Lectures on Deformations of Singularities

By

Michael Artin

Tata Institute of Fundamental Research
Bombay
1976

Printed in India
Anil D. Ved At Prabhat Printers. Bombay 400 004
And Published By
The Tata Institute of Fundamental Research.

Introduction

These notes are based on a series of lectures given at the Tata Institute in January-February, 1973. The lectures are centered about the work of M. Scahlessinger and R. Elkik on infinitesimal deformations. In general, let X be a flat scheme over a local Artin ring R with residue field k. Then one may regard X as an infinitesimal deformation of the closed fiber $x_0 = X \underset{\mathrm{Spec}(R)}{\times} \mathrm{Spec}(k)$. Schlessinger's main result proven in part (for more information see his Harvard Ph.D. thesis) is the construction, under certian hypotheses, of a "versal deforamtion space" for X_0. He shows that $\exists$ a *complete* local k-algebra $A = \lim A/m_A^n$ and a sequence of deformations X_n over $\mathrm{Spec}(A/m_A^n)$ such that the formal A-prescheme $\mathscr{X} = \varinjlim X_n$ is versal in this sense: For all Artin local rings R, every deformation $S/S\,pec(R)$ of X_0 may be obtained from some homomorphism $A \to R$ by setting $X = \mathscr{X} \underset{\mathrm{Spec}(A)}{\times} \mathrm{Spec}(R)$.

Note that by "versal deformation" we do not mean that there is in fact a deformation of X_0 over A. The versal deforamtion is given only as a formal scheme. However, Elkik has proven (cf. "Algebrisation du module formel d'une singularite isolée" Séminaire, E.N.S., 1971-72) that such a deformation of X_0 over A does exist when X_0 is a affine and has isolated singularities. We give a proof of this result in these lectures.

Finally, some of the work of M. Schaps, A. Iarrobino, and H. Pinkham is considered here. We prove schaps's result that every Cohen-Macaulay affine scheme of pure codimension 2 is determinantal. Moreover, we outline the proof of her result that every unmixed Cohen-Macaulay scheme of codimension 2 in an affine space of dimension < 6 has nonsingular deformations. For more details see her Harvard

Ph. D. thesis, "Deforamtions of Cohen-Macaulay schemes of codimension 2 and non-singular deformation of space curves". We also reproduce Iarrobino's counterexample (cf. "Reducibility of the families of 0-dimensional schemes on a variety", University of Texas, 1970) that not every 0-dimensional projective scheme (in $\mathbb{P}$ for $n > 2$) has a smooth deformation. Finally, we give some of Pinkham's results on deformations of cones over rational curves (cf. his Harvard Ph.D. thesis, "Deformations of algebraic varieties with $\mathbb{G}_m$ action).

There is of course much more literature in this subject. Two papers relevant to these notes are Mumford's "Pathologies-IV" (Amer. J. Math., Vol. XCVII, p. 847-849) in which expanding on Iarrobino's methods he proves that not every 1-dimensional scheme has nonsingular deformations, and M. Artin's "Versal deformations and algebraic stacks" (Inventions mathematicae, vol. 27, 1974), in which is shown that formal versality is an open condition.

Thanks are due to the Institute for Advanced study for providing excellent facilities in getting this manuscript typed.

Contents

<u>**Deformations of Singularities**</u>

Part 1

Formal Theory and Computations

1 Definition of deformations

1

We work over an algebraically closed field k.

Let X be an affine scheme, $X \hookrightarrow \mathbb{A}^n$. Let A be finite (i.e., finite-dimensional/k) local algebra over k, so that $A \approx k[t_1, \ldots, t_r]/\mathfrak{a}$ with $\sqrt{\mathfrak{a}} = (t_1, \ldots, t_r)$. Let $T = \operatorname{Spec} A$.

Definition 1.1. An *infinitesimal deformation* X_T(or X_A) of X is a scheme *flat*/T together with a k-isomorphism $X_T \times_T \operatorname{Spec} k \xrightarrow{\sim} X$.

More generally, suppose we are given a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \overline{X} \arrow[d, "\text{flat}"] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

where R is a ring of finite type over k and $\overline{X}$ is a scheme with closed fiber isomorphic to X. We then say that $\overline{X}$ is a *family of deformations* or a *deformation* of X over R.

Remark 1.1. X_T is necessarily affine. In fact $\exists$ a closed immersion $X_T \hookrightarrow \mathbb{A}^n_T(= \operatorname{Spec} A[X_1, \ldots, X_n])$ such that its base change with respect to the morphism $\operatorname{Spec} k \to T$ (representing the closed point of $\operatorname{Spec} A$ is the immersion $X \hookrightarrow \mathbb{A}^n (= \mathbb{A}^n_k)$.

Let $\mathscr{O}$ = coordinate ring of X and $\mathscr{O} = k[X_1, \ldots, X_n]/I$, with x_i the
2 canonical images of X_i in $\mathscr{O}$. To prove the remark, it suffices to prove that if $X_A \hookrightarrow \mathbb{A}^n_A$ is an affine scheme (over A) imbedded in $\mathbb{A}^n_A, A'$ is a finite local algebra/k such that

$$0 \to J \to A' \to A \to 0 \quad \text{is exact}$$

with $J^2 = 0$ (J is an ideal of square 0) and $X_{A'}$ is a scheme/A' such that $X_{A'} \times_{\operatorname{Spec} A'} \operatorname{Spec} A \simeq X_A$, then the immersion $X_A \hookrightarrow \mathbb{A}^n_A$ can be lifted to an immersion $X_{A'} \hookrightarrow \mathbb{A}^n_{A'}$. (This reduction is immediate since the maximal ideal is nilpotent. Say that $m^{\rho}_{A'} = 0$, and take $J = m^{\rho-1}_{A'}, \ldots$). We have an exact sequence

$$0 \to I \to \mathscr{O}_{X_{A'}} \to \mathscr{O}_{X_A} \to 0,$$

where $\mathscr{O}_{X_{A'}}$, denotes the structure sheaf of $X_{A'}$. Now $I^2 = 0$ since $J^2 = 0$. This implies that I, which is *a priori* a (sheaf of) $\mathscr{O}_{X_{A'}}$ module(s), is in fact a module over $\mathscr{O}_{X_{A'}}/I$, i.e., it acquires a canonical structure of coherent $\mathscr{O}_{X_A}$ -module. Since X_A is affine, it follows that $H^1(X_{A'}, I) = 0$ (for it is $= H^1(X_A, I)$). Hence

$$0 \to H^o(X_{A'}, I) \to H^o(X_{A'}, X_{A'}) \to H^o(X, X_A) \to 0$$

is exact, i.e., in particular $H^o(X_{A'}, \mathscr{O}_{X_{A'}}) \to H^o(X, \mathscr{O}_{X_A}) \to 0$ is exact. Let x_i be the coordinate functions on X_A which define $X_A \hookrightarrow \mathbb{A}^n_A$. The x_i can be lifted to $\xi_i \in H^o(X_{A'}, \mathscr{O}_{X_{A'}})$. Then the ξ_i define a morphism $\xi : X_{A'} \to \mathbb{A}^n_{A'}$. It follows at first that ξ is a *local* immersion; for this
3 it suffices to prove that ξ_1 generate the local ring $\mathscr{O}_{X_{A',x}}$ at every closed point x of $X_{A'}$. Let I_x be the stalk of the ideal sheaf I at a closed point x of X. We have $0 \to I_x \to \mathscr{O}_{A',x} \to \mathscr{O}_{A,x} \to 0 (\mathscr{O}_{A',x'} \mathscr{O}_{A,x}$ represent the local rings at x of $X_{A'}$ and X_A respectively). We have $I_x = J \cdot \mathscr{O}_{A',x'}$. Now $j \cdot \theta_1 = j \cdot \theta_2$ for $j \in J$ and θ_1, θ_2 in $\mathscr{O}_{A',x}$ such that their canonical images in $\mathscr{O}_{A,x}$ are the same. Let S be the subalgebra of $\mathscr{O}_{A',x}$ generated by ξ_i over A'. Then we see that $I_x = JS$. Since $J \subset A'$ it follows that $I_x \subset S$.

Since S maps onto $\mathcal{O}_{A,x'}$ given $\lambda \in \mathcal{O}_{A',x'} \exists s \in S$ such that $\lambda - s \in I_x$. This implies that $\lambda \in S$. This proves $S = \mathcal{O}_{A,x'}$. We conclude then that $\xi : X_{A'} \hookrightarrow \mathbb{A}^n_{A'}$ is a local immersion. But ξ is a proper *injective* map (since $X_A \hookrightarrow^n_A$ is a closed immersion). From this it follows that ξ is a closed immersion. This proves the Remark.

Note that in the above proof we have *not used the fact* that $X_{A'}$ is *flat*/A'.

Given the closed subscheme $X \hookrightarrow \mathbb{A}^n_k$ let us define the following two functors on the category of finite local algebras over k.

(Def. X) : (Finite local alg) $\to$ (Sets)
||
{Deformations of X/A}$\mapsto${isom. classes (over A) of schemes X flat/A
and suct that $X_A \otimes k \simeq X$}

(Emb. def. X):(Finite local alg)$\to$(Sets)

{Embedded deformations/A $\mapsto$ closed subschemes X_A of $\mathbb{A}^n_A$ flat/A, such that $X_A \hookrightarrow \mathbb{A}^n_A$ by base change is the given $X_A \hookrightarrow \mathbb{A}^n_k$}

These should be called respectively *infinitesimal deformations* of X **4**
and *infinitesimal embedded deformations* of X. Then we have a canonical morphism of functors

$$(Emb.def.X) \xrightarrow{f} (Def.X).$$

The above Remark says that f is *formally smooth*; that is, if $A' \to A \to 0$ is exact in (Finite local alg), we have

$$\begin{tikzcd}
(Emb.def.X)(A') \arrow[r] \arrow[d] & (Def.X)(A') \arrow[d] \\
(Emb.def.X)(A') \arrow[r] & (Def.X)(A)
\end{tikzcd}$$

(by definition of a morphism of functors), and the canonial map

$$(Emb.def.X)(A') \to (Def.X)(A') \times_{(def.X)(A)} (Emb.def.X)(A)$$

is *surjective*.

2 Iarrobino's example of a 0-dimensional scheme which is not a specialization of d distinct points

Given X as above, we can ask whether it can be "deformed" into a non-singular scheme. Here by a *deformation* we do not mean an infinitesimal one, but a general family of deformations. Let us consider the simplest case of Krull dimension 0. Then $X = \operatorname{Spec}\mathscr{O}$ where $\mathscr{O}$ is a k-algebra of finite dimension d. If $d = 1$, $\mathscr{O} \approx k$. If $d = 2$, $\mathscr{O} \approx k \times k$ or $\mathscr{O} \approx k[t]/(t^2)$. If $d = 3$, we have

$$\mathscr{O} = \begin{cases} k^3 \quad \text{or} \quad k \times k[t]/t^2 \\ \text{or} \quad k[t]/t^3 \quad \text{or} \quad k[X,Y]/(X,Y)^2 \end{cases}$$

In our particular case the question is whether X can be deformed into d distinct points of $\mathbb{A}^n$. Now $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$ and we see easily that this deformation implies a "deformation" of closed subschemes of $\mathbb{P}^n$, i.e., if X can be deformed into d "distinct points" we see (without much difficulty) that this can be done as an "embedded deformation" in $\mathbb{A}^n$ and in fact as an embedded deformation in $\mathbb{P}^n$. Let Hilb_d denote the Hilbert scheme of 0-dimensional subschemes $Z \hookrightarrow \mathbb{P}^n$ such that if $Z = \operatorname{Spec} B$ then B is of dim d over k. Then it is known that Hilb_d is projective/k. Let U_d denote the *open* sub-scheme of Hilb_d corresponding to d distinct points, i.e., those closed sub-schemes of $\mathbb{P}^n$ corresponding to points of Hilb_d which are smooth. We see that U_d is irreducible; in fact it is d-fold symmetric product of $\mathbb{P}^n$ minus the "diagonals". Now if every 0-dimensional subscheme can be deformed into a nonsingular one, then U_d is dense in Hilb_d and it follow that *$Hilb_d$ is irreducible*.

We shall now give the counterexample (due to Iarrobino) where Hilb_d is *not* irreducible. It follows therefore that a 0-dimensional scheme cannot in general be deformed to a smooth one.

Theorem 2.1. *Let $Hilb_{d,n}$ denote the Hilbert scheme of closed* 0*-dimensional subschemes of $\mathbb{P}^n$ of length d. Then $Hilb_{d,n}$ is* irreducible *for $n \leq 2$. For $n > 2$, $\mathrm{Hilb}_{d,n}$ is* not *in general* irreducible.

Proof. We give only the counter example for the case $n > 2$. Let $\mathscr{O}' = k[X_1, \dots, X_n]/(X_1, \dots, X_n)^{r+1}$. Let I be the ideal in $(X)^r/(X)^{r+1}$ (where

$(X) = (X_1, \ldots, X_n))$. Then I is a vector space over k and

$$\begin{aligned} \text{Rank of} \quad I/k &= \text{Polynomial of degree} \quad (n-1) \quad \text{in} \quad r \\ \text{Rank of} \quad \mathscr{O}'/k &= \text{Polynomial of degree} \quad n \quad \text{in} \quad r. \end{aligned}$$

$\square$ **6**

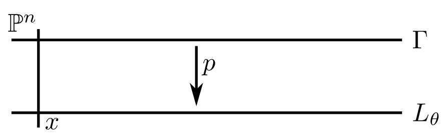
Now I is an ideal in $\mathscr{O}'$ and in particular an $\mathscr{O}'$ module. Moreover, if $\lambda \in$ max ideal of $\mathscr{O}'$ then $\lambda \cdot I = 0$. Hence $\mathscr{O}'$ operated on I through its residue field. In particular, it follows that *any linear* subspace (over k) of I is an ideal in $\mathscr{O}'$ and hence defines a closed subscheme of Spec $\mathscr{O}'$. Take now $\theta = \frac{1}{2}$ Rank I or $\frac{1}{2}$ Rank $I - \frac{1}{2}$ according as Rank I is even or odd and $d = \text{Rank}(\mathscr{O}'/V)$, where V is a subspace of I of rank θ. Hence $\theta =$ Polynomial of degree$(n-1)$ in r and $d =$ polynomial of degree n in r. Let us now count the dimension of the set L_θ of all linear subspaces of I of rank $= \theta$. Then it is a Grassmannian and $dimL_\theta = \left(\frac{1}{2}\text{ Rank } I\right)^2$ or $\left(\frac{1}{2}\text{ Rank } I - \frac{1}{2}\right)\left(\frac{1}{2}\text{ Rank } I + \frac{1}{2}\right)$ according as Rank I is even or odd=(Polynomial of *deg* $n-1)^2$ in $r =$ *Polynomial of degree* $(2n-2)$ *in* r. Now if U_d is the subscheme of Hilb_d of "d distinct points" as above, then

$$\dim U_d = d.n = n(\text{Polynomial of degree} \quad n \quad \text{in} \quad r).$$

Now if $r \gg 0$, we see that

$$\dim L_\theta > \dim U_d.$$

Since L_θ can be identified as a subscheme of Hilb_d, it follows now that U_d is *not* dense in Hilb_d. To see that L_θ is a subscheme of *Hilb*$_d$, note that Spec $\mathscr{O}/I$ as a point set consists only of one point. The family of subschemes of $\mathbb{P}^n$ parametrized by L_θ as a point given by $(x_o \times L_\theta)$. It suffices to check that on $(x \times L_\theta)$, we have a natural structure of a scheme Γ such that



Γ is a closed subscheme of $\mathbb{P}^n \times L_\theta$ and $p^{-1}(x)$ is the subscheme of Spec $\mathscr{O}/I$ defined by the linear subspace of I corresponding to x. Then 7
we see that $p : \Gamma \to L_\theta$ is flat, for p is a finite morphism over L_θ such that $\mathscr{O}_\Gamma$ is a sheaf of $\mathscr{O}_{L_\theta}$ -algebras; in particular, $\mathscr{O}_\Gamma$ becomes a coherent sheaf over $\mathscr{O}_{L_\theta}$. At every $x \in L_\theta$, for $\mathscr{O}_\Gamma \otimes \mathscr{O}_{L_\theta}, x/\text{maxideal}(= \mathscr{O}_\Gamma \otimes k)$, the rank is the same. L_θ is reduced, this implies that $\mathscr{O}_\Gamma$ is locally free over $\mathscr{O}_{L_\theta}$. In particular, Γ is flat/L_θ.

3 Meaning of flatness in terms of relations

A module M over a ring A is said to be *flat* if the functor $N \mapsto M \otimes_A$ is *exact*.

$$\Leftrightarrow Tor_q^A(M, N) = 0 \quad \forall N/A.$$
$$\Leftrightarrow Tor_l^A(M, N) = 0 \quad \forall \text{finitely generated} N/A.$$

Let us now consider the case when A is a finite *local k-algebra*. Then if N is an A-module of finite type, there is a composition series

$$N = N_0 \supset N_1 \supset \ldots \supset N_\ell = 0, \quad \text{such that} \quad N_i/N_i + 1 \approx k.$$

From this it follows immediately that

$$M flat/A \Leftrightarrow Tor_1(M, k) = 0 \quad (\text{if } A \text{ finite local } alg/k).$$

Let X_A = Spec $\mathscr{O}_A$, A *finite local algebra* and $\mathscr{O}_A$ an A-algebra of *Iinite type* so that we have

$$0 \to I_A \to P_A \to \mathscr{O}_A \to 0$$

exact with $P_A A[X](X = (X_1, \ldots, X_n))$. Tensor this with k. Then we have

$$o \to Tor_1(\mathscr{O}_A, k) \to I_A \otimes k \to P_k \to \mathscr{O}_k \to 0$$

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Let $X = \operatorname{Spec} \mathcal{O}_k$, $X \approx X_A \otimes k$, I ideal of X in $\mathbb{A}^n$. Then

$$\begin{aligned} X_A \text{is flat}/A &\Leftrightarrow Tor_1(\mathcal{O}_A, k) = 0, \\ &\Leftrightarrow I_A \otimes k = I. \end{aligned}$$

Take a presentation for the ideal I_A in P_A, i.e., an exact sequence

$$(*) \qquad \begin{tikzcd} P_A^\ell \arrow[r] & P_A^m \arrow[rr] \arrow[dr] & & P_A \arrow[r] & \mathcal{O}_A \arrow[r] & 0 \\ & & I_A \arrow[ur] \arrow[dr] & & & \\ & 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

Then we have (because of the above facts): $\mathcal{O}_A$ is A-*flat* $\Leftrightarrow$ *the above presentation for* I_A tensored by k, *gives a presentation for* I, i.e., *tensored by* k *gives again an exact sequence*

$$(**) \qquad \begin{tikzcd} P_k^\ell \arrow[r] & P_k^m \arrow[rr] \arrow[dr] & & P_k \arrow[r] & \mathcal{O} \arrow[r] & 0 \\ & & I \arrow[ur] \arrow[dr] & & & \\ & 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

Suppose we are given:

$\mathcal{O} = K[X]/(f_1, \ldots, f_m)$ and liftings (f_i') of (f_i) to elements in $A[X]$. Let $I = (f_1, \ldots, f_m)$, $I_A = (f_1', \ldots, f_m')$ and $\mathcal{O}_A = A[X](f_1', \ldots, f_m')$. These data are equivalent to giving a lifting $P_A^m \to P_A \to \mathcal{O}_A \to 0$ of the exact sequence $P^m \to P \to \mathcal{O} \to 0$, i.e., to giving an *exact sequence* $P_A^m \to P_A \to \mathcal{O}_A \to 0$ such that its $\otimes_A k$ is the given exact sequence 9
$P^m \to P \to \mathcal{O} \to 0$. *Note that this need not imply that* $I_A \otimes k = I$ *if* $I_A = \operatorname{Ker}(P_A \to \mathcal{O}_A)$ *and* $I = \operatorname{Ker}(P \to \mathcal{O})$.

Suppose we are now given a "complete set of relations" (or a presentation for I) between f_i's, i.e., an *exact sequence*

$$(*) \qquad P^\ell \to P^m \to P \to \mathcal{O} \to 0.$$

Then *giving a lifting of these relations* to that of $f_i^{'}$'s (or I_A) is to give

$$(**) \qquad P_A^\ell \to P_A^m \to P_A \to \mathcal{O}_A \to 0$$

which extends the exact sequence $P_A^m \to P_A \to \mathcal{O}_A \to 0$, which is a *complex* at P_A^m, i.e., $ImP_A^\ell \subset \mathrm{Ker}(P_A^m \to P_A)$ and such that $(**)$ lifts $(*)$. In this situation we have the following

Proposition 3.1. *Suppose*

$$(*) \qquad P^\ell \to P^m \to P \to \mathcal{O} \to 0$$

is exact *and*

$$(**) \qquad P_A^\ell \to P_A^m \to P_A \to \mathcal{O}_A \to 0$$

is a complex *such that the part*

$$P_A^m \to P_A \to \mathcal{O}_A \to 0$$

is exact and (**) $\otimes_A k = (*)$. *Then* $\mathcal{O}_A$ *is A-flat.*

Proof. Suppose first that

$$(*) \qquad P_A^\ell \to P_A^m \to P_A \to \mathcal{O}_A \to 0$$

10 is exact not merely a "complex at $P_A^{m_{11}}$. Then we claim that the flatness of $\mathcal{O}_A$ over A follows easily. For then (**) can be split up as:

$$\left.\begin{array}{c} P_A^\ell \to L_A \to 0, \quad 0 \to L_A \to P_A^m \to I_A \to 0 \\ 0 \to I_A \to P_A \to \mathcal{O}_A \to 0 \end{array}\right\} \quad \text{exact.}$$

Therefore $P_A^\ell \otimes k \to L_A \otimes k \to 0$ and $L_A \otimes k \to P_A^m \otimes k \to 0$ are exact. This implies that $I_A \otimes k = \mathrm{coker}(k \otimes P_A^\ell \to k \otimes P_A^m)$m i.e., cokernel is pre-served by $k \otimes_A$. On the other hand, $I = \mathrm{Coker}(P^\ell \to P^m)$. Hence $I_A \otimes k = I$. *From this it follow that* $\mathcal{O}_A$ *is flat/A as remarked before*. Now the hypotheses of our proposition amount to the fact all relations in I can be lifted to I_A. Given a relation in I_A, i.e., an m-tuple $(\lambda_1', \ldots, \lambda_m')$ such that $\sum \lambda_i' f_i' = 0$, this descends to a relation in 1 by taking the canonical images of $\lambda_i^{'}$ in P. Take a complete set of relations for I_A, i.e., an exact sequence

$$\text{(i)} \qquad P_A^{\ell'} \to P_A^m \to P_A \to \mathcal{O}_A \to 0 \qquad (\ell^{'} \quad \text{need not be} \quad \ell),$$

then from our above argument this descends to a complete set of relations for I, i.e., tensoring (i) by k we get an exact sequence

$$P^{\ell^{'}} \to P^m \to P \to \mathcal{O} \to 0.$$

Consequently, we have already shown in this situation that we must have $\mathcal{O}_A$ to be A-flat. □

The criterion for flatness can given be formulated as follows:

Corollary . *Let $\mathcal{O} = k[X]/(f_1, \ldots, f_m)$ and $\mathcal{O}_A = A[X]/(f_1^{'}, \ldots, f_m^{'})$ where $f_i^{'}$ are liftings of f_i. Then $\mathcal{O}_A$ is A-flat $\Leftrightarrow$ every relation among $(f_1, \ldots, f_m)$ lifts to a relation among $(f_1^{'}, \ldots, f_m^{'})$.* **11**

Remark 3.1. It is seen immediately that

$$\mathcal{O}_A \text{flat}/A \Rightarrow I_A \text{flat}/A.$$

For, the exact sequence $0 \to I_A \to P_A \to \mathcal{O}_A \to 0$ by tensoring by k gives

$$\begin{array}{ccccccccccccc} 0 & & & & 0 & & 0 & & & & & & \\ \| & & & & \| & & \| & & & & & & \\ \mathrm{Tor}_2(\mathcal{O}_A, k) & \to & \mathrm{Tor}_1(I_A, k) & \to & \mathrm{Tor}_1(P_A, k) & \to & \mathrm{Tor}_1(\mathcal{O}_A, k) & \to & 1 & \to & P_k & \to \mathcal{O} \to & 0 \end{array}$$

This implies that $\mathrm{Tor}_1(I_A, k) = 0$, hence that I_A is flat/A, by a previous Remark. Repeating the procedure for $\mathcal{O}_A$, we see by succesive reasoning that any *resolution* for $\mathcal{O}$ lifts to one for $\mathcal{O}_A$.

