_n^R(\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \\ &\longrightarrow L_n^{S^+}(\mathbb{C}_{X^+}(\mathbb{A})) \oplus L_n^{S^-}(\mathbb{C}_{X^-}(\mathbb{A})) \longrightarrow \varinjlim_b L_n^{S^+, S^-}(\Delta_b) \\ &\xrightarrow{\partial} \varinjlim_b L_{n-1}^R(\mathbb{C}_{\mathcal{N}_b(X^+, X^-, X)}(\mathbb{A})) \longrightarrow \dots, \end{aligned}$$

and define inverse isomorphisms

$$\begin{aligned} U &: \varinjlim_b L_n^{S^+, S^-}(\Delta_b) \longrightarrow L_n^T(\mathbb{C}_X(\mathbb{A})) ; (V^+, V^-) \longrightarrow V^+ \cup V^-, \\ V &: L_n^T(\mathbb{C}_X(\mathbb{A})) \longrightarrow \varinjlim_b L_n^{S^+, S^-}(\Delta_b) ; (E, \theta) \longrightarrow (V^+, V^-). \end{aligned}$$

□

§15. L -theory of $\mathbb{C}_1(\mathbb{A})$

The identification $K_1(\mathbb{C}_1(\mathbb{A})) = K_0(\mathbb{P}_0(\mathbb{A}))$ of §7 will now be generalized to L -theory, and it will be proved that

$$L_n(\mathbb{C}_1(\mathbb{A})) = L_{n-1}(\mathbb{P}_0(\mathbb{A}))$$

for any filtered additive category with involution \mathbb{A} .

The isomorphisms $L_{n-1}(\mathbb{P}_0(\mathbb{A})) \longrightarrow L_n(\mathbb{C}_1(\mathbb{A}))$ will be defined by the following construction, which is a direct generalization of the product (Ranicki [61], [62]) of a quadratic complex and the symmetric Poincaré complex of S^1 .

DEFINITION 15.1 (i) Given a finite chain complex (C, p) in $\mathbb{P}_0(\mathbb{A})$ define a finite chain complex in $\mathbb{A}[z, z^{-1}]$

$$(C, p) \otimes \mathbb{R} = C(1 - pz : C[z, z^{-1}] \longrightarrow C[z, z^{-1}]) .$$

(ii) Given an $(n-1)$ -dimensional quadratic (Poincaré) complex (C, p, ψ) in $\mathbb{P}_0(\mathbb{A})$ define an n -dimensional quadratic (Poincaré) complex in $\mathbb{A}[z, z^{-1}]$

$$(C, p, \psi) \otimes \mathbb{R} = ((C, p) \otimes \mathbb{R}, \psi \otimes 1 \in Q_n((C, p) \otimes \mathbb{R}))$$

by

$$\begin{aligned}
 (\psi \otimes \mathbb{R})_s &= \begin{pmatrix} 0 & (-)^s z \psi_s \\ (-)^{n-r-1} \psi_s & (-)^{n-r+s} T \psi_{s+1} \end{pmatrix} \\
 : ((C, p) \otimes \mathbb{R})^{n-r-s} &= C^{n-r-s}[z, z^{-1}] \oplus C^{n-r-s-1}[z, z^{-1}] \\
 \longrightarrow ((C, p) \otimes \mathbb{R})_r &= C_r[z, z^{-1}] \oplus C_{r-1}[z, z^{-1}] \quad (s \geq 0) .
 \end{aligned}$$

□

In order to apply the L -theory Mayer-Vietoris exact sequences of §14 express the bounded \mathbb{R} -graded category as

$$\mathbb{C}_{\mathbb{R}}(\mathbb{A}) = \mathbb{C}_{O(X^+ \cup X^-)}(\mathbb{A}) ,$$

with

$$X^+ \cup X^- = S^0 = \{+1, -1\} , \quad X^\pm = \{\pm 1\} , \quad O(X^\pm) = \mathbb{R}^\pm ,$$

$$O(X^+ \cap X^-) = O(X^+) \cap O(X^-) = \{0\} ,$$

$$O(X^+ \cup X^-) = O(X^+) \cup O(X^-) = \mathbb{R}$$

as in §6, using the equivalence $\mathbb{C}_1(\mathbb{A}) \approx \mathbb{C}_{\mathbb{R}}(\mathbb{A})$ to identify

$$L_*(\mathbb{C}_1(\mathbb{A})) = L_*(\mathbb{C}_{\mathbb{R}}(\mathbb{A})) .$$

A Mayer-Vietoris presentation of a finite chain complex E in $\mathbb{C}_1(\mathbb{A})$ is the exact sequence

$$\mathbb{E} : 0 \longrightarrow E^+ \cap E^- \xrightarrow{i} E^+ \oplus E^- \xrightarrow{j} E \longrightarrow 0$$

determined by subcomplexes $E^+, E^- \subseteq E$ such that for some $b \geq 0$

$$E_r^+ \subseteq \sum_{j=-b}^{\infty} E_r(j) , \quad E_r^- \subseteq \sum_{j=-\infty}^b E_r(j) \quad (r \geq 0) .$$

THEOREM 15.2 *The quadratic L -groups of the bounded \mathbb{Z} -graded category $\mathbb{C}_1(\mathbb{A})$ are such that up to natural isomorphism*

$$L_n(\mathbb{C}_1(\mathbb{A})) = L_{n-1}(\mathbb{P}_0(\mathbb{A})) \quad (n \in \mathbb{Z}) .$$

The map

$$B : L_n(\mathbb{C}_1(\mathbb{A})) = L_n(\mathbb{C}_{\mathbb{R}}(\mathbb{A})) \longrightarrow L_{n-1}(\mathbb{P}_0(\mathbb{A})) ;$$

$$(E, \theta) \longrightarrow \partial((E, \theta)/E^+)$$

defined using any Mayer-Vietoris presentation \mathbb{E} of E is an isomorphism with inverse

$$B^{-1} = - \otimes \mathbb{R} : L_{n-1}(\mathbb{P}_0(\mathbb{A})) \longrightarrow L_n(\mathbb{C}_1(\mathbb{A})) ;$$

$$(C, p, \psi) \longrightarrow (C, p, \psi) \otimes \mathbb{R} .$$

PROOF The natural flasque structures on the categories $\mathbb{C}_{O(X^\pm)}(\mathbb{A}) = \mathbb{C}_{\mathbb{R}^\pm}(\mathbb{A})$ are compatible with the involutions, so that

$$L_*(\mathbb{C}_{\mathbb{R}^\pm}(\mathbb{A})) = 0 .$$

The exact sequence of 14.4 includes

$$0 \longrightarrow L_n(\mathbb{C}_1(\mathbb{A})) \xrightarrow{\partial} L_{n-1}(\mathbb{P}_0(\mathbb{A})) \longrightarrow 0 ,$$

so that $B = \partial$ is an isomorphism. For any $(n-1)$ -dimensional quadratic Poincaré complex (C, p, ψ) in $\mathbb{P}_0(\mathbb{A})$ let

$$(E, \theta) = (C, p, \psi) \otimes \mathbb{R}$$

and define a Mayer-Vietoris presentation \mathbb{E} for E by

$$E^+ = C(1-zp : C[z] \longrightarrow C[z]) , \quad E^- = C(1-zp : C[z^{-1}] \longrightarrow zC[z^{-1}]) .$$

The $\mathbb{C}_0(\mathbb{A})$ -finitely dominated $(n-1)$ -dimensional quadratic Poincaré complex defined in $\mathbb{C}_1(\mathbb{A})$ by $\partial((E, \theta)/E^+)$ is homotopy equivalent to $(C, p, \psi) \otimes \mathbb{R}$, so that

$$B(E, \theta) = (C, p, \psi) \in L_{n-1}(\mathbb{P}_0(\mathbb{A}))$$

and

$$B^{-1}(C, p, \psi) = (E, \theta) = (C, p, \psi) \otimes \mathbb{R} \in L_n(\mathbb{C}_1(\mathbb{A})) .$$

□

The isomorphism $B : K_1(\mathbb{C}_1(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_0(\mathbb{A}))$ of §7 is such that

$$B* = -*B : K_1(\mathbb{C}_1(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_0(\mathbb{A})) ,$$

since the inverse

$$B^{-1} = \bar{B} : K_0(\mathbb{P}_0(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_1(\mathbb{A})) ;$$

$$[M, p] \longrightarrow \tau(1 - p + zp : M[z, z^{-1}] \longrightarrow M[z, z^{-1}])$$

is such that

$$\begin{aligned} * \bar{B}[M, p] &= \tau(1 - p^* + z^{-1}p^* : M^*[z, z^{-1}] \longrightarrow M^*[z, z^{-1}]) \\ &= \tau((1 - p^* + zp^*)^{-1} : M^*[z, z^{-1}] \longrightarrow M^*[z, z^{-1}]) \\ &= -\bar{B} * [M, p] \in K_1(\mathbb{A}[z, z^{-1}]) . \end{aligned}$$

It follows that

$$\hat{H}^*(\mathbb{Z}_2; K_1(\mathbb{C}_1(\mathbb{A}))) = \hat{H}^{*-1}(\mathbb{Z}_2; K_0(\mathbb{P}_0(\mathbb{A}))) .$$

COROLLARY 15.3 (i) *The intermediate quadratic L-groups of $\mathbb{C}_1(\mathbb{A})$ are such that up to natural isomorphism*

$$L_n^S(\mathbb{C}_1(\mathbb{A})) = L_{n-1}^{\partial S}(\mathbb{P}_0(\mathbb{A}))$$

*for any *-invariant subgroup $S \subseteq K_1(\mathbb{C}_1(\mathbb{A}))$ (resp. $Wh(\mathbb{C}_1(\mathbb{A}))$), with $\partial S \subseteq K_0(\mathbb{P}_0(\mathbb{A}))$ (resp. $\tilde{K}_0(\mathbb{P}_0(\mathbb{A}))$) the isomorphic image of S in the class group.*

(ii) *For any *-invariant subgroups $S \subseteq T \subseteq K_1(\mathbb{C}_1(\mathbb{A}))$ the exact sequence*

$$\begin{aligned} \dots \longrightarrow L_n^S(\mathbb{C}_1(\mathbb{A})) \longrightarrow L_n^T(\mathbb{C}_1(\mathbb{A})) \longrightarrow \hat{H}^n(\mathbb{Z}_2; T/S) \\ \longrightarrow L_{n-1}^S(\mathbb{C}_1(\mathbb{A})) \longrightarrow \dots \end{aligned}$$

is naturally isomorphic to the exact sequence

$$\begin{aligned} \dots \longrightarrow L_{n-1}^{\partial S}(\mathbb{P}_0(\mathbb{A})) \longrightarrow L_{n-1}^{\partial T}(\mathbb{P}_0(\mathbb{A})) \longrightarrow \widehat{H}^{n-1}(\mathbb{Z}_2; \partial T / \partial S) \\ \longrightarrow L_{n-2}^{\partial S}(\mathbb{P}_0(\mathbb{A})) \longrightarrow \dots \end{aligned}$$

Similarly for $S \subseteq T \subseteq Wh(\mathbb{C}_1(\mathbb{A}))$.

PROOF (i) As for 15.2 (the special case $S = K_1(\mathbb{C}_1(\mathbb{A}))$), but using 14.6 and 14.7 instead of 14.4.

(ii) Immediate from (i). □

In particular, 15.3 gives

$$\begin{aligned} L_n^{\{0\} \subseteq K_1(\mathbb{C}_1(\mathbb{A}))}(\mathbb{C}_1(\mathbb{A})) &= L_{n-1}^{\{0\} \subseteq K_0(\mathbb{P}_0(\mathbb{A}))}(\mathbb{P}_0(\mathbb{A})) = L_{n-1}^r(\mathbb{A}), \\ L_n^s(\mathbb{C}_1(\mathbb{A})) &= L_n^{\{0\} \subseteq Wh(\mathbb{C}_1(\mathbb{A}))}(\mathbb{C}_1(\mathbb{A})) \\ &= L_{n-1}^{\text{im}(K_0(\mathbb{A})) \subseteq K_0(\mathbb{P}_0(\mathbb{A}))}(\mathbb{P}_0(\mathbb{A})) = L_{n-1}(\mathbb{A}) \end{aligned}$$

with L_*^r the round L -groups.

REMARK 15.4 The total surgery obstruction groups $\mathbb{S}_*(X)$ of a space X are defined in Ranicki [60], [69] to fit into an exact sequence

$$\dots \longrightarrow H_n(X; \mathbb{L}_.) \longrightarrow L_n(\mathbb{Z}[\pi]) \longrightarrow \mathbb{S}_n(X) \longrightarrow H_{n-1}(X; \mathbb{L}_.) \longrightarrow \dots$$

with $\pi = \pi_1(X)$ and $\pi_*(\mathbb{L}_.) = L_*(\mathbb{Z})$. The total surgery obstruction $s(X) \in \mathbb{S}_n(X)$ of a finite n -dimensional geometric Poincaré complex X with $n \geq 5$ is such that $s(X) = 0$ if and only if X is homotopy equivalent to a compact n -dimensional topological manifold. The total projective surgery obstruction groups $\mathbb{S}_*^p(X)$ are defined to fit into an exact sequence

$$\dots \longrightarrow H_n(X; \mathbb{L}_.) \longrightarrow L_n^p(\mathbb{Z}[\pi]) \longrightarrow \mathbb{S}_n^p(X) \longrightarrow H_{n-1}(X; \mathbb{L}_.) \longrightarrow \dots$$

with a Rothenberg-type sequence

$$\begin{aligned} \dots \longrightarrow \widehat{H}^{n+1}(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi])) \longrightarrow \mathbb{S}_n(X) \longrightarrow \mathbb{S}_n^p(X) \\ \longrightarrow \widehat{H}^n(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi])) \longrightarrow \dots \end{aligned}$$

The following conditions on a finitely dominated n -dimensional geometric Poincaré complex X with $n \geq 5$ are equivalent:

- (i) the total projective surgery obstruction $s^p(X) \in \mathbb{S}_n^p(X)$ (Pedersen and Ranicki [52]) is such that $s^p(X) = 0$,
- (ii) $X \times S^1$ is homotopy equivalent to a compact $(n+1)$ -dimensional topological manifold,
- (iii) there exists an open $(n+1)$ -dimensional manifold W with an \mathbb{R} -bounded homotopy equivalence $h : W \simeq X \times \mathbb{R}$.

Thus if X is a finite n -dimensional geometric Poincaré complex with

$s^p(X) = 0$ the total surgery obstruction is such that

$$s(X) \in \ker(\mathbb{S}_n(X) \longrightarrow \mathbb{S}_n^p(X)) = \text{im}(\widehat{H}^{n+1}(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi])) \longrightarrow \mathbb{S}_n(X)) .$$

The corresponding element of $\widehat{H}^{n+1}(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi]))$ is determined by a choice of \mathbb{R} -bounded homotopy equivalence $h : W \simeq X \times \mathbb{R}$, and may be regarded as either the class of the Siebenmann end obstruction $[W]_+ \in \widetilde{K}_0(\mathbb{Z}[\pi])$ or the bounded Whitehead torsion $\tau(h) \in Wh(\mathbb{C}_1(\mathbb{Z}[\pi]))$, as follows.

In the first instance, note that for any finite n -dimensional geometric Poincaré complex X the isomorphism $B : Wh(\mathbb{C}_1(\mathbb{Z}[\pi])) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\pi])$ sends the torsion $\tau(X \times \mathbb{R})$ to

$$\begin{aligned} B\tau(X \times \mathbb{R}) &= [X \times \mathbb{R}]_+ + (-)^n([X \times \mathbb{R}]_+)^* = 0 \in \widetilde{K}_0(\mathbb{Z}[\pi]) \\ &(\pi = \pi_1(X)) , \end{aligned}$$

so that

$$\tau(X \times \mathbb{R}) = 0 \in Wh(\mathbb{C}_1(\mathbb{Z}[\pi]))$$

and $X \times \mathbb{R}$ is a simple $(n+1)$ -dimensional \mathbb{R} -bounded geometric Poincaré complex. Given an \mathbb{R} -bounded homotopy equivalence $h : W \simeq X \times \mathbb{R}$ make it transverse regular at $X \times \{0\} \subset X \times \mathbb{R}$, so that there is defined a normal map

$$(f, b) = h| : M = h^{-1}(X \times \{0\}) \longrightarrow X$$

from a compact n -dimensional manifold M . By definition, h splits if there exists a normal bordism

$$(g; h, h') : (V; W, W') \longrightarrow X \times \mathbb{R} \times ([0, 1]; \{0\}, \{1\})$$

with $(V; W, W')$ a bounded h -cobordism and $h' : W' \longrightarrow X \times \mathbb{R}$ an \mathbb{R} -bounded homotopy equivalence such that the restriction

$$(f', b') = h'| : M' = h'^{-1}(X \times \{0\}) \longrightarrow X$$

is a homotopy equivalence. By analogy with the splitting theorems of Farrell and Hsiang [24] and Wall [84, §12B] the obstruction to splitting h for $n \geq 5$ can be identified with the Tate \mathbb{Z}_2 -cohomology class of the bounded torsion

$$\tau(h) \in \widehat{H}^n(\mathbb{Z}_2; Wh(\mathbb{C}_1(\mathbb{Z}[\pi]))) ,$$

using the isomorphic exact sequences

$$\begin{aligned} \dots &\longrightarrow L_{n+2}^s(\mathbb{C}_1(\mathbb{Z}[\pi])) \longrightarrow L_{n+2}(\mathbb{C}_1(\mathbb{Z}[\pi])) \\ &\longrightarrow \widehat{H}^{n+2}(\mathbb{Z}_2; Wh(\mathbb{C}_1(\mathbb{Z}[\pi]))) \longrightarrow L_{n+1}^s(\mathbb{C}_1(\mathbb{Z}[\pi])) \longrightarrow \dots , \\ \dots &\longrightarrow L_{n+1}(\mathbb{Z}[\pi]) \longrightarrow L_{n+1}^p(\mathbb{Z}[\pi]) \\ &\longrightarrow \widehat{H}^{n+1}(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi])) \longrightarrow L_n(\mathbb{Z}[\pi]) \longrightarrow \dots . \end{aligned}$$

The primary obstruction to splitting h is the simple bounded surgery obstruction of Ferry and Pedersen [28], which is determined by the torsion $\tau(h) \in Wh(\mathbb{C}_1(\mathbb{Z}[\pi]))$

$$\sigma_*^s(h) = [\tau(h)] \in \text{im}(\widehat{H}^{n+2}(\mathbb{Z}_2; Wh(\mathbb{C}_1(\mathbb{Z}[\pi]))) \longrightarrow L_{n+1}^s(\mathbb{C}_1(\mathbb{Z}[\pi])))$$

and is just the finite surgery obstruction of (f, b)

$$\sigma_*^s(h) = \sigma_*(f, b) \in L_{n+1}^s(\mathbb{C}_1(\mathbb{Z}[\pi])) = L_n(\mathbb{Z}[\pi]) .$$

Working as in Pedersen and Ranicki [52] it is possible to identify X up to homotopy equivalence with the homotopy inverse limit of a system of neighbourhoods of the end $W^+ = h^{-1}(\mathbb{R}^+)$ of W

$$X \simeq \varprojlim_k h^{-1}([k, \infty))$$

and to extend (f, b) to a normal map of finitely dominated n -dimensional geometric Poincaré bordisms

$$(e, a) : (W^+; M, X) \longrightarrow X \times ([0, 1]; \{0\}, \{1\})$$

with $e| = \text{id} : X \longrightarrow X$. The finite bounded surgery obstruction of h and the projective surgery obstruction of (f, b) are both zero

$$\sigma_*(h) = \sigma_*^p(f, b) = 0 \in L_{n+1}(\mathbb{C}_1(\mathbb{Z}[\pi])) = L_n^p(\mathbb{Z}[\pi]) .$$

The finite surgery obstruction of (f, b) is determined by the bounded torsion

$$\tau(h) \in Wh(\mathbb{C}_1(\mathbb{Z}[\pi])) ,$$

which is the Siebenmann end obstruction of W^+

$$B\tau(h) = [W]_+ = [W^+] \in \widetilde{K}_0(\mathbb{Z}[\pi]) .$$

Both W and $X \times \mathbb{R}$ are simple \mathbb{R} -bounded geometric Poincaré complexes

$$\tau(W) = \tau(X \times \mathbb{R}) = 0 \in Wh(\mathbb{C}_1(\mathbb{Z}[\pi])) ,$$

and

$$\tau(X \times \mathbb{R}) - \tau(W) = \tau(h) + (-)^{n+1} \tau(h)^* \in Wh(\mathbb{C}_1(\mathbb{Z}[\pi])) ,$$

$$B\tau(h) = -[W]_+ \in \widetilde{K}_0(\mathbb{Z}[\pi]) ,$$

so that

$$B(\tau(W) - \tau(X \times \mathbb{R})) = [W]_+ + (-)^{n-1}([W]_+)^* = 0 \in \widetilde{K}_0(\mathbb{Z}[\pi]) .$$

The finite surgery obstruction of (f, b) is

$$\begin{aligned} \sigma_*(f, b) &= [W]_+ \in \text{im}(\widehat{H}^{n+1}(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi])) \longrightarrow L_n(\mathbb{Z}[\pi])) \\ &= \ker(L_n(\mathbb{Z}[\pi]) \longrightarrow L_n^p(\mathbb{Z}[\pi])) . \end{aligned}$$

If this vanishes then (f, b) is normal bordant to a homotopy equivalence; let

$$(L; M, M') \longrightarrow X \times ([0, 1]; \{0\}, \{1\})$$

be such a normal bordism, restricting to a homotopy equivalence $M' \simeq X$. The open n -dimensional manifold

$$W' = W^- \cup_M L \cup_{M'} -L \cup_M W^+$$

is equipped with a split bounded homotopy equivalence $h' : W' \longrightarrow X$, related to h by a normal bordism

$$(g, c) : (V; W, W') \longrightarrow X \times ([0, 1]; \{0\}, \{1\})$$

such that the finite bounded rel ∂ surgery obstruction

$$\sigma_*(g, c) \in L_{n+2}(\mathbb{C}_1(\mathbb{Z}[\pi])) = L_{n+1}^p(\mathbb{Z}[\pi])$$

has image

$$[\sigma_*(g, c)] = \tau(h) = [W]_+$$

$$\in \hat{H}^{n+2}(\mathbb{Z}_2; Wh(\mathbb{C}_1(\mathbb{Z}[\pi]))) = \hat{H}^{n+1}(\mathbb{Z}_2; \tilde{K}_0(\mathbb{Z}[\pi])) .$$

For $n \geq 5$ it is possible to make a choice of (L, M, M') with $(V; W, W')$ a bounded h -cobordism if and only if

$$\begin{aligned} \sigma_*(g, c) \in \text{im}(L_{n+2}^s(\mathbb{C}_1(\mathbb{Z}[\pi])) \longrightarrow L_{n+2}(\mathbb{C}_1(\mathbb{Z}[\pi]))) \\ = \text{im}(L_{n+1}(\mathbb{Z}[\pi])) \longrightarrow L_{n+1}^p(\mathbb{Z}[\pi])) , \end{aligned}$$

giving the secondary obstruction. (This is just the appropriate version of the trick used by Browder [10] for dealing with an embedding problem by first solving it up to cobordism). In the bounded version of the terminology of Wall [84, §12B] the bounded splitting obstruction is

$$\begin{aligned} \tau(h) = [W]_+ \in LS_n(\Phi) = \hat{H}^{n+2}(\mathbb{Z}_2; Wh(\mathbb{C}_1(\mathbb{Z}[\pi]))) \\ = \hat{H}^{n+1}(\mathbb{Z}_2; \tilde{K}_0(\mathbb{Z}[\pi])) . \end{aligned}$$

□

REMARK 15.5 By the realization theorem of Ferry and Pedersen [28] for a finitely presented group π and $n \geq 6$ every element $x \in L_n(\mathbb{C}_1(\mathbb{Z}[\pi]))$ is the bounded surgery obstruction $x = \sigma_*(e, a)$ of an n -dimensional bounded normal map

$$(e, a) : (W, \partial W) \longrightarrow (X, \partial X) \times \mathbb{R}$$

with $(W, \partial W)$ an \mathbb{R} -bounded n -dimensional manifold, $(X, \partial X)$ a compact $(n-1)$ -dimensional manifold with boundary such that $\pi_1(\partial X) = \pi_1(X) = \pi$ and $h = e| : \partial W \longrightarrow \partial X \times \mathbb{R}$ an \mathbb{R} -bounded homotopy equivalence. Making (e, a) transverse regular at $(X, \partial X) \times \{0\} \subset (X, \partial X) \times \mathbb{R}$ there is obtained an $(n-1)$ -dimensional normal map of pairs

$$(f, b) = e| : (M, \partial M) = e^{-1}((X, \partial X) \times \{0\}) \longrightarrow (X, \partial X) .$$

As in 15.4 it is possible to extend $\partial f : \partial M \longrightarrow \partial X$ to a normal map of finitely dominated $(n-1)$ -dimensional geometric Poincaré bordisms

$$(g, c) : (\partial W^+; \partial M, \partial X) \longrightarrow \partial X \times ([0, 1]; \{0\}, \{1\})$$

with $g| = \text{id.} : \partial X \longrightarrow \partial X$, and $\partial W^+ = e^{-1}(\partial X \times \mathbb{R}^+) \subset \partial W$. The K -theory isomorphism

$$B : Wh(\mathbb{C}_1(\mathbb{Z}[\pi])) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\pi])$$

sends the bounded torsion $\tau(h) \in Wh(\mathbb{C}_1(\mathbb{Z}[\pi]))$ to the end obstruction

$$B\tau(h) = [\partial W^+] \in \widetilde{K}_0(\mathbb{Z}[\pi]) .$$

The L -theory isomorphism

$$B : L_n(\mathbb{C}_1(\mathbb{Z}[\pi])) \longrightarrow L_{n-1}^p(\mathbb{Z}[\pi])$$

sends the bounded surgery obstruction $x = \sigma_*(e, a) \in L_n(\mathbb{C}_1(\mathbb{Z}[\pi]))$ to the projective surgery obstruction of the normal map of finitely dominated $(n-1)$ -dimensional geometric Poincaré pairs

$$(f', b') = (f, b) \cup (g, c) : (M \cup \partial W^+, \partial X) \longrightarrow (X, \partial X) ,$$

with the identity on the boundary, that is

$$B\sigma_*(e, a) = \sigma_*^p(f', b') \in L_{n-1}^p(\mathbb{Z}[\pi]) .$$

□

§16. L -theory of $\mathbb{A}[z, z^{-1}]$

The splitting of $K_1(\mathbb{A}[z, z^{-1}])$ obtained in §10 will now be generalized to L -theory, and it will be proved that up to natural isomorphism

$$L_n(\mathbb{A}[z, z^{-1}]) = L_n(\mathbb{A}) \oplus L_{n-1}(\mathbb{P}_0(\mathbb{A}))$$

for any additive category with involution \mathbb{A} .

The duality involutions act on the split exact sequence of §10

$$\begin{aligned} 0 \longrightarrow K_1(\mathbb{A}) \xrightarrow{i_!} K_1(\mathbb{A}[z, z^{-1}]) &\xrightarrow{B \oplus N_+ \oplus N_-} \\ &K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \longrightarrow 0 \end{aligned}$$

by

$$i_!^* = *i_! , \quad B^* = - * B , \quad N_{\pm}^* = *N_{\mp} ,$$

so there is induced a split exact sequence of the Tate \mathbb{Z}_2 -cohomology groups

$$\begin{aligned} 0 \longrightarrow \widehat{H}^n(\mathbb{Z}_2; K_1(\mathbb{A})) &\xrightarrow{i_!} \widehat{H}^n(\mathbb{Z}_2; K_1(\mathbb{A}[z, z^{-1}])) \\ &\xrightarrow{B} \widehat{H}^{n-1}(\mathbb{Z}_2; K_0(\mathbb{P}_0(\mathbb{A}))) \longrightarrow 0 . \end{aligned}$$

Similarly for the Whitehead group $Wh(\mathbb{A}[z, z^{-1}])$, with K_0 replaced by \widetilde{K}_0 . In particular, for any group π there is defined a split exact sequence

$$\begin{aligned} 0 \longrightarrow \widehat{H}^n(\mathbb{Z}_2; Wh(\pi)) &\xrightarrow{i_!} \widehat{H}^n(\mathbb{Z}_2; Wh(\pi \times \mathbb{Z})) \\ &\xrightarrow{B} \widehat{H}^{n-1}(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi])) \longrightarrow 0 . \end{aligned}$$

The geometric model for the L -theory splitting is the Künneth theorem for manifold bordism

$$\Omega_n(X \times S^1) = \Omega_n(X) \oplus \Omega_{n-1}(X)$$

and its analogues for geometric Poincaré bordism. Here, X is any reasonable space (such as a finite CW complex) and $\Omega_n(X)$ is the bordism group of pairs (M, f) with M a compact n -dimensional manifold and $f : M \rightarrow X$ a reference map. The manifolds can be in any category with transversality, such as smooth, PL or topological, and can be taken to be oriented if required. The splitting of the manifold bordism groups is given by a direct sum system

$$\Omega_n(X) \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{j} \end{array} \Omega_n(X \times S^1) \begin{array}{c} \xrightarrow{B} \\ \xleftarrow{C} \end{array} \Omega_{n-1}(X)$$

with

$$\begin{aligned} i : \Omega_n(X) &\longrightarrow \Omega_n(X \times S^1) ; (M, f) \longrightarrow (M, qf) \\ (q : X &\longrightarrow X \times S^1 ; x \longrightarrow (x, *) \text{ for a base point } * \in S^1) , \\ B : \Omega_n(X \times S^1) &\longrightarrow \Omega_{n-1}(X) ; (N, g) \longrightarrow (g^{-1}(X \times \{*\}), |g|) \\ (\text{assuming } g : N &\longrightarrow X \times S^1 \text{ transverse at } X \times \{*\} \subset X \times S^1) , \\ j : \Omega_n(X \times S^1) &\longrightarrow \Omega_n(X) ; (N, g) \longrightarrow (N, pg) \\ (p = \text{projection} : X \times S^1 &\longrightarrow X) , \\ C : \Omega_{n-1}(X) &\longrightarrow \Omega_n(X \times S^1) ; (M, f) \longrightarrow (M \times S^1, f \times 1) . \end{aligned}$$

Let $\Omega_n^h(X)$ (resp. $\Omega_n^p(X)$) be the bordism group of pairs (M, f) with M a finite (resp. finitely dominated) n -dimensional geometric Poincaré complex and $f : M \rightarrow X$ a reference map. For $n \geq 5$ the finite and finitely dominated geometric Poincaré bordism groups are related by the exact sequence of Pedersen and Ranicki [52]

$$\begin{aligned} \dots &\longrightarrow \Omega_n^h(X) \longrightarrow \Omega_n^p(X) \longrightarrow \hat{H}^n(\mathbb{Z}_2; \tilde{K}_0(\mathbb{Z}[\pi])) \\ &\longrightarrow \Omega_{n-1}^h(X) \longrightarrow \dots \quad (\pi = \pi_1(X)) . \end{aligned}$$