4 Deformations of complete intersections

Let $X \hookrightarrow \mathbb{A}^n$ be a complete intersection. Let $d = \dim X$ (Krull dim) and $I = (f_1, \ldots, f_{n-d})$ the ideal of X. Let $\mathcal{O} = P/I$, where $P = P_k =$

$k[X_1, \ldots, X_n]$ Then it is well known that the "Koszul complex" gives a resolution for $\mathscr{O}$, i.e.,

$$\overset{2}{\wedge} P^{n-d} \to \overset{1}{\wedge} P^{n-d} (= P^{n-d}) \to P \to \mathscr{O} \to 0$$

12 the homomorphisms being interior multiplication by the vector

$$(f_1, \ldots, f_{n-d}) \in P^{n-d},$$

(e.g., $P^{n-d} \to P$ is the map $(\lambda_1, \ldots, \lambda_{n-d}) \mapsto \sum_i \lambda_i f_i$). The image of $\overset{2}{\wedge} P^{n-d}$ in P^{n-d} gives the realtions in I, which shows that the relations among the f_i are (generated by) the obvious ones, i.e., $f_i z_j - f_j z_i = 0$.

Let $A = k[t]/(t^2)$ (*which we write* $A = k + kt$ *with* $t^2 = 0$). *Then deformations of X over A are called first order deformations*. Lert I_A be the ideal in $A[X_1, \ldots, X_n]$.

$$I_A = ((f_1 + g_1 t), \ldots, (f_{n-d} + g_{n-d} t))$$

where $g_i \in P_k$. We *claim* that *for arbitrary choice of* $g_i \in P_k, \mathscr{O}_A = P_A/I_A$ is *flat* over A (of course we have seen that any deformation X_A of X can be defined by I_A for sutiable choice of g_i). This is an immediate consequence of the fact that above *explicit relations* between f_i can obviously be lifted to relations between $(f_i + g_i t)$. This proves the claim.

Thus to classify embedded first order deformations it suffices to write down conditions on $(g_i), (g_i^{'})$ in P so that in P_A the ideals $((f_i + g_i t))$ and $((f_i + g_i^{'} t))$ are the same. *We claim*:

$$((f_i + g_i t)) = ((f_i + g_i^{'} t)) \Leftrightarrow g_i - g_i^{'} \in I.$$

To prove this we proceed as follows:

$$\begin{aligned} ((f_i + g_i^{'} t)) \subset ((f_i + g_i t)) \Leftrightarrow & \qquad (\text{Set} \quad r = n - d.) \\ f_i + g_i^{'} t = \sum_{j=1}^{n-d} & (\alpha_{ij} + \beta_{ij} t)(f_j + g_j t) \\ &= \left(\sum_{j=1}^{r} \alpha_{ij} f_j \right) + t \left(\sum_{j=1}^{r} \alpha_{ij} g_j + \sum_{j=1}^{r} \beta_{ij} f_j \right) \Leftrightarrow \end{aligned}$$

13 $\exists\ (r \times r)$ matrices (α_{ij}) and (β_{ij}) with coefficients in P such that

(a) $(\alpha_{ij})\begin{bmatrix} f_1 \\ \vdots \\ f_r \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_r \end{bmatrix}$, i.e.,$(\alpha_{ij} - Id)\begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} = (0)$, and

(b) $(\alpha_{ij})\begin{bmatrix} g_1 \\ \vdots \\ g_r \end{bmatrix} + (\beta_{ij})\begin{bmatrix} f_1 \\ \vdots \\ f_r \end{bmatrix} = \begin{bmatrix} g'_1 \\ \vdots \\ g'_r \end{bmatrix}$.

Since the coordinates of the relation vectors are in I it follows from (a) above that the element of $(\alpha_{ij} - Id)$ are in I, i.e., $(a) \Rightarrow (\alpha_{ij}) \equiv (Id)$ mod I). Then (b) implies that

$$\begin{bmatrix} g_1 \\ \vdots \\ g_r \end{bmatrix} \equiv \begin{bmatrix} g'_1 \\ \vdots \\ g'_r \end{bmatrix} (\text{mod}\, I).$$

Hence $((f_i + g'_i)) \subset ((f_i + g_i t) \Rightarrow (g_i - g'_i) \in I$. Hence $((f_i + g_i t)) = ((f_i + g'_i t)) \Rightarrow (g_i - g'_i) \in I$. Conversely, suppose that $(g_i - g'_i) \in I$. Then there is a matrix (β_{ij}) such that

$$(\beta_{ij}\begin{bmatrix} f_1 \\ \vdots \\ f_r \end{bmatrix} = \begin{bmatrix} g'_1 \\ \vdots \\ g'_r \end{bmatrix} - \begin{bmatrix} g_1 \\ \vdots \\ g_r \end{bmatrix}$$

Hence, $(Id)\begin{bmatrix} g_1 \\ \vdots \\ g_r \end{bmatrix} + (\beta_{ij})\begin{bmatrix} f_1 \\ \vdots \\ f_r \end{bmatrix} = \begin{bmatrix} g'_1 \\ \vdots \\ g'_r. \end{bmatrix}$

Taking $(\alpha_{ij}) = Id$ we see that the conditions $(a), (b)$ are satisfied, which implies that $((f_i + g_i t)) = ((f_i + g'_i t))$. This proves the claim and thus we have classified *all embedded (in $\mathbb{A}^n$) first order deformations of* **14**
X.

Now to classify first order deformations of X we have only to write down the condition when two embedded deformations X_A, X'_A are isomorphic over A. Let θ be an isomorphism $X_A \xrightarrow{\sim} X'_A$. By assumption, $X \otimes k = X'_A \otimes k = X$. i.e., their fibres over the closed point of Spec A are

$X \subset \mathbb{A}^n$. We denote (of course) by X_ν the canonical coordinate functions of $X_A \hookrightarrow \mathbb{A}^n_A = \operatorname{Spec} A[X_1, \ldots, X_n]$. Let $X^{'}_\nu = \theta^\star(X_\nu)$. Then we have

$$X^{'}_\nu = X_\nu + \varphi_\nu(X)t$$

for some polynomials $\varphi_\nu(X)$. Hence to identify two embedded deformations of X we have to consider the identification by the above change of coordinates. Then

$$f_i + g_i t \mapsto f_i(X_\nu + \varphi_\nu(t)) + g_i((X_\nu + \varphi_\nu(t))t.$$

By Taylor expansion up to the first order, we get

$$\begin{aligned} &= f_i(X) + t\left\{\sum_\nu \frac{\partial f_i}{\partial X}\varphi_\nu(X)\right\} + tg_i(X) \\ &= f_i(X) + t\left\{g_i(X) + \sum_{\nu=1}^{n} \frac{\partial f_i}{\partial X}\varphi_\nu(X)\right\} \end{aligned}$$

Hence $(f_i + g_i t)$ and $(f_i + g^{'}_i t)$ define the same deformation of the first order up to change of coordinates above, which is equivalent to the fact that there exists $(\varphi_1, \ldots, \varphi_n)$ such that

$$(*) \qquad \begin{bmatrix} g_1 \\ \vdots \\ g_r \end{bmatrix} - \begin{bmatrix} g^{'}_1 \\ \vdots \\ g^{'}_r \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial X_1} \cdots \frac{\partial f_1}{\partial X_n} \\ \cdots \\ \frac{\partial f_r}{\partial X_1} \cdots \frac{\partial f_r}{\partial X_n} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix}.$$

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Recall that $\mathscr{O} = P/I$ and $X = \operatorname{Spec} \mathscr{O}$. Then embedded first order deformatios are classified by $\mathscr{O}^{n-d} = \mathscr{O} \oplus \ldots \oplus \mathscr{O}$ ($n - d$ times), which is an $\mathscr{O}$-module.

To classified all deformations, consider the homomorphism of P modules

$$\mathscr{O} \overset{\text{Jac}}{\to} \mathscr{O}^{n-d} \text{ whose matrix is } \left(\frac{\partial f_i}{\partial X_j}\right); \frac{\partial f_i}{\partial X_j} \text{ the images of } \frac{\partial f_i}{\partial X_j} \text{ in } P/I.$$

Let us call the euotient $\mathscr{O}^{n-d}/Im\mathscr{O}^n$ by this map T. This is an $\mathscr{O}$-module and we see that its support is located at the singular points of

X. In particular, if X has isolated singularities, T is a finite dimensional vector space/k.

For example, consider the case that X is of codimension one, i.e., defined by one equation f. Then

$$T = k[X]/\left(f, \frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n}\right).$$

The cone in 3-space has equation $f = Z^2 - XY$, and if chark $\neq 2$,

$$T = k[X, Y, Z]/(f, -Y, -X, 2Z) \cong k.$$

Thus a *universal* first order deformation is given by

$$Z^2 - XY + t = 0.$$

5 The case of Cohen-Macaulay varieties of codim 2 in $\mathbb{A}^n$ (Hilbert, schaps)

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The theorem that we shall prove now was essentially found by Hilbert. This has been studied recently by Mary schaps.

Let P be as usual the polynomial ring $P = k[X_1, \dots, X_n]$. Let (g_{ij}) be an $(r \times r-1)$ matrix over P $\begin{bmatrix} g_{11} \cdots g_{r-1} \\ g_{r,1} \cdots g_{r,r-1} \end{bmatrix}$. Let $\delta_i = (-1)^i det((r-1) \times (r-1)$ minor with i^{th} row deleted).

Then $(\delta_1, \dots, \delta_r) \begin{bmatrix} g_{1,1} \cdots g_{1,r-1} \\ g_{r,1} \cdots g_{r,r-1} \end{bmatrix} \begin{bmatrix} g_{1,1} \cdots g_{1,r-1}, g_{1,1} \\ g_{r,1} \cdots g_{r,r-1}, g_{r,1} \end{bmatrix} = 0$ etc. This implies that the sequence

$$P^{r-1} \xrightarrow[(g_{ij})]{} P^r \xrightarrow[(\delta_1, \dots, \delta_r)]{} P$$

is a *complex*. Note that $P^{r-1} \to P^r$ is injective if over the quotient field of P. (g_{ij}) has rnak $(r-1)$, or equivalently, $\exists$ some $x_o \in \mathbb{A}^n$ such that $(g_{ij}(x_o))$ is of rank $(r-1)$.

Theorem 5.1 (Hilbert, Schaps)**.** (1) *Let (g_{ij}) be an $r \times (r-1)$ matrix over P and δ_i its minors as defined above. Let J be the ideal $(\delta_1, \ldots, \delta_r)$. Assume that $V(J) = V(\delta_1, \ldots, \delta_r)$ is of codim ≥ 2 in $\mathbb{A}^n$. Then $X = V(J)$ is Cohen-Macaulay, precisely of codim 2 in $\mathbb{A}^n$. Further, the sequence*

$$0 \to P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(\delta_1,\ldots,\delta_r)} P \to P/J \to 0$$

17 *is exact, i.e., it gives resolution for P/J.*

(2) *Conversely, suppose given a Cohen-Macaulay closed subscheme $X \hookrightarrow \mathbb{A}^n$ of codim 2. Let $X = V/(J)$, then P/J has a resolution of length 3 which will be of the form $0 \to P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(f_1,\ldots,f_r)} P \to P/J \to 0$, because $hd_P P/J = 2$. (Depth $P/J + hd_p P/J = n$, depth $P/J = n-2, \Rightarrow hg P/J = 2$). Note that f_i need not be δ_i as defined above. Then we claim that we have as isomorphism*

$$\begin{tikzcd}
0 \arrow[r] & P^{r-1} \arrow[r, "(g_{ij})"] \arrow[d, equal] & P^r \arrow[r, "(f_1,\ldots,f_r)"] \arrow[d, equal] & P \arrow[d, "\wr"] \\
0 \arrow[r] & P^{r-1} \arrow[r, "(g_{ij})"] & P^r \arrow[r, "(\delta_1,\ldots,\delta_r)"] & P
\end{tikzcd}$$

i.e., $\exists$ a unit $u \in P$ such that $f_i = u\delta_i$.

(3) *The map of functors (Deformations of (g_{ij}))$\to$ $(Def \quad X)$ is* smooth, *i.e., -(i) deforming (g_{ij}) gives a deformation of X, (ii) any deformation of X can be obtained by deforming g_{ij}, and (iii) given a deformtion X_A of X defined by (g_{ij}) over $A[X]$, $\quad A' \to A \to 0$ exact and a deformation X_A of X inducing X_A, $\exists (g'_{ij})$ over $A'[X]$ which defines X_A and the canonical image of (g'_{ij}) in A is (g_{ij}).*

Proof. (1) Let $(g_{ij}), \delta_i$, etc., be as in (l). We shall first prove that

$$0 \to P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(\delta_i)} P \to P/J \to 0$$

is *exact*, assuming codim $X \geq 2$. This will complete the proof of (l). For, it follows $hd_P P/J < 2$. On the other hand, since codim $X \geq 2$, depth $P/J \leq (n-2)$.

18 But depth $P/J + hd_P P/J = n$. This implies that $\dim P/J =$ depth $P/J = (n-2)$ and $hd_P P/J = 2$, which shows that X is Cohen-Macaulay of codim 2 in $\mathbb{A}^n$.

Since $V(J) \neq \mathbb{A}^n$, the δ_i's are not *all* identically 0, hence $0 \to P^{r-1} \to P^r$ is *exact*. Further we note that any $x_o \notin V(J), (g_{ij}(x_o))$ is of rank $(r-1)$ and in fact one of $(\delta_1, \ldots, \delta_r)$ is nonzero at x_o and hence a *unit* locally at x_o. This implies that $P^{r-1} \to P^r \to P$ *split exact* locally at x_o (i.e., if B is the local ring of $\mathbb{A}^n$ at x_o, then tensoring by B gives a split exact sequence). Because of our hypothesis that codim $V(J) \geq 2$, if x is a point of $\mathbb{A}^n$ represented by a prime ideal of height one and B its local ring. then tensoring by B makes $P^{r-1} \to P^r \to P$ exact (i.e., the sequence is exact in codim 1). Let $K = \operatorname{Ker}(P^r \to P)$. We have $K \subset P^r$ such that $0 \to P^{r-1} \to K$, and $0 \to K \to P^r \to P$ exact. Tensoring by B as above, it follows that $P^{r-1} \otimes B \to K \otimes B$ is an isomorphism (tensoring by B is a localization); i.e., the inclusion $P^{r-1} \subset K$ is in fact an isomorphism in codim 1. Since P^{r-1} is free sections of P^{r-1} defined in codimension 1 exrtend to global sections. Moreover, K is torsion-free. Therefore, the fact that $P^{r-1} \subset K$ is an isomorphism in codimension 1 implies that it is an *isomorphism everywhere*. Therefore, $P^{r-1} \to P^r \to P$ is exact, and this completes the proof of (1).

(2) Let $I = (f_1, \ldots, f_r)$ be a Cohen-Macaulay codim 2 ideal in P. Since $hd_P P/I = 2$, there is a resolution

$$(*) \qquad 0 \to P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(f_1,\ldots,f_r)} P \to P/I \to 0.$$

(Here we should take the warning about using free resolutions instead of using projective resolutions.) Let the complex (**) be defined by

$$(**) \qquad 0 \to P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(\delta_1 \ldots,\delta_r)} P \to P/J \to 0$$

δ_i being as before. 19

The sequence (*) is split exact at every point $x \notin V(I)$. This implies that some δ_i is a unit at x, and hence by direct calculation that (**)

is also split exact, and $P/J = 0$, at x. This means

$$V(J) \subseteq V(I),$$

hence codim $V(J) \geq 2$. Hence by (l) it follows that (**) is exact.

We shall now show that $\exists$ a unit u such that $f_i = u\delta_i$. Take the dual of (*) and (**) *dual:* $\mathrm{Hom}_P(M, P) = M^*$), i.e.,

$$(*)^* \qquad (P^{r-1})^* \xleftarrow[(g_{ij})^t]{} (P^r)^* \xleftarrow[(f)^t]{} P^* \leftarrow 0$$

$$(**)^* \qquad (P^{r-1})^* \xleftarrow[(g_{ij})^t]{} (P^r)^* \xleftarrow[(\delta_i)^t]{} P^* \leftarrow 0.$$

We claim that these sequences are *exact*. The required assertion about the existence of u is an immediate consequence of this. For then $P^* \xrightarrow{(f)^*} (P^*)^r$ and $P^* \xrightarrow{(\delta_i)^*} (P^*)^r$ are injective maps into the same submodule of $(P^*)^r$ of rank 1 which implies that (f) and (δ) differ by a unit. We note that $P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(f_1,\ldots,f_r)} P$ and $P^{r-1} \to P^r \xrightarrow{(\delta_1,\ldots\delta_r)} P$ are split exact in codimension 1. Consequently. it follows from this and the fact that $\mathrm{Hom}_P(P/I, P) = \mathrm{Hom}_P(P/J, P) = 0$ that we have sequences

$$0 \to P^* \xrightarrow{(f_i)^t} P^{r^*} \xrightarrow{(g_{ij})^t} P^{r-1^*}$$

$$0 \to P^* \xrightarrow{(\delta_i)^*} P^{rt} \xrightarrow{(g_{ij})^t} P^{r-1^*}$$

20 and we must prove exactness at the P^{r^*} module. But we have that $Im f_i^* \subset \mathrm{Ker}(g_{ij})^t$ and $Im(\delta_i)^* \subset \mathrm{Ker}(g_{ij})^t$ with equality at the localization of each prime ideal of height 1. Consequently, $Im f_i^* = \mathrm{Ker}(g_{ij})^t, Im(\delta_i)^t = \mathrm{Ker}(g_{ij})^t$ (same argument as was used above). This completes the proof of (2).

(3) Let A be an Artin local ring over k, and $P_A = A[X_1, \ldots, X_n]$. Let (g'_{ij}) be an $r \times (r-1)$ matrix over P_A and (g_{ij}) the matrix over P such that $g'_{ij} \mapsto g_{ij}$ by the canonical homomorphism $A[X] \to k[X]$. Define δ'_i analogous to δ_i. Suppose that codim $V(\delta'_i)$ in $\mathbb{A}^n_A$ is of

codim $\geq 2 \Leftrightarrow$ codim$V(\delta_i)$ in $\mathbb{A}_k^n$ is of codim $\geq 2 \Leftrightarrow$ condim$V(\delta_i) =$ 2 because of (1). Consider

$$(**) \qquad 0 \to P_A^{r-1} \xrightarrow{(g'_{ij})} P_A^r \xrightarrow{(\delta'_i)} P_A \to P_A/I_A \to 0$$

$$(*) \qquad 0 \to P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(\delta_i)} P \to P/I \to 0.$$

Now (**) is lifting of (*). Of course (*) is exact and $P_A^r \to P_A \to P_A/J_A \to 0$ is exact. Besides, (**) is a complex. This implies by proposition 3.1 that $P_A^{r-1} \to P_A^r \to P_A \to P_A/J_A \to 0$ is *exact* and P_A/I_A is A-*flat* and by Remark 3.1 (**) is exact/ (The exactness of (**) can also be proved by a direct argument generalizing (1).) This shows that any (infinitesimal) deformation (g'_{ij}) of (g_{ij}) as in (1) gives a (flat) deformation X_A of $X = \operatorname{Spec} P/I$ and that X_A is "presented" in the same way as X.

Conversely, let X_A be an infinitesimal deformation of $X = V(\delta_1, \ldots,$ **21**
$\delta_r)$. Lift the generators for I to I_A say $(f_l, \ldots, f_r)$. Then as we remarked before the exact sequence (*) can be lifted to an *exact sequence*

$$(**)' \qquad 0 \to P_A^{r-1} \xrightarrow{(g'_{ij})} P_A^r \xrightarrow{(f_i)} P_A \to P_A/I_A \to 0$$

[Note that f_i need not be (*a priori*) the determinants of minors of (g'_{ij}).] Let δ'_i be the minors of (g'_{ij}), J_A the ideal $(\delta'_l, \ldots \delta'_r)$. Then as we saw above

$$(**) \qquad 0 \to P_A^{r-1} \xrightarrow{(g'_{ij})} P_A^r \xrightarrow{(\delta'_i)} P_A \to P_A/J_A \to 0$$

is *exact* and P_A/J_A is A-flat. Taking the duals of (**) and $(**)'$ as before, it follows that there is a unit u in P_A such that $f_i = u\delta'_i$. This shows that $I_A = J_A$. Thus any deformation is obtained by a diagram of type (**). The assertion of smoothness also follows from this argument. This completes the proof of the theorem.

$\square$

Remark 5.1. Any Cohen-Macaulay $X \hookrightarrow \mathbb{A}^n$ of codim 2 is defined by the minors of an $r \times (r-1)$ matrix (g_{ij}) (this is local, note the warning about using free modules instead of projective ones). The g_{ij}'s define a morphism

$$\mathbb{A}^n \xrightarrow{\Phi} \mathbb{A}^{r(r-1)}.$$

Let $Q = k[X_{ij}]$, $1 \le i \le r$, $1 \le j \le r-1$, so that $\mathbb{A}^{r(r-1)} = \operatorname{Spec} k[X_{ij}]$. Then $\Phi^*(X_{ij}) = g_{ij}$. Consider (X_{ij}) as an $r \times (r-1)$ matrix over Q, and let $\triangle_i = (-1)^i$ det(minor of X_{ij} with i^{th} row deleted). Then the variety
22 $V = V(\triangle_1, \ldots, \triangle_r) \to \mathbb{A}^{r(r-1)}$ is called the *generic determinantal variety* defined by $r \times (r-1)$ matrices. It is Cohen-Macaulay and of codim 2 in $\mathbb{A}^{r(r-1)}$. Then $\Phi^{-1}(x) = X$. This means that any Cohen-Macaulay codim 2 subscheme is obtained as the inverse image by $\mathbb{A}^n \to \mathbb{A}^{r(r-1)}$ of the generic determinantal variety V of type $r \times (r-1)$.

Remark 5.2. Other simple examples of codim 2, Cohen-Macaulay X are

(a) any O-dimensional subscheme in $\mathbb{A}^2$,

(b) any 1-dimensional reduced X in $\mathbb{A}^3$, and

(c) any normal 2-dimensional X in $\mathbb{A}^4$.

More about the generic determinantal variety.

Now let $X \subset \mathbb{A}^{r(r-1)}$ denote the determinantal variety $V(\triangle_i) = V$ defined above. Then any infinitesimal deformation X_A of X is obtained by the minors of a matrix $(X_{ij} + m_{ij})$ where $m_{ij} \in m_A[X_{ij}]$, where m_A is the maximal ideal of A. Set $X'_{ij} = X_{ij} + m_{ij}$. We see that $X_{ij} \mapsto X'_{ij}$ is just change of coordinates in $\mathbb{A}^n_A$, i.e., $A[X_{ij}] = A[X'_{ij}]$. *This implies that X is rigid*, i.e., every deformation of X is trivial.