The map

$$\Omega_n^p(X) \longrightarrow \hat{H}^n(\mathbb{Z}_2; \tilde{K}_0(\mathbb{Z}[\pi])) ; (M, f) \longrightarrow [M]$$

sends a finitely dominated Poincaré bordism class to the Tate \mathbb{Z}_2 -cohomology of its projective class, and the map

$$\hat{H}^n(\mathbb{Z}_2; \tilde{K}_0(\mathbb{Z}[\pi])) \longrightarrow \Omega_{n-1}^h(X) ; x \longrightarrow (N, g)$$

sends the Tate \mathbb{Z}_2 -cohomology of a self $(-)^n$ -dual projective class

$$x = (-)^n x^* \in \tilde{K}_0(\mathbb{Z}[\pi])$$

to the bordism class (N, g) of any finite $(n-1)$ -dimensional Poincaré complex N with a map $g : N \rightarrow X$ for which there exists a finitely

dominated Poincaré null-bordism $(\delta N, \delta g)$ with projective class $[\delta N] = x$. Geometric Poincaré complexes do not have transversality: if N is a finite n -dimensional geometric Poincaré complex and $g : N \rightarrow X \times S^1$ is a map it is not in general possible to make g Poincaré transverse at $X \times \{*\} \subset X \times S^1$, with $g^{-1}(X \times \{*\}) \subset N$ a finite codimension 1 Poincaré subcomplex. For $n \geq 6$ the obstruction to the existence of such a subcomplex up to bordism is the image of the Tate \mathbb{Z}_2 -cohomology class of the torsion $\tau(N) \in \hat{H}^n(\mathbb{Z}_2; Wh(\pi \times \mathbb{Z}))$ under the projection

$$B : \hat{H}^n(\mathbb{Z}_2; Wh(\pi \times \mathbb{Z})) \longrightarrow \hat{H}^{n-1}(\mathbb{Z}_2; \tilde{K}_0(\mathbb{Z}[\pi])) .$$

The obstruction vanishes for

$$(N, g) \times S^1 = (N \times S^1, g \times 1) \in \Omega_{n+1}^h(X \times S^1 \times S^1) ,$$

and there exists a finite codimension 1 Poincaré subcomplex

$$Q = (g \times 1)^{-1}(X \times \{*\} \times S^1) \subset N \times S^1$$

such that $Q \simeq P \times S^1$ with $P = \overline{Q}$ the infinite cyclic cover of Q classified by

$$Q \longrightarrow N \times S^1 \xrightarrow{\text{proj.}} S^1 .$$

Now P is a finitely dominated $(n-1)$ -dimensional geometric Poincaré complex with a codimension 1 Poincaré embedding $P \subset N$, and the projective class is such that

$$[P] = B\tau(N) \in \hat{H}^{n-1}(\mathbb{Z}_2; \tilde{K}_0(\mathbb{Z}[\pi])) .$$

The construction defines a map

$$B : \Omega_n^h(X \times S^1) \longrightarrow \Omega_{n-1}^p(X) ; (N, g) \longrightarrow (P, h) \quad (h = g|)$$

which fits into a split exact sequence

$$0 \longrightarrow \Omega_n^h(X) \xrightarrow{i} \Omega_n^h(X \times S^1) \xrightarrow{B} \Omega_{n-1}^p(X) \longrightarrow 0$$

with

$$i : \Omega_n^h(X) \longrightarrow \Omega_n^h(X \times S^1) ; (M, f) \longrightarrow (M, qf)$$

as in the manifold case. In accordance with the distinction established in Ranicki [66] there are two natural choices for splitting this sequence, given by the ‘algebraically significant’ direct sum system

$$\Omega_n^h(X) \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{j} \end{array} \Omega_n^h(X \times S^1) \begin{array}{c} \xrightarrow{B} \\ \xleftarrow{\bar{B}} \end{array} \Omega_{n-1}^p(X)$$

and the ‘geometrically significant’ direct sum system

$$\Omega_n^h(X) \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{j'} \end{array} \Omega_n^h(X \times S^1) \begin{array}{c} \xrightarrow{B} \\ \xleftarrow{\bar{B}'} \end{array} \Omega_{n-1}^p(X) ,$$

with

$$j : \Omega_n^h(X \times S^1) \longrightarrow \Omega_n^h(X) ; (N, g) \longrightarrow (N, pg) ,$$

$$\bar{B}' : \Omega_{n-1}^p(X) \longrightarrow \Omega_n^h(X \times S^1) ; (M, f) \longrightarrow (M \times S^1, f \times 1)$$

using the mapping torus trick of Mather [46] to replace $M \times S^1$ by a homotopy equivalent finite n -dimensional geometric Poincaré complex in the definition of \bar{B}' . The difference between the algebraically and geometrically significant split injections \bar{B} , \bar{B}' is given by

$$\begin{aligned} \bar{B}' - \bar{B} : \Omega_{n-1}^p(X) &\longrightarrow \hat{H}^{n-1}(\mathbb{Z}_2; \tilde{K}_0(\mathbb{Z}[\pi])) = \hat{H}^{n+1}(\mathbb{Z}_2; \tilde{K}_0(\mathbb{Z}[\pi])) \\ &\longrightarrow \Omega_n^h(X) \xrightarrow{i} \Omega_n^h(X \times S^1) . \end{aligned}$$

The bordism groups $\Omega_n^s(X)$ of pairs $(M, f : M \longrightarrow X)$ with M a simple n -dimensional geometric Poincaré complex fit into an exact sequence

$$\dots \longrightarrow \Omega_n^s(X) \longrightarrow \Omega_n^h(X) \longrightarrow \hat{H}^n(\mathbb{Z}_2; Wh(\pi)) \longrightarrow \Omega_{n-1}^s(X) \longrightarrow \dots$$

and a split exact sequence

$$0 \longrightarrow \Omega_n^s(X) \xrightarrow{i} \Omega_n^s(X \times S^1) \xrightarrow{B} \Omega_{n-1}^h(X) \longrightarrow 0 .$$

Again, B has both an algebraically significant splitting \bar{B} and a geometrically significant splitting \bar{B}' , with

$$\begin{aligned} \bar{B}' - \bar{B} : \Omega_{n-1}^h(X) &\longrightarrow \hat{H}^{n-1}(\mathbb{Z}_2; Wh(\pi)) = \hat{H}^{n+1}(\mathbb{Z}_2; Wh(\pi)) \\ &\longrightarrow \Omega_n^s(X) \xrightarrow{i} \Omega_n^s(X \times S^1) . \end{aligned}$$

The chain complex proof in Milgram and Ranicki [47] of the L -theory splitting theorem for the Laurent polynomial extension $A[z, z^{-1}]$ of a ring with involution A

$$L_n^h(A[z, z^{-1}]) = L_n^h(A) \oplus L_{n-1}^p(A)$$

depended on an algebraic transversality result: every free quadratic Poincaré complex over $A[z, z^{-1}]$ has a ‘fundamental domain’ projective quadratic Poincaré pair over A , by an algebraic mimicry of the geometric transversality by which every infinite cyclic cover of a compact manifold has a compact fundamental domain. The same method works for

$$L_n(\mathbb{A}[z, z^{-1}]) = L_n(\mathbb{A}) \oplus L_{n-1}(\mathbb{P}_0(\mathbb{A}))$$

for any additive category with involution \mathbb{A} , using the locally finite \mathbb{Z} -graded category with involution

$$\mathbb{F}_1(\mathbb{A}) = \mathbb{F}_{\mathbb{Z}}(\mathbb{A}) .$$

DEFINITION 16.1 (i) An n -dimensional quadratic cobordism in \mathbb{A}

$$X = ((f \ g) : C \oplus C' \longrightarrow D, (\delta\psi, \psi \oplus -\psi'))$$

is *fundamental* if

$$(C', \psi') = (C, \psi) .$$

(ii) The *union* of a fundamental n -dimensional quadratic cobordism X in \mathbb{A} is the n -dimensional quadratic complex in $\mathbb{A}[z, z^{-1}]$

$$U(X) = (E, \theta)$$

with

$$\begin{aligned} d_E &= \begin{pmatrix} d_D & (-)^{r-1}(f - gz) \\ 0 & d_C \end{pmatrix} : \\ E_r &= D_r[z, z^{-1}] \oplus C_{r-1}[z, z^{-1}] \\ &\longrightarrow E_{r-1} = D_{r-1}[z, z^{-1}] \oplus C_{r-2}[z, z^{-1}] , \\ \theta_s &= \begin{pmatrix} \delta\psi_s & (-)^s g\psi_s z \\ (-)^{n-r-1}\psi_s f^* & (-)^{n-r-s} T\psi_{s+1} \end{pmatrix} : \\ E^{n-r-s} &= D^{n-r-s}[z, z^{-1}] \oplus C^{n-r-s-1}[z, z^{-1}] \\ &\longrightarrow E_r = D_r[z, z^{-1}] \oplus C_{r-1}[z, z^{-1}] . \end{aligned}$$

(iii) A fundamental cobordism X in $\mathbb{F}_1(\mathbb{A})$ is *finitely balanced* if C and D are $(\mathbb{G}_1(\mathbb{A}), \mathbb{C}_0(\mathbb{A}))$ -finitely dominated and

$$[C] = [D] \in \tilde{K}_0(\mathbb{P}_0(\mathbb{A})) ,$$

in which case

$$[U(X)] = [D[z, z^{-1}]] - [C[z, z^{-1}]] = 0 \in \tilde{K}_0(\mathbb{P}_0(\mathbb{A}[z, z^{-1}]))$$

and the union $U(X)$ is homotopy $\mathbb{A}[z, z^{-1}]$ -finite. □

The union of a fundamental cobordism X can be viewed as the ‘infinite cyclic cover’

$$U(X) = \bigcup_{j=-\infty}^{\infty} z^j X$$

obtained by glueing together a countable number of copies of X .

$z^{j-1}D$	$z^j D$	$z^{j+1}D$
$z^{j-1}C$	$z^j C$	$z^{j+1}C$

If X is a fundamental quadratic Poincaré cobordism in \mathbb{A} (resp. $\mathbb{F}_1(\mathbb{A})$) then the union $U(X)$ is a quadratic Poincaré complex in $\mathbb{A}[z, z^{-1}]$ (resp. $\mathbb{F}_1(\mathbb{A})[z, z^{-1}]$).

THEOREM 16.2 *The quadratic L -groups $L_*(\mathbb{A}[z, z^{-1}])$ of a Laurent polynomial extension category $\mathbb{A}[z, z^{-1}]$ fit into a split exact sequence*

$$0 \longrightarrow L_n(\mathbb{A}) \xrightarrow{i_!} L_n(\mathbb{A}[z, z^{-1}]) \xrightarrow{B} L_{n-1}(\mathbb{P}_0(\mathbb{A})) \longrightarrow 0 ,$$

with B the composite of the map $L_n(\mathbb{A}[z, z^{-1}]) \longrightarrow L_n(\mathbb{C}_1(\mathbb{A}))$ induced by the inclusion $\mathbb{A}[z, z^{-1}] \subset \mathbb{C}_1(\mathbb{A})$ and the isomorphism of 15.2

$$B : L_n(\mathbb{C}_1(\mathbb{A})) \cong L_{n-1}(\mathbb{P}_0(\mathbb{A})) .$$

PROOF Let $L_n(\nabla)$ be the cobordism group of fundamental n -dimensional quadratic Poincaré cobordisms in $\mathbb{P}_0(\mathbb{A})$

$$X = ((f, g) : C \oplus C \longrightarrow D, (\delta\psi, \psi \oplus -\psi))$$

which are finitely balanced. The groups $L_*(\nabla)$ fit into split exact sequences

$$0 \longrightarrow L_n(\mathbb{A}) \longrightarrow L_n(\nabla) \longrightarrow L_{n-1}(\mathbb{P}_0(\mathbb{A})) \longrightarrow 0 ,$$

with

$$L_n(\mathbb{A}) \longrightarrow L_n(\nabla) ; (D, \delta\psi) \longrightarrow (0 \longrightarrow D, (\delta\psi, 0)) ,$$

$$L_n(\nabla) \longrightarrow L_{n-1}(\mathbb{P}_0(\mathbb{A})) ; X \longrightarrow (C, \psi) .$$

It is now possible to work exactly as for the case $\mathbb{A} = \mathbb{B}^f(A)$ in §§1-4 of Milgram and Ranicki [47], using the L -theory of the locally finite additive category with involution $\mathbb{F}_1(\mathbb{A})$. Specifically, it is claimed that the union defines an isomorphism

$$U : L_n(\nabla) \longrightarrow L_n(\mathbb{A}[z, z^{-1}]) ; X \longrightarrow U(X) .$$

For any finite n -dimensional quadratic Poincaré complex (E, θ) in $\mathbb{A}[z, z^{-1}]$ and any Mayer-Vietoris presentation of E in $\mathbb{C}_1(\mathbb{A})$

$$\mathbb{E} : 0 \longrightarrow E^+ \cap E^- \xrightarrow{i} E^+ \oplus E^- \xrightarrow{j} E \longrightarrow 0$$

there exists a finitely balanced fundamental n -dimensional quadratic Poincaré cobordism X in $\mathbb{F}_1(\mathbb{A})$ with

$$(C, \psi) = \partial((E, \theta)/E^+) ,$$

such that (E, θ) is homotopy equivalent to $U(X)$. The inverse of U is defined by

$$U^{-1} : L_n(\mathbb{A}[z, z^{-1}]) \longrightarrow L_n(\nabla) ; (E, \theta) \longrightarrow X$$

□

For a ring with involution A and $\mathbb{A} = \mathbb{B}^f(A)$ 16.2 recovers the split exact sequence of Ranicki [57]

$$0 \longrightarrow L_n^h(A) \xrightarrow{i_!} L_n^h(A[z, z^{-1}]) \xrightarrow{B} L_{n-1}^p(A) \longrightarrow 0 .$$

As for $K_1(\mathbb{A}[z, z^{-1}])$ and $\Omega_*^h(X \times S^1)$ there are two distinct natural

choices of splitting $L_n(\mathbb{A}[z, z^{-1}])$, given by the *algebraically significant* direct sum system

$$\begin{array}{ccccc} L_n(\mathbb{A}) & \xrightleftharpoons[i_!]{j_!} & L_n(\mathbb{A}[z, z^{-1}]) & \xrightleftharpoons[B]{\bar{B}} & L_{n-1}(\mathbb{P}_0(\mathbb{A})) \end{array}$$

and the *geometrically significant* direct sum system

$$\begin{array}{ccccc} L_n(\mathbb{A}) & \xrightleftharpoons[i_!]{j_!} & L_n(\mathbb{A}[z, z^{-1}]) & \xrightleftharpoons[B]{\bar{B}'} & L_{n-1}(\mathbb{P}_0(\mathbb{A})) \end{array}$$

with $j_!$ induced by $j : \mathbb{A}[z, z^{-1}] \longrightarrow \mathbb{A}; L[z, z^{-1}] \longrightarrow L$ and

$$\begin{aligned} \bar{B}' &= - \otimes \mathbb{R} : L_{n-1}(\mathbb{P}_0(\mathbb{A})) \longrightarrow L_n(\mathbb{A}[z, z^{-1}]) ; \\ (C, p, \psi) &\longrightarrow (C, p, \psi) \otimes \mathbb{R} . \end{aligned}$$

See Ranicki [66] for further details concerning the two types of splitting.

REMARK 16.3 Assume now that \mathbb{A} has a a stable canonical structure which is compatible with the involution, so that an n -dimensional quadratic Poincaré complex (E, θ) in $\mathbb{A}[z, z^{-1}]$ has a torsion

$$\tau(E, \theta) = \tau(\theta_0 : E^{n-*} \longrightarrow E) \in Wh(\mathbb{A}[z, z^{-1}]) .$$

The proof of 16.2 gives a finitely balanced fundamental quadratic Poincaré cobordism in $\mathbb{F}_1(\mathbb{A})$

$$X = ((f \ g) : C \oplus C \longrightarrow D, (\delta\psi, \psi \oplus -\psi))$$

such that $(E, \theta) \simeq U(X)$ and

$$B\tau(E, \theta) = [C] = [D] \in \tilde{K}_0(\mathbb{P}_0(\mathbb{A})) .$$

The Tate \mathbb{Z}_2 -cohomology class $\tau(E, \theta) \in \hat{H}^n(\mathbb{Z}_2; Wh(\mathbb{A}[z, z^{-1}]))$ is such that

$$B\tau(E, \theta) = 0 \in \hat{H}^{n-1}(\mathbb{Z}_2; \tilde{K}_0(\mathbb{P}_0(\mathbb{A})))$$

if and only if there exists a fundamental quadratic Poincaré cobordism X in \mathbb{A} such that $(E, \theta) \simeq U(X)$.

□

As usual, there are also versions of the splittings

$$L_n(\mathbb{A}[z, z^{-1}]) = L_n(\mathbb{A}) \oplus L_{n-1}(\mathbb{P}_0(\mathbb{A}))$$

for the intermediate and round L -groups, such as

$$L_n^{rs}(\mathbb{A}[z, z^{-1}]) = L_n^{rs}(\mathbb{A}) \oplus L_{n-1}^r(\mathbb{A}) ,$$

$$L_n^s(\mathbb{A}[z, z^{-1}]) = L_n^s(\mathbb{A}) \oplus L_{n-1}(\mathbb{A})$$

with $L_*^s = L_*^{\{0\} \subseteq Wh}$ the simple L -groups and $L_*^{rs} = L_*^{\{0\} \subseteq K_1}$ the round simple L -groups.

§17. Lower L -theory

The lower quadratic L -groups $L_*^{\langle -m \rangle}(\mathbb{A})$ ($m \geq 0$) are defined for any additive category with involution \mathbb{A} , and the connections with Laurent polynomial extensions and the Rothenberg exact sequences are obtained.

By convention, write

$$L_n^{\langle 1 \rangle}(\mathbb{A}) = L_n(\mathbb{A}) \quad (n \in \mathbb{Z}) .$$

DEFINITION 17.1 The *lower quadratic L -groups* of \mathbb{A} are defined for $m \geq 0$ by

$$L_n^{\langle -m \rangle}(\mathbb{A}) = \text{coker}(i_! : L_{n+1}^{\langle -m+1 \rangle}(\mathbb{A}) \longrightarrow L_{n+1}^{\langle -m+1 \rangle}(\mathbb{A}[z, z^{-1}])) \quad (n \in \mathbb{Z}) .$$

□

It is an immediate consequence of the 4-periodicity of the quadratic L -groups $L_*(\mathbb{A}) = L_{*+4}(\mathbb{A})$ that the lower quadratic L -groups are also 4-periodic

$$L_n^{\langle -m \rangle}(\mathbb{A}) = L_{n+4}^{\langle -m \rangle}(\mathbb{A}) \quad (m \geq 0, n \in \mathbb{Z}) .$$

The duality involutions act on the split exact sequence of §10

$$0 \longrightarrow K_1(\mathbb{A}) \xrightarrow{i_!} K_1(\mathbb{A}[z, z^{-1}]) \xrightarrow{B \oplus N_+ \oplus N_-} K_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \longrightarrow 0$$

by

$$i_!^* = *i_! , \quad B^* = - * B , \quad N_{\pm}^* = *N_{\mp} ,$$

so there is induced a split exact sequence of the Tate \mathbb{Z}_2 -cohomology groups

$$0 \longrightarrow \widehat{H}^n(\mathbb{Z}_2; K_1(\mathbb{A})) \xrightarrow{i_!} \widehat{H}^n(\mathbb{Z}_2; K_1(\mathbb{A}[z, z^{-1}])) \xrightarrow{B} \widehat{H}^{n-1}(\mathbb{Z}_2; K_0(\mathbb{P}_0(\mathbb{A}))) \longrightarrow 0 .$$

Define the duality involution on the lower K -groups

$$* : K_{-m}(\mathbb{A}) \longrightarrow K_{-m}(\mathbb{A})$$

to agree with $(-)^m$ the projective class duality

$$(-)^m * : K_0(\mathbb{P}_m(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_m(\mathbb{A})) ,$$

or equivalently with $(-)^{m+1}$ the torsion duality

$$(-)^{m+1} * : K_1(\mathbb{C}_{m+1}(\mathbb{A})) \longrightarrow K_1(\mathbb{C}_{m+1}(\mathbb{A})) .$$

With this convention there are defined split exact sequences

$$0 \longrightarrow \widehat{H}^n(\mathbb{Z}_2; K_{1-m}(\mathbb{A})) \xrightarrow{i_!} \widehat{H}^n(\mathbb{Z}_2; K_{1-m}(\mathbb{A}[z, z^{-1}])) \xrightarrow{B} \widehat{H}^{n-1}(\mathbb{Z}_2; K_{-m}(\mathbb{P}_0(\mathbb{A}))) \longrightarrow 0$$

for all $m \geq 0$.

Define the *reduced lower K -groups* of \mathbb{A} by

$$\begin{aligned}\tilde{K}_{-m}(\mathbb{P}_0(\mathbb{A})) &= \text{coker}(K_{-m}(\mathbb{A}) \longrightarrow K_{-m}(\mathbb{P}_0(\mathbb{A}))) \\ & (= K_{-m}(\mathbb{A}) \text{ for } m \geq 1) .\end{aligned}$$

THEOREM 17.2 *The lower quadratic L -groups are such that for all $m \geq -1$ and $n \in \mathbb{Z}$*

$$\begin{aligned}L_n^{\langle -m \rangle}(\mathbb{A}[z, z^{-1}]) &= L_n^{\langle -m \rangle}(\mathbb{A}) \oplus L_{n-1}^{\langle -m-1 \rangle}(\mathbb{A}) , \\ L_n^{\langle -m \rangle}(\mathbb{C}_1(\mathbb{A})) &= L_{n-1}^{\langle -m-1 \rangle}(\mathbb{A}) , \\ L_n^{\langle -m \rangle}(\mathbb{A}) &= L_{m+n}(\mathbb{P}_m(\mathbb{A})) \\ &= L_{m+n+1}(\mathbb{C}_{m+1}(\mathbb{A})) = L_{m+n+2}^s(\mathbb{C}_{m+2}(\mathbb{A})) ,\end{aligned}$$

and there are defined Rothenberg exact sequences

$$\begin{aligned}\dots \longrightarrow L_n^{\langle -m+1 \rangle}(\mathbb{A}) &\longrightarrow L_n^{\langle -m \rangle}(\mathbb{A}) \longrightarrow \hat{H}^n(\mathbb{Z}_2; \tilde{K}_{-m}(\mathbb{P}_0(\mathbb{A}))) \\ &\longrightarrow L_{n-1}^{\langle -m+1 \rangle}(\mathbb{A}) \longrightarrow \dots .\end{aligned}$$

PROOF By induction on m , with the initial case $m = 0$ already obtained in §16. As usual, there are two preferred ways of splitting the exact sequence

$$\begin{aligned}0 \longrightarrow L_n^{\langle -m+1 \rangle}(\mathbb{A}) &\xrightarrow{i!} L_n^{\langle -m+1 \rangle}(\mathbb{A}[z, z^{-1}]) \xrightarrow{B} L_{n-1}^{\langle -m \rangle}(\mathbb{P}_0(\mathbb{A})) \longrightarrow 0 , \\ \text{an algebraically significant one compatible with the functor } j : \mathbb{A}[z, z^{-1}] &\longrightarrow \mathbb{A} \text{ and a geometrically significant one involving the split injection} \\ \bar{B}' = - \otimes \mathbb{R} : L_{n-1}^{\langle -m \rangle}(\mathbb{P}_0(\mathbb{A})) &\longrightarrow L_n^{\langle -m+1 \rangle}(\mathbb{A}[z, z^{-1}]) .\end{aligned}$$

□

PROPOSITION 17.3 *For any $m \geq 1$ the quadratic L -groups $L_* = L_*^{\langle 1 \rangle}$ of the m -fold Laurent polynomial extension $\mathbb{A}[\mathbb{Z}^m]$ of \mathbb{A} are such that up to natural isomorphism*

$$L_n(\mathbb{A}[\mathbb{Z}^m]) = \sum_{i=0}^m \binom{m}{i} L_{n-i}^{\langle 1-i \rangle}(\mathbb{A}) \quad (n \in \mathbb{Z}) .$$

PROOF Iterate the case $m = 1$ obtained in §16. Again, the natural isomorphism can be chosen to be either algebraically or geometrically significant. The sum of the algebraically significant summands for $i \geq 1$ is the subgroup

$$\ker(j_! : L_n(\mathbb{A}[\mathbb{Z}^m]) \longrightarrow L_n(\mathbb{A})) \subseteq L_n(\mathbb{A}[\mathbb{Z}^m])$$

with

$$j : \mathbb{A}[\mathbb{Z}^m] \longrightarrow \mathbb{A} ; L[\mathbb{Z}^m] \longrightarrow L$$

the augmentation functor.

□

The geometrically significant summand

$$\mathrm{im}(\bar{B}'_1 \bar{B}'_2 \dots \bar{B}'_m = - \otimes \mathbb{R}^m : L_{n-m}^{\langle -m+1 \rangle}(\mathbb{A}) \longrightarrow L_n(\mathbb{A}[\mathbb{Z}^m])) \subseteq L_n(\mathbb{A}[\mathbb{Z}^m])$$

will be identified in §18 below with the subgroup of the T^m -transfer invariant elements.

The lower quadratic L -groups of a ring with involution A

$$L_n^{K-m(A)}(A) = L_n^{\langle -m \rangle}(A) \quad (m \geq 0, n \in \mathbb{Z})$$

were defined in Ranicki [57] inductively to fit into the algebraically significant splittings

$$L_n^{\langle -m \rangle}(A[z, z^{-1}]) = L_n^{\langle -m \rangle}(A) \oplus L_{n-1}^{\langle -m-1 \rangle}(A) \quad (m \geq -1) .$$

Here, $A[z, z^{-1}]$ is the Laurent polynomial extension ring $A[z, z^{-1}]$, with the involution extended by

$$\bar{z} = z^{-1} .$$

For $m = 0, 1$ these are the projective and free L -groups of A

$$L_*^{\langle 0 \rangle}(A) = L_*^p(A) , \quad L_*^{\langle 1 \rangle}(A) = L_*^h(A) .$$

EXAMPLE 17.4 The lower quadratic L -groups of a ring with involution A are the lower quadratic L -groups of the additive category with involution $\mathbb{B}^f(A)$ of based f.g. free A -modules

$$L_*^{\langle -m \rangle}(A) = L_*^{\langle -m \rangle}(\mathbb{B}^f(A)) \quad (m \geq -1) .$$

This is immediate from 17.2 and the identifications

$$\begin{aligned} \mathbb{C}_m(A) &= \mathbb{C}_1(\mathbb{C}_{m-1}(A)) , \\ \mathbb{C}_m(A[z, z^{-1}]) &= \mathbb{C}_m(A)[z, z^{-1}] . \end{aligned}$$

□

REMARK 17.5 Given a ring with involution A let $A[x]$, $A[x^{-1}]$, $A[x, x^{-1}]$ be the polynomial extension rings with the involution extended by

$$\bar{x} = x .$$

The lower L -groups $L_*^{\langle -m \rangle}(A)$ ($m \geq -1$) were shown in Ranicki [59, 4.5] to fit into split exact sequences

$$\begin{aligned} 0 \longrightarrow L_n^{\langle -m \rangle}(A) &\longrightarrow L_n^{\langle -m \rangle}(A[x]) \oplus L_n^{\langle -m \rangle}(A[x^{-1}]) \\ &\longrightarrow L_n^{\langle -m \rangle}(A[x, x^{-1}]) \longrightarrow L_n^{\langle -m-1 \rangle}(A) \longrightarrow 0 , \end{aligned}$$

by analogy with the lower K -theory split exact sequences

$$\begin{aligned} 0 \longrightarrow K_{-m}(A) &\longrightarrow K_{-m}(A[x]) \oplus K_{-m}(A[x^{-1}]) \\ &\longrightarrow K_{-m}(A[x, x^{-1}]) \longrightarrow K_{-m-1}(A) \longrightarrow 0 . \end{aligned}$$

□

DEFINITION 17.6 The *intermediate lower quadratic L -groups* $L_*^S(\mathbb{A})$ are defined for a $*$ -invariant subgroup $S \subseteq K_{-m}(\mathbb{A})$ ($m \geq 1$) by

$$L_n^S(\mathbb{A}) = L_{m+n}^{S_0}(\mathbb{P}_m(\mathbb{A})) = L_{m+n+1}^{S_1}(\mathbb{C}_{m+1}(\mathbb{A})) \quad (n \in \mathbb{Z})$$

with $S_0 \subseteq K_0(\mathbb{P}_m(\mathbb{A}))$, $S_1 \subseteq K_1(\mathbb{C}_{m+1}(\mathbb{A}))$ the $*$ -invariant subgroups isomorphic to S . □

The intermediate lower L -groups associated to $*$ -invariant subgroups $S \subseteq S' \subseteq K_{-m}(\mathbb{A})$ are related by the usual Rothenberg exact sequence $\dots \longrightarrow L_n^S(\mathbb{A}) \longrightarrow L_n^{S'}(\mathbb{A}) \longrightarrow \widehat{H}^n(\mathbb{Z}_2; S'/S) \longrightarrow L_{n-1}^S(\mathbb{A}) \longrightarrow \dots$, and

$$L_*^{K_{-m}(\mathbb{A})}(\mathbb{A}) = L_*^{\langle -m \rangle}(\mathbb{A}), \quad L_*^{\{0\} \subseteq K_{-m}(\mathbb{A})}(\mathbb{A}) = L_*^{\langle -m-1 \rangle}(\mathbb{A}).$$

DEFINITION 17.7 The *ultimate lower quadratic L -groups* of an additive category with involution \mathbb{A} are defined by the direct limits

$$L_n^{\langle -\infty \rangle}(\mathbb{A}) = \lim_{m \rightarrow \infty} L_n^{\langle -m \rangle}(\mathbb{A}) \quad (n \in \mathbb{Z}),$$

with $L_n^{\langle -m \rangle}(\mathbb{A}) \longrightarrow L_n^{\langle -m-1 \rangle}(\mathbb{A})$ the forgetful maps. □

The ultimate lower L -groups are such that

$$L_*^{\langle -\infty \rangle}(\mathbb{A}) = L_*^{\langle -\infty \rangle}(\mathbb{P}_0(\mathbb{A})) = L_{*+1}^{\langle -\infty \rangle}(\mathbb{C}_1(\mathbb{A})).$$

The following identification of the functor $X \longrightarrow L_{*+1}^{\langle -\infty \rangle}(\mathbb{C}_{O(X)}(\mathbb{A}))$ with a non-connective delooping of L -theory was obtained by Ferry and Pedersen [28] using geometric bounded surgery methods. (A lower L -theory delooping was previously obtained by Yamasaki [89] in the context of controlled surgery). See Ranicki [69, Appendix C] for the connection with the assembly map in algebraic L -theory.