The singularity of X: X can be identified with the subset of $\mathbb{A}^{r(r-1)}$ $\mathrm{Hom}_{\text{linear}}(\mathbb{A}^r, \mathbb{A}^{r-1})$ of linear maps of rank $\le r-2$. Now $G = GL(r) \times GL(r-1)$ operates on $\mathbb{A}^{r(r-1)}$ in a natural manner. Lt X_k denote the subset of $\mathbb{A}^{r(r-1)}$ of linear maps of rank equal to $(r-k)$. *We note that X_k is an orbit under G* and hence is a smooth, irreducible locally closed subset

of $\mathbb{A}^{r(r-1)}$. To compute its dimension we proceed as follows: If $\varphi \in$ Hom$(\mathbb{A}^r, \mathbb{A}^{r-1})$ and $\varphi \in X_k$, then $im\varphi$ is a k-dimensional space. Hence **23**
φ is determined by an arbitrary k-dimensional subspace of $\mathbb{A}^r(= \ker\varphi)$ an $(r-k)$-dimensional subspace of $\mathbb{A}^{(r-1)}(= Im\varphi)$ and an arbitrary linear map of an $(r-k)$-dimensional linear space onto an $(r-k)$-dimensional linear space.

Hence

$$\begin{aligned}\dim X_k &= (r-k)k + (r-k)[(r-1)-(r-k)] + (r-k)^2\\ &= (r-k)k + (r-k)(k-1) + (r-k)^2.\end{aligned}$$

Suppose $k = 2$, i.e., consider linear maps of rank $(r-2)$. Then X_2 is obviously open in X and

$$\begin{aligned}\dim X_2 &= (r-2)2 + (r-2) + (r-s)^2\\ &= (r-2)2 + 1 + (r-s)\\ &= (r-2)(r+1)\end{aligned}$$

It follows that codim $X_2 = 2$.

It is clear that X_2 is dense in X (easily seen that every $x_o \in X$ is a specialization of some $x \in X_2$). The implies that X is *irreducible*. Further more, X_2 is *smooth*, and in particular reduced. By the unmixedness theorem, since X is Cohen-Macaulay, it follows that X is reduced. Hence X is a subvariety of $\mathbb{A}^{r(r-1)}$.

Let $X_3' = X_3 \cup X_4 \ldots$ be the set of all linear maps of rank $\leq (r-3)$. Clearly X_3' is a closed subset of X. To compute $\dim X_3$, as we see easily that x_3 is a dense open set in X'_3. Now

$$\dim X_3 = (r-3)3 + (r-3)2 + (r-3)^2 = (r-3)(r+2).$$

Hence codim X_3 (in $\mathbb{A}^{r(r-1)}$)= 6, for $r \geq 3$. (If $r = 2$ it is a complete intersection; X is the intersection of two linear spaces and hence X is smooth.)

Hence codim X_3 *in* X *is* 4. We claim also that every point of X_3' is **24**
singular on X. To prove this, suppose for example $\theta \in X_3$. Since G acts

transitively on X_3, we can assume it is the point

$$\theta = \begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \\ & & & \\ & 0 & & 0 \\ & & & \end{bmatrix}$$

Then $\left.\dfrac{\partial \triangle_i}{\partial X_{kl}}\right|_{\theta} = 0\ \forall k, l$. Thus $\theta \in X$ is smooth $\Leftrightarrow \theta \in X_2$.

Thus the generic determinantal variety has an isolated singularity if and only if $r = 3$, in which case $\dim X = 4$, $X \subseteq \mathbb{A}^6 (= \mathbb{A}^{r(r-1)})$. We thus get an example of a *rigid isolated singularity* (a 4-dimensional isolated Cohen-Macaulay singularity).

Proposition 5.1. *(Schaps). Let* $X_0 \hookrightarrow \mathbb{A}^d (d \leq 5)$ *be of codim* 2 *and Cohen-Macaulay. Then* X_0 *can be deformed into a smooth variety.*

Proof. We give only an outline. Since X_0 is determinantal, it is obtained as the inverse image of X in $\mathbb{A}^{r(r-1)}$ by some map

$$\varphi = (g_{ij}) : \mathbb{A}^d \to \mathbb{A}^{r(r-1)}$$

Now codim X in $\mathbb{A}^{r(r-1)}$ is 6. "Perturbing" (g_{ij}) to $(g'_{ij}) = \varphi'$, the map $\mathbb{A}^d \to \mathbb{A}^{r(r-1)}$, defined by (g'_{ij}) can be made "transversal" to X. This implies that $\varphi'(\mathbb{A}^d) \cap \mathrm{Sing} X = \emptyset$[Sing$X$ is of condim 6] and in fact that $\varphi'^{-1}(X)$ is smooth. □

25 **Remark 5.3.** It has been proved by Svanes that if X is the generic determinantal variety defined by determinants of $(r \times r)$ minors of an $(m \times n)$ matrix, then X is rigid, except for the case $m = n = r$.

6 First order deformations of arbitrary X and Schelessinger's T^1

Let $X = V(I)$ and $A = k[t]/(t^2)$. Let $I = (f_1, \ldots, f_m)$. Fix a presentation for I, i.e., an exact sequence

$$(*) \qquad P^\ell \xrightarrow{(r_{ij})} P^m \xrightarrow{(f_i)} P \to P/I \to 0.$$

Let $I_A = (f_1', \ldots, f_m')$ where $f_i^{'} = (f_i + tg_i), g_i \in P$. Then $X_A = V(I_A)$ is A-flat if if (*) lifts to an exact sequence

$$(**) \qquad P_A^\ell \xrightarrow{(r_{ij}')} P_A^m \xrightarrow{(f_i')} P_A \to P_A/I_A \to 0.$$

In fact we have seen (cf. Proposition 3.1) that in order that (**) be exact it suffices that (**) be a *complex* at P_A^m, i.e., $X_A = V(I_A)$ is A-flat iff there is a matrix (r_{ij}') over P_A extending (r_{ij}), such that

$$(f_1', \ldots, f_m') \begin{pmatrix} r_{11}' \cdots r_{11}' \\ r_{m1}' \cdots r_{m1}' \end{pmatrix} = 0$$

Set $r_{ij}^{'} = r_{ij} + ts_{ij}$, where $s_{ij} \in P$. Then X_A is A-flat iff $exists$(s_{ij} over P such that the matrix product

$$(f + gt)(r + st) = fr + t(gr + fs) = 0$$

where $f = (f_i), \ldots, r = (r_{ij})$. Now $fr = 0$ since (*) is exact. Thus **26**
the flatness of X_A over A is equivalent with the existence of a matrix is $s = (s_{ij})$ over P such that

$$(*) \qquad (gr) + (fs) = 0.$$

Consider the homomorphism

$$(g) : P^m \to P$$

defined by (g_{ij}). Then condition (*) implies that (g) maps ImP^ℓ under the homomorphism ($r_{ij} : P^\ell \to P^m$ into the ideal I. Hence (g) induces a homomorphism

$$(\overline{g}) : P^m/Im(r) = I \to P/I,$$

i.e., $\overline{g}$ is an element of $\mathrm{Hom}_P(I, P/I) \simeq \mathrm{Hom}_{p/I}(I/I^2, P/I) = \mathrm{Hom}_{\mathscr{O}_X}(I/I^2, \mathscr{O}_X)$ = the dual of the coherent $\mathscr{O}_X$ module I/I^2 on X. The sheaf $\underline{\mathrm{Hom}}_{\mathscr{O}_{\mathscr{X}}}(I/I^2, \mathscr{O}_X) = N_X$ is called the *normal sheaf* to X in $\mathbb{A}^n$. Thus $\overline{g}$ is a global section of N_X. Conversely, given a homomorphism $\underline{g}$: $\mathrm{Hom}_P(I, P/I)$ and a lifting of $\underline{g}$ to $g = P^m \to P$, then (g) satisfies $\overline{(*)}$. This shows that we have a surjective map of the set of first order deformations onto $H^o(X, N_X)$. Suppose we are given two liftings $(f + tg_1)$ and $(f + g_2 t)$ such that g_1, g_2 define the same homomorphisms of I/I^2 into P/I, i.e., $\overline{g_1} = \overline{g_2}$. *We claim* that if $I_A = ((f_i + tg_1, i)$ and $J_A = ((f_i + tg_{2,i}))$, then $I_A = J_A$, i.e., the two liftings define the same sub-scheme of $\mathbb{A}^n_A$. This will prove that the canonical map

$$j : (\text{First order def. of} X) \to H^o(X, N_X)$$

27 is injective, which implies, since j is surjective, that j is needed *bijective*. The proof that $I_A = J_A$ is immediate; from the computation in § 4 we see that

$$((f_i + tg_{2,i})) \subset ((f_i + tg_{1,i})) \Leftrightarrow (J_A \subset I_A)$$

$\Leftrightarrow \exists (m \times m)$ matrices (α_{ij}) and (β_{ij}) over P such that

(a) $(\alpha_{ij} - id)\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = 0$

(b) $(\alpha_{ij})\begin{pmatrix} g_{1,1} \\ \vdots \\ g_{1,m} \end{pmatrix} + (\beta_{ij})\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} g_{2,1} \\ \vdots \\ g_{2,m} \end{pmatrix}.$

As was done in § 4, if $(\overline{g_1}) = (\overline{g_2})$, i.e.,

$$g_{1i} - g_{2i} \equiv 0 (mod\, I)\ \forall i,$$

Then there exists (β_{ij}) such that (a) and (b) are satisfied with $(\alpha_{ij}) = Id$. This implies that $J_A \subset I_A$. In a similar manner $I_A \subset J_A$, which proves that $J_A = I_A$. Thus we have proved

Theorem 6.1. *The set of* The set first order embedded deformations *of* X *in* $\mathbb{A}^n$ *is canonically in one-one correspondence with* N_X *(or* $H^o(N_X)$)– *the normal bundle to* X *of* $X \subset \mathbb{A}^n$.

Remark 6.1. Suppose now we are given

$$0 \to \in A' \to A' \to A \to 0 \quad \text{exact}$$

where A', A are finite Artin local rings such that $rk_k(\epsilon A') = 1$ (in partic- **28**
ular it follows easily that ϵ is an element of square 0). We see easily that $\epsilon A'$ has a natural structure of an A-module and in fact $\epsilon A' \simeq A/m_A$ as A-module. (Given *any* surjective homomorphism $A' \to A$ of Artin local rings by successive steps this can be reduced to this situation.) Suppose now that X_A is a lifting of X defined by $V(I_A), I_A = (f'_1, \ldots, f'_m)$ with f'_is as liftings of $f_i, X = V(I)$, $I = (f_1, \ldots, f_m)$. We observe that even if A is *not* of the form $k[t]/t^2$, in order that X_A be flat/A and be a lifting of X it is neccessary and sufficient that there exists a matrix (r'_{ij}) such that

$$(f'_1, \ldots, f'_m)(r'_{ij}) = 0 \quad (f'_i \text{are liftings of} f_i).$$

Suppose now are given two liftings X^1_A, and X^2_A, of X_A, flat over A defined by $V(I^1_{A'})$ and $V(I^2_{A'})$:

$$I^1_{A'} = (g_1, \ldots, g_m), I^2_A, = (g'_1, \ldots, g'_m).$$

Now X_A is defined by an exact sequence

$$(*) \qquad P^1_A \xrightarrow{(r'_{ij})} P^m_A \to P_A \to P_A/IA \to 0$$

such that $\otimes_A K$ gives the presentation for X. An easy extension of the argument given before (cf. Proposition 3.1) for characterization of flatness by lifting of generators for I shows that $X'_{A'}$, and $X^2_{A'}$ are flat/A. They are presented respectively by

$$(\alpha) \qquad P^1_{A'} \xrightarrow{(s_{ij})} P^m_{A'} \xrightarrow{(g_i)} P_{A'}/I^1_{A'}, \to 0 \quad \text{exact}$$

$$(\beta) \qquad P^1_{A'} \xrightarrow{(s'_{ij})} P^m_{A'} \xrightarrow{(g'_i)} P_{A'} \to P_{A'}/I^2_{A'}, \to 0$$

such that $(\alpha \otimes_A, A)$ and $(\beta \otimes_A, A)$ coincide with (*) and in fact it suffices 29
that $(g)(s)$ and $(g')(s')$ are zero and s, s' are liftings of r. We can write

$$\begin{aligned} g'_i &= g_i + \epsilon h_i, \quad \text{with} \quad h_i \in P, \quad \text{and} \\ s'_{ij} &= s_{ij} + \epsilon t_{ij}, \quad \text{with} \quad t_{ij} \in P. \end{aligned}$$

Then $(g')(s') = 0$ iff $(g_i)(t_{ij}) + (h_i)(s_{ij}) = 0$, and by the remark about $\epsilon A'$ as an A-module (or A'-module) it follows that the right hand side above is 0 if and only if

$$(\dagger) \qquad (f_i)(t_{ij}) + (h_i)(r_{ij}) = 0,$$

i.e., $\exists (t_{ij})$ and (h_i) over p such that this holds. Conversely, we see that given X^1_A, flat/A' and (†), we can construct $X^2_{A'}$, flat/A' by defining $g'_i = g_i + \epsilon h_i$ and lifting of relations by $s'_{ij} = s_{ij} + \epsilon t_{ij}$. Now (†) gives rise to N_X as before. Thus *fixing* an X^1_A, flat/A' which is a lifting of X_A, flat/A the set of liftings $X_{A'}$ of X_A over A are in one-one correspondence with N_X (or $H^o(N_X)$) *or in other words the set of flat liftings $X_{A'}$ over X_A is either empty or is a principal homogeneous space under N_X (or $H^o(N_X)$). In the case $A = k, A' = k[t]/(t^2)$ (or more generally $A' = A[\epsilon]$) we use $X^1_{A'} = X \otimes_k A'$ (resp. $X^1_{A'} = X_A \otimes_A A'$) as the canonical base point, and using this base point the set of first order deformations gets identified with $N_X$or $H^o(N_X)$.*

Remark 6.2. Compare the previous proof (cf. § 4) for the calculation
30 of first order (embedded) deformations of $X - -a$ complete intersection in $\mathbb{A}^n$. In that proof we required the knowledge of the relations between f'_i where $X = V(I), I = (f_1, \ldots, f_m)$ whereas this id *not required* in the present proof (we assume X arbitrary). This is because in the proof for the complete intersection we obtained a little more, namely, (First order embedded deformations of X)$\leftrightarrow \{g_i\}, g_i \in \mathscr{O}_X$. The previous proof gives also the fact that N_X *is free over $\mathscr{O}_X$ of rank=codim X*. Hence a better proof for the case of the complete intersection would be to prove first the general case and then show that I/I^2 *is free of over $\mathscr{O}_X$ of rank= codim$X$ (in $\mathbb{A}^n$)* (this implies that $N_X = \mathrm{Hom}_{\mathscr{O}_X}(I/I^2, \mathscr{O}_X)$ is free of rank=codim X).

Remark 6.3. The above proof also generalizes to the computation for first order deformations of the Hilbert scheme or more generally the Quotient scheme (in the sense of Grothendieck) as well as the consideration of Remark 6.1 above.

Schlessinger's T^1:

We have computed above the *first order embedded deformations* of X. Now to compute the first order deformation of X (as we did in § 4) we have to identify two first order embedded deformations $X_A, X_{A'}(\subset \mathbb{A}^n_A)A = k[t]/(t^2)$ when there is an isomorphism $\theta : X_A \simeq X'_A$ which induces the identity auto-morphism $X \to X$. As we saw before, the isomorphism θ is induced by a change of coordinates in $\mathbb{A}^n_A$, i.e., if X_ν are the canonical coordinates of $\mathbb{A}^n$ we started with and $\theta^*(X_\nu) = X'_\nu$ then we have

$$X'_\nu = X_\nu + \varphi_\nu(X)t, \nu = 1, \ldots, n(X \subset \mathbb{A}^n).$$

If a first order embedded deformation of $X = V(I), I = (f_i)$ is given by 31
$(f_i + g_i t)$, then

$$\begin{aligned}
f_i + g_i t \mapsto\ & f_i((X_\nu + t\varphi_\nu) + g((X_\nu + \varphi_\nu t))t \\
=\ & f_i(X) + t\left\{\sum \frac{\partial f_i}{\partial X_i}\varphi_\nu(x)\right\} + tg_i(X) \\
=\ & f_i(X) + t\left\{g_i(X) + \sum_{\nu=1}^{n} \frac{\partial f_i}{\partial X}\varphi_\nu(X)\right\}.
\end{aligned}$$

Let λ be the canonical image of (g_i) in N_X nad λ' the canonical image of $g_i + \sum_{\nu=1}^{n} \frac{\partial f_i}{\partial X_\nu}\varphi_\nu(X)$ in N_X. Now $\varphi_l, \ldots, \varphi_n$ are arbitrary elements of the coordinate ring of $\mathbb{A}^n$, and the canonicla image of $\sum_{\nu=1}^{n} \frac{\partial f_i}{\partial X_\nu}\varphi_\nu(X)$ in N_X is precisely the image under the canonical homomorphism

$$\Theta_{\mathbb{A}^n}\big|_X \to N_X$$

where $\Theta_{\mathbb{A}^n}$ reprsents the tangent bundle of $\mathbb{A}^n$ and $\big|_X$ denotes its restriction to X. We have a natural exact sequence

$$I/I^2 \to \Omega^1_{\mathbb{A}^n}\big|_X \to \Omega^1_X \to 0$$

where Ω^1_Z denotes the K ahler differentials of order one on a scheme X/k. The dual of this exact sequence gives an exact sequence

$$0 \to \Theta_X \to \Theta^1_{\mathbb{A}^n}\big|_X \to N_X,$$

32 where $\Theta^1_{\mathbb{A}^n}\big|_X \to N_X$ is the homomorphism defined above. We define T^1_X to be the cokernel of this homomorphism. So that we have

$$0 \to \Theta_X \to \Theta^1_{\mathbb{A}^n}\big|_X \to N_X \to T^1_X \to 0.$$

Thus we have proved

Theorem 6.2. *The first order deformation of X are in one-one correspondence with T^1_X.*

Remark 6.4. Suppose that $A' = A[\epsilon]$ (i.e., $A' = A\oplus \in k$). Then the argument which is a combination of that of the theorem and Remark 6.1 above shows that

$$\mathrm{Def}(A') = \mathrm{Def}(A) \times \mathrm{Def}(k[\epsilon]),$$

i.e., the set of deformations of X over A' which extend a given deformation over A are in one-one correspondence with the first order deformation of X. For the proof of this, we remark that a similar observation has been proved above for embedded deformations. Now identifying two embedded deformations $X_{A'}$ and $X'_{A'}$ which reduce to the same embedded deformation X_A over A, the argument is the same as in the discussion preceding Theorem 6.2, and the above remark then follows.

Remark 6.5. Deformations over A' which extend a given deformation over A form a prinipal homogeneous space under $\mathrm{Def}(k[\epsilon])$, or else form the empty set.

33 **Remark 6.6.** *Note that if X is smooth, then $T^1_X = 0$. This implies that any two embedded deformations $X_A, X'_A (A = k[\epsilon])$ of X are isomorphic over A (in fact obtainable by a change of coordinates in $\mathbb{A}^n_A$).*

Remark 6.7. If X has isolated singularities, T^1_X as a vector space over k has finite dimension. For, it is a finite $\mathscr{O}_X$ -module with a finite set as support.

Proposition 6.1. *Suppose that $X = X_{red}$. Then*

$$T^1_X \simeq \mathrm{Ext}^1_{\mathscr{O}_X}(\Omega^1_X, \mathscr{O}_X)(X \subset \mathbb{A}^n).$$

Proof. Let $X = V(I)$. Then the exact sequence

$$I/I^2 \to \Omega^1_{\mathbb{A}^n}\big|_X \to \Omega^1_X \to 0$$

can be split into exact sequence as follows:

(i) $0 \to \in \to I/I^2 \to F \to 0$

(ii) $0 \to F \to \Omega^1_{\mathbb{A}^n}\big|_X \to \Omega^1_X \to 0.$

Since $X = X_{\text{red}}$, the set of smooth points of X is dense open in X, so that ϵ, being concentrated at the nonsmooth points, is a torsion sheaf, in particular, $\mathrm{Hom}_{\mathscr{O}_X}(\epsilon, \mathscr{O}_X) = 0$. Writing the exact sequence $\mathrm{Hom}(\cdot, \mathscr{O}_X)$ for (i), we get that

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathscr{O}_X}(F, \mathscr{O}_X) \arrow[r] & \mathrm{Hom}(I/I^2, \mathscr{O}_X) \arrow[r] \arrow[d, equal] & 0 \text{ is exact .} \\ & & N_X & \end{tikzcd}$$

Writing the exact sequence $\mathrm{Hom}(\cdot, \mathscr{O}_X)$ for (ii), we get that **34**

$$0 \to \Theta_X \to \Theta_{\mathbb{A}^n}\big|_X \to F^* \to Ext^1_{\mathscr{O}_X}(\Omega^1_X, \mathscr{O}_X) \to 0 \text{is exact}$$

(since $\mathrm{Ext}^1_{\mathscr{O}_X}\left(\Omega_{\mathbb{A}}n\big|_X, \mathscr{O}_X\right) = 0, \Omega_{\mathbb{A}^n}\big|_X$ being free).

Now $T^1_X = \mathrm{coker}\left(\Theta_{\mathbb{A}^n}\big|_X \to N_X\right)$, and coker $\left(\Theta_{\mathbb{A}^n}\big|_X \to F^*\right) = \mathrm{Ext}^1_{\mathscr{O}_X}$ $(\Omega^1_X, \mathscr{O}_X)$. Above $F^* \simeq N_X$, and so

$$T^1_X \simeq \mathrm{Ext}^1_{\mathscr{O}_X}(\Omega^1_X, \mathscr{O}_X).$$

$\square$

7 Versal deformations and Schlessinger's theorem

Let R be a complete local k-algebra with k as residue field (k-alg. closed as before). We write $R_n = R/\mathfrak{m}_R^{n+1}$ where $\mathfrak{m}_R = \mathfrak{m}$ is the maximal ideal of R. We are given a closed subscheme X of $\mathbb{A}^n$ (note that definitions similar to the following could be given for more general X).

Definition 7.1. A *formal deformation* X_R of X is: (i) a sequence $\{X_n\}$, $X_n = X_{R_n}$ is a deformation of X over R_n, and (ii) iso-morphisms $X_n \otimes_{R_n} R_{n-1} \simeq X_{n-1}$ for each n.

Suppose that A is a finite-dimensional local k-algebra. Then a k-algebra homomorphism $\varphi : R \to A$ is equivalent to giving a *compatible sequence of homomorphisma* $\varphi_n : R_n \to A$ for n sufficiently large. This is so because a (local) homomorphism $\varphi : R \to S$ of two complete local rings R, S is equivalent to a sequence of compatible homomorphisms
35 $\varphi_n : R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$, and in our case $\mathfrak{m}_A^n = 0$ for $n \gg 0$. Given a formal deformation X_R of X and a homomorphism $\varphi : R \to A; X_n \otimes_{R_n} A$ (via $\varphi_n : R \to A$ as above) is up to isomorphism the same for $\gg 0$. It is a deformation of X over A. *We define this to be* $X_R \otimes A$ (*base change of* X_R *by* $\operatorname{Spec} A \to \operatorname{Spec} R$).

Definition 7.2. A formal deformation X_R of X is said to be *versal* if the following conditions hold: Given a deformation X_A of X over a finite dimensional local k-algebra A, there exists a homomorphism $\varphi : R \to A$ and an isomorphism $X_R \otimes A \simeq X_A$; in fact, we demand a stronger condition as follows: Given a surjective homomorphism $A' \xrightarrow{\theta} A$ of local k-algebras, a deformation $X_{A'}$ over A' a homomorphism $\varphi : R \to A$ and an isomorphism $X_A \otimes_R A \simeq X_{A'} \otimes A$, there is a homomorphism $\varphi' : R \to A'$ such that φ' lifts φ and $X_R \otimes_R A'$ is isomorphic to X'_A. (Note that it suffices to assume the lifting property in the case $\ker\theta$ is of rank 1 over k.)