PROPOSITION 17.8 *For any filtered additive category with involution \mathbb{A} the functor*

$$\{\text{compact polyhedra}\} \longrightarrow \{\mathbb{Z}\text{-graded abelian groups}\};$$

$$X \longrightarrow L_{*+1}^{\langle -\infty \rangle}(\mathbb{C}_{O(X)}(\mathbb{A}))$$

is a reduced generalized homology theory with a 4-periodic non-connective coefficient spectrum $\mathbb{L}_^{\langle -\infty \rangle}(\mathbb{A})$ such that*

$$\pi_*(\mathbb{L}_*^{\langle -\infty \rangle}(\mathbb{A})) = L_*^{\langle -\infty \rangle}(\mathbb{A}),$$

$$L_{*+1}^{\langle -\infty \rangle}(\mathbb{C}_{O(X)}(\mathbb{A})) = \widetilde{H}_*(X; \mathbb{L}_*^{\langle -\infty \rangle}(\mathbb{A})).$$

PROOF For homotopy invariance it suffices to consider the ultimate lower L -theory of $\mathbb{C}_{O(CX)}(\mathbb{A})$, with $O(CX)$ the open cone on an actual cone

CX of a compact polyhedron X . The natural flasque structure defined on $\mathbb{C}_{O(CX)}(\mathbb{A})$ in 3.3 is compatible with the involution. For each $m \geq 0$ the category $\mathbb{C}_m(\mathbb{C}_{O(CX)}(\mathbb{A}))$ has a natural flasque structure which is compatible with the involution, so that

$$L_*^{\langle -m \rangle}(\mathbb{C}_{O(CX)}(\mathbb{A})) = L_{m+*+1}(\mathbb{C}_m(\mathbb{C}_{O(CX)}(\mathbb{A}))) = 0 ,$$

and passing to the limit

$$L_*^{\langle -\infty \rangle}(\mathbb{C}_{O(CX)}(\mathbb{A})) = \lim_{m \rightarrow \infty} L_{m+*+1}^{\langle -m \rangle}(\mathbb{C}_m(\mathbb{C}_{O(CX)}(\mathbb{A}))) = 0 .$$

For exactness let $X = X^+ \cup X^-$. For each $m \geq 0$ define the $*$ -invariant subgroup

$$\begin{aligned} Y_{-m} &= \text{im}(\partial : K_{-m}(\mathbb{C}_{O(X^+ \cup X^-)}(\mathbb{A})) \longrightarrow K_{-m-1}(\mathbb{P}_{O(X^+ \cap X^-)}(\mathbb{A}))) \\ &\subseteq K_{-m-1}(\mathbb{P}_{O(X^+ \cap X^-)}(\mathbb{A})) , \end{aligned}$$

which is naturally isomorphic to the $*$ -invariant subgroup

$$\begin{aligned} Y'_{-m} &= \text{im}(\partial : K_1(\mathbb{C}_{O(X'^+ \cup X'^-)}(\mathbb{A})) \longrightarrow K_0(\mathbb{P}_{O(X'^+ \cap X'^-)}(\mathbb{A}))) \\ &\subseteq K_0(\mathbb{P}_{O(X'^+ \cap X'^-)}(\mathbb{A})) \end{aligned}$$

with $X' = \Sigma^{m+1}X$, $X'^{\pm} = \Sigma^{m+1}X^{\pm}$ the $(m+1)$ -fold reduced suspensions. By 14.6 there is defined a Mayer-Vietoris exact sequence

$$\begin{aligned} \dots &\longrightarrow L_n^{Y_{-m}}(\mathbb{P}_{O(X^+ \cap X^-)}(\mathbb{A})) \\ &\longrightarrow L_n^{\langle -m \rangle}(\mathbb{C}_{O(X^+)}(\mathbb{A})) \oplus L_n^{\langle -m \rangle}(\mathbb{C}_{O(X^-)}(\mathbb{A})) \\ &\longrightarrow L_n^{\langle -m \rangle}(\mathbb{C}_{O(X^+ \cup X^-)}(\mathbb{A})) \xrightarrow{\partial} L_{n-1}^{Y_{-m}}(\mathbb{P}_{O(X^+ \cap X^-)}(\mathbb{A})) \longrightarrow \dots \end{aligned}$$

Passing to the limit as $m \rightarrow \infty$ and identifying

$$\begin{aligned} \lim_{\overrightarrow{m}} L_*^{Y_{-m}}(\mathbb{P}_{O(X^+ \cap X^-)}(\mathbb{A})) &= L_*^{\langle -\infty \rangle}(\mathbb{P}_{O(X^+ \cap X^-)}(\mathbb{A})) \\ &= L_*^{\langle -\infty \rangle}(\mathbb{C}_{O(X^+ \cap X^-)}(\mathbb{A})) \end{aligned}$$

gives the ultimate lower L -theory exact sequence

$$\begin{aligned} \dots &\longrightarrow L_n^{\langle -\infty \rangle}(\mathbb{C}_{O(X^+ \cap X^-)}(\mathbb{A})) \longrightarrow \\ &L_n^{\langle -\infty \rangle}(\mathbb{C}_{O(X^+)}(\mathbb{A})) \oplus L_n^{\langle -\infty \rangle}(\mathbb{C}_{O(X^-)}(\mathbb{A})) \longrightarrow \\ &L_n^{\langle -\infty \rangle}(\mathbb{C}_{O(X^+ \cup X^-)}(\mathbb{A})) \xrightarrow{\partial} L_{n-1}^{\langle -\infty \rangle}(\mathbb{C}_{O(X^+ \cap X^-)}(\mathbb{A})) \longrightarrow \dots \end{aligned}$$

□

§18. Transfer in L -theory

In §12 the lower K -group $K_{1-m}(\mathbb{P}_0(\mathbb{A}))$ ($= K_{1-m}(\mathbb{A})$ for $m \geq 2$) was identified with the subgroup $K_1(\mathbb{A}[\mathbb{Z}^m])^{INV}$ of the T^m -transfer invariant elements in $K_1(\mathbb{A}[\mathbb{Z}^m])$. The same is now done for L -theory, identifying the lower L -group $L_n^{\langle 1-m \rangle}(\mathbb{A})$ ($m \geq 1$) with the subgroup

$L_{m+n}(\mathbb{A}[\mathbb{Z}^m])^{INV}$ of the T^m -transfer invariant elements in $L_{m+n}(\mathbb{A}[\mathbb{Z}^m])$. Again, start with the case $m = 1$.

As in §16 consider first the case of manifold bordism. The q -fold covering map $q : S^1 \longrightarrow S^1; z \longrightarrow z^q$ induces transfer maps

$$q^! : \Omega_n(X \times S^1) \longrightarrow \Omega_n(X \times S^1) ; (W, f) \longrightarrow (W^!, f^!)$$

using the pullback diagram

$$\begin{array}{ccc} W^! & \xrightarrow{f^!} & X \times S^1 \\ \downarrow & & \downarrow 1 \times q \\ W & \xrightarrow{f} & X \times S^1 . \end{array}$$

If $f : W \longrightarrow X \times S^1$ is transverse regular at $X \times \{*\} \subset X \times S^1$ with inverse image the codimension 1 framed submanifold

$$M = f^{-1}(X \times \{*\}) \subset W$$

then $f^! : W^! \longrightarrow X \times S^1$ is also transverse regular at $X \times \{*\} \subset X \times S^1$ with inverse image a copy of M

$$M \cong (f^!)^{-1}(X \times \{*\}) \subset W^! ,$$

so that

$$Bq^!(W, f) = B(W^!, f^!) = (M, f|) = B(W, f) \in \Omega_{n-1}(X) .$$

Moreover, if $f : W \longrightarrow X \times \{*\} \longrightarrow X \times S^1$ then

$$q^!(W, f) = (W^!, f^!) = \coprod_q (W, f) = q(W, f) \in \Omega_n(X \times S^1) .$$

Thus the bordism transfer map fits into a morphism of split exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_n(X) & \xrightarrow{i} & \Omega_n(X \times S^1) & \xrightarrow{B} & \Omega_{n-1}(X) \longrightarrow 0 \\ & & \downarrow q & & \downarrow q^! & & \downarrow 1 \\ 0 & \longrightarrow & \Omega_n(X) & \xrightarrow{i} & \Omega_n(X \times S^1) & \xrightarrow{B} & \Omega_{n-1}(X) \longrightarrow 0 \end{array}$$

and the image of the split injection

$$C : \Omega_{n-1}(X) \longrightarrow \Omega_n(X \times S^1) ; (M, f) \longrightarrow (M \times S^1, f \times 1)$$

coincides with the subgroup of transfer-invariant bordism classes

$$\Omega_n(X \times S^1)^{INV} =$$

$$\{(W, f) \in \Omega_n(X \times S^1) \mid q^!(W, f) = (W, f) \text{ for all } q \geq 1\} .$$

The identity

$$\text{im}(C : \Omega_{n-1}(X) \longrightarrow \Omega_n(X \times S^1)) = \Omega_n(X \times S^1)^{INV}$$

allows the identification

$$\Omega_n(X \times S^1)^{INV} = \Omega_{n-1}(X) .$$

Similar considerations apply to geometric Poincaré bordism, with a morphism of split exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_n^h(X) & \xrightarrow{i} & \Omega_n^h(X \times S^1) & \xrightarrow{B} & \Omega_{n-1}^p(X) & \longrightarrow & 0 \\ & & \downarrow q & & \downarrow q^! & & \downarrow 1 & & \\ 0 & \longrightarrow & \Omega_n^h(X) & \xrightarrow{i} & \Omega_n^h(X \times S^1) & \xrightarrow{B} & \Omega_{n-1}^p(X) & \longrightarrow & 0 . \end{array}$$

It follows that there is an identity

$$\mathrm{im}(\bar{B}' : \Omega_{n-1}^p(X) \longrightarrow \Omega_n^h(X \times S^1)) = \Omega_n^h(X \times S^1)^{INV}$$

with \bar{B}' the geometrically significant split injection, allowing the identification

$$\Omega_n^h(X \times S^1)^{INV} = \Omega_{n-1}^p(X) .$$

For an additive category with involution \mathbb{A} and any integer $q \geq 1$ the q -fold transfer on the bounded \mathbb{Z} -graded category

$$q^! : \mathbb{C}_1(\mathbb{A}) \longrightarrow \mathbb{C}_1(\mathbb{A}) ; M \longrightarrow q^!M , \quad q^!M(j) = \sum_{k=0}^{q-1} M(qj+k)$$

is a functor of additive categories with involution. The q -fold transfer functor on the Laurent polynomial extension category

$$q^! : \mathbb{A}[z, z^{-1}] \longrightarrow \mathbb{A}[z, z^{-1}] ;$$

$$L[z, z^{-1}] \longrightarrow q^!(L[z, z^{-1}]) = \left(\sum_{k=0}^{q-1} L \right)[z, z^{-1}]$$

induces the q -fold transfer maps in the algebraic L -groups

$$q^! : L_n(\mathbb{A}[z, z^{-1}]) \longrightarrow L_n(\mathbb{A}[z, z^{-1}]) .$$

By analogy with 12.1 and 12.2:

DEFINITION 18.1 The S^1 -transfer invariant subgroup of $L_n(\mathbb{A}[z, z^{-1}])$ is

$$L_n(\mathbb{A}[z, z^{-1}])^{INV} = \{x \in L_n(\mathbb{A}[z, z^{-1}]) \mid q^!(x) = x \text{ for every } q \geq 2\} .$$

□

PROPOSITION 18.2 (i) The effect of $q^!$ on the geometrically significant direct sum decomposition of $L_n(\mathbb{A}[z, z^{-1}])$ is given by the commutative

diagram

$$\begin{array}{ccc} L_n(\mathbb{A}) \oplus L_{n-1}(\mathbb{P}_0(\mathbb{A})) & \xrightarrow{(i_! \bar{B}')} & L_n(\mathbb{A}[z, z^{-1}]) \\ q \oplus 1 \downarrow & & \downarrow q^! \\ L_n(\mathbb{A}) \oplus L_{n-1}(\mathbb{P}_0(\mathbb{A})) & \xrightarrow{(i_! \bar{B}')} & L_n(\mathbb{A}[z, z^{-1}]) . \end{array}$$

(ii) The image of the geometrically significant split injection

$$\bar{B}' : L_{n-1}(\mathbb{P}_0(\mathbb{A})) \longrightarrow L_n(\mathbb{A}[z, z^{-1}])$$

is the subgroup of the S^1 -transfer invariant elements

$$L_{n-1}(\mathbb{P}_0(\mathbb{A})) = \text{im}(\bar{B}') = L_n(\mathbb{A}[z, z^{-1}])^{INV} .$$

(iii) The effect of $q^!$ on $L_n(\mathbb{C}_1(\mathbb{A}))$ is

$$q^! = 1 : L_n(\mathbb{C}_1(\mathbb{A})) \longrightarrow L_n(\mathbb{C}_1(\mathbb{A})) .$$

PROOF (i) By the proof of 16.2 it is possible to regard $L_n(\mathbb{A}[z, z^{-1}])$ as the cobordism group of n -dimensional quadratic Poincaré cobordisms in $\mathbb{P}_0(\mathbb{A})$

$$X = ((f_- \ f_+) : C \oplus C \longrightarrow D, (\delta\psi, \psi \oplus -\psi))$$

such that

$$[C] = [D] \in \tilde{K}_0(\mathbb{P}_0(\mathbb{A})) .$$

The q -fold transfer of X is the union of q copies of X

$$q^!X = \bigcup_q X = ((f'_- \ f'_+) : C \oplus C \longrightarrow D', (\delta\psi', \psi \oplus -\psi))$$

with $D' = C(g)$ the algebraic mapping cone of the chain map

$$g = \begin{pmatrix} f_+ & 0 & 0 & \dots & 0 \\ -f_- & f_+ & 0 & \dots & 0 \\ 0 & -f_- & f_+ & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & f_+ \\ 0 & 0 & 0 & \dots & -f_- \end{pmatrix}$$

$$: \sum_{q-1} C = C \oplus C \oplus \dots \oplus C \longrightarrow \sum_q D = D \oplus D \oplus \dots \oplus D .$$

The relations $\bar{B}'q^! = \bar{B}'$, $i_!q^! = i_!q$ are now clear.

(ii)+(iii) Immediate from (i).

□

Let $m \geq 1$. For any m -tuple $Q = (q_1, q_2, \dots, q_m)$ of integers $q_j \geq 1$ the Q -fold transfer functor of additive categories with involution

$$Q^! : \mathbb{A}[\mathbb{Z}^m] \longrightarrow \mathbb{A}[\mathbb{Z}^m] ; L[\mathbb{Z}^m] \longrightarrow \left(\sum_{j_1=0}^{q_1-1} \sum_{j_2=0}^{q_2-1} \dots \sum_{j_m=0}^{q_m-1} L \right) [\mathbb{Z}^m]$$

induces transfer maps in the L -groups

$$Q^! : L_*(\mathbb{A}[\mathbb{Z}^m]) \longrightarrow L_*(\mathbb{A}[\mathbb{Z}^m]) .$$

By analogy with 12.3 and 12.4:

DEFINITION 18.3 The T^m -transfer invariant subgroup of $L_n(\mathbb{A}[\mathbb{Z}^m])$ is

$$L_n(\mathbb{A}[\mathbb{Z}^m])^{INV} = \{x \in L_n(\mathbb{A}[\mathbb{Z}^m]) \mid Q^!(x) = x \text{ for every } Q\} .$$

□

PROPOSITION 18.4 For each $m \geq 1$ the geometrically significant split injection

$$\bar{B}'_1 \bar{B}'_2 \dots \bar{B}'_m : L_n^{(1-m)}(\mathbb{A}) \longrightarrow L_{m+n}(\mathbb{A}[\mathbb{Z}^m])$$

maps the lower quadratic L -group to the subgroup of the transfer invariant elements in $L_{m+n}(\mathbb{A}[\mathbb{Z}^m])$

$$L_n^{(1-m)}(\mathbb{A}) = \text{im}(\bar{B}'_1 \bar{B}'_2 \dots \bar{B}'_m) = L_{m+n}(\mathbb{A}[\mathbb{Z}^m])^{INV} .$$

PROOF This follows from the case $m = 1$ dealt with in 18.2 and the factorization of the Q -fold transfer for $Q = (q_1, q_2, \dots, q_m)$ as the m -fold composition of the single transfers

$$Q^! = (q_1)^!(q_2)^! \dots (q_m)^! : \mathbb{A}[\mathbb{Z}^m] \longrightarrow \mathbb{A}[\mathbb{Z}^m] ,$$

with $(q_j)^!$ acting on the j th coordinate ($1 \leq j \leq m$).