Let F : (Fin.loc.k − alg) $\to$ (Sets) be the functor defining deformations of X, i.e., $F(A)$ = (isomorphism classes/A of deformations of X/A). Given a formal deformation X_R of X, let

$$G : (\text{Fin.loc.}k - \text{alg.}) \to (\text{Sets})$$

be defined by $G(A) = \mathrm{Hom}_{k-\mathrm{alg}}.(R, A)$. Then we have a morphism $j : G \to F$ of functors defined by

$$\varphi \in \mathrm{Hom}_{k-\mathrm{alg}}.(R, A) \to X_R \otimes_R A.$$

That a formal deformation is versal is equivalent to saying that the functor j is formally smooth. **36**

Theorem 7.1. *(Schlessinger). Let F :(Finite, local, k-alg.)→ (Sets) be a (convaraint) functor. Then there is a formally smooth functor (as above) i.e.,*

$$Hom_{K-alg}.(R, \cdot) \to F(\cdot),$$

where R is acomplete local k-algebra with residue filed k, if

(1) *$F(k)$ = a single point.*

(2) *Given $(\epsilon) \to A' \to A \to 0$ with A', A finite local k-algebras and $(\epsilon) = \mathrm{Ker}(A' \to A)$ of rank 1 over k and a homomorphism $\varphi : B \to A$ let $B' = A' \times_A B\{(\alpha, \beta) \in A' \times B$ such that their canonical images in A are equal $\}$. ($\mathrm{Spec}\, B'$ is the "gluing" of $\mathrm{Spec}\, B$ and $\mathrm{Spec}\, A'$ along $\mathrm{Spec}\, A$ by the morphisms $\mathrm{Spec}\, A \to \mathrm{Spec}\, A'$. If φ is surjective, i.e., $\mathrm{Spec}\, A \to \mathrm{Spec}\, B$ is also a closed immersion, this is a true gluing.) Then we demand that the canonical homomorphism*

$$F(B') \to F(A') \otimes_{F(A)} F(B)$$

is surjective. (Note that B' is also a finite-dimensional local k-algebra.)

(3) *In (2) above, take the particular case $A = k$ and $A' = k[\epsilon]$ (ring of dual numbers), then the canonical map defined as in (*) above*

$$\begin{tikzcd}
F(B') \arrow[r] \arrow[d, equal] & F(k[\epsilon]) \times_{F(k)} F(B) \arrow[d, equal] \\
(F(B[\epsilon]) \arrow[r] & F(k[\epsilon]) \times F(B))
\end{tikzcd}$$

is bijective, *not merely surjective.*

(4) *$F(k[\in])$ is a finite-dimensional vector space over k.* 37

Remark 7.1. One first notes that $F(k[\epsilon])$ has a natural structure of a vector space over k without assuming the axiom (4) above: Given $c \in k$, we have an isomorphism $k[\epsilon] \to k[\epsilon]$ defined by $\epsilon \to c \cdot \epsilon$ (of course $1 \to 1$) which defines a bijective map

$$c_* : F(k[\epsilon]) \to F(k[\epsilon]).$$

By this we define "multiplication by $c \in k$" on $F(k[\epsilon])$. By the third axiom we have a bijection

$$F(k[\epsilon_1, \epsilon_2]) \simeq F(k[\epsilon_1]) \times_{pt} F(k[\epsilon_2])$$

($k[\epsilon_1, \epsilon_2]$ being the 4-dimmensional k-algebra with $\epsilon_1^2 = \epsilon_2^2 = 0$ and basis $1, \epsilon_1, \epsilon_2, \epsilon_1\epsilon_2$). We have canonical homomorphism $k[\epsilon_1, \epsilon_2] \to k[\epsilon]$ defined by $\epsilon_1 \to \epsilon, \epsilon_2 \to \epsilon$ which gives a map $F(k[\epsilon_1, \epsilon_2]) \to F(k[\epsilon])$, so that we get a canonical map

$$F(k[\epsilon]) \times_{pt} F(k[\epsilon]) \to F(k[\epsilon]).$$

Here we use the fact that

$$F(k[\epsilon] \otimes_k k[\epsilon]) = F(k[\epsilon]) \times_{pt} F(k[\epsilon])$$

This follows by axiom (3), and

$$k[\epsilon] \otimes_k k[\epsilon] \simeq k[\epsilon, \epsilon'].$$

We define addition in $F(k[\epsilon])$ by this map, and then we check that this map is bilinear. This gives a natural structure of a k-vector space on
38 $F(k[\epsilon])$ if axioms (1), (3) hold.

Remark 7.2. The versal R can be constructed with $\mathfrak{m}_R/\mathfrak{m}_R^2 \simeq F(k[\epsilon])$. Then with this condition R is uniquely determined up to iso morphism. (Note: We do not claim that given X_A the homomorphism $R \to A$ is unique.) The functor represented by R on (finite local k-alg.) is called the *hull* of F which is therefore uniquely determined up to automorphism.

Proof of Schlessinger's Theorem 1. Let $\mathscr{C}_n$ denotes the full subcategory of (fin.loc.k-alg.) consisting of the rings A such that $\mathfrak{m}_A^{n+1} = 0$. This category is closed under fibered products. We will show the existence of R by finding a versal R_n for $F \mid \mathscr{C}_n$, for all n inductively.

Take the case $n = 1$. $\mathscr{C}_1$ is just the category of rings of the form $A = k \oplus V$ where V is a finite-dimensional vector space. So, $\mathscr{C}_1$ is equivalent to the category of finite-dimensional vector spaces. Let V be the dual space to $F(K[\epsilon])$, and put $R_1 = k \oplus V$. Then a map $R_1 \to k[\epsilon]$ is given by a map $V \to \epsilon k$. i.e., an element of $F(k[\epsilon])$. So, $\mathrm{Hom}(R_1, k[\epsilon]) = F(k[\epsilon])$. Since, by axiom (3), F is compatiable with products of vector spaces, R_1 represents $F \mid \mathscr{C}_1$.

Suppose now that R_{n-1} is given, versal for F/C_{n-1}, and let $u_{n-1} \in F(R_{n-1})$ be the versal element. Let P be a power series ring mapping onto R_{n-1}

$$0 \to J_{n-1} \to P \to R_{n-1} \to 0.$$

Choose an ideal J_n,

$$J_{n-1} \supset J_n \supset \mathfrak{m} J_{n-1},$$

which is minimal with respect to the property that u_{n-1} lifts to $u_n \in$ 39
$F(P/J_n)$, and put $R_n = P/J$. We test (R_n, u_n) for versality. Let $A' \to A$ be a surjection with length 1, kerel $\mathscr{C}_n$, and let a test situation be given:

$$\begin{tikzcd} & A' \arrow[d] \\ R_n \arrow[r] & A, \end{tikzcd} \qquad \begin{tikzcd} & a' \in F(A') \arrow[d, rightsquigarrow] \\ u_n \arrow[r, rightsquigarrow] & a \end{tikzcd}$$

Form $R' = R_n \times_A A'$:

$$\begin{tikzcd} & R' \arrow[r] \arrow[d] & A' \arrow[d] \\ P \arrow[ur, dashed] \arrow[r] & R_n \arrow[r] & A \end{tikzcd} \qquad \begin{tikzcd} u' \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & a' \arrow[d, rightsquigarrow] \\ u_n \arrow[r, rightsquigarrow] & a \end{tikzcd}$$

Since P is smooth, a dotted arrow exists. By axiom (2), there is a $u' \in F(R')$ mapping to u_n and a'. Since J_n was minimal, R' cannot be a quotient of P. Hence $im(P \to R') \approx im(P \to R_n) = R_n$, and so $R' \to R_n$ splits:

$$R' \approx R_n[\epsilon] = R_n \times_k k[\epsilon].$$

Let v' be the element of $F(R')$ induced from u_n by the splitting. The versality will be checked if $u' = v'$, for then the map $R_n \to R' \to A'$ is the required one.

We can still change the splitting, and the permissible changes are by elements of $\mathrm{Hom}(R_n, k[\epsilon]) = \mathrm{Hom}(R_1, k[\epsilon]) = F(k[\epsilon])$. By axiom (3), $F(R') = F(R_n) \times F(k[\epsilon])$. Both u' and v' have the same image u_n in $F(R_n)$. So, we can make the required adjustment. This completes the proof of Schlessinger's theorem.

8 Existence formally versal deformations

40 **Theorem 8.1.** *Let $X \subset \mathbb{A}^n$ be an affine scheme over k with* isolated *singularities. Then for the functor $F = Def.X = Def$ the conditions of Schles-singer's theorem are satisfied. In particular X admits a versal deformation.*

Proof. Since X has isolated singularities, we have $\mathrm{rank}_k(\mathrm{Def}.X)[k[\epsilon]] = \mathrm{rank}_k(T^1_X) < \infty$. Hence it remains only to check the axioms (2) and (3).

Axiom (2): Given $(\epsilon) \to A' \to A, \varphi' : B \to A$ a deformation X_A of X and two deformations $X_{A'}, X_B$ over X_A

$$\begin{tikzcd} X_{A'} \arrow[dr, rightsquigarrow] & & X_B \arrow[dl, rightsquigarrow] \\ & X_A & \end{tikzcd}$$

Let $B' = A' \times_A B$. We need to find a deformation $X_{B'}$ over B' induc-ing the given deformations $X_{A'}$ and X_B over A' and B respectively. As usual we write $\mathscr{O}_{X_{A'}} = \mathscr{O}_{A'}, \ldots$, etc. We now set $\mathscr{O}_{B'} = \mathscr{O}_{A'} \times_{\mathscr{O}_A} \mathscr{O}_B$. We have canonical homomorphisms $B' \to B$ and $B' \to A'$. It is easily seen that $\mathscr{O}_{B'} \otimes_{B'} B \approx \mathscr{O}_B$ and $\mathscr{O}_{B'} \otimes_{B'} A' \approx \mathscr{O}_{A'}$. The only serious point to check is that $\mathscr{O}_{B'}$ is flat/B'. Since $B' = A' \times_A B$, we have a diagram

$$0 \to (\epsilon) \to B' \to B \to 0$$
$$0 \to (\epsilon) \to A' \to A \to 0$$

with exact rows. (ϵ) is of rank 1 over k, so $(\epsilon) \approx k$ as B' modules. Similarly,

$$\begin{tikzcd}
0 \arrow[r] & \in \mathscr{O}_k \arrow[r] \arrow[d, equal] & \mathscr{O}_{B'} \arrow[r] \arrow[d] & \mathscr{O}_B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \in \mathscr{O}_k \arrow[r] & \mathscr{O}_{A'} \arrow[r] & \mathscr{O}_A \arrow[r] & 0.
\end{tikzcd}$$

It follows easily that the exact sequence $o \to \in \mathscr{O}_k \to \mathscr{O}_{B'} \to \mathscr{O}_B \to 0$ **41**
is obtained by tensorting $0 \to (\epsilon) \to B' \to B \to 0$ wiht $\mathscr{O}_{B'}$, and that $\epsilon \cdot \mathscr{O}_k \approx \mathscr{O}_k$. Now the faltness of $\mathscr{O}_{B'}$ is a consequence of $\square$

Lemma 8.1. *Suppose that* $0 \to (\epsilon) \to B' \to B \to 0$ *is exact with* B', B *finite local k-algebras and* $rk_k(\epsilon) = 1$ *(so that* $(\epsilon) \approx k$ *as* B' *module). Suppose that* $X_{B'} = \operatorname{Spec} \mathscr{O}_{B'}$*, is a scheme over* B' *such that* $\mathscr{O}_{B'} \otimes_{B'} B = \mathscr{O}_B$ *is* flat *over B,with* $X_B = \operatorname{Spec} \mathscr{O}_B$ *a deformation of* $X = \operatorname{Spec} \mathscr{O}_k$*, and that* $\ker(\mathscr{O}_{B'} \to \mathscr{O}_B)$ *is isomorphic to* $\mathscr{O}_k$ *(as* $\mathscr{O}_{B'}$ *module). Then* $\mathscr{O}_{B'}$ *is* B' *flat; in particular,* $X_{B'} = \operatorname{Spec} \mathscr{O}_{B'}$ *is a deformation of* X *over* B'.

Conversely, if $\mathscr{O}_{B'}$ *is* B' *flat giving a deformation of* $X = \operatorname{Spec} \mathscr{O}_k$, $\mathscr{O}_{B'}$ *has an exact sequence representation as in the lemma.*

Proof. We have an exact sequence

$$0 \to \epsilon\mathscr{O}_k \to \mathscr{O}_{B'} \to \mathscr{O}_B \to 0,$$

with $\epsilon\mathscr{O}_k \approx \mathscr{O}_k$. We claim that this is obtained by tensoring

$$(*) \qquad 0 \to (\epsilon) \to B' \to B \to 0$$

by $\mathscr{O}_{B'}$ and in fact that it remains exact because of our hypothesis that $\operatorname{Ker}(\mathscr{O}_{B'} \to \mathscr{O}_l)$ is $\approx \mathscr{O}_k$ as B'-module. For, tensoring (*) by $\mathscr{O}_{B'}$ we have

$$\epsilon \otimes \mathscr{O}_{B'} \to \mathscr{O}_{B'} \to \mathscr{O}_B \to 0 \quad \text{exact.}$$

But $\epsilon \otimes \mathcal{O}_{B'} \approx \in \cdot \mathcal{O}_k \approx \mathcal{O}_k$. It follows then that the canonical - homomor- 42
phism $\in \otimes \mathcal{O}_{B'} \to \mathcal{O}_{B'}$, is injective, i.e., (*) remains exact when tensorted by $\mathcal{O}_{B'}$. Consider

$$\begin{tikzcd} & & & 0 \arrow[d] & 0 \arrow[d] & \\ (R1) \arrow[r] & 0 \arrow[r] & (\epsilon) \arrow[r] \arrow[d, equal] & \mathfrak{m}_{B'} \arrow[r] \arrow[d] & \mathfrak{m}_B \arrow[r] \arrow[d] & 0 \\ (R2) \arrow[r] & 0 \arrow[r] & (\epsilon) \arrow[r] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\ & & & k \arrow[d] & k \arrow[d] & \\ & & & (C1) & (C2) & \end{tikzcd} \otimes_{B'} \mathcal{O}_{B'}.$$

Tensoring $(C1)$ by $\mathcal{O}_{B'}$ (over B') and using the Tor sequence we find that $\mathrm{Tor}_1^{B'}(\mathcal{O}_{B'}, k) = 0$ iff $\mathcal{O}_{B'}$ flat iff the canonical homomorphism $\mathcal{O}_{B'} \otimes_{B'} \mathfrak{m}_{B'} \to \mathcal{O}_{B'} \otimes_{B'} B' = \mathcal{O}_{B'}$ is injective. (We use the fact $\mathrm{Tor}_1^{B'}$ $(\mathcal{O}_{B'}, B') = 0$.) Now $(C2)$ is an exact sequence of B-modules, and tensoring it by $\mathcal{O}_{B'}$ (over B') amounts to tensoring it by $\mathcal{O}_B$ (over B). Hence $(C2) \otimes_{B'} \mathcal{O}_{B'}$ stays exact since $\mathcal{O}_B$ is B-flat. Finally

$$\begin{tikzcd} (R1) \otimes_{B'} \mathcal{O}_{B'} & & \mathcal{O}_{B'} \otimes_{B'} (\epsilon) \arrow[r, "i"] \arrow[d, equal] & \mathcal{O}_{B'} \otimes_{B'} \mathfrak{m}_{B'} \arrow[r] \arrow[d, "\alpha'"'] & \mathcal{O}_{B'} \otimes_{B'} \mathfrak{m}_B \arrow[r] \arrow[d, hook, "\alpha"'] & 0 & \text{exact} \\ (R2) \otimes_{B'} \mathcal{O}_{B'} & 0 \arrow[r] & \mathcal{O}_{B'} \otimes_{B'} (\epsilon) \arrow[r] & \mathcal{O}_{B'} \otimes_{B'} \mathfrak{m}_{B'} B' \arrow[r] & \mathcal{O}_{B'} \otimes_{B'} B \arrow[r] & 0 & \text{exact} \end{tikzcd}$$

$(R2) \otimes_{B'} \mathcal{O}_{B'}$ is exact as we observed above. To prove that $\mathcal{O}_{B'}$ is B' flat, it is equivalent to proving that α' is injective, i.e., $\mathrm{Ker}\alpha' = 0$. Note that $\mathcal{O}_{B'} \otimes_{B'} B \approx \mathcal{O}_B$ and $\mathcal{O}_{B'} \otimes_{B'} \mathfrak{m}_B \approx \mathcal{O}_B \otimes_B \mathfrak{m}_B$. Since $\mathcal{O}_B$ is B-flat, α is injective, and form the diagram it follows that α' is also injective.
43 *Further discussion of Axiom (2) and Axiom (3):* Suppose we are given $0 \to (\epsilon) \to A' \to A \to 0$ with $rk_k(\epsilon) = 1$ and a k-algebra homomorphism $\varphi : B \to A$. Set $B' = B \times_A A'$. Let us consider the canonical map (*) in Axiom (2) of Schlessinger's theorem for the functor Def.(X) in more detail, (i.e., the map $\mathrm{Def.}(B') \to \mathrm{Def.}(A') \times_{\mathrm{Def.}(A)} \mathrm{Def.}(B)$). Suppose we are given *deformations* (not merely isomorphism classes) $X_{B'}$

over B', $X_{A'}$ over A' and X_B over B and isomorphisms

$$X_{B'} \otimes_{B'} B \xrightarrow{\nu_1} X_B, \otimes_{B'} A' \xrightarrow{\nu_2} X_A.$$

Now ν_i induce isomorphisms

$$(X_{B'} \otimes_{B'} B) \otimes_B A \xrightarrow{\nu_1 \otimes_B 1_A} X_B \otimes_B A,$$
$$(X_{B'} \otimes_{B'} A') \otimes_{A'} A \xrightarrow{(\nu_2 \otimes_{A'} 1_A)} X_{A'} \otimes_{A'} A.$$

But now there is a canonical isomorphism

$$(X_{B'} \otimes_{B'} B) \otimes_B A \to (X_{B'} \otimes_{B'} A') \otimes_{A'} A.$$

Hence the ν_i determine an isomorphism

(*) $$(X_B \otimes_B A) \xrightarrow{\theta} (X_{A'} \otimes_{A'} A)$$

By the universal property of the "join", we get a morphism

$$f : X_{B'} \to Z_{B'}$$

where $Z_{B'} = \operatorname{Spec}(\mathscr{O}_B \times_{\mathscr{O}_A} \mathscr{O}_{A'})(X_A = \operatorname{Spec} \mathscr{O}_A$ is chosen to be one of the objects in (*) and the homomorphisms $\mathscr{O}_B \to \mathscr{O}_A, \mathscr{O}_{A'} \to \mathscr{O}_A$ are then **44**
defined uniquely but the fibre product $\mathscr{O}_B \times_{\mathscr{O}_A} \mathscr{O}_{A'}$ is independent of the choice for X_A: It is fibre product of $\mathscr{O}_B$ and $\mathscr{O}_{A'}$ by

$$\begin{tikzcd}
\mathscr{O}_B \arrow[dr] & & \mathscr{O}_{A'} \arrow[dl] \\
& \mathscr{O}_B \otimes_B A \underset{\sim}{\overset{\theta}{-}} \mathscr{O}_{A'} \otimes_{A'} A &
\end{tikzcd}$$

and thus well defined). *We claim that f is is an isomorphism*. From this claim it follows as a consequence that given deformations X_B and $X_{A'}$ of X such that $X_B \otimes_B A$ is isomorphic to $X_{A'} \otimes_{A'} A$ i.e., given point $\xi \in (\mathrm{def}.X)(B) \times_{\mathrm{Def}(X)(A)} (\mathrm{Def}.X)(A')$, a deformation $X_{B'}$, which lifts X_B and $X_{A'}$ depends (up to isomorphism) only on the chioce of the isomorphism $X_B \otimes_B A \approx X_{A'} \otimes_{A'} A$, so in particular, we have a *surjective map*

(*) $$\operatorname{Isom}(X_B \otimes_B A \to X_{A, \otimes A}) \to \lambda^{-1}(\xi)$$

where $\lambda : (\mathrm{Def}.X)(B') \to (\mathrm{Def}.X)(B)\times_{(\mathrm{Def}.X)(A)}(\mathrm{Def}.X)(A')$ is the canonical map of Axiom (2). ($(\mathrm{Def}.X)(B)$ by definition=isomorphism classes of deformations of X, i.e., all deformations of $X_{B'}$ of X over B' modulo isomorphisms which induce the identity map on X.) Take the particular case $A = k, A' = k[\epsilon], \varphi : B \to A$; then $X_B \otimes_B A \xrightarrow[\text{can}]{\sim} X, X_{A'} \otimes_{A'} A \xrightarrow[\text{can}]{\sim} X$ and then the left hand side of (*) consists of a unique element, namely, the one induced by the identity map $X \to X$. This implies that $\lambda^{-1}(\xi)$ consists of a unique element, and completes the verification of Axiom (3).Thus, finally it suffices to prove that the morphism $f : X_{B'} \to Z_{B'}$
45 is an isomorphism as claimed above. Note that $(f \otimes k)$ is the identity map $X \to X$ and that $X_{B'}, Z_{B'}$ are flat over B. Hence our claim is a consequence of □

Lemma 8.2. *Let $X^1_A = \mathrm{Spec}\,\mathcal{O}^1_A$ and $X^2_A = \mathrm{Spec}\,\mathcal{O}^2_A$ be two deformations of $X = \mathrm{Spec}\,\mathcal{O}_k$ and $f^* : X^2_A \to X^1_A$ ($f : \mathcal{O}^1_A \to \mathcal{O}^2_A$ a homomorphism of k-algebras) a morphism such that $f^* \otimes k$ is the identity ($f \otimes k$ is the identity). Then is an isomorphism. (In the proof, it would suffice to assume $\mathcal{O}^2_A$ flat/A).*

Proof. The $\mathcal{O}^i_A$ can be realized as embedded deformations of $X = \mathrm{Spec}\,\mathcal{O}_k$, so that we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & I^1_A \arrow[r] & P_A \arrow[r] & \mathcal{O}^1_A \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & I^2_A \arrow[r] & P_A \arrow[r] & \mathcal{O}^2_A \arrow[r] & 0 \end{tikzcd}$$

Let $X'_\nu = f(X_\nu)$ where X_ν are the variables in P_n. We have $X'_\nu = X_\nu + \varphi_\nu$ where $\varphi_\nu \in \mathfrak{m}_A(X_1, \ldots, X_n)(\mathfrak{m}_A = \text{max .ideal of}A)$. It follows easily since A is finite over k (as we have seen before) that $X_\nu \mapsto X'_\nu$ is just a change of coordinates in $\mathbb{A}^n_A$, so that f is induced by an isomorphism $P_A \xrightarrow{\sim} P_A$. We can assume without loss of generality that this is the identity. Then it follows that f is induced by an inclusion $I^1_A \subset I^2_A$. This implies that f is surjective. Let $J = \mathrm{Ker} f$ so that

$$0 \to J \to \mathcal{O}^1_A \to \mathcal{O}^2_A \to 0 \quad \text{is exact.}$$

Since $\mathscr{O}_A^2$ is A-flat, it follows that

$$0 \to J \otimes_A k \to \mathscr{O}_A^1 \otimes k \xrightarrow{f \otimes k} \mathscr{O}_A^2 \otimes k \to 0 \quad \text{is exact.}$$

Since $f \otimes k$ is the identity, it follows that $(J \otimes k) = 0$ Since $\mathfrak{m}_A$ is in **46**
the radical of $\mathscr{O}_A^1$, by Nakayama's lemma, it follows that $J = 0$; hence f is an isomorphism. This completes the proof of the theorem. □

9 The case that X is normal

Let $X \hookrightarrow \mathbb{A}^n$ be normal of dimension ≥ 2. Let $U = X - \text{Sing}X$ (Sing X=Singular points of X). We use the well-known

Proposition 9.1. *Let Z be any smooth (not necessarily affine) scheme over k. Then the set of first order deformations of Z is in one-to-one correspondence with $H^1(Z, \Theta_Z)$, i.e.,*

$$(Def.Z)(k[\epsilon]) \approx H^l(Z, \Theta_Z) \qquad (\Theta_Z \quad \textit{tangent bundle of} Z).$$

Proof. If Z is affine, we have seen that any first order deformation of Z is trivial, i.e., it is isomorphic to base change by $k[\epsilon]$ (this was a con sequence of the fact that when $Z \hookrightarrow \mathbb{A}^n$ and Z is smooth, $T_Z^1 = (0)$). Hence any first order deformation of Z is locally base change by $k[\epsilon]$. Hence if $\{U_i\}$ is an affine covering of Z, a first order deformation of Z is given by $\{\varphi_{ij}\}$

$$\varphi_{ij} : (U_i \cap U_j) \otimes k[\epsilon] \to (U_i \cap U_j) \otimes k[\epsilon],$$

$\{\varphi_{ij}\}$ being automorphisms of $(U_i \cap U_j) \otimes k[\epsilon]$ satisfying the cocycle condition. It is easy to see that first order deformations correspond to cohomology classes. Now it is easy to see that for an affine scheme **47**
$W/k, (W = \text{Spec}\, B)$

$$\text{Aut}_{k[\epsilon]}(W \otimes k[\epsilon]) = \text{Derivations of } B/k (= H^o(W, \Theta_W)).$$

The proposition now follows. □

Lemma 9.1. *Let X_A^1, X_A^2 be two deformations of X (not necessarily of the first order, A =local, finite over k). Let $U^i = X_A^i\big|U$ and let $\varphi : U_A^1 \to U_A^2$ be an isomorphism over A. Then φ extends to an isomorphism (unique) $X_A^1 \to X_A^2$.*

Proof. If X_A is a deformation of X as above, we shall prove

$$(*) \qquad H^o(X_A, \mathscr{O}_A) = H^o(U_A, \mathscr{O}_{U_A})(U_A = X_A\big|_U).$$

The proposition is an immediate consequence of (*), for φ induces a homomorphism

$$\begin{tikzcd} \varphi^* : H^o(U_A^2\mathscr{O}_A) \arrow[r] \arrow[d, equal] & H^o(U_A^1\varphi_A) \arrow[d, equal] \\ H^o(X_A^2, \mathscr{O}_{X_A^2}) \arrow[r] & H^o(X_A^1, \mathscr{O}_{X_A^1}). \end{tikzcd}$$

This implies that φ is induced by a morphism $\psi : X_A^1 \to X_A^2$. It is easily seen that $(\psi \otimes k)$ is the identity, and this implies easily that ψ is an isomorphism.

To prove (*), we note first that if $A = k$, it is well known. In the general case, we have a representation

$$0 \to (\epsilon) \to A \to A_o \to 0 \quad \text{exact,} \quad rk_k(\epsilon) = 1$$

48 By induction it suffices to prove (*) assuming its truth for $X_{A_o} = X_A \otimes_A A_o$. Then we have

$$0 \to \epsilon\mathscr{O}_{X_A} \to \mathscr{O}_{X_A} \to \mathscr{O}_{X_{A_o}} \to 0 \quad \text{exact,}$$

with $\epsilon \cdot \mathscr{O}_{X_A} \approx \mathscr{O}_k$. This is because X_A is flat over A. As a sheaf its restriction to U is also exact. Then we get a diagram

$$\begin{tikzcd} 0 \arrow[r] & H^o(X, \mathscr{O}_k) \arrow[r] \arrow[d, "\wr"'] & H^o(X_A, \mathscr{O}_A) \arrow[r] \arrow[d] & H^o(X_{A_o}, \mathscr{O}_{A_o}) \arrow[r] \arrow[d, "\wr"'] & 0 \\ 0 \arrow[r] & H^o(U, \mathscr{O}_k) \arrow[r] & H^o(U_A, \mathscr{O}_A) \arrow[r] & H^o(U_{A_o}, \mathscr{O}_{A_o}). \end{tikzcd}$$

The first and the last vertical arrows are isomorphisms, and it follows then that the middle one is also an isomorphism. This proves the proposition. □

Remark 9.1. The proposition is equivalent to saying that the morphism of functors

$$\text{(Def. X)} \to \text{(Def. U)}$$

obtained by restriction to U is a *monomorphism*.

Lemma 9.2. *Suppose that X has depth ≥ 3 at all the points (not merely closed points) of Sing X (e.g., $\dim X \geq 3$ and X is Cohen-Macaulay) (X necessarily normal). Then every deformation U_A of U extends to a deformation X_A of X.*

Proof. Define X_A by $X_A = \operatorname{Spec} \mathscr{O}_{X_A}$ with $\mathscr{O}_A = \mathscr{O}_{X_A} = H^o(U_A, \mathscr{O}_A)$. Suppose we have a presentation $0 \to (\epsilon) \to A \to A_o \to 0$ $rk_k(\epsilon) = 1$. We prove this lemma again by an induction as in the previous proposi- 49
tion. This induces an exact sequence of sheaves by flatness

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{O}_U \arrow[r] \arrow[d] & \mathscr{O}_{U_A} \arrow[r] & \mathscr{O}_{U_{A_o}} \arrow[r] & 0 \\ & (\epsilon \cdot \mathscr{O}_U) & . & & \end{tikzcd}$$

Since depth of $\mathscr{O}_{X,x}$ at $x \in \operatorname{Sing} X \geq 3$, by local cohomology the maps

$$\left.\begin{array}{l} H^1(X, \mathscr{O}_K) \to H^1(U, \mathscr{O}_K) \\ H^o(X, \mathscr{O}_K) \to H^o(U, \mathscr{O}_K) \end{array}\right\} \quad \text{are isomorphisms}$$

The isomorphisms $H^i(X, \mathscr{O}_k) \xrightarrow{\sim} H^i(U, \mathscr{O}_k)$ follow from Theorem **??**, p. 44, in Hartshorne's Local cohomology. Indeed, we have the following exact sequence $0 \to \mathscr{O}_X \to \mathscr{O}_U \mathscr{F} \to 0$. We must prove that $H^o(\mathscr{F}) = H^1(\mathscr{F}) = 0$. Theorem **??** (loc. cit) states that for X a locally Noetherian prescheme, Y a closed pre-scheme and $\mathscr{D}$ a coherent sheaf on X, the following conditions are equivalent: (i) $H^i_y(\mathscr{D}) = 0$, $i < n$. (ii) $\operatorname{depth}_Y \mathscr{D} \geq n$. Taking $\mathscr{D} = \mathscr{O}_X$ and $Y = \operatorname{Sing} X$, the fact that $H^o(\mathscr{F}) = H^1(\mathscr{F}) = 0$ follows immediately. In particular, $H^1(U, \mathscr{O}_k) = 0$. Hence we get

$$0 \to H^o(U, \mathscr{O}_U) \to H^o(U, \mathscr{O}_{U_A}) \to H^o(U, \mathscr{O}_{U_{A_o}}) \to 0 \quad \text{is exact.}$$

Now $H^o(U, \mathscr{O}_U) = H^o(X, \mathscr{O}_k) = \mathscr{O}_k$ and

$$H^o(U, \mathscr{O}_{U_{A_o}}) = \mathscr{O}_{A_o}\text{flat}/A_o \quad \text{by induction hypothesis}.$$

50 Hence

$$0 \to \mathscr{O}_k \to \mathscr{O}_A \to \mathscr{O}_{A_o} \to 0 \quad \text{is exact}$$

and $\mathscr{O}_{A_o}$ flat/A_o. By proposition 3.1 it follows that $\mathscr{O}_A$ is flat/A and so X_A represents a deformation of X. ◻

Proposition 9.2. *Let $X \hookrightarrow \mathbb{A}^n$ be such that X is of depth ≥ 3 at $\forall x \in$ Sing X. Then the morphism of functors*

$$Def(X) \to Def(U), \quad U = X - SingX$$

obtained by restriction is an isomorphism, i.e.,

$$(Def\, X)(A) \to (Def\, U)(A) \quad \textit{is an isomorphism } \forall_A \quad \textit{local finite over } .k.$$

Proof. Immediate conseqence of the above two lemmas. ◻

Remark 9.2. Suppose $X \hookrightarrow \mathbb{A}^n$ is normal (with isolated singularities). Then if Θ_X is of depth ≥ 3 at $\forall x \in$ Sing X is rigid. For, it suffices to prove that first order deformations of X are trivial and by the preceding to prove this for U. We have an isomorphism

$$H^1(U, \Theta_U) \leftarrow H^1(X, \Theta_X)$$

because of our hypothesis. But $H^1(X, \Theta_X) = 0$ Since X is affine. This implies the assertion.

10 Deformation of a quotient by a finite group action

51

Theorem 10.1. *(Schlessinger–Inventiones'70). Let Y be a smooth affine variety over k with char$k = 0$. Suppose we are given an action of a finite group G on Y such that the isotropy group is trivial except at a finte number of points of Y, so that the normal affine varirty $X = Y/G$ has* *only isolated singularities (at most the images of the points where the isotropy is not trivial).* Then if $dim Y = dim X \geq 3, X$ is rigid. *(It suffices to assume that the set of points where isotropy is not trivial is of codim≥ 3 ion Y. Of course, the singular points of X need not be isolated, but even then X is rigid.)*

Proof. Let $U = X - \mathrm{Sing}X$, and $V = Y$-(points at which isotropy is not trivial). Then the canonical morphism $V \to U$ is an etale Galois covering with Galois group G. From the foregoing discussion, it suffices to prove that $H^1(U, \Theta_U) = 0$ We have the Cartan-Leray spectral sequence

$$H^p(G, H^q(V, \theta_V)) \Rightarrow H^{p+q}(U, f_*^G(\theta_V))$$

where $f_*^G(\theta_V)$ denotes the G-invariant subsheaf of the direct image of the sheaf of tangent vectors on V. We have $\theta_U = f_*^G(\theta_V)$. Hence the above spetral sequence gives the spectral sequence

$$H^p(G, H^q(V, \theta_V)) \Rightarrow H^{p+q}(U, \theta_U).$$

Now $H^P(G, H^q(V, \theta_V)) = 0$ for $p \geq 1$ since G is finite and the characteristic is zero. Hence the spectal sequence degenerates and we have

$$H^o(G, H^q(V, \theta_V)) \xrightarrow{\text{isom.}} H^q(U, \theta_U).$$

In particular, $H^1(U, \Theta_U) \simeq H^o(G, H^1(V, \Theta_V))$. Now Θ_Y is a vector bundle, and since $\dim Y \geq 3$ and $(Y - V)$ is a finite number of points, Θ_Y is of depth ≥ 3 at every point of $(Y - V)$ (or because codim $(Y - V) \geq 3$). Hence $H^1(V, \Theta_V) = H^1(Y, \Theta_Y) = 0$ It follows then that $H^1(U, \Theta_U) = 0$, which proves the theorem. ◻ **52**

Remark 10.1. Let Y be the (x, y)-plane, $G = \mathbb{Z}/2$ operating by $(x, y) \mapsto (-x, -y)$. Then $X = Y/G$ can be identified with the image of the (x, y) plane in 3 space by the mapping $(x, y) \mapsto (x^2, xy, y^2)$, so that X can be identified with the cone $v^2 = uw$ in the 3 space (u, v, w). It has an isolated singularity at the origin, but it is *not rigid* (cf., the computation of T_X^1 which has been done for the case of a complete intersection, § 4 and § 6).

11 Deformations of cones

Let X be a scheme and F a locally free $\mathscr{O}_X$-modlue of constant rank r. The vector bundle $V(F)$ associated to F is by definition Spec $S(F)$, i.e., "generalized spec" of $S(F)$–the sheaf of symmetric algebras of the $\mathscr{O}_X-$ module F. We get a canonical affine morphism $p : V(F) \to X$. The sections of $V(F)$ over X (morphisms $s = X \to V(F)$ such that $p \circ s = Id_X$ can be identified with

$$\mathrm{Hom}_{\mathscr{O}_X} - alg(S(F), \mathscr{O}_X) \approx \mathrm{Hom}_{\mathscr{O}_X} - \mathrm{mod}, (F, \mathscr{O}_X) \approx H^o(X, F^*),$$

where F^* is the dual of F. *With this convention, sections of the vetor bundle $V(F)$ considered as a scheme over X* are $\approx$ sections of F^* over X.

Let $\mathbb{P}^{N+1} - (0, \dots, 0, 1) \xrightarrow{\pi} \mathbb{P}^N$ be the "projection of $\mathbb{P}^{N+1}$ onto $\mathbb{P}^N$ from the point $(0, \dots 0, 1)$", i.e., π is the morphism obtained by dropping
53 the last coordinate. Let $L = \mathbb{P}^{N+1} - (0, \dots, 0, 1)$. Then we see that L has a natural structure of a line bundle over $\mathbb{P}^N$, and in fact

$$L \simeq \mathrm{Spec}\left(\bigoplus_{n=0}^{\infty} \mathscr{O}_{\mathbb{P}^N}(-n)\right).$$

Proof that $L \simeq \mathrm{Spec}\left(\bigoplus_{n=0}^{\infty} \mathscr{O}_{\mathbb{P}^N}(-n)\right)$: We first remark that in general given F a locally free $\mathscr{O}_X$-module of constant rank r, the vector bundle $V(F) = \mathrm{Spec}\, S(F)$ associated to F (in the definition of Grothendieck) is such that the geometric points of $V(F)$ correspond to the dual of the vector bundle $\mathbb{F}$ associated to F in the "usual" way, that is, F becomes the sheaf of sections of $\mathbb{F}$. Consequently, in order to prove that $L \simeq \mathrm{Spec}\left(\bigoplus_{n=0}^{\infty} \mathscr{O}_{\mathbb{P}^N}(-n)\right)$ we must show that the invertible sheaf corresponding to $L \to \mathbb{P}^N$ is $\mathscr{O}_{\mathbb{P}^N}(1)$. We do this by calculating the transition functions. If we denote the standard covering of $\mathbb{P}^N$ by $u_0, \dots, u_N$, then we see for $\mathbb{P}^{N+1} - (0, 0, \dots, 0, 1) \xrightarrow{\pi} \mathbb{P}^N$ we have $\pi^{-1}((X_0, \dots, X_N))$ is of form $(X_0, \dots, X_N, \lambda)$ and hence the patching data is of the form $\pi^{-1}(u_i \cap u_j) \simeq \mathbb{C} \times (u_i \cap u_j) \xrightarrow{X_i/X_j} \mathbb{C} \times (u_i \cap u_j) \simeq \pi^{-1}(u_i \cap u_j)$ (where X_i are the standard coordinate functions on $\mathbb{P}^N$). This is precisely the patching data for $\mathscr{O}_{\mathbb{P}^N}(1)$. Consequently, $\mathscr{O}_{\mathbb{P}^N}(1)$ is the invertible sheaf associated to $\mathbb{P}^{N+1} - (0, \dots, 0, 1) \xrightarrow{\pi} \mathbb{P}^N$. The set of points

$\mathbb{P}^N \approx S = (*, \ldots, *, 0) \subset \mathbb{P}^{N+1}$ give a section of L over $\mathbb{P}^N$; we can identify this as the 0-section of the line bundle L. We have a canonical isomorphism $\mathbb{A}^{N+1} \xleftarrow{\sim} (\mathbb{P}^{N+1} - S)$ sending $(0, \ldots, 0) \leftarrow (0, \ldots, 0, 1)$, and hence L-(0-section)$\approx \mathbb{A}^{N+1} - (0, \ldots, 0)$. Futher more, it is easily seen that L-(0-section)= $\operatorname{Spec}\left(\bigoplus_{-\infty}^{\infty} \mathscr{O}_{\mathbb{P}^N}(-n)\right)$ and it is the $\mathbb{G}_m$ bundle associated to L. **54**

Suppose now that Y is a closed subscheme of $\mathbb{P}^N$. Let $L_Y = \pi^{-1}(Y)$. Then L_Y is a line bundle over Y and from the preceding we have

$$L_Y = \operatorname{Spec}\left(\bigoplus_{n=0}^{\infty} \mathscr{O}_Y(-n)\right).$$

Let C be the cone over Y, i.e., if P is the canonical morphism $P : (\mathbb{A}^{N+1} - (0, \ldots, 0)) \to \mathbb{P}^N$ induced by the isomorphism $\mathbb{A}^{N+1} \leftarrow (\mathbb{P}^{N+1} - S)$ defined above, we define $C' = P^{-1}(Y)$ and then $C' = C - (0, \ldots, 0)$. The point $(0, \ldots, 0)$ is the vertex of the cone C. So, $C =$ Closure ofC' in $\mathbb{A}^{N+1}$. As before we have

$$L_Y - (0 - \mathit{section}) = \operatorname{Spec}\left(\bigoplus_{n=-\infty}^{\infty} \mathscr{O}_Y(n)\right).$$

Let $\overline{C}$ be the closure of C in $\mathbb{P}^{N+1}$ (C being identified iin $\mathbb{P}^{N+1}$ as above). Then we see that

$$\overline{C} = L_Y \cup (0, \ldots, 0, 1), C = L_Y - (0 - \text{section}).$$

We call $(0, 0, \ldots, 1)$ the vertex of the projective cone $\overline{C}$, for $(0, \ldots 0, 1)$ goes to the vertex of C under the canonical isomorphism $(\mathbb{P}^{N+1} - S) \to \mathbb{A}^{N+1}$. If Y is smooth, $\overline{C}$ is smooth at every point except (possibly) at the vertex.

Consider $\pi : L \to \mathbb{P}^N$. Let T denote the line bundle on L consisting of tangent vectors tangent to the fibers of $L \to \mathbb{P}^N$. Then we have

$$0 \to T \to \Theta_L \to \pi^* \Theta_{\mathbb{P}^N} \to 0 \quad \text{quad}$$

Let us now suppose that Y is a smooth closed subscheme of $\mathbb{P}^N$. Let $\pi_Y : L_Y \to Y$ be the canonical morphism. Then we have a similar **55** exact sequence. Let T_Y denote the bundle of tangents along the fibres of $L_Y \to Y$, so that we have a commutative diagram

(A)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_Y \arrow[r] \arrow[d] & \Theta_{L_Y} \arrow[r] \arrow[d] & \pi_Y^*(\Theta_Y) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T|_{L_Y} \arrow[r] \arrow[d] & \Theta_L|_{L_Y} \arrow[r] \arrow[d] & \pi^*((\Theta_{\mathbb{P}^N})|_Y) \arrow[r] \arrow[d] & 0 \\
 & 0 & N_{L_Y} \arrow[r, "\sim"] \arrow[d] & \pi^*(N_Y) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where N_{L_Y} = normal bundle for the immersion $L_Y \hookrightarrow \mathbb{P}^{N+1}$, N_Y = normal bundle for the immersion $Y \hookrightarrow \mathbb{P}^N$. We have $T_Y \simeq T|_{L_Y}$ from which it follows that

$$\operatorname{Coker}(\Theta_{L_Y} \to \Theta_{L|_{L_Y}}) \xrightarrow{\sim} \operatorname{Coker}[(\pi_Y^*(\Theta_Y)) \to \pi^*(\mathscr{O}_L)_{L_Y}],$$

i.e.,
$$N_{L_Y} \xrightarrow{\simeq} \pi^*(N_Y).$$

Let $U = C$-(vertex). Then if N_U is the normal bundle for $U \hookrightarrow \mathbb{A}^{N+1}$, it is immediate that $N_U = j^*(N_{L_Y})$ where $j : U \to L_Y$ is the canonical inclusion. If we denote by the same π the canonical morphism $\pi : U \to Y$, we deduce that $N_U = \pi^*(N_Y)$. Then restricting the bundles in (A) to U we get

$$
\text{(B)}\quad
\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
\text{(B1)} & 0 \arrow[r] & T_Y|_U \arrow[r] \arrow[d, equal] & \Theta_U \arrow[r] \arrow[d] & \pi^*(\Theta_Y) \arrow[r] \arrow[d] & 0 \\
\text{(B2)} & 0 \arrow[r] & T_Y|_U \arrow[r] & \Theta_{\mathbb{A}^{N+1}-(0)}|_U \arrow[r] \arrow[d] & \pi^*(\Theta_{\mathbb{P}^N}|_Y) \arrow[r] \arrow[d] & 0 \\
 & & & N_U \arrow[r, phantom, "\simeq"] \arrow[d] & \pi^*(N_Y) \arrow[d] & \\
 & & & 0 & 0 &
\end{tikzcd}
$$

The exact sequence (B2) is obtained by restriction to U of the exact **56**
sequence

$$
(**)\qquad 0 \to T\Big|_{\mathbb{A}^{N+1}-(0)} \to \Theta_{\mathbb{A}^{N+1}-(0)} \to \pi^*(\Theta_{\mathbb{P}^N}) \to 0
$$

where $T\Big|_{\mathbb{A}^{N+1}-(0)}$ is the bundle of tangent vectors tangent to the fibres of $\pi : \mathbb{A}^{N+1} - (0) \to \mathbb{P}^N$. We shall now show that we have

$$
\text{(C)}\quad
\begin{tikzcd}
0 \arrow[r] & T|_U \arrow[r] \arrow[d, phantom, "\wr"] & \Theta_{\mathbb{A}^{N+1}-(0)}|_U \arrow[r] \arrow[d, phantom, "\wr"] & \pi^*(\Theta_{\mathbb{P}^N}|_Y) \arrow[r] \arrow[d, phantom, "\wr"] & 0 \\
0 \arrow[r] & \pi^*(\mathscr{O}_{\mathbb{P}^N}) \arrow[r] & \pi^*(\mathscr{O}_{\mathbb{P}^N}(1))^{N+1} \arrow[r] & \pi^*(\Theta_{\mathbb{P}^N}) \arrow[r] & 0,
\end{tikzcd}
$$

where the second row is the pull-back π^* of the well-known sequence

$$
0 \to \mathscr{O}_{\mathbb{P}^N} \to (\mathscr{O}_{\mathbb{P}^N}(1))^{N+1} \to \Theta_{\mathbb{P}^N} \to 0
$$

on $\mathbb{P}^N$ (the middle term of this exact sequence is the direct sum of $\mathscr{O}_{\mathbb{P}^N}(1)$ taken $(N + 1)$ times). Let us examine the canonical homomorphism $h : T|_U \to \Theta_{\mathbb{A}^{N+1}-(0)}|_U$. Let $(z_0, \ldots, z_N)$ be the coordinate of $\mathbb{A}^{N+1}$. Then $H^0(\mathbb{A}^{N+1}, \Theta_{\mathbb{A}^{N+1}})$ is a free module M over $P = k[z_0, \ldots, z_{N+1}]$ **57**
with basis $\dfrac{\partial}{\partial z_0}, \ldots, \dfrac{\partial}{\partial z_{N+1}}$. We see easily that h is defined by restriction to $\mathbb{A}^{N+1} - (0)$ of

$$
\varphi : P \to M
$$

where $\varphi(1) = \Sigma z_i \dfrac{\partial}{\partial z_i}$ (1 is generator of P over P). Hence φ is a graded homomorphism, and is therefore defined by a homomorphism of sheaves on $\mathbb{P}^N$. We see indeed $\varphi = \pi^*(\varphi_0)$, where

$$\varphi_0 : \mathscr{O}_{\mathbb{P}^N} \to (\mathscr{O}_{\mathbb{P}^N}(1))^{N+1}.$$

($\mathscr{O}_{\mathbb{P}^N}(1)$ is defined by homogeneous elements of degree ≥ 1 in P considered as a module over P). From this the assertion (C) follows easily, and we leave these details to the reader.