□

§19. Symmetric L -theory

The symmetric L -groups $L^*(\mathbb{A})$ of an additive category with involution \mathbb{A} are defined in Ranicki [68] to be the cobordism groups of symmetric Poincaré complexes in \mathbb{A} . The symmetric L -groups of $\mathbb{A} = \mathbb{B}^f(A)$ for a ring with involution A are the symmetric L -groups $L^*(A)$ of Ranicki [61]. Except for 4-periodicity all the results on the quadratic L -groups $L_*(\mathbb{A})$ of §13-§18 have symmetric L -theory analogues, with entirely analogous proofs. In particular, there are defined lower symmetric L -groups $L_{(-m)}^n(\mathbb{A})$ ($m \geq 0$, $n \in \mathbb{Z}$). The one noteworthy feature is that for $n \leq -3$ the lower symmetric L -groups coincide with the lower quadratic L -groups $L_n^{(-m)}(\mathbb{A})$.

Let $\epsilon = \pm 1$. Given a finite chain complex C in \mathbb{A} let \mathbb{Z}_2 act on $C \otimes_{\mathbb{A}} C$ by the ϵ -transposition involution

$$T_{\epsilon} : C \otimes_{\mathbb{A}} C \longrightarrow C \otimes_{\mathbb{A}} C ; \phi \longrightarrow \epsilon T \phi .$$

The $\begin{cases} \epsilon\text{-symmetric} \\ \epsilon\text{-quadratic} \end{cases}$ Q -groups of C are defined by

$$\begin{cases} Q^n(C, \epsilon) = H_n(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_{\mathbb{A}} C)) \\ Q_n(C, \epsilon) = H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes_{\mathbb{A}} C)) . \end{cases}$$

See Ranicki [68] for the definition of the n -dimensional $\begin{cases} \epsilon\text{-symmetric} \\ \epsilon\text{-quadratic} \end{cases}$ L -group $\begin{cases} L^n(\mathbb{A}, \epsilon) \\ L_n(\mathbb{A}, \epsilon) \end{cases}$ ($n \geq 0$) as the cobordism group of n -dimensional $\begin{cases} \epsilon\text{-symmetric} \\ \epsilon\text{-quadratic} \end{cases}$ Poincaré complexes $\begin{cases} (C, \phi \in Q^n(C, \epsilon)) \\ (C, \psi \in Q_n(C, \epsilon)) \end{cases}$ in \mathbb{A} , with

$$\begin{cases} \phi_0 : C^{n-*} \longrightarrow C \\ (1 + T_\epsilon)\psi_0 : C^{n-*} \longrightarrow C \end{cases}$$

a chain equivalence. $\begin{cases} L^0(\mathbb{A}, \epsilon) \\ L_0(\mathbb{A}, \epsilon) \end{cases}$ (resp. $\begin{cases} L^1(\mathbb{A}, \epsilon) \\ L_1(\mathbb{A}, \epsilon) \end{cases}$) is the Witt group of

nonsingular $\begin{cases} \epsilon\text{-symmetric} \\ \epsilon\text{-quadratic} \end{cases}$ forms (resp. formations) in \mathbb{A} . The skew-suspension maps

$$\begin{cases} \bar{S} : L^n(\mathbb{A}, \epsilon) \longrightarrow L^{n+2}(\mathbb{A}, -\epsilon) ; (C, \phi) \longrightarrow (SC, \bar{S}\phi) \\ \bar{S} : L_n(\mathbb{A}, \epsilon) \longrightarrow L_{n+2}(\mathbb{A}, -\epsilon) ; (C, \psi) \longrightarrow (SC, \bar{S}\psi) \end{cases}$$

are isomorphisms in the ϵ -quadratic case, but not (in general) in the ϵ -symmetric case. For $\epsilon = 1$ write

$$L^*(\mathbb{A}, 1) = L^*(\mathbb{A}) \quad , \quad L_*(\mathbb{A}, 1) = L_*(\mathbb{A}) \quad .$$

DEFINITION 19.1 An n -dimensional ϵ -symmetric complex (C, ϕ) in \mathbb{A} is *even* if there exists $\psi_0 \in \text{Hom}_{\mathbb{A}}(C^n, C_n)$ such that

$$\phi_0 = \psi_0 + \epsilon\psi_0^* : C^n \longrightarrow C_n \quad .$$

□

(Let $L\langle v_0 \rangle^n(\mathbb{A}, \epsilon)$ be the cobordism group of n -dimensional even ϵ -symmetric Poincaré complexes in \mathbb{A} . The skew-suspension maps

$$\bar{S} : L^n(\mathbb{A}, \epsilon) \longrightarrow L\langle v_0 \rangle^{n+2}(\mathbb{A}, -\epsilon) ; (C, \phi) \longrightarrow (SC, \bar{S}\phi)$$

are isomorphisms for all $n \geq 0$).

The unified L -theory of Ranicki [63, §1.8] (originally for $\mathbb{A} = \mathbb{P}(A)$) allows n -dimensional ϵ -symmetric complexes to be defined for all $n \leq -1$ in any additive category with involution \mathbb{A} , namely

$2i$ -dimensional ϵ -symmetric complex

$$= 0\text{-dimensional} \begin{cases} \text{even } (-\epsilon)\text{-symmetric} \\ (-)^i \epsilon\text{-quadratic} \end{cases} \text{ complex for } \begin{cases} i = -1 \\ i \leq -2 \end{cases} \quad ,$$

$(2i + 1)$ -dimensional ϵ -symmetric complex

$$= 1\text{-dimensional} \begin{cases} \text{even } (-\epsilon)\text{-symmetric} \\ (-)^i \epsilon\text{-quadratic} \end{cases} \text{ complex for } \begin{cases} i = -1 \\ i \leq -2 \end{cases} \quad .$$

The cobordism groups $L^n(\mathbb{A}, \epsilon)$ of n -dimensional ϵ -symmetric complexes in \mathbb{A} are defined for all $n \in \mathbb{Z}$ to be the cobordism groups of n -dimensional

ϵ -symmetric Poincaré complexes in \mathbb{A} , with

$$L^n(\mathbb{A}, \epsilon) = L_n(\mathbb{A}, \epsilon) \text{ for } n \leq -3 .$$

Only the most important of the symmetric L -theory versions of the results of §13-§18 are spelled out:

PROPOSITION 19.2 (i) *For a compact polyhedron $X = X^+ \cup X^- \subseteq S^k$ the ϵ -symmetric L -groups of the bounded $O(X^+ \cup X^-)$ -graded category $\mathbb{C}_{O(X^+ \cup X^-)}(\mathbb{A})$ of a filtered additive category with involution \mathbb{A} fit into a Mayer-Vietoris exact sequence*

$$\begin{aligned} \dots \longrightarrow L_Y^n(\mathbb{P}_{O(X^+ \cap X^-)}(\mathbb{A}), \epsilon) &\longrightarrow L^n(\mathbb{C}_{O(X^+)}(\mathbb{A}), \epsilon) \oplus L^n(\mathbb{C}_{O(X^-)}(\mathbb{A}), \epsilon) \\ &\longrightarrow L^n(\mathbb{C}_{O(X^+ \cup X^-)}(\mathbb{A}), \epsilon) \xrightarrow{\partial} L_Y^{n-1}(\mathbb{P}_{O(X^+ \cap X^-)}(\mathbb{A}), \epsilon) \longrightarrow \dots \\ &\quad (n \in \mathbb{Z}) , \end{aligned}$$

with

$$\begin{aligned} Y &= \ker(\tilde{K}_0(\mathbb{P}_{O(X^+ \cap X^-)}(\mathbb{A})) \longrightarrow \tilde{K}_0(\mathbb{P}_{O(X^+)}(\mathbb{A})) \oplus \tilde{K}_0(\mathbb{P}_{O(X^-)}(\mathbb{A}))) \\ &= \operatorname{im}(\tilde{\partial} : \operatorname{Wh}_{\mathbb{C}_{O(X)}}(\mathbb{A})) \longrightarrow \tilde{K}_0(\mathbb{P}_{O(X^+ \cap X^-)}(\mathbb{A})) \\ &\quad \subseteq \tilde{K}_0(\mathbb{P}_{O(X^+ \cap X^-)}(\mathbb{A})) . \end{aligned}$$

(ii) *The ϵ -symmetric L -groups of $\mathbb{C}_1(\mathbb{A})$ and $\mathbb{A}[z, z^{-1}]$ are such that for all $n \in \mathbb{Z}$*

$$\begin{aligned} L^n(\mathbb{C}_1(\mathbb{A}), \epsilon) &= L^{n-1}(\mathbb{P}_0(\mathbb{A}), \epsilon) , \\ L^n(\mathbb{A}[z, z^{-1}], \epsilon) &= L^n(\mathbb{A}, \epsilon) \oplus L^{n-1}(\mathbb{P}_0(\mathbb{A}), \epsilon) . \end{aligned}$$

□

By convention, write

$$L_{\langle 1 \rangle}^n(\mathbb{A}, \epsilon) = L^n(\mathbb{A}, \epsilon) \quad (n \in \mathbb{Z}) .$$

DEFINITION 19.3 The *lower ϵ -symmetric L -groups* of \mathbb{A} are defined by

$$\begin{aligned} L_{\langle -m \rangle}^n(\mathbb{A}, \epsilon) &= \operatorname{coker}(i! : L_{\langle -m+1 \rangle}^{n+1}(\mathbb{A}, \epsilon) \longrightarrow L_{\langle -m+1 \rangle}^{n+1}(\mathbb{A}[z, z^{-1}], \epsilon)) \\ &\quad (m \geq 0 , \quad n \in \mathbb{Z}) . \end{aligned}$$

□

Note that for $n \leq -3$

$$L_{\langle -m \rangle}^n(\mathbb{A}, \epsilon) = L_n^{\langle -m \rangle}(\mathbb{A}, \epsilon) \quad (m \geq -1) ,$$

with the lower $(-)^k \epsilon$ -quadratic L -group $L_{n+2k}^{\langle -m \rangle}(\mathbb{A}, (-)^k \epsilon)$ defined for any k such that $n + 2k \geq 0$.

PROPOSITION 19.4 *The lower ϵ -symmetric L -groups are such that for all $m \geq 0$, $n \in \mathbb{Z}$*

$$\begin{aligned}
L^n(\mathbb{A}[\mathbb{Z}^m], \epsilon) &= \sum_{i=0}^m \binom{m}{i} L_{\langle 1-i \rangle}^{n-i}(\mathbb{A}, \epsilon) , \\
L_{\langle -m \rangle}^n(\mathbb{A}[z, z^{-1}], \epsilon) &= L_{\langle -m \rangle}^n(\mathbb{A}, \epsilon) \oplus L_{\langle -m-1 \rangle}^{n-1}(\mathbb{A}, \epsilon) , \\
L_{\langle -m \rangle}^n(\mathbb{C}_1(\mathbb{A}), \epsilon) &= L_{\langle -m-1 \rangle}^{n-1}(\mathbb{A}, \epsilon) , \\
L_{\langle -m \rangle}^n(\mathbb{A}, \epsilon) &= L^{m+n}(\mathbb{P}_m(\mathbb{A}), \epsilon) = L^{m+n+1}(\mathbb{C}_{m+1}(\mathbb{A}), \epsilon) \\
&= L_s^{m+n+2}(\mathbb{C}_{m+2}(\mathbb{A}), \epsilon) , \\
L_{\langle 1-m \rangle}^n(\mathbb{A}, \epsilon) &= L^{m+n}(\mathbb{A}[\mathbb{Z}^m], \epsilon)^{INV} ,
\end{aligned}$$

and there are defined Rothenberg exact sequences

$$\begin{aligned}
\cdots \longrightarrow L_{\langle -m+1 \rangle}^n(\mathbb{A}, \epsilon) &\longrightarrow L_{\langle -m \rangle}^n(\mathbb{A}, \epsilon) \longrightarrow \hat{H}^n(\mathbb{Z}_2; \tilde{K}_{-m}(\mathbb{P}_0(\mathbb{A}))) \\
&\longrightarrow L_{\langle -m+1 \rangle}^{n-1}(\mathbb{A}, \epsilon) \longrightarrow \cdots .
\end{aligned}$$

□

DEFINITION 19.5 The *ultimate lower ϵ -symmetric L -groups* are defined by the direct limits

$$L_{\langle -\infty \rangle}^n(\mathbb{A}, \epsilon) = \lim_{m \rightarrow \infty} L_{\langle -m \rangle}^n(\mathbb{A}, \epsilon) \quad (n \in \mathbb{Z}) ,$$

with $L_{\langle -m \rangle}^n(\mathbb{A}, \epsilon) \longrightarrow L_{\langle -m-1 \rangle}^n(\mathbb{A}, \epsilon)$ the forgetful maps.

□

PROPOSITION 19.6 *For any \mathbb{A}, ϵ the functor*

$$\{\text{compact polyhedra}\} \longrightarrow \{\mathbb{Z}\text{-graded abelian groups}\} ;$$

$$X \longrightarrow L_{\langle -\infty \rangle}^{*+1}(\mathbb{C}_{O(X)}(\mathbb{A}), \epsilon)$$

is a reduced generalized homology theory with a non-connective coefficient spectrum $\mathbb{L}_{\langle -\infty \rangle}(\mathbb{A}, \epsilon)$ such that

$$\begin{aligned}
\pi_*(\mathbb{L}_{\langle -\infty \rangle}(\mathbb{A}, \epsilon)) &= L_{\langle -\infty \rangle}^*(\mathbb{A}, \epsilon) , \\
L_{\langle -\infty \rangle}^{*+1}(\mathbb{C}_{O(X)}(\mathbb{A}), \epsilon) &= \tilde{H}_*(X; \mathbb{L}_{\langle -\infty \rangle}(\mathbb{A}, \epsilon)) .
\end{aligned}$$

□

§20. The algebraic fibering obstruction

The chain complex approach to the K - and L -theory of the Laurent polynomial extension category $\mathbb{A}[z, z^{-1}]$ developed in §10 and §16 will now be used to give an abstract algebraic treatment of the obstruction theory for fibering n -dimensional manifolds over S^1 for $n \geq 6$. Following the positive results of Stallings [79] for $n = 3$ and Browder and

Levine [11] for $n \geq 6$ and $\pi_1 = \mathbb{Z}$ a general fibering obstruction theory for $n \geq 6$ was developed by Farrell [21], [22] and Siebenmann [74], [76], with obstructions in the Whitehead group of an extension by an infinite cyclic group. See Kearton [38] and Weinberger [85] for examples of non-fibering manifolds in dimensions $n = 4, 5$ with vanishing Whitehead torsion fibering obstruction.

The *mapping torus* of a self-map $h : F \longrightarrow F$ is defined by

$$T(h) = F \times [0, 1] / \{(x, 0) = (h(x), 1) | x \in F\} ,$$

as usual. If F is a compact $(n-1)$ -dimensional manifold and $h : F \longrightarrow F$ is a self homeomorphism then $T(h)$ is a compact n -dimensional manifold such that

$$T(h) \longrightarrow I/(0=1) = S^1 ; (x, t) \longrightarrow [t]$$

is the projection of a fibre bundle over S^1 with fibre F and monodromy h .

A *CW complex band* is a finite *CW* complex X with a finitely dominated infinite cyclic cover \bar{X} . Let $\zeta : \bar{X} \longrightarrow \bar{X}$ be a generating covering translation. For the sake of simplicity we shall only consider *CW* complex bands with

$$\zeta_* = 1 : \pi_1(\bar{X}) \longrightarrow \pi_1(\bar{X}) ,$$

so that

$$\pi_1(X) = \pi_1(\bar{X}) \times \mathbb{Z} , \quad \mathbb{Z}[\pi_1(X)] = \mathbb{Z}[\pi_1(\bar{X})][z, z^{-1}] .$$

However, there is also a version of the theory which allows a non-trivial monodromy $\zeta_* : \pi_1(\bar{X}) \longrightarrow \pi_1(\bar{X})$ in the fundamental group, corresponding to the exact sequences of Farrell and Hsiang [23] and Ranicki [58] for the K - and L -groups of α -twisted Laurent polynomial rings $A_\alpha[z, z^{-1}]$ with $az = z\alpha(a)$ for an automorphism $\alpha : A \longrightarrow A$

$$\begin{aligned} K_1(A) &\xrightarrow{1-\alpha} K_1(A) \longrightarrow K_1(A_\alpha[z, z^{-1}]) \\ &\longrightarrow K_0(A) \oplus \widetilde{\text{Nil}}_0(A, \alpha) \oplus \widetilde{\text{Nil}}_0(A, \alpha^{-1}) \xrightarrow{(1-\alpha) \oplus 0 \oplus 0} K_0(A) , \\ \dots &\longrightarrow L_n^J(A) \xrightarrow{1-\alpha} L_n(A) \longrightarrow L_n(A_\alpha[z, z^{-1}]) \longrightarrow L_{n-1}^J(A) \longrightarrow \dots \end{aligned}$$

with $J = \ker(1 - \alpha : \tilde{K}_0(A) \longrightarrow \tilde{K}_0(A))$ and $\bar{z} = z^{-1}$.

A *finite structure* on a topological space X is an equivalence class of pairs

$$(\text{finite } CW \text{ complex } K , \text{ homotopy equivalence } f : K \longrightarrow X)$$

subject to the equivalence relation

$$(K, f) \sim (K', f') \text{ if } \tau(f'^{-1}f : K \longrightarrow K') = 0 \in Wh(\pi_1(X)) .$$

The mapping torus $T(\zeta)$ of a generating covering translation $\zeta : \bar{X} \rightarrow \bar{X}$ for a CW band X has a preferred finite structure (Mather [46], Ferry [27] and Ranicki [67]), represented by $(T(f\zeta g), h)$ for any finite domination of X

$$(K, f : K \rightarrow X, g : X \rightarrow K, gf \simeq 1 : K \rightarrow K),$$

with $h : T(f\zeta g) \simeq T(gf\zeta) \simeq T(\zeta)$ defined as in [67, §6]. Let $p : \bar{X} \rightarrow X$ be the covering projection, and define homotopy equivalences

$$q^+ : T(\zeta) \rightarrow X ; (x, t) \rightarrow p(x),$$

$$q^- : T(\zeta^{-1}) \rightarrow X ; (x, t) \rightarrow p(x).$$

The intrinsic finite structure on X is not in general compatible with either of the extrinsic finite structures X inherits via q^+, q^- from the preferred finite structures on $T(\zeta), T(\zeta^{-1})$.

The *geometric fibering obstructions* of a CW band X with respect to a choice of generating covering translation $\zeta : \bar{X} \rightarrow \bar{X}$ are the torsions

$$\Phi^+(X) = \tau(q^+ : T(\zeta) \rightarrow X),$$

$$\Phi^-(X) = \tau(q^- : T(\zeta^{-1}) \rightarrow X) \in Wh(\pi_1(X))$$

measuring the difference between the intrinsic and the two extrinsic finite structures on X . The effect on the fibering obstructions of the opposite choice of generating covering translation ζ is given by

$$\Phi^+(X^{op}) = \Phi^-(X), \quad \Phi^-(X^{op}) = \Phi^+(X).$$

Further below, the fibering obstruction of a manifold X with finitely dominated infinite cyclic cover \bar{X} will be identified with the geometric fibering obstructions of the underlying CW band – in the manifold case there is a Poincaré duality $\Phi^-(X) = \pm \Phi^+(X)^*$, so the two fibering obstructions determine each other.

Let $(\mathbb{B}, \mathbb{A} \subseteq \mathbb{B})$ be a pair of additive categories. Given a self chain equivalence $h : C \rightarrow C$ of an \mathbb{A} -finitely dominated chain complex C in \mathbb{B} define the *mixed torsion-class* invariant

$$[C, h] = \tau(-zh : C[z, z^{-1}] \rightarrow C[z, z^{-1}]) \in Wh(\mathbb{A}[z, z^{-1}]).$$

The mixed invariant has components

$$[C, h] = (\tau(h), [C], 0, 0)$$

$$\in Wh(\mathbb{A}[z, z^{-1}]) = Wh(\mathbb{A}) \oplus \tilde{K}_0(\mathbb{P}_0(\mathbb{A})) \oplus \tilde{\text{Nil}}_0(\mathbb{A}) \oplus \tilde{\text{Nil}}_0(\mathbb{A})$$

with respect to the geometrically significant direct sum decomposition of $Wh(\mathbb{A}[z, z^{-1}])$. The mixed invariant of a self-homotopy equivalence $h : X \rightarrow X$ of a finitely dominated CW complex X with $h_* = 1 : \pi_1(X) = \pi \rightarrow \pi$ is defined by

$$[X, h] = [C(\tilde{X}), \tilde{h}] \in Wh(\pi \times \mathbb{Z}),$$

with $\tilde{h} : C(\tilde{X}) \longrightarrow C(\tilde{X})$ the induced self chain equivalence of the finitely dominated cellular $\mathbb{Z}[\pi]$ -module chain complex of the universal cover \tilde{X} . In Ranicki [67] the mixed invariant was identified with the torsion

$$[X, h] = \tau(h \times -1 : X \times S^1 \longrightarrow X \times S^1) \in Wh(\pi \times \mathbb{Z}) .$$

Assume now that \mathbb{A} has a stable canonical structure. An \mathbb{A} -finite structure on a chain complex C in \mathbb{B} is an equivalence class of pairs (D, f) with D a finite chain complex in \mathbb{A} and $f : D \longrightarrow C$ a chain equivalence, subject to the equivalence relation

$$(C, f) \sim (C', f') \text{ if } \tau(f'^{-1}f : D \longrightarrow D') = 0 \in Wh(\mathbb{A}) .$$

Given a self chain equivalence $h : C \longrightarrow C$ of an \mathbb{A} -finitely dominated chain complex C in \mathbb{B} (as before) define a preferred $\mathbb{A}[z, z^{-1}]$ -finite structure (E, k) on the algebraic mapping torus

$$T^+(h) = T(h) = C(1 - zh : C[z, z^{-1}] \longrightarrow C[z, z^{-1}])$$

as in Ranicki [67, §6], using any \mathbb{A} -finite domination $(D, f : C \longrightarrow D, g : D \longrightarrow C, gf \simeq 1)$ of C , with $E = T(fhg : D \longrightarrow D)$. Define similarly a preferred $\mathbb{A}[z, z^{-1}]$ -finite structure on the opposite algebraic mapping torus

$$T^-(h) = C(1 - z^{-1}h : C[z, z^{-1}] \longrightarrow C[z, z^{-1}]) .$$

For any chain homotopy inverse $h^{-1} : C \longrightarrow C$ of h let

$$r : T^-(h^{-1}) \longrightarrow T^+(h)$$

be the chain equivalence defined by

$$r = \begin{pmatrix} -zh & e \\ 0 & 1 \end{pmatrix} : T^-(h^{-1})_n = C_n \oplus C_{n-1} \longrightarrow T^+(h)_n = C_n \oplus C_{n-1}$$

for the appropriate $e : C_{n-1} \longrightarrow C_n$. The torsion of r with respect to the preferred $\mathbb{A}[z, z^{-1}]$ -finite structures is the mixed invariant of h

$$\tau(r) = [C, h] \in Wh(\mathbb{A}[z, z^{-1}]) .$$

For any CW band X there is defined a homotopy commutative diagram

$$\begin{array}{ccc} T(\zeta^{-1}) & \xrightarrow{r} & T(\zeta) \\ & \searrow q^- & \swarrow q^+ \\ & X & \end{array}$$

with

$$r : T(\zeta^{-1}) \longrightarrow T(\zeta) ; (x, t) \longrightarrow (x, 1 - t) .$$

With the conventions of 8.15 the cellular $\mathbb{Z}[\pi \times \mathbb{Z}]$ -module chain complex

of the universal cover $\tilde{T}(\zeta)$ of $T(\zeta)$ is the algebraic mapping torus of $\tilde{\zeta}^{-1} : C(\tilde{X}) \longrightarrow C(\tilde{X})$

$$\begin{aligned} C(\tilde{T}(\zeta)) &= T^+(\tilde{\zeta}^{-1}) \\ &= C(1 - z\tilde{\zeta}^{-1} : C(\tilde{X})[z, z^{-1}] \longrightarrow C(\tilde{X})[z, z^{-1}]) . \end{aligned}$$

Now r induces the isomorphism of fundamental groups

$$r_* : \pi_1(T(\zeta^{-1})) = \pi \times \mathbb{Z} \longrightarrow \pi_1(T(\zeta)) = \pi \times \mathbb{Z} ; (g, n) \longrightarrow (g, -n) ,$$

and the induced $\mathbb{Z}[\pi \times \mathbb{Z}]$ -module chain equivalence

$$\tilde{r} : r_* C(\tilde{T}(\zeta^{-1})) \longrightarrow C(\tilde{T}(\zeta))$$

is given algebraically by

$$r : T^-(\tilde{\zeta} : C(\tilde{X}) \longrightarrow C(\tilde{X})) \longrightarrow T^+(\tilde{\zeta}^{-1} : C(\tilde{X}) \longrightarrow C(\tilde{X}))$$

with torsion the mixed invariant

$$\tau(r) = [C(\tilde{X}), \tilde{\zeta}^{-1}] = [\bar{X}, \zeta^{-1}] \in Wh(\pi \times \mathbb{Z}) .$$

Applying the sum formula for Whitehead torsion gives

$$\Phi^-(X) - \Phi^+(X) = \tau(r) = [\bar{X}, \zeta^{-1}] \in Wh(\pi \times \mathbb{Z}) .$$

Let now E be a chain complex band in $\mathbb{A}[z, z^{-1}]$, that is a finite chain complex such that $i^!E$ is $\mathbb{C}_0(\mathbb{A})$ -finitely dominated in $\mathbb{G}_1(\mathbb{A})$. The positive and negative end invariants $[E]_{\pm}$ are such that

$$[E]_+ + [E]_- = [i^!E] \in \tilde{K}_0(\mathbb{P}_0(\mathbb{A})) .$$

The universal Mayer-Vietoris presentation

$$\mathbb{E}\langle \infty \rangle : 0 \longrightarrow i_! i^! E \xrightarrow{1 - z\zeta^{-1}} i_! i^! E \xrightarrow{p} E \longrightarrow 0$$

determines chain equivalences in $\mathbb{G}_1(\mathbb{A})[[z, z^{-1}]]$

$$q^+ = (p \ 0) : T^+(\zeta^{-1}) = C(1 - z\zeta^{-1}) \longrightarrow E ,$$

$$q^- = (p \ 0) : T^-(\zeta) = C(1 - z^{-1}\zeta) \longrightarrow E .$$

Working as in Ranicki [67, §6] use a $\mathbb{C}_0(\mathbb{A})$ -finite domination of $i^!E$ in $\mathbb{G}_1(\mathbb{A})$

$$(D, f : i^!E \longrightarrow D, g : D \longrightarrow i^!E, gf \simeq 1)$$

to define a preferred $\mathbb{A}[z, z^{-1}]$ -finite structure (C, h) on $T^+(\zeta^{-1})$ with

$$C = T(f\zeta^{-1}g : D \longrightarrow D) ,$$

and similarly for $T^-(\zeta)$. The chain equivalence

$$r : T^-(\zeta) \longrightarrow T^+(\zeta^{-1})$$

is defined by

$$r = \begin{pmatrix} -z\zeta^{-1} & 0 \\ 0 & 1 \end{pmatrix} :$$

$$T^-(\zeta)_n = i^!E_n \oplus i^!E_{n-1} \longrightarrow T^+(\zeta^{-1})_n = i^!E_n \oplus i^!E_{n-1} ,$$

with torsion the mixed invariant

$$\tau(r : T^-(\zeta) \longrightarrow T^+(\zeta^{-1})) = [i^!E, \zeta^{-1}] \in Wh(\mathbb{A}[z, z^{-1}]) .$$

The *algebraic fibering obstructions* of a chain complex band E in $\mathbb{A}[z, z^{-1}]$ are defined by

$$\Phi^+(E) = \tau(q^+ : T^+(\zeta^{-1}) \longrightarrow E) ,$$

$$\Phi^-(E) = \tau(q^- : T^-(\zeta) \longrightarrow E) \in Wh(\mathbb{A}[z, z^{-1}]) ,$$

and are such that

$$\begin{aligned} \Phi^-(E) - \Phi^+(E) &= \tau(r : T^-(\zeta) \longrightarrow T^+(\zeta^{-1})) \\ &= [i^!E, \zeta^{-1}] \in Wh(\mathbb{A}[z, z^{-1}]) . \end{aligned}$$

The geometric fibering obstructions of a CW band X with $\pi_1(X) = \pi \times \mathbb{Z}$ are just the algebraic fibering obstructions of the cellular $\mathbb{Z}[\pi \times \mathbb{Z}]$ -module chain complex $C(\tilde{X})$ of the universal cover \tilde{X}

$$\Phi^\pm(X) = \Phi^\pm(C(\tilde{X})) \in Wh(\pi \times \mathbb{Z}) .$$

The algebraic mapping torus of a self chain equivalence $h : F \longrightarrow F$ of a finite chain complex F in \mathbb{A}

$$T(h) = C(1 - zh : F[z, z^{-1}] \longrightarrow F[z, z^{-1}])$$

has $\overline{T(h)} \simeq F$, $\zeta \simeq h^{-1}$, so that the algebraic fibering obstructions are given by

$$\begin{aligned} \Phi^+(T(h)) &= \tau(1 : T(h) \longrightarrow T(h)) = 0 \in Wh(\mathbb{A}[z, z^{-1}]) , \\ \Phi^-(T(h)) &= \tau(r : T^-(h^{-1}) \longrightarrow T^+(h)) = [F, h] = i_! \tau(h) \\ &\in \text{im}(i_! : Wh(\mathbb{A}) \longrightarrow Wh(\mathbb{A}[z, z^{-1}])) . \end{aligned}$$

A chain complex band E in $\mathbb{A}[z, z^{-1}]$ is simple chain equivalent to $T(h)$ for a simple self chain equivalence $h : F \longrightarrow F$ of a finite chain complex F in \mathbb{A} if and only if

$$\Phi^+(E) = \Phi^-(E) = 0 \in Wh(\mathbb{A}[z, z^{-1}]) .$$

$\Phi^+(E)$ has image

$$\begin{aligned} (B \oplus N_+ \oplus N_-) \Phi^+(E) \\ = (-[E]_-, [i^!E/\zeta^{-N^+}E^+, \zeta], [i^!E/\zeta^{N^-}E^-, \zeta^{-1}]) \\ \in \tilde{K}_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) , \end{aligned}$$

for any $N = (N^+, N^-) \in \mathbb{N}^f(E)$. Similarly, $\Phi^-(E)$ has image

$$\begin{aligned} (B \oplus N_+ \oplus N_-) \Phi^-(E) &= ([E]_+, [i^!E/\zeta^{N^-}E^-, \zeta^{-1}], [i^!E/\zeta^{-N^+}E^+, \zeta]) \\ &\in \tilde{K}_0(\mathbb{P}_0(\mathbb{A})) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) \oplus \widetilde{\text{Nil}}_0(\mathbb{A}) . \end{aligned}$$