Let $\mathfrak{F}$ be a coherent $\mathscr{O}_{\mathbb{P}^N}$ module. Then we have

$$H^p(L, \pi^*\mathfrak{F}) \simeq H^p(\mathbb{P}^N, \pi_*\pi^*\mathfrak{F}).$$

Now $\pi^*\mathfrak{F}$ is defined by the sheaf of $\mathscr{O}_{\mathbb{P}^N}$ modules

$$\bigoplus_0^\infty \mathfrak{F}(-n),$$

considered as a sheaf of modules over $\bigoplus_0^\infty \mathscr{O}_{\mathbb{P}^N}(-n)$), so that we find $\pi_*\pi^*\mathfrak{F} = \bigoplus_0^\infty \mathfrak{F}(-n)$ and hence

$$H^p(L, \pi^*\mathfrak{F}) = \bigoplus_0^\infty H^p(\mathbb{P}^N, \mathfrak{F}(-n)).$$

58 Similarly, if $\mathfrak{F}$ is a coherent $\mathscr{O}_Y$-module, we get

$$H^p(L_Y, \pi^*\mathfrak{F}) = \bigoplus_0^\infty H^p(Y, \mathfrak{F}(-n)),$$

$$H^p(U, \pi^*\mathfrak{F}) = \bigoplus_{-\infty}^\infty H^p(Y, \mathfrak{F}(n)) \quad (U = C - (0)) \quad \text{and}$$

$$H^p(\mathbb{A}^{N+1} - (0), \pi^*\mathfrak{F}) = \bigoplus_{-\infty}^\infty H^p(\mathbb{P}^N, \mathfrak{F}(n)).$$

Let us now suppose that Y is a smooth closed subvariety of $\mathbb{P}^N$ (or dimension ≥ 1) and that it is projectively normal, i.e., C is normal.

We have then

$$T^1_C = \operatorname{Coker}(\Theta_{\mathbb{A}^{N+1}|_C} \to N_C).$$

But since C is normal (*of dimension* ≥ 2)

$$H^0(\Theta_{\mathbb{A}^{N+1|C}}) \simeq H^0(U, \Theta_{\mathbb{A}^{N+1}-(0)}|_C).$$

For, $\Theta_{\mathbb{A}^{n+1}}|_C$ is a trivial vector bundle and hence this follows from the fact $H^0(U, \mathscr{O}_U) = H^0(C, \mathscr{O}_C)$. Now N_C is a *reflexive* $\mathscr{O}_C$-module since $N_C = \operatorname{Hom}_{\mathscr{O}_C}(I/I^2, \mathscr{O}_C)$, where I is the defining ideal of C in $\mathbb{A}^{N+1}$. Because of this it follows that

$$H^0(U, N_U) = H^0(C, N_C).$$

Hence we have

$$T^1_C = \operatorname{Coker}(H^0(U, \Theta_{\mathbb{A}^{N+1}-(0)}|_U) \to H^0(U, N_U)).$$

Now from (B) we get **59**

$$\begin{array}{ccc} H^0(U, \Theta_{\mathbb{A}^{N+1}-(0)}|_U) & \xrightarrow{p} & H^0(U, \pi^*(\Theta_{\mathbb{P}^N}|_Y)) \\ \alpha\downarrow & & \downarrow\beta \\ H^0(U, N_U) & \xrightarrow[q]{\sim} & H^0(U, \pi^* N_Y). \end{array}$$

Hence $T^1_C = \operatorname{Coker}\alpha = \operatorname{Coker}(\operatorname{Im} p \xrightarrow{\beta} H^0(U, \pi^* N_Y))$. Now β is a graded homomorphism of graded modules over $\bigoplus\limits_{n=0} H^0(Y, \mathscr{O}_Y(n))$, namely, the gradings are

$$\begin{array}{ccc} H^0(U, \pi^*(\Theta_{\mathbb{P}^N}|_Y)) & = & \bigoplus\limits_{-\infty}^{\infty} H^0(Y, (\Theta_{\mathbb{P}^N}|_Y)(n)) \\ & & \downarrow \\ H^0(U, \pi^* N_Y) & = & \bigoplus\limits_{-\infty}^{\infty} H^0(Y, N_Y(n)). \end{array}$$

From (C), identifying $\Theta_{\mathbb{A}^{N+1}-(0)}|_U \approx \pi^*(\mathscr{O}_{\mathbb{P}^N}(1))^{N+1}$, we get

$$H^0(U, \Theta_{\mathbb{A}^{N+1}-(0)}|_U) = \bigoplus_{n=-\infty}^{\infty} H^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(n+1))$$

and then by (C), p is also a graded homomorphism. Hence $(Im\ p)$ is a graded submodule of $H^0(U, \pi^*(\Theta_{\mathbb{P}^N}|_Y))$. From this it follows that T^1_C has a canonical structure of a graded module over $k[z_0, \ldots, z_{N+1}]$ and in fact that it is a quotient of the graded module

$$N_C = H^0(C, N_C) = \bigoplus_{n=-\infty}^{\infty} H^0(Y, N_Y(n)).$$

Thus we get

60 **Proposition 11.1.** *Let Y be a smooth projective subvariety of $\mathbb{P}^N$ (of dimension ≥ 1) such that the cone C over Y is normal (we have only to suppose that C is normal at its vertex). Then T^1_C has a canonical structure of a graded module over $k[z_0, \ldots, z_{N+1}]$, in fact it is a quotient of the graded module $\bigoplus_{-\infty}^{\infty} H^0(Y, N_Y(n))$.*

12 Theorems of Pinkham and Schlessinger on deformations of cones

Theorem 12.1. *Let Y be as above, i.e., Y is smooth closed $\hookrightarrow \mathbb{P}^N$ and* $\dim Y \geq 1$. *Then*

(1) *(Pinkham). Suppose that T^1_C is negatively graded, i.e., $T^1_C(m) = 0$, $m > 0$. Then the functor*

$$\mathrm{Hilb}(\overline{C}) \to \mathrm{Def}(C)$$

is formally smooth.

(2) *(Schlessinger). Suppose that T^1_C is concentrated in degree 0. Then we have a canonical functor*

$$\mathrm{Hilb}(Y) \to \mathrm{Def}(C),$$

and it is formally smooth. In particular, every deformation of C is a cone.

Recall that $\mathrm{Hilb}(\overline{C})$ *is the functor such that for local finite k-algebras A* $\mathrm{Hilb}(\overline{C})(A) = \{Z \subset \mathbb{P}^{n+1} \times A, Z$ *closed subscheme, Z is flat over A and Z represents an embedded deformation of $\overline{C}$ over A*$\}$.

Proof. (1) Given $0 \to (\epsilon) \to A' \to A \to 0$ exact with $rk_k(\epsilon) = 1$ and A', A finite local over k, we have to prove that the canonical map

$$(*) \qquad \mathrm{Hilb}(\overline{C})(A') \to \mathrm{Def}(C)(A') \times_{\mathrm{Def}(C)(A)} \mathrm{Hilb}(\overline{C})(A)$$

is surjective. Take the case $A' = k[\epsilon]$, $A = k$. In particular $(*)$ implies that **61**

(a) $\mathrm{Hilb}(\overline{C})(k[\epsilon]) \to \mathrm{Def}(C)(k[\epsilon])$ is surjective, and

(b) given $\xi \in \mathrm{Hilb}(\overline{C})(A)$, let $\overline{\xi}$ be the canonical image of ξ in $\mathrm{Def}(C)(A)$.

Then, if $\overline{\xi}$ can be extended to a deformation of C over A', then $\exists \eta \in \mathrm{Hilb}(\overline{C})(A')$ such that $\eta \mapsto \xi$ (the difference between this and $(*)$ above is that we do not insist that $\overline{\eta} \mapsto \overline{\xi}$).

We claim now that (a) and (b) $\Rightarrow$ $(*)$. (In particular, to prove (1) it suffices to check (a) and (b).) Given ξ as above, the set of all $\eta \in \mathrm{Hilb}(\overline{C})(A')$ such that $\eta \mapsto \xi$ (provided there exists an η_0 such that $\eta_0 \mapsto \xi$) has a structure of a principal homogeneous space under $\mathrm{Hilb}(C)(k[\epsilon])$ (see Remark 6.1). [This can be proved in a way similar to proving that $\mathrm{Hilb}(\overline{C})(k[\epsilon]) \simeq H^0(\overline{C}, \mathrm{Hom}(I, \mathscr{O}_{\overline{C}}))$, where I is the ideal sheaf defining $\overline{C}$ in $\mathbb{P}^{N+1}$, cf. Remark 6.2.] Similarly, all deformations $\overline{\eta}$ which extend $\overline{\xi}$ form a principal homogeneous space under $\mathrm{Def}(C)(k[\epsilon])$, provided there exists one. Hence, if (b) is satisfied and $\overline{\eta}$ is a deformation extending $\overline{\xi}$, because of (a) there exists in fact η, $\eta \mapsto \xi$ and $\eta \mapsto \overline{\eta}$. This completes the proof of the above claim.

Now $\mathrm{Hilb}(\overline{C})(k[\epsilon]) = H^0(\overline{C}, N_{\overline{C}})$. We have $\overline{C} = L_Y \cup$ (vertex) and $\overline{C}$ is normal at its vertex. Hence

$$H^0(\overline{C}, N_{\overline{C}}) = H^0(L_Y, N_{L_Y}) = \bigoplus_{n=0}^{\infty} H^0(Y, N_Y(-n)).$$

If T^1_C is negatively graded, it follows that the canonical map

$$\begin{tikzcd} \bigoplus_{n=0}^{\infty} H^0(Y, N_Y(-n)) \arrow[d] & \subset & \bigoplus_{n=-\infty}^{\infty} H^0(Y, N_Y(n)) \\ T^1_C & & \end{tikzcd}$$

62 is surjective. Hence we have checked that

$$\mathrm{Hilb}(\overline{C})(k[\epsilon]) \to (\mathrm{Def}\, C)(k[\epsilon])$$

is surjective. It remains to check (b). Given $\xi \in \mathrm{Hilb}(\overline{C})(A)$, suppose that ξ is locally extendable, i.e., given $\xi : \overline{C}_A \to \mathbb{P}^{N+1}_A$, (i) ξ can be extended locally to deformation over A' at every point of $\overline{C}_A$, and (ii) this extension can be embedded in $\mathbb{P}^{N+1}_{A'}$, so as to extend ξ locally. We observe that it is superfluous to assume (ii), for we have seen that in the *affine* case ($X \subset \mathbb{A}^n$)

$$(\text{Embedded Def})(X) \to \mathrm{Def}(X)$$

is formally smooth. Then we see that to extend ξ to an $\eta \in \mathrm{Hilb}(\overline{C})(A')$, we get an obstruction in $H^1(\overline{C}, N_{\overline{C}})$ (this is an immediate consequence of the fact already observed, that extensions form a principal homogeneous space, cf. Remark 6.5). We observe that for ξ the property (i) is satisfied. Since there is an $\overline{\eta} \in \mathrm{Def}(C)(A')$ with $\overline{\eta} \mapsto \xi$ ($\xi \in \mathrm{Def}(C)(A), \xi \to \overline{\xi}$), the condition (i) is satisfied for all $x \in C$. But now $\overline{C}$ is smooth at every point of $\overline{C} - C$. In this case we have already remarked before that (i) is satisfied (cf., Proof of Proposition 9.1). Since $\overline{C}$ is normal, $\overline{C}$ is of depth ≥ 2 at its vertex, so that by local cohomology we get

$$H^1(\overline{C}, N_{\overline{C}}) \hookrightarrow H^1(L_Y, N_{L_Y}).$$

We have

$$H^1(L_Y, N_{L_Y}) = \bigoplus_{n=0}^{\infty} H^1(Y, N_Y(-n)) \subset \bigoplus_{n=-\infty}^{\infty} H^1(Y, N_Y(n)) = H^1(U, N_U),$$

63 Now by hypothesis $\overline{\xi} \in \mathrm{Def}(C)(A)$ can be extended to $\overline{\eta} \in \mathrm{Def}(C)(A')$; in particular, $\overline{\eta}|_U$ gives an extension of $\overline{\xi}|_U$. Hence the canonical image of this obstruction in $H^1(U, N_U)$ is zero. (We see that as above, extending an embedded deformation of U gives an obstruction in $H^1(U, N_U)$.) This implies that there is an $\eta \in \mathrm{Hilb}(\overline{C})(A')$ such that $\eta \mapsto \xi$. This checks (b), and the proof of (1) is now complete.

(2) Given a deformation of Y, we get canonically a deformation of $U = C - (0)$. To get a canonical functor $\mathrm{Hilb}(Y) \to \mathrm{Def}(C)$, it suffices to prove that a deformation of U can be extended to a deformation (which is unique by an earlier consideration). If depth of C at its vertex is ≥ 3 this follows by an earlier result, but we shall prove this without using it. Let $0 \to (\epsilon) \to A' \to A \to 0$ be as usual, and let $Y_{A'} \rightsquigarrow Y_A$ be deformations of Y in $\mathbb{P}^N$. Then we have an exact sequence of sheaves.

$$0 \to \mathscr{O}_Y \to \mathscr{O}_{Y_{A'}} \to \mathscr{O}_{Y_A} \to 0, \quad \mathscr{O}_Y \approx \epsilon \cdot \mathscr{O}_{Y_{A'}},$$

$\mathscr{O}_{Y_{A'}}$ being A' flat, etc. Similarly for $Y = \mathbb{P}^N$. Then we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
H^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(n)) \arrow[r] \arrow[d] & H^0(Y, \mathscr{O}_Y(n)) \arrow[r] \arrow[d] & 0 \\
H^0(\mathbb{P}^N_{A'}, \mathscr{O}_{\mathbb{P}^N_{A'}}(n)) \arrow[r] \arrow[d] & H^0(Y_{A'}, \mathscr{O}_{Y_{A'}}(n)) \arrow[r] \arrow[d] & 0 \\
H^0(\mathbb{P}^N_A, \mathscr{O}_{\mathbb{P}^N_A}(n)) \arrow[r] \arrow[d] & H^0(Y_A, \mathscr{O}_{Y_A}(n)) \arrow[r] \arrow[d] & 0 \\
0 & 0 & \\
(C1) & (C2) &
\end{tikzcd}$$

where $n \geq 0$. It is immediate that (C1) and (C2) are exact. From this 64 commutative diagram, by induction on $rk_k A'$, it follows that

$$H^0(\mathbb{P}^N_{A'}, \mathscr{O}_{\mathbb{P}^N_{A'}}(n)) \to H^0(Y_{A'}, \mathscr{O}_{Y_{A'}}(n)) \to 0$$

is exact because (C1) and (C2) are exact and the first and third rows are exact. Then as an easy exercise it follows that the second row is exact. Then it follows that $H^0(Y_{A'}, \mathscr{O}_{Y_{A'}}, (n)) \to H^0(Y_A, \mathscr{O}_{Y_A}(n)) \to 0$ is exact. Define $\mathscr{O}_{C_{A'}} = \bigoplus_{n=0}^{\infty} H^0(Y_{A'}, \mathscr{O}_{Y_{A'}}(n))$ (resp. $\mathscr{O}_{C_A}$). Then we get an exact sequence

$$0 \to \mathscr{O}_C \to \mathscr{O}_{C_{A'}} \to \mathscr{O}_{C_A} \to 0, \quad \mathscr{O}_C \simeq \epsilon \cdot \mathscr{O}_{C_{A'}}.$$

Again by induction on $rk\, A'$ it follows that $\mathscr{O}_{C_{A'}}$ is A'-flat (for, $\mathscr{O}_{C_A}$ is A-flat by induction hypothesis and by, an earlier lemma (Lemma 8.1) this claim follows). Hence Spec $\mathscr{O}_{C_{A'}}$ provides a deformation of C, i.e., it extends the deformation of $U = C - (0)$ which is given by the deformation $Y_{A'}$ of Y in $\mathbb{P}^N$. This proves the required assertion. Since depth of C at its vertex is ≥ 3 by an earlier result this is the case. Hence we get a canonical functor

$$\mathrm{Hilb}(Y) \to \mathrm{Def}(C).$$

From this stage the proof is similar to (1) above. As before it suffices to check assertions similar to (a) and (b) above. We see that Hilb(Y) ($k[\epsilon]$) $\simeq H^0(Y, N_Y)$. We have

$$\begin{tikzcd} H^0(Y, N_Y) \subset \displaystyle\bigoplus_{n=-\infty}^{\infty} & H^0(Y, N_Y(n)) \arrow[d] \\ & T^1_C. \end{tikzcd}$$

65

The hypothesis that T^1_C consists only of degree 0 elements, implies that $H^0(Y, N_Y) \to T^1_C$ is surjective, i.e.,

$$\mathrm{Hilb}(Y)(k[\epsilon]) \to (\mathrm{Def}\, C)(k[\epsilon])$$

is surjective. This checks (a). To check (b), take $0 \to (\epsilon) \to A' \to A \to 0$ as usual, $\xi \in \mathrm{Hilb}(Y)(A)$, $\xi \mapsto \eta \in (\mathrm{Def}\, C)(A)$. Suppose we are given $\eta' \in (\mathrm{Def}\, C)(A)$ extending η, then we have to find $\xi' \in \mathrm{Hilb}(Y)(A')$ extending ξ. Since Y is smooth, the condition for local extension over A is satisfied as above, and hence we get an obstruction element $\lambda \in H^1(Y, N_Y)$. We have

$$H^1(Y, N_Y) \hookrightarrow \bigoplus_{-\infty}^{\infty} H^1(Y, N_Y(n)).$$

In a similar way, the element $\eta|_U$ defines an obstruction element $\mu \in H^1(U, N_U) = \bigoplus\limits_{-\infty}^{\infty} H^1(Y, N_Y(n))$. But by hypothesis this obstruction is zero. By functorially and the fact that $H^1(Y, N_Y) \hookrightarrow \bigoplus\limits_{-\infty}^{\infty} H^1(Y, N_Y(n))$ it follows that $\lambda = 0$. This proves the existence of ξ' and the theorem is proved. $\square$

Examples where the hypothesis of the above theorem are satisfied.

Lemma 12.1. *Let $Y \hookrightarrow \mathbb{P}^N$ be a smooth projective variety such that*

$$\begin{cases} H^1(Y, \mathscr{O}_Y(n)) = 0, & \forall n \neq 0, \quad n \in \mathbb{Z} \\ H^1(Y, \Theta_Y(n)) = 0, & \forall n \neq 0, \quad n \in \mathbb{Z}, \end{cases}$$

and Y is projectively normal. Then T^1_C consists only of degree 0 elements ($\dim Y \geq 2$, follows from the hypothesis). 66

The hypothesis $H^1(Y, \mathscr{O}_Y(n)) = 0$, $n \neq 0$ implies that

$$T^1_C = \mathrm{Coker}(\Theta_{\mathbb{A}^{N+1}-(0)}|_U \to N_U) = \mathrm{Coker}(\pi^*(\Theta_{\mathbb{P}^N}|_Y) \to \pi^*(N_Y))$$

[as graded modules modulo elements of degree 0]. To prove this we have only to write the cohomology exact sequence for (B2) of § 11 as well as use (C) of § 11. To compute $\mathrm{Coker}(\pi^*(\Theta_{\mathbb{P}^N}|_Y) \to \pi^*(N_Y))$, use the exact sequence

$$0 \to \pi^*(\Theta_Y) \to \pi^*(\Theta_{\mathbb{P}^N}|_Y) \to \pi^*(N_Y) \to 0.$$

Now $H^1(\pi^*(\Theta_Y)) = \bigoplus_{-\infty}^{\infty} H^1(Y, \Theta_Y(n))$. Since by hypothesis $H^1(Y, \Theta_Y(n)) = 0$ for $n \neq 0$, it follows by writing the cohomology exact sequence for the above, that $\mathrm{Coker}(\pi^*(\Theta_{\mathbb{P}^N}|_Y) \to \pi^*(N_Y)) = T^1_C$ has only degree 0 elements. This proves the lemma.

Lemma 12.2. *Let $Y \subset \mathbb{P}^N$ be a smooth projective variety such that*

$$H^1(Y, \mathcal{O}_Y(n)) = 0, \quad \forall n > 0$$
$$H^1(Y, \Theta_Y(n)) = 0, \quad \forall n > 0.$$

Then T^1_C has only elements in degree ≤ 0.

Proof. The proof is the same as for (2) above. □

67 **Remark 12.1.** Given a smooth projective Y and an ample line bundle L on Y the conditions in (1) (resp. (2)) above are satisfied for the projective embeddings of Y defined by nL, $n \gg 0$ if $\dim Y \geq 1$ (resp. $\dim Y \geq 2$).

Exercise 12.1. Let $Y_0 = 3$ *collinear* points in $\mathbb{P}^2$ and $Y_1 = 3$ non-collinear points in $\mathbb{P}^2$. Take a deformation of Y_0 to Y_1, which obviously exists. Show that this deformation cannot be extended to a deformation of the affine cone C_0 over Y_0 to the affine cone C_1 over Y_1. [For, if such a deformation exists, we see in fact that $\exists$ a deformation of $\overline{C}_0$ to $\overline{C}_1$, closures respectively of C_0 and C_1 in $\mathbb{P}^3$. We have a (finite) morphism $\varphi : \overline{C}_1 \to \overline{C}_0$ which is an isomorphism outside the vertex and not an isomorphism at the vertex. We have

$$0 \to \mathcal{O}_{\overline{C}_0} \to \varphi_*(\mathcal{O}_{\overline{C}_1}) \to k \to 0.$$

This implies that the arithmetic genus of $\overline{C}_1$ = (arithmetic genus of $\overline{C}_0 + 1$), but if there existed a deformation, they would be equal.]

13 Pinkham's computation for deformations of the cone over a rational curve in $\mathbb{P}^n$

Lemma 13.1. *Let $Y \subset \mathbb{P}^n$ be a connected curve of degree n, not contained in any hyperplane. The $Y = Y_1 \cup \ldots \cup Y_r$ where Y_i are smooth* *rational curves of degree n_i such that $n = \Sigma n_i$, each Y_i spans a linear subspace of $\mathbb{P}^n$ of dimension n_i and the intersections of Y_i are transversal.*

Proof. Let Y be an irreducible curve of degree $< n$ in $\mathbb{P}^n$. Then we claim that there is a hyperplane H such that $Y \subset H$. Choose n distinct points on Y. There exists a hyperplane H passing through these n points. **68**
We claim that H contains Y for if H does not contain Y it follows that $\deg(H \cdot Y) > n$, contradicting the fact $\deg Y < n$. It then follows that if Y is an irreducible curve of degree r, $r < n$, then there is a linear subspace $H \simeq \mathbb{P}^s$ in $\mathbb{P}^n$ such that $Y \subset H \simeq \mathbb{P}^s$.

Now let Y be a curve in $\mathbb{P}^n$ of degree n not contained in any hyperplane. Let $Y_i (1 \leq i \leq r)$ be the irreducible components of Y. Then if $n_i = \deg Y_i$, we have $n = \Sigma n_i$. By the foregoing, if H_i is the linear subspace of $\mathbb{P}^n$ generated by Y_i, then $\dim H_i \leq n_i$. Since Y_i's generate $\mathbb{P}^n$, *a fortiori* the H_i's generate $\mathbb{P}^n$. Since Y is connected, $\cup H_i$ is also connected, i.e., we can write a sequence $H_1, \ldots, H_r$ such that $H_j \cap H_{j+1} \neq \phi$. We find easily that if L_j is the linear subspace generated by $H_1, \ldots, H_j$, then $\dim L_j \leq n_1 + \cdots + n_j$. Since $n = \sum_{i=1}^{r} n_i$ and $\dim L_r = n$ it follows that $\dim L_j = n_1 + \cdots + n_j$. It follows in particular that $\dim H_i = n_i$ and that $L_j \cap H_j =$ one point. It remains to prove that Y_i is smooth and rational (the assertions about transversality are immediate by the foregoing), and this is a consequence of the following □

Lemma 13.2. *Let Y be an irreducible curve in $\mathbb{P}^n$ of degree n not contained in any hyperplane. Then Y is a smooth rational curve, in fact parametrized by $t \mapsto (1, t, t^2, \ldots, t^n)$.*

Proof. Let x_1 be a singular point of Y. Choose n distinct points $x_1, \ldots, x_n$ on Y. Then there is a hyperplane H such that $x_i \in H$. Now x_1 cannot be smooth in $H \cap Y$, for if it were so it would follow that x_1 is also smooth on Y. From this it follows easily **69**

$$\mathrm{Deg}(H \cdot Y) > n$$

which is a contradiction. Hence every point of Y is smooth. □

To prove the assertion about parametric representation and ratiorality of Y, project Y from a point p in Y into $\mathbb{P}^{n-1}$ (see § 11). We get an irreducible curve $Y \subset \mathbb{P}^{n-1}$ of degree $(n-1)$. Then it is smooth and by induction hypothesis Y' can be parametrized as $(1, t, \ldots, t^{n-1})$. The projection can be identified as the mapping obtained by dropping the last coordinate. Hence the parametric form of Y can be taken as $(1, t, \ldots, t^{n-1}, f(t))$ where $f(t)$ is a polynomial. Take the hyperplane H as $x_n = 0$; then by the hypothesis that Y is of degree n, it follows that $f(t)$ polynomial of degree n. Then by change of coordinates we see easily that the parametric form of Y is $t \mapsto (1, t, t^2, \ldots, t^n)$.

Lemma 13.3. *Let Y be an irreducible curve in $\mathbb{P}^n$ of degree n, not contained in any hyperplane, or equivalently, a (smooth) curve parametrized by $(1, t, \ldots, t^n)$. Then Y is projectively normal.*

Proof. Let $L = \mathscr{O}_{\mathbb{P}^n}(1)|_Y$. It is well known that projective normality of $Y \subseteq \mathbb{P}^n$ is equivalent to the fact that the canonical mapping

$$\varphi_\nu : H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\nu)) \to H^0(Y, L^\nu) \quad \text{is surjective.}$$

□

This follows from the following

70 **Sublemma 13.1.** *Let X be a normal projective variety. Then X is projectively normal, i.e., $\hat{X}$ (the cone over X) is normal if and only if $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\nu)) \to H^0(X, L^\nu)$, $L = \mathscr{O}_{\mathbb{P}^n}(1)|_X$, $\nu \geq 0$, is surjective.*

Proof. In general we have $A = \{\text{functions on } \hat{X} - (0)\} = \bigoplus_{\nu \in \mathbb{Z}} H^0(X, L^\nu)$ (see discussion in § 11). Now since L is ample we have $H^0(X, L^\nu) = 0$, $\nu < 0$, and therefore $A = \bigoplus_{\nu \geq 0} H^0(X, L^\nu)$. Now suppose $\hat{X}$ is normal. Then $\{\text{functions on } \hat{X} - (0)\} = \{\text{functions on } \hat{X}\}$. Also, $\hat{X} \subset \hat{\mathbb{P}}^n = \mathbb{A}^{n+1}$, and using the same reasoning as before $B = \{\text{functions on } \mathbb{A}^{n+1}\} = \bigoplus_{\nu \geq 0} H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\nu))$. But any function on $\hat{X}$ can be extended to $\mathbb{A}^{n+1}$ which implies the natural map $B \xrightarrow{\gamma} A$ is surjective. Therefore $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\nu)) \to H^0(X, L^\nu)$ is surjective $\forall \nu \geq 0$.

In general we have $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\nu)) \to H^0(X, L^\nu)$ is surjective for ν large. Also $\oplus H^0(X, L^\nu)$ is normal since X is. Thus $\bigoplus_{\nu \geq 0} H^0(X, L^\nu)$ is the integral closure of $\operatorname{Im}\gamma$. Hence if we assume $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\nu)) \to H^0(X, L^\nu)$ is surjective $\forall \nu \geq 0$, we have that $\oplus H^0(X, L^\nu)$ is normal. This means X is normal.

Now returning to the proof of the lemma, $Y \simeq \mathbb{P}^1$ and $L \simeq \mathscr{O}_{\mathbb{P}^1}(n)$ so that $L^\nu \simeq \mathscr{O}_{\mathbb{P}^1}(n\nu)$. It is clear that φ_1 is injective if and only if Y is not contained in any hyperplane. On the other hand, we have

$$\begin{aligned} \dim H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)) &= \dim H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)) \\ &= n + 1. \end{aligned}$$

Hence φ_1 is an isomorphism, i.e., $Y \hookrightarrow \mathbb{P}^n$ is the immersion defined 71
by the complete linear system associated to $\mathscr{O}_{\mathbb{P}^1}(n)$. It is seen easily that $S^\nu(H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)) \to H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n\nu))$ is surjective. This implies that φ_ν is surjective for all ν, and the lemma is proved. □

Lemma 13.4. *Let Y be as in the previous lemma. Then*

$$\begin{aligned} H^1(Y, L^\nu) = 0, &\quad \nu \geq 0 \\ H^1(Y, \Theta_Y(\nu)) = 0, &\quad \nu \geq 0 \end{aligned}$$

(the conditions in Lemma 12.1 are satisfied).

Proof. It is well known that $\Theta_Y \simeq \mathscr{O}_{\mathbb{P}^1}(2)$ and $H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(\nu)) = 0$, $\nu \geq 0$. This proves the lemma. □

Proposition 13.1. *Let Y be the nonsingular rational curve in $\mathbb{P}^n$ of degree n parametrized by $t \mapsto (1, t, \ldots, t^n)$. Let C be the cone in $\mathbb{A}^{n+1}$ over Y and $\overline{C}$ its closure in $\mathbb{P}^{n+1}$. Then the function*

$$\operatorname{Hilb}(\overline{C}) \to \operatorname{Def}(C)$$

is formally smooth. (Note by $\operatorname{Hilb}(\overline{C})$ *we mean the restriction of the usual* Hilb *to (local alg. fin./k) and defining deformations of $\overline{C}$.)*

Proof. This is an immediate consequence of Lemma 12.1. □

Let R be the complete local ring associated to the versal deformation of C (with tangent space = $\mathrm{Def}(C)(k[\epsilon])$ and R' the completion of the local ring at the point of the scheme $\mathrm{Hilb}(\overline{C})$ corresponding
72 to $\overline{C}$. Let $\rho : R \to R'$ be the k-algebra homomorphism defined by $\mathrm{Hilb}(\overline{C}) \to \mathrm{Def}(C)$ to define this homomorphism we need not use the representability of $\mathrm{Hilb}(\overline{C})$, it suffices to note that $\mathrm{Hilb}(\overline{C})$ also satisfies Schlessinger's axioms). From the above proposition it follows that the homomorphism ρ is formally smooth. Hence we can conclude that R is reduced (resp. integral, etc.) iff R' is reduced (integral, etc.).

To study $\mathrm{Hilb}(\overline{C})$ we note that $\overline{C}$ is of degree n in $\mathbb{P}^{n+1}$ and that it is smooth outside its vertex.

Proposition 13.2. *Let $X \hookrightarrow \mathbb{P}^{n+1}$ be a* smooth *projective surface of degree n in $\mathbb{P}^{n+1}$, not contained in any hyperplane.* Then X specializes to $\overline{C}$, i.e., there is a 1-parameter flat family connecting X and $\overline{C}$.

Proof. Let H_∞ be the hyperplane at ∞, with equation $x_{n+1} = 0$ (coordinates $(x_0, \dots, x_{n+1})$). Let X be any closed subscheme of $\mathbb{P}^{n+1}$. Then it is easy to see that X "specializes set theoretically" to the cone over $X \cdot H_0$ (with vertex $(0, \dots, 0, 1)$ and base $X \cdot H_\infty$); in fact we define the 1-parameter family X_t, $X_1 = X$, X_0 = cone over $X \cdot H_\infty$ (as point set) as follows.

$$(x_0, \dots, x_{n+1}) \in X_t \Longleftrightarrow (x_0, \dots, tx_{n+1}) \in X.$$

This family is obtained as follows: Consider the morphism

$$\mathbb{A}^{n+2} \times \mathbb{A}^1 \to \mathbb{A}^{n+2}$$

defined by $((x_0, \dots, x_{n+1}), t) \mapsto (x_0, \dots, x_n, tx_{n+1})$. We see that it defines rational morphism

$$: \mathbb{P}^{n+1} \times \mathbb{A}^1 \to \mathbb{P}^{n+1}$$

73 and that φ is indeed a morphism outside the point $x_0 = ((0, \dots, 0, 1), 0)$.
Let us denote by φ_t the morphism (resp. rational for $t = 0$)

$$\varphi_t : \mathbb{P}^{n+1} \to \mathbb{P}^{n+1};\ \varphi_t = \varphi | \mathbb{P}^{n+1} \times \{t\}.$$

Then for all $t \neq 0$, φ_t is an isomorphism. Let $Z' = \varphi^{-1}(X)$ (scheme theoretic inverse image). We see that Z' is a closed subscheme of $\mathbb{P}^{n+1} \times$

$(\mathbb{A}^1 - x_0)$. There exists a closed subscheme Z of $\mathbb{P}^{n+1} \times \mathbb{A}^1$ which extends Z' and such that Z is the closure of Z' as point set. It is easy to see that $Z = Z' \cup (0, \ldots, 0, 1)$ (as sets). Let $p : Z \to \mathbb{A}^1$ denote the canonical morphism. Then $\forall t \in \mathbb{A}^1$, $t \neq 0$, $p^{-1}(t) \simeq X_t$ and for $t = 0$, $p^{-1}(0) \simeq$ cone over $X \cdot H_\infty$ (as sets).

The map $(x_0, \ldots, x_{n+1}) \to (x_0, \ldots, x_n, tx_{n+1})$ $(t \neq 0)$ defines an automorphism of $\mathbb{P}^{n+1}$, the image of X under this is denoted by X_t. The scheme X_t specializes to a scheme $X_0 \hookrightarrow \mathbb{P}^{n+1}$ (this follows for example by using the fact that Hilb is proper). From the above discussion it follows that X_0 = cone over $H \cdot X_\infty$ (as point sets). It follows from this argument that we can choose Z so that $p : Z \to \mathbb{A}^1$ is *flat* (and then Z is uniquely determined).

Suppose now that X is smooth, $X \cdot H_\infty$ is smooth and that the cone over $H_\infty \cdot X$ (scheme theoretic intersection) is normal. Then we will show that $p^{-1}(0) = X_0 =$ (cone over $X \cdot H_\infty$) scheme theoretically. For this, let I be the homogeneous ideal in $k[x_0, \ldots, x_{n+1}]$ of all polynomials vanishing on X. Let $I = (f_1, \ldots, f_q)$ where $f_i = f_i(x_0, \ldots, x_{n+1})$ are homogeneous polynomials. Then $X_t = V(I_t)$ where $I_t(f_i(x_0, \ldots, x_n, tx_{n+1}))$ for $t \neq 0$. Let $X'_0 = V(I'_0)$ where I'_0 is the ideal $I'_0 = (f_i(x_0, \ldots, x_n, 0))$. **74**
Let I_0 be the ideal of X_0. Then clearly $I'_0 \subset I_0$. Also it is easy to see X'_0 is the scheme theoretic cone over $X \cdot H_\infty$. This means $(X'_0)\,\mathrm{red} = (X_0)\,\mathrm{red}$. For, since $I'_0 \subset I_0$ we have $X'_0 \supset X_0$. But by assumption X'_0 is normal hence in particular reduced. Therefore $X'_0 = X_0$ as required.

Let us now return to our particular case, where X is a smooth projective surface in $\mathbb{P}^{n+1}$ of degree n not contained in any hyperplane. Then we can choose a suitable hyperplane and call it H_∞ such that $H_\infty \cdot X$ is smooth (Bertini) of degree n in $\mathbb{P}^n$, and not contained in any hyperplane.

(The ideal defined by $X \cap H_\infty$ is $\mathscr{O}_X(-1)$. We have

$$0 \to \mathscr{O}_X \to \mathscr{O}_X(1) \to \mathscr{O}_{X \cap H_\infty}(1) \to 0$$

is exact. This gives

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, \mathscr{O}_X) \arrow[r] & H^0(\mathscr{O}_X(1)) \arrow[r] & H^0(X \cap H_\infty, \mathscr{O}_{X \cdot H_\infty}(1)) & \text{exact} \\
0 \arrow[r] & H^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}) \arrow[r] \arrow[u, "\text{surjection}"'] & H^0(\mathscr{O}_{\mathbb{P}_{n+1}}(1)) \arrow[r] \arrow[u, "\text{injective}"] & H^0(\mathscr{O}_{\mathbb{P}^n}, \mathscr{O}_{\mathbb{P}^n}(1)) \arrow[r] \arrow[u] & 0 \text{ exact.}
\end{tikzcd}$$

The second row is exact, and the first vertical arrow is a surjection. The second vertical line is injective. This implies (via diagram chasing) that the third arrow is an injection which implies $X \cap H_\infty$ is not contained in any hyperplane. (We make this fuss because *a priori* degree does not imply having the same Hilbert polynomial.))

Then by the preceding lemmas it follows that $H_\infty \cdot X$ is a rational curve in $\mathbb{P}^n$ parametrized by $(1, t, \ldots, t^n)$ and then by the above dis-
75 cussion it follows that there is a 1-parameter flat family of closed subschemes deforming X to $\overline{C}$ ($\overline{C}$ is the closure in $\mathbb{P}^{n+1}$ of the cone associated to $X \cdot H_\infty$). This comples the proof of the proposition. $\square$

Remark 13.1. Let P be the Hilbert polynomial of $\overline{C} \hookrightarrow \mathbb{P}^{n+1}$. Let H denote the Hilbert scheme of all closed subschemes of $\mathbb{P}^{n+1}$ with Hilbert polynomial P. Then H is known to be projective. The ring associated with $\mathrm{Hilb}(\overline{C})$ above, is the completion of the local ring of H at the point corresponding to $\overline{C}$. Let H_s be the open subscheme of H of points $h \in H$ such that (a) the associated subscheme X_h of $\mathbb{P}^{n+1}$ is smooth and (b) X_h is not contained in any hyperplane in $\mathbb{P}^{n+1}$. It is easy to see that (a) and (b) define an open condition; that (a) defines an open condition is well known, and that (b) also defines an open condition is easily checked. The foregoing shows that as a point set H_s corresponds to the set of smooth surfaces in $\mathbb{P}^{n+1}$ of degree n not contained in any hyperplane and further $\overline{H}_s$ contains points associated to projective cones over smooth irreducible rational curves of degree n in $\mathbb{P}^n$.

The varieties in $\mathbb{P}^{n+1}$ corresponding to points of H_s have been classified by the following

Theorem 13.1 (Nagata)**.** *Let $X \hookrightarrow \mathbb{P}^{n+1}$ be a closed smooth (irreducible) surface of degree n, not contained in any hyperplane. Then either (a) X is a* rational scroll*, i.e., it is a ruled surface where the rulings are lines in $\mathbb{P}^{n+1}$ and it is $\mathbb{P}^1$ bundle over $\mathbb{P}^1$; in fact $X \approx F_{n-2}$; where F_r we denote a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ with a section B having self-*
76 *intersection $(B)^2 = -r$, and then it is embedded by the linear system $|B + (n-1)L|$, where L denotes the line bundle corresponding to the ruling, or (b) $n = 4$ and $X = \mathbb{P}^2$ given by the Veronese embedding of $\mathbb{P}^2$ in*

$\mathbb{P}^5$, *i.e. the embedding defined by* $\mathcal{O}_{\mathbb{P}^2(2)}$*:*

$$(x_0, x_1, x_2) \mapsto (x_0^2, x_1^2, x_2^2, x_1x_2, x_2x_0, x_0x_1), \quad \textit{or}$$

(c) $n = 1$, $X = \mathbb{P}^2$.

Remark 13.2. Let $X = \mathbb{P}^2$, and take the Veronese embedding, with

$$\begin{aligned} U_0 = x_0^2,\ U_1 = x_1^2,\ U_2 = x_2^2 \\ V_0 = x_1x_2,\ V_1 = x_2x_0,\ V_2 = x_0x_1. \end{aligned}$$

Then X is a *determinantal variety*, defined by (2×2) minors of the matrix

$$\begin{bmatrix} U_0 & V_2 & V_1 \\ V_2 & U_1 & V_0 \\ V_1 & V_0 & U_2 \end{bmatrix}$$

The cone $\overline{C}$ over the rational quartic curve has a determinantial representation defined, for instance, by letting U_2 specialize to V_2, i.e., substituting V_2 for U_2 in the above matrix.

Remark 13.3. The general scroll can be checked to be the determinantal variety defined by (2×2) minors of the $(2 \times n)$ matrix

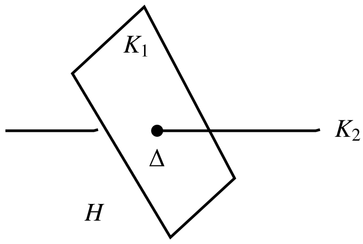
$$\begin{bmatrix} x_0 & \dots & (x_{n-1} + x_{n+1}) \\ x_1 & \dots & x_n \end{bmatrix},$$

and the cone $\overline{C}$ is obtained by setting $x_{n+1} = 0$ in this matrix. **77**

Let us now take the case $n = 4$. Any two scrolls (resp. Veronese surfaces) in $\mathbb{P}^5$ are equivalent under the projective group. Hence H_s split up into two orbits under the projective group: say, $H_s = K_1 \cup K_2$. We note also that there exists no flat family of closed subschemes of $\mathbb{P}^{n+1}$ connecting a scroll and a Veronese. For if there existed one, the Veronese would be topologically isomorphic to a scroll (say, we are over $\mathbb{C}$). This is not the case: One can see this, for example, by the fact that H^2 (scroll, $\mathbb{Z}$) $= \mathbb{Z}^2$, $H^2(\mathbb{P}^2, \mathbb{Z}) = \mathbb{Z}$. Since K_i are orbits under $PGL(5)$, they are locally closed in H_s, and the nonexistence of a flat family connecting a member of K_1 to K_2 shows that $K_i (i = 1, 2)$ are in fact open in H_s, so that H_s is the union of disjoint open sets K_1 and K_2; in particular, H_s is reducible. Let ξ be the point of H determined by $\overline{C}$ [we fix a particular $\overline{C}$, namely, the cone in $\mathbb{P}^{n+1}$ with vertex $(0, \ldots, 0, 1)$ and base the rational curve in $\mathbb{P}^n$ parametrized by $(1, t, t^2, \ldots, t^n)$]. We saw above that the closure of $\overline{H}_s$ contains ξ. This implies that H is reducible at ξ.

We shall now show that any "generalization" of $\overline{C}$ is again a projective cone over a smooth rational curve in $\mathbb{P}^n$ not contained in any hyperplane. In other words, consider a 1-parameter flat family of closed subschemes whose special member is $\overline{C}$ and whose generic member is W. Then W has only isolated (normal) singularities, is of degree n and not contained in any hyperplane. We shall now prove more generally that a surface W in $\mathbb{P}^{n+1}$ (which is not smooth) having these properties is a cone of the type $\overline{C}$. To prove this let θ be a singular point of W.
78 Then we can find a hyperplane L throught θ such that $(L \cdot W)$ is smooth outside θ (Bertini's theorem). Now $L \approx \mathbb{P}^n$, $L \cdot W$ is *connected*, $(L \cdot W)$ is not contained in any hyperplane (same argument as for $X \cap H_\infty$ in the proof of Proposition 13.2), and it is of degree n. Since it has only one singular point, it follows by Lemmas 13.1 and 13.2 that $(L \cdot W)$ consists of n lines meeting at θ. This happens for almost all hyperplanes L passing through θ. We take coordinates in $\mathbb{P}^{n+1}$ so that $\theta = (0, \ldots, 0, 1)$ and take projection on the hyperplane $M = \{x_{n+1} = 0\}$. We can suppose that the choice of L is so made that $L \cdot W$ is smooth and not contained in any hyperplane, so that $L \cdot W$ is the rational curve parametrized by $(1, t, \ldots, t^n)$ with respect to suitable coordinates in $L \approx \mathbb{P}^n$. Let B be the projected variety in M. It follows that almost all the lines joining θ to points of B are in W from which it follows immediately that in fact all these lines are in W. In particular, we have $B = W \cdot M$ and W is the cone over B. This proves the required assertion.

Thus any "generalization" of $\overline{C}$ is either a cone of the same type as $\overline{C}$, a scroll, or a Veronese. It follows then that in the neighborhood of ξ, H consists of only (parts) of three orbits under $PGL(5)$, namely, K_1, K_2 and Δ, where Δ is the orbit under $PGL(5)$ determined by $\overline{C}$. In a neighborhood of ξ, $\overline{K}_1$ and $\overline{K}_2$ are patched along Δ.



It follows that $\operatorname{Spec} R$ where R is the complete local ring defined by $\mathrm{Def}(C)$ is again reducible and has only two irreducible components.

Remark 13.4. It has been shown by Pinkham that R is reduced, that 79
the irreducible component corresponding to scrolls is of dimension 2 and the one corresponding to Veronese is of dimension 1, that they are smooth, and that they intersect at the unique point corresponding to $\overline{C}$ having normal crossings at this point.

Remark 13.5. The above argument can be extended to show that $\operatorname{Spec} R$ for $n \geq 4$ is *irreducible*. Pinkham shows $\operatorname{Spec} R$ has an embedded component at the point corresponding to $\overline{C}$ and outside this it corresponds to scrolls.

Remark 13.6. The reason that for $n = 4$ we have got two components is that in this case the cone over the rational quartic is a determinantal variety in two ways, namely, it can be defined by (2×2) minors of

$$\begin{bmatrix} x_0 & \dots & x_3 \\ x_1 & \dots & x_4 \end{bmatrix} \quad \text{or of} \quad \begin{bmatrix} x_0 & x_2 & x_4 \\ x_2 & x_1 & x_3 \\ x_4 & x_3 & x_2 \end{bmatrix}$$

Remark 13.7. *Case $n \leq 4$.* Exercise: Discuss.

Part 2

Elkik's Theorems on Algebraization

1 Solutions of systems of equations

Let A be a commutative noetherian ring with 1 and B a commutative finitely generated A-algebra, i.e., $B = A[X_1, \ldots, X_N]/(f_1, \ldots, f_q)$, $F = (f_1, \ldots, f_q)$. Then finding a *solution* in A to $F(x) = 0$, i.e., finding a vector $x = (a_1, \ldots, a_N) \in A^N$ such that **80**

$$f_i(x) = 0 \quad (1 \leq i \leq q),$$

is equivalent to finding a sectin for $\operatorname{Spec} B$ over $\operatorname{Spec} A$.

Let J be the Jacobian matrix of the f_i's defined by

$$J = \begin{bmatrix} \dfrac{\partial f_1}{\partial X_1} & \cdots & \dfrac{\partial f_1}{\partial X_N} \\ \vdots & & \\ \dfrac{\partial f_q}{\partial X_N} & & \dfrac{\partial f_q}{\partial X_N} \end{bmatrix} \quad (q \times N) \text{ matrix.}$$

We recall that at a point $z \in \operatorname{Spec} B \hookrightarrow \mathbb{A}^N$ represented by a prime ideal $\mathscr{P}$ in $A[X_1, \ldots, X_N]$, $\operatorname{Spec} B$ is *smooth* over $\operatorname{Spec} A$ at z if and only if: There is a subset $(a) = (a_1, \ldots, a_p)$ of $(1, 2, \ldots, q)$ such that

(i) there exists a $(p \times p)$ minor M of the $(p \times N)$ matrix $\dfrac{\partial f_{a_i}}{\partial X_j}$ such that $M \not\equiv 0 (\operatorname{mod} \mathscr{P})$, and

(ii) $(f_1, \ldots, f_q)$ and $(f_{a_1}, \ldots, f_{a_p})$ generate the same ideal at z.

If the conditions (i) and (ii) are satisfied at z, then $\operatorname{Spec} B$ is of relative codimension p in $\mathbb{A}^N$ (i.e., relative to $\operatorname{Spec} A$).

81 Let $F_{(\alpha)}$ be the ideal $(f_{a_1}, \ldots, f_{a_p})$. The condition (ii) above is equivalent to the following: There is a $g \in A[X_1, \ldots, X_n]$ such that $z \notin V(g)$ and

$$(F_{(a)})_g = (F)_g.$$

(The subscript g means localization with respect to the multiplicative set generated by g. This implies that $g^r(F) \subset (F_{(a)})$ for some r. Conversely suppose given a $g \in A[X_1, \ldots, X_N]$, such that

$$g(F) \subset F_{(a)}$$

(i.e., $g \in$ conductor of F in F_a, $(F_{(a)} : F)$).