The algebraic fibering obstructions are absolute invariants of chain

complexes which determine the Whitehead torsion, by analogy with Reidemeister torsion. (In the geometric context this was already observed by Siebenmann [74], [76]). Specifically, the Whitehead torsion of a chain equivalence $f : E' \longrightarrow E$ of chain complex bands in $\mathbb{A}[z, z^{-1}]$ is given by

$$\begin{aligned}\tau(f) &= \Phi^+(E) - \Phi^+(E') \\ &= \Phi^-(E) - \Phi^-(E') \in Wh(\mathbb{A}[z, z^{-1}]) ,\end{aligned}$$

since there are defined commutative squares of chain equivalences

$$\begin{array}{ccc} T^\pm((\zeta')^{\mp 1}) & \longrightarrow & T^\pm(\zeta^{\mp 1}) \\ (q')^\pm \downarrow & & \downarrow q^\pm \\ E' & \xrightarrow{f} & E \end{array}$$

with

$$\tau(T^\pm((\zeta')^{\mp 1}) \longrightarrow T^\pm(\zeta^{\mp 1})) = 0 \in Wh(\mathbb{A}[z, z^{-1}]) .$$

In particular, if E is a contractible finite chain complex in $\mathbb{A}[z, z^{-1}]$ the fibering obstructions are given by the torsion

$$\Phi^+(E) = \Phi^-(E) = \tau(E) \in Wh(\mathbb{A}[z, z^{-1}]) .$$

If \mathbb{A} has an involution then the n -dual of a chain complex band E in $\mathbb{A}[z, z^{-1}]$ is a chain complex band E^{n-*} with algebraic fibering obstructions given by

$$\Phi^\pm(E^{n-*}) = (-)^{n-1} \Phi^\mp(E)^* \in Wh(\mathbb{A}[z, z^{-1}]) .$$

For a pair $(\mathbb{B}, \mathbb{A} \subseteq \mathbb{B})$ of additive categories with involution the dual of the mixed invariant $[C, h] \in Wh(\mathbb{A}[z, z^{-1}])$ of a self chain equivalence $h : C \longrightarrow C$ of an \mathbb{A} -finitely dominated chain complex C in \mathbb{B} is given by

$$\begin{aligned}[C, h]^* &= \tau(-zh : C[z, z^{-1}] \longrightarrow C[z, z^{-1}])^* \\ &= \tau(-z^{-1}h^* : C^*[z, z^{-1}] \longrightarrow C^*[z, z^{-1}]) \\ &= -[C^*, (h^*)^{-1}] \in Wh(\mathbb{A}[z, z^{-1}]) .\end{aligned}$$

If (E, θ) is an n -dimensional symmetric Poincaré complex in $\mathbb{A}[z, z^{-1}]$ with E a chain complex band then $(i^!E, i^!\theta)$ is an \mathbb{A} -finitely dominated $(n-1)$ -dimensional symmetric Poincaré complex in $\mathbb{F}_1(\mathbb{A})$. The ‘covering translation’

$$\zeta : i^!E \longrightarrow i^!E ; x \longrightarrow zx$$

defines a self chain equivalence

$$\zeta : (i^!E, i^!\theta) \longrightarrow (i^!E, i^!\theta)$$

such that

$$\begin{aligned} [i^!E, \zeta^{-1}]^* &= -[i^!E]^*, \zeta^*] = (-)^n [i^!E, \zeta^{-1}] \in Wh(\mathbb{A}[z, z^{-1}]) , \\ \tau(\zeta)^* &= (-)^n \tau(\zeta) \in Wh(\mathbb{A}) , \quad [i^!E]^* = (-)^{n-1} [i^!E] \in \widetilde{K}_0(\mathbb{P}_0(\mathbb{A})) . \end{aligned}$$

The torsion

$$\tau(E, \theta) = \tau(\theta_0 : E^{n-*} \longrightarrow E) \in Wh(\mathbb{A}[z, z^{-1}])$$

is such that

$$\begin{aligned} \tau(E, \theta) &= \Phi^+(E) - \Phi^+(E^{n-*}) = \Phi^+(E) + (-)^n \Phi^-(E)^* \\ &= \Phi^+(E) + (-)^n \Phi^+(E)^* + [i^!E, \zeta^{-1}] \in Wh(\mathbb{A}[z, z^{-1}]) , \\ B\tau(E, \theta) &= [E]_+ + (-)^n ([E]_-)^* \in \widetilde{K}_0(\mathbb{P}_0(\mathbb{A})) , \\ N_{\pm} \tau(E, \theta) &= N_{\pm} \Phi^+(E) + (-)^n (N_{\mp} \Phi^+(E))^* \\ &= -([i^!E / \zeta^{\mp N^{\pm}} E^{\pm}, \zeta^{\pm 1}] + (-)^n [i^!E / \zeta^{\pm N^{\mp}} E^{\pm}, \zeta^{\mp 1}]^*) \\ &\quad \in \widetilde{\text{Nil}}_0(\mathbb{A}) . \end{aligned}$$

The algebraic mapping torus $T(\zeta)$ is an n -dimensional symmetric Poincaré complex in $\mathbb{F}_1(\mathbb{A})[z, z^{-1}]$ with a preferred $\mathbb{A}[z, z^{-1}]$ -finite structure, with respect to which

$$\tau(T(\zeta)) = [i^!E, \zeta^{-1}] = (-)^n [i^!E, \zeta^{-1}]^* \in Wh(\mathbb{A}[z, z^{-1}]) .$$

Any chain equivalence $E \simeq T(h)$ to the algebraic mapping torus of a self chain equivalence $h : F \longrightarrow F$ of an \mathbb{A} -finitely dominated chain complex F can be lifted to a chain equivalence $i^!E \simeq F$, and F supports an $(n-1)$ -dimensional symmetric Poincaré structure ϕ preserved by h , such that there is defined a homotopy equivalence of homotopy \mathbb{A} -finite n -dimensional symmetric Poincaré complexes

$$(E, \theta) \simeq T(h : (F, \phi) \longrightarrow (F, \phi)) .$$

Thus $\Phi^+(E) = \Phi^-(E) = 0 \in Wh(\mathbb{A}[z, z^{-1}])$ if and only if (E, θ) is simple homotopy equivalent to the algebraic mapping torus $T(h)$ of a simple self homotopy equivalence $h : (F, \phi) \longrightarrow (F, \phi)$ of an $(n-1)$ -dimensional symmetric Poincaré complex (F, ϕ) in \mathbb{A} , in which case (E, θ) (but not necessarily (F, ϕ)) is simple.

For a simple n -dimensional symmetric Poincaré complex (E, θ) in $\mathbb{A}[z, z^{-1}]$

$$\tau(E, \theta) = \Phi^+(E) + (-)^n \Phi^-(E)^* = 0 \in Wh(\mathbb{A}[z, z^{-1}]) ,$$

and

$$N_+ \tau(E, \theta) = N_+ \Phi^+(E) + (-)^n (N_- \Phi^+(E))^* = 0 \in \widetilde{\text{Nil}}_0(\mathbb{A}) ,$$

so the two $\widetilde{\text{Nil}}$ -components of $\Phi^+(E)$ are related by

$$N_+ \Phi^+(E) = (-)^{n-1} (N_- \Phi^+(E))^* \in \widetilde{\text{Nil}}_0(\mathbb{A}) .$$

Since

$$\Phi^+(E) = (-)^{n-1} \Phi^-(E)^* \in Wh(\mathbb{A}[z, z^{-1}])$$

there is redundancy in the two algebraic fibering conditions

$$\Phi^+(E) = \Phi^-(E) = 0 \in Wh(\mathbb{A}[z, z^{-1}]) ,$$

and $\Phi^+(E) = 0 \in Wh(\mathbb{A}[z, z^{-1}])$ if and only if (E, θ) is simple homotopy equivalent to the algebraic mapping torus $T(h)$ of a simple self homotopy equivalence $h : (F, \phi) \longrightarrow (F, \phi)$ of an $(n-1)$ -dimensional symmetric Poincaré complex (F, ϕ) in \mathbb{A} . For any such (F, ϕ) the Tate \mathbb{Z}_2 -cohomology class of the torsion

$$\tau(F, \phi) = (-)^{n-1} \tau(F, \phi)^* \in Wh(\mathbb{A})$$

is the image of the Wh_2 -invariant

$$\tau_2(E, \theta) \in \hat{H}^n(\mathbb{Z}_2; Wh_2(\mathbb{A}[z, z^{-1}]))$$

under the split surjection induced by the Wh_2 -analogue

$$B_2 : Wh_2(\mathbb{A}[z, z^{-1}]) \longrightarrow Wh(\mathbb{A})$$

of $B : Wh(\mathbb{A}[z, z^{-1}]) \longrightarrow \tilde{K}_0(\mathbb{P}_0(\mathbb{A}))$

$$\tau(F, \phi) = B_2 \tau_2(E, \theta) \in \hat{H}^{n-1}(\mathbb{Z}_2; Wh(\mathbb{A})) .$$

There exists a simple homotopy equivalence

$$(E, \theta) \simeq T(h : (F, \phi) \longrightarrow (F, \phi))$$

with

$$\tau(h : F \longrightarrow F) = \tau(F, \phi) = 0 \in Wh(\mathbb{A})$$

if and only if

$$\Phi^+(E) = 0 \in Wh(\mathbb{A}[z, z^{-1}]) ,$$

$$B_2 \tau_2(E, \theta) = 0 \in \hat{H}^{n-1}(\mathbb{Z}_2; Wh(\mathbb{A})) .$$

The Wh_2 -invariant is implicit in the way Wall [84, §12B] used fibering obstruction theory to identify the codimension 1 splitting obstruction groups LS_* in the two-sided non-separating case with $\hat{H}^{*+1}(\mathbb{Z}_2; Wh(\pi))$. Indeed, the groups M_* of [84, Thm. 12.6] are the higher L -groups decorated by $\text{im}(i! : Wh_2(\pi) \longrightarrow Wh_2(\pi \times \mathbb{Z}))$. The fibering Wh_2 -invariant is closely related to the pseudo-isotopy Wh_2 -invariant of Hatcher and Wagoner [33], and appears explicitly in the work of Kinsey [41] on the uniqueness of fibre bundle structures over S^1 .

If X is an n -dimensional geometric Poincaré complex which is a CW band then the torsion of $T(\zeta)$ with respect to the preferred finite struc-

ture is such that

$$\begin{aligned}\tau(T(\zeta)) &= \tau(r : T(\zeta^{-1}) \longrightarrow T(\zeta)) = [\bar{X}, \zeta^{-1}] = (-)^n [\bar{X}, \zeta^{-1}]^* , \\ \tau(X) &= \tau(T(\zeta)) + \Phi^+(X) + (-)^n \Phi^+(X)^* \\ &= \Phi^+(X) + (-)^n \Phi^-(X)^* \in Wh(\pi_1(X)) .\end{aligned}$$

A ‘candidate for fibering’ in the sense of Siebenmann [76] is a compact n -dimensional manifold X with a finitely dominated infinite cyclic cover \bar{X} . Again, only the case $\zeta_* = 1 : \pi = \pi_1(\bar{X}) \longrightarrow \pi_1(\bar{X})$ is considered here, so that $\pi_1(X) = \pi \times \mathbb{Z}$. For $n \neq 4$ X is a CW band, since it is a CW complex via the handlebody decomposition, but in any case X has a preferred intrinsic finite structure. Any finite CW complex Y in the preferred simple homotopy type of X is a CW band which is a simple n -dimensional geometric Poincaré complex. The algebraic fibering obstruction of X is defined by

$$\Phi^+(X) = \Phi^+(C(\tilde{X})) \in Wh(\pi \times \mathbb{Z}) ,$$

interpreting $C(\tilde{X})$ as $C(\tilde{Y})$ if $n = 4$. $\Phi^+(X)$ is the total fibering obstruction of [76] and Farrell [22], such that $\Phi^+(X) = 0$ if (and for $n \geq 6$ only if) X fibers over S^1 , i.e. is homeomorphic to the mapping torus $T(h)$ of a self-homeomorphism $h : F \longrightarrow F$ of a compact $(n-1)$ -dimensional manifold F , with the infinite cyclic cover \bar{X} \mathbb{Z} -homeomorphic to $T(h) = F \times \mathbb{R}$. The total fibering obstruction $\Phi^+(X)$ is related to the original two-stage fibering obstruction theory of Farrell [21] by the geometrically significant split exact sequence

$$\begin{aligned}0 \longrightarrow Wh(\pi) &\xrightarrow{i_!} Wh(\pi \times \mathbb{Z}) \xrightarrow{B \oplus N_+ \oplus N_-} \\ &\tilde{K}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\text{Nil}}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\text{Nil}}_0(\mathbb{Z}[\pi]) \longrightarrow 0 .\end{aligned}$$

The images

$$B\Phi^+(X) = -[\bar{X}^-] = -[i^! C(\tilde{X}) / \zeta^{-N^+} C(\tilde{X})^+] \in \tilde{K}_0(\mathbb{Z}[\pi]) ,$$

$$N_+ \Phi^+(X) = [i^! C(\tilde{X}) / \zeta^{-N^+} C(\tilde{X})^+, \zeta] \in \widetilde{\text{Nil}}_0(\mathbb{Z}[\pi])$$

are the primary obstructions of [21], being the components of the splitting obstruction of Farrell and Hsiang [23] (cf. 10.9); if these vanish and $n \geq 6$ there exists a fundamental domain $(V; U, \zeta U)$ for \bar{X} which is an h -cobordism, with the torsion $\tau(V; U, \zeta U) \in Wh(\pi)$ the secondary obstruction, such that

$$\begin{aligned}\Phi^+(X) &= i_! \tau(V; U, \zeta U) \\ &\in \ker(B \oplus N_+ \oplus N_- : Wh(\pi \times \mathbb{Z}) \\ &\longrightarrow \tilde{K}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\text{Nil}}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\text{Nil}}_0(\mathbb{Z}[\pi])) \\ &= \text{im}(i_! : Wh(\pi) \longrightarrow Wh(\pi \times \mathbb{Z})) .\end{aligned}$$

Thus $\Phi^+(X) = 0$ if and only if $(V; U, \zeta U)$ is an s -cobordism, which by the s -cobordism theorem is homeomorphic to a product $U \times ([0, 1]; \{0\}, \{1\})$. Since X is simple

$$\tau(X) = \Phi^+(X) + (-)^n \Phi^+(X)^* + [\bar{X}, \zeta^{-1}] = 0 \in Wh(\pi \times \mathbb{Z}) ,$$

$$B\tau(X) = [\bar{X}^+] + (-)^n [\bar{X}^-]^* = 0 \in \tilde{K}_0(\mathbb{Z}[\pi]) .$$

The identity

$$[\bar{X}^+] = (-)^{n-1} [\bar{X}^-]^* \in \tilde{K}_0(\mathbb{Z}[\pi])$$

is a generalization of the end obstruction duality theorem of Siebenmann [73, §11]. Since X is a manifold $\tau_2(X) = 0 \in Wh_2(\pi \times \mathbb{Z})$, so $\Phi^+(X) \in Wh(\pi \times \mathbb{Z})$ is the only algebraic obstruction to X being simple homotopy equivalent to the mapping torus $T(h)$ of a simple self homotopy equivalence $h : F \longrightarrow F$ of a simple $(n-1)$ -dimensional geometric Poincaré complex F .

Siebenmann, Guillou and Hähl [77, 5.13] developed a version of the fibering obstruction theory for Hilbert cube manifolds. In the corrected version of this theory (Chapman and Siebenmann [19, p.208]) there are two independent obstructions, which are precisely the two algebraic fibering obstructions $\Phi^+(E)$, $\Phi^-(E)$ of the associated chain complex band E .

It is possible to extend the algebraic fibering obstruction theory to fibrations over manifolds other than S^1 , which in the non-simply-connected case necessarily involves both lower K - and lower L -theory. For the geometric fibering obstruction theory see Casson [17], Quinn [55], Burghelea, Lashof and Rothenberg [14] and Levitt [43]. For fibrations over T^n part of the obstruction concerns the expression of a non-compact manifold as a product $M \times \mathbb{R}^n$ - see Bryant and Pacheco [13] for the corresponding obstruction theory. There is also a connection with the geometric theory developed by Hughes, Taylor and Williams [37] for relating manifold approximation fibrations and bundles.

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