Then at all points $z \in \operatorname{Spec} B$ such that $g(z) \neq 0$, (F) and $F_{(a)}$ generate the same ideal (since $F_{(a)} \subset F$). Hence the condition (ii) above can be expressed as:

(ii) There is an element $g \in K_{(a)}$ = conductor of F in $F_{(a)}$ (i.e., the set of elements g such that $gF \subset F_{(a)}$) such that $g(z) \neq 0$.

Let $\Delta_{(a)}$ = ideal generated by the determinants of the $(p \times p)$ minors of the $(p \times N)$ matrix $\left(\dfrac{\partial f_{a_i}}{\partial X_j}\right)$. Let H be the ideal in $A[X_1, \ldots, X_N]$ defined by

$$H = \sum_{(a)} K_{(a)} \Delta_{(a)}$$

i.e., the ideal generated by the ideals $\{K_{(a)}\Delta_{(a)}\}$ where (a) ranges over all subsets of $(1, \ldots, q)$. Then we see that at a point $z \in \operatorname{Spec} B \hookrightarrow \mathbb{A}^N$, $\operatorname{Spec} B$ is smooth over $\operatorname{Spec} A \Leftrightarrow H$ generates the unit ideal at $z \Leftrightarrow z \notin V(H)$. Hence we conclude:

$z \in \operatorname{Spec} B$ is *smooth* over $\operatorname{Spec} A$ if and only if $z \notin V(H) \cap \operatorname{Spec} B$.

2 Existence of solutions when A is t-adically complete

With the notations as in the above § 80, we have 82

Theorem 2.1. *Suppose further that A is complete with respect to the (t)-adic topology, (t) = principal ideal generated by $t \in A$ (i.e., $A = \varprojlim A/(t)^n$). Let I be an ideal in A. Then there is a positive integer q_0 such that whenever*

$$F(a) \equiv 0(mod\ t^n I),\ n \geq n_0,\ n > r$$

(i.e., $f_i(a) = 0(mod\ t^n I)$, $\forall 1 \leq i \leq q$, $n \geq n_0$, $n > r$) and

$$t^r \in H(a)$$

($H(a)$ is the ideal in A generated by H evaluated at a, or equivalently, the closed subscheme of Spec A *obtained as the inverse image of $V(H)$ by the section* Spec $A \xrightarrow{s} \mathbb{A}^N$ *defined by $(a_1, \ldots, a_N)$), then we can find $(a'_1, \ldots, a'_N) \in A^N$ such that*

$$F(a') = 0 \text{ and } a' \equiv a(mod\ t^{n-r} I).$$

Remark 2.1. The condition $t^r \in H(a)$ implies that $V((t)) \supset V(H(a))$. So the section $s : \operatorname{Spec} A \hookrightarrow \mathbb{A}^N$ defined by $(a_1, \ldots, a_N)$ does not pass through $V(H)$ except for the points $t = 0$. Roughly speaking, the above theorem says that a section s of $\mathbb{A}^N_A$ over Spec A which is an approximate section of Spec B over Spec A and not passing through $V(H)$ except over $t = 0$ can be approximated by a true section of Spec B over Spec A.

Proof of Theorem 2.1. (1) *We claim that it suffices to prove the following*: $\exists h \in A^N$ (represented as a column vector) such that

$$(*) \qquad \begin{cases} F(a) \equiv J(a)h(\text{mod } t^{2n-r} I), & \text{and} \\ h \equiv 0(t^{n-r} I). \end{cases}$$

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To prove this claim we use the Taylor expansion

$$F(a-h) = F(a) - J(a)h + O(h^2) \quad \text{(error terms quadratic in } h\text{)}.$$

Now $h^2 \in t^{2(n-r)}I$. We note further that

$$t^{2n-r}I = t^r \cdot t^{2n-2r}I \subset t^{(2n-2r)}I.$$

We have $F(a) - J(a)h \in t^{2n-r}I$ and hence

$$F(a) - J(a)h \in t^{2n-2r}I,$$

i.e., $(*)$ implies

$$\begin{cases} F(a-h) \in t^{2(n-r)}I, & \text{and} \\ h \in t^{(n-r)}I. \end{cases}$$

Set $a_1 = a - h$, $a_0 = a$. Then this gives

$$F(a_1) \in t^{2(n-r)}I,$$
$$a_1 - a_0 \in t^{(n-r)}I.$$

Hence by iteration we can find $a_i \in A$ such that

(i) $F(a_i) \in t^{2^i(n-r)}I$, $i \geq 0$, and

(ii) $(a_i - a_{i-1}) \in t^{2^{i-1}(n-r)}I$, $i \geq 1$.

Now by (ii), $a' = \varprojlim a_i$ exists, and $a' \equiv a (\operatorname{mod} t^{n-r}I)$. Further (i) implies that $F(a') = 0$. This completes the proof of the claim.

(2) *We claim that it suffices to prove that there is a $z \in A^N$ such that*

$$t^rF(a) \equiv J(a)z (\operatorname{mod} t^{2n}I), \quad \text{and}$$
$$z \equiv 0(t^nI).$$

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For, we see easily that there is an $h \in A^N$ such that $t^rh = z$ and $h \in t^{n-r}I$. Then, if $J = J_F$ denotes the Jacobian for F, we have

$$t^r(F(a) - J_F(a)h) \equiv 0(\operatorname{mod} t^{2n}I),$$

$$h \equiv 0 (\text{mod } t^{n-r} I).$$

Set $t^r F = G$; then these relations can be expressed as

$$\begin{cases} G(a) - J_G(a)h \equiv 0(\text{mod } t^{2n} I), & \text{and} \\ h \equiv 0(\text{mod } t^{n-r} I). \end{cases}$$

In particular we certainly have

$$(**) \qquad \begin{cases} G(a) - J_G(a)h \equiv 0(\text{mod } t^{2n-r} I), & \text{and} \\ h \equiv 0(\text{mod } t^{n-r} I). \end{cases}$$

These are just the same as the relations $(*)$ as in Step (1) above, with F replaced by G. Hence we conclude (as for F) that there is an a' such that

$$\begin{cases} G(a') = 0, & \text{and} \\ a' \equiv a(\text{mod } t^{n-r} I). \end{cases}$$

Thus we conclude that there is an a' such that

$$\begin{cases} t^r F(a') = 0, & \text{and} \\ a' \equiv a(\text{mod } t^{n-r} I). \end{cases}$$

Note that $F(a') \in t^{n-r} I$ since

$$F(a') = F(a) + J_F(a)(a' - a)(\text{mod } t^{2(n-r)} I)$$

and $F(a) \in t^n I$. Thus we have

$$t^r F(a') = 0, \quad \text{and} \quad F(a') \in t^{n-r} I.$$

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We would like to conclude that $F(a') = 0$; this may not be true; however, we have

Lemma 2.1. *Let $T_q = \{a \in A | t^q a = 0\}$ and let q_0 be an integer such that $T_{q_0} = T_{q_0+1} = T_{q_0+2} = \ldots$, etc., (note that $T_q \subset T_{q'}$ for $q' \geq q$ and A being noetherian, the sequence $T_1 \subset T_2 \ldots$ terminates). Then*

$$T_s \cap (t^m)A = (0) \quad \textit{for} \quad m \geq q_0.$$

Proof. Let $a \in T_{q_0} \cap t^m A$. Then $a = t^m a' = t^{q_0}(t^{m-q_0}a')$. Since $t^{q_0}a = 0$, we have $t^{n+q_0}a' = 0$. But by the choice of q_0, we have in fact $t^{q_0}a' = 0$. Hence $a = t^{q_0}t^{m-q_0}a' = t^{m-q_0}t^{q_0}a' = 0$. This proves the lemma □

By the lemma if $(n-r) \geq q_0$, then $t^r F(a') = 0$, and $F(a') \in t^{n-r}I$ implies that $F(a') = 0$. Thus the claim (2) is proved.

(3) Let $(\beta) = (\beta_1, \ldots, \beta_p)$ denote a subset of p elements of $(1, \ldots, N)$. Given $(\alpha) = (\alpha_1, \ldots, \alpha_p) \subset (1, \ldots, q)$, let $\delta_{\alpha,\beta}$, be the $p \times p$ minor $\left(\dfrac{\partial f_{\alpha_i}}{\partial X_{\alpha_j}}\right)$ of the Jacobian matrix J. Then we claim that it suffices to prove the following:

Given $k \in K_{(\alpha)}$ and $\delta_{\alpha\beta} \in \Delta_\alpha$ ($\delta_{\alpha\beta}$ defined as above) then there exists a $z \equiv 0 (\text{mod } t^n I)$ such that

$$k(a)\delta_{\alpha\beta}(a)F(a) \equiv G(a)z (\text{mod } t^{2n}I).$$

For, we have $t^r = \sum\limits_{\alpha,\beta} \lambda_{\alpha\beta}k_\alpha(a)\delta_{\alpha\beta}(a)$. Then by hypothesis, given k_α and $\delta_{\alpha\beta}$, we have $z_{\alpha\beta}$ such that $z_{\alpha\beta} = 0 (\text{mod } t^n I)$ and

$$k_\alpha(a)\delta_{\alpha\beta}(a)F(a) \equiv J(a)z_{\alpha\beta} (\text{mod } t^{2n}I);$$

86 then we see that if we set $z = \sum\limits_{\alpha\beta} \lambda_{\alpha\beta} z_{\alpha\beta}$, we have

$$t^r \cdot F(a) \equiv J(a)z (\text{mod } t^{2n}I), \quad \text{and}$$
$$z \equiv 0 (\text{mod } t^n I),$$

which is the claim (2).

(4) We can assume without loss of generality that $(\alpha) = (1, \ldots, p)$ and $(\beta) = (1, \ldots, p)$ so that $\delta_{\alpha,\beta} = \det M$ where $M = \left(\dfrac{\partial f_i}{\partial X_j}\right), \dfrac{1 \leq i \leq p}{1 \leq j \leq p}$. Then we have

$$J = \begin{bmatrix} M & * \\ * & * \end{bmatrix}.$$

Let N be the matrix formed by the determinants of the $(p-1)\times(p-1)$ minors of M so that we have

$$MN = \delta \cdot \text{Id},\ \delta = \delta_{\alpha\beta},\ \alpha, \beta \quad \text{as above.}$$

We denote by $\begin{bmatrix} N \\ 0 \end{bmatrix}$ the $(N \times p)$ matrix by adding zeros to N.

Let $k \in K_{(\alpha)}$, (α) as above and δ be as above. *Then we claim that if z is defined by*

$$Z = k \begin{bmatrix} N \\ 0 \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_p \end{bmatrix}, \ Z \text{ is an } (N \times 1) \text{ matrix;}$$

then if $z = Z(a)$, we have

$$k(a)\delta(a)F(a) \equiv J(a)z(\text{mod } t^{2n}I), \quad \text{and}$$
$$z \equiv 0(\text{mod } t^n I).$$

By the foregoing, if we prove this claim, then the proof of the theorem would be completed.

To prove this claim, we observe that the relation $z \equiv 0(\text{mod } t^n I)$ is **87**
immediate. Since $k \in K_{(\alpha)}$, (α) as above, we have

$$\text{(I)} \qquad k f_j = \sum_{i=1}^{p} \lambda_{ij} f_i, \ 1 \leq j \leq q.$$

This gives $k\dfrac{\partial f_j}{\partial X_1} = \sum\limits_{i=1}^{p} \lambda_{ij} \dfrac{\partial f_i}{\partial X_1}(\text{mod } F)$, $1 \leq j \leq q$.

Substituting (a), we get

$$k(a)\frac{\partial f_j}{\partial X_1}(a) \equiv \sum_{i=1}^{p} \lambda_{ij} \frac{\partial f_i}{\partial X_1}(a)(\text{mod } F(a))$$
$$\equiv \sum_{i=1}^{p} \lambda_{ij}(a) \frac{\partial f_i}{\partial X_1}(a)(\text{mod } t^n I).$$

Let $u \in A^N$ be an element such that $u \equiv 0(\text{mod } \mathcal{V})$, where $\mathcal{V}$ is some ideal of A. Let $v = J(a) \cdot u \in A^N$. Then

$$k(a)v_j = k(a) \sum_{\ell=1}^{N} \frac{\partial f_j}{\partial X_\ell} u_\ell$$

$$\equiv \sum_{\ell=1}^{N}\left(\sum_{i=1}^{p}\lambda_{ij}(a)\frac{\partial f_i}{\partial X_1}(a)\right)u_\ell (\text{mod } t^n I\mathcal{V})$$
$$\equiv \sum_{i=1}^{p}\lambda_{ij}(a)\cdot\left(\sum_{\ell=1}^{N}\frac{\partial x_i}{\partial X_1}(a)u_\ell\right)(\text{mod } t^n I\mathcal{V})$$
$$\equiv \sum_{i=1}^{p}\lambda_{ij}(a)v_i (\text{mod } t^n I\mathcal{V}).$$

In other words,

$$\text{(II)} \qquad k(a)v_j = \sum_{i=1}^{p}\lambda_{ij}(a)v_i (\text{mod } t^n I\mathcal{V}), \quad \text{for} \quad 1 \le j \le q,$$

88 where $u \in A^N$, $v = J(a)u$, and $u \equiv 0(\text{mod } \mathcal{V})$.

Now take for $u \in A^N$ and v the elements

$$u = \begin{bmatrix} N \\ 0 \end{bmatrix}\begin{bmatrix} f_1 \\ \vdots \\ f_p \end{bmatrix}(a), \quad v = J(a)\begin{bmatrix} N(a) \\ 0 \end{bmatrix}\begin{bmatrix} f_1(a) \\ \vdots \\ f_p(a) \end{bmatrix}.$$

Then $u \equiv 0(\text{mod } t^n I)$ and we have

$$k(a)v_j = \sum_{i=1}^{p}\lambda_{ij}(a)v_i (\text{mod } t^{2n} I).$$

Moreover,

$$v = J(a)\begin{bmatrix} N(a) \\ 0 \end{bmatrix}\begin{bmatrix} f_1(a) \\ \vdots \\ f_p(a) \end{bmatrix} = \begin{bmatrix} M(a) & * \\ * & * \end{bmatrix}\begin{bmatrix} N(a) \\ 0 \end{bmatrix}\begin{bmatrix} f_1(a) \\ \vdots \\ f_p(a) \end{bmatrix}$$

$$= \begin{bmatrix} M & N \\ * & N \end{bmatrix}(a)\begin{bmatrix} f_1(a) \\ \vdots \\ f_p(a) \end{bmatrix} = \begin{bmatrix} \delta(a)f_1(a) \\ \vdots \\ \delta(a)f_p(a) \\ \vdots \\ w_{p+1} \\ \vdots \\ w_N \end{bmatrix}$$

Hence $v_i = \delta(a) f_i(a)$, $1 \leq i \leq p$. By (II),

$$k(a)v_i \equiv \sum_{i=1}^{p} \lambda_{ij}(a) v_i (\text{mod } t^{2n} I), \text{ so that}$$

$$k(a)v_i \equiv \sum_{i=1}^{p} \lambda_{ij}(a) f_i(a)\delta(a)(\text{mod } t^{2n} I).$$

By (I),

$$\sum_{i=1}^{p} \lambda_{ij}(a) f_i(a)\delta(a) = k(a)\delta(a) f_j(a).$$

Hence it follows that

$$k(a)J(a)\begin{bmatrix} N \\ 0 \end{bmatrix}\begin{bmatrix} f_i \\ f_p \end{bmatrix} \equiv k(a)\delta(a)\begin{bmatrix} f_1(a) \\ f_q(a) \end{bmatrix} \quad (\text{mod } t^{2n} I)$$

which is precisely the claim in Step (4) above. This completes the proof 89
of the theorem as remarked before.

Remark 2.2. A better proof of the theorem is along the following lines: Introduce a set of relations for F so that we have an exact sequence

$$P^{\ell} \xrightarrow{R} P^q \xrightarrow{F} P \to P/F \to 0.$$

(Here F denotes $\begin{bmatrix} f_1 \\ \vdots \\ f_q \end{bmatrix}$ as well as the ideal generated by f_i and $P = A[X_1, \ldots, X_N]$, and R is the matrix of relations of F; it has entries in P.) In matrix notation, we have $F \cdot R = 0$. Differentiating with respect to X_1, we obtain

$$JR \equiv 0(\text{mod } F),$$

where J is the Jacobian matrix $\left(\dfrac{\partial f_i}{\partial X_j}\right)$ (it is an $N \times q$ matrix and is therefore the transpose of the matrix J introduced in the theorem above). Let $\overline{P} = P/F$ and define $\overline{J}$, $\overline{R}$ similarly. Then we obtain a *complex*

$$(*) \qquad \overline{P}^{\ell} \xrightarrow{\overline{R}} \overline{P}^{q} \xrightarrow{\overline{J}} \overline{P}^{N}$$

(or: $P^\ell \xrightarrow{R} P^q \xrightarrow{J} P^N$ is a *complex* mod F). It is not difficult to see that at a point $x \in \operatorname{Spec}(P/F)$, the complex $(*)$ is homotopic to the identity (in particular, is exact), i.e., denoting by a subscript x the localization at x, there is a diagram

$$\begin{tikzcd} \overline{P}_x^1 \arrow[r, "\overline{R}"] & \overline{P}_x^q \arrow[r, "\overline{J}"] \arrow[l, bend right, "r"'] & \overline{P}_x^N, \arrow[l, bend right, "j"'] \end{tikzcd} \quad \text{with } \overline{R}r + j\overline{J} = \mathrm{Id}$$

90 if and only if $\operatorname{Spec}(P/F)$ is smooth at x. For example, suppose that $\operatorname{Spec}(P/F)$ is smooth at x. Then we can assume without loss of generality that $\det\left(\dfrac{\partial f_i}{\partial X_i}\right)$, ${}_{\substack{1\le i\le p\\ 1\le j\le p}}$, is a unit in $\overline{P}_x$. From this it follows easily that there is a *direct summand* $Q_1 \hookrightarrow \overline{P}_x^q$, $Q_1 \approx \overline{P}_x^p$ such that $\operatorname{Im}(\overline{J}) = \operatorname{Im}(\overline{J}|Q_1)$ and $\overline{J}|Q_1 : Q_1 \to \operatorname{Im}(\overline{J})$ is an isomorphism ($J|O_1$ denotes the restriction to Q_1). Indeed, we can take Q_1 to be the submodule generated by the first p-coordinates. (Note we have $\overline{J}(e_i) = \sum\limits_{\ell=1}^{N} \dfrac{\partial f_i}{\partial X_1}\xi_1$, with (e_i) a basis of $\overline{P}^q$, and (ξ_i) a basis of $\overline{P}^N$). We see also that $\operatorname{Im}(\overline{R})$ is of rank $(q-p)$ and is a direct summand in $\overline{P}^q$. In fact the relations

$$f_j = \sum_{i=1}^{p} \lambda_{ij} f_i, \quad j \ge p+1$$

give elements of the form

$$e_j = \sum_{i=1}^{p} \lambda_{ij} e_i, \quad j \ge p+1,$$

in $\operatorname{Im}\overline{R}$. Suppose now $\sum\limits_{i=1}^{p} \mu_i f_i$ is a relation; then as we have seen before (on relations for a complete intersection), $\mu_i \in (f_1, \dots, f_i)$ so that $\overline{\mu}_i = 0$. Hence we conclude that $(\operatorname{Im}\overline{R})$ is precisely the submodule generated by $(e_j - \sum\limits_{j=1}^{p} \overline{\lambda}_{ij} e_i)$, $j \ge p+1$, which shows that $\operatorname{Im}\overline{R}$ is of rank $(q-p)$ and is a direct summand in $\overline{P}^q$. *Now we see easily that the complex* $(*)$

is homotopic to the identity at x if and only if, (i) ($*$) is exact and (ii) $\operatorname{Im}\overline{R}$ is a direct summand which admits a complement Q_1 such that $\overline{J}|Q_1$ is an isomorphism and $\operatorname{Im}\overline{J} \simeq \operatorname{Im}(\overline{J}|Q_1)$. Now we have checked these conditions when $\operatorname{Spec}(\overline{P})$ is smooth at x. Conversely if (i) and (ii) are **91**
satisfied (at x), it can be checked that $\operatorname{Spec}(\overline{P})$ is smooth at x.

Suppose more generally that we are given a complex

$$\overline{P}^1 \xrightarrow{\overline{R}} \overline{R}^q \xrightarrow{\overline{J}} \overline{P}^N$$

(we keep the same notation). Then we can define an ideal $\mathscr{H}$ in $\overline{P}$ which measures the nonsplitting of the complex as follows: $\mathscr{H}$ is the ideal generated by elements h such that there are maps r, j:

$$\begin{tikzcd} \overline{P}^1 \arrow[r, "\overline{R}"] & \overline{P}^q \arrow[r, "\overline{J}"] \arrow[l, bend right, "r"'] & \overline{P}^N \arrow[l, bend right, "j"'] \end{tikzcd}, \quad \text{such that } \overline{R}r + j\overline{J} = \mathrm{Id}.h.$$

It can be shown that $\mathscr{H}$ is the following ideal: Take a $(p \times p)$ minor M in $\overline{J}$ and a "*complementary*" $(q - p) \times (q - p)$ minor K in $\overline{R}$ (involving "complementary indices"); then $\mathscr{H}$ is the ideal generated by the elements $(\det M)(\det K)$.

In our case, $\operatorname{Spec}\overline{P} - V(\mathscr{H})$ is the open subscheme of smooth points. It follows then that $\mathscr{H}$ and the ideal $\overline{H}$ (H is the ideal defined before and $\overline{H}$ denotes the image of H in $B = P/F$) have the same radical. In our case we have then

$$Rr + jJ \equiv \widetilde{h} (\operatorname{mod} F), \quad \text{for some} \quad \widetilde{h} \in P \quad \text{with image} \quad h \text{ in } \mathscr{H}.$$

Multiplying by F (where F is a vector now), we have

$$FRr + FjJ \equiv F\widetilde{h} (\operatorname{mod} F^2).$$

Since $FR = 0$, this gives

$$FjJ \equiv F\widetilde{h} (\operatorname{mod} F^2).$$

We have used the transposes of the original J; setting $z = j^t F^t$ and taking "evaluation at a", we get **92**

$$h(a)F(a) \equiv J(a)z(\text{mod } t^{2n}I).$$

Now a power of h is contained in $\overline{H}$ since $\overline{H}$ and $\mathscr{H}$ have the same radical. This is the crucial step in the proof of the theorem above. Hence from this the proof of the theorem follows easily.

Before going to the next theorem, let us recall the following facts about $\mathfrak{a}$-adic rings (cf. Serre, *Algebre Locale*, Chap. II, A). Let $(A, \mathfrak{a})$ be a Zariski ring, i.e., A is a noetherian ring, is an ideal contained in the Jacobson radical Rad A of A, and A is endowed with the $\mathfrak{a}$-adic topology, i.e., a fundamental system of neighborhoods of 0 is formed by $\mathfrak{a}^n$. Since $\bigcap\limits_n \mathfrak{a}^n = (0)$, it follows that this topology is Hausdorff. Let $\overline{A}$ denote the $\mathfrak{a}$-adic completion. Then $\overline{A}$ is noetherian. If M is an A-module of finite type we can consider the $\mathfrak{a}$-adic topology on M and similarly it is Hausdorff and if $\overline{M}$ denotes the $\mathfrak{a}$-adic completion of M, we have $\overline{M} = M \otimes_A \overline{A}$. In fact the functor $M \mapsto \overline{M}$ is exact. Suppose now that $M = \overline{M}$. Then we note that any submodule N of M is *closed* with respect to the $\mathfrak{a}$-adic topology (for, the quotient topology in M/N is the $\mathfrak{a}$-adic topology and since M/N is of finite type and $\mathfrak{a} \subset \operatorname{Rad} A$, this topology is Hausdorff and hence N is closed). If $A = \overline{A}$, i.e., A is complete, then any module of finite type is complete for the adic topology so that we don't have to assume further that $M = \overline{M}$.

Let $A = \overline{A}$, M be as usual and $t \in \mathfrak{a}$. Then M/tM is complete for the $\mathfrak{a}/t\mathfrak{a}$-adic topology, for this is simply the $\mathfrak{a}$-adic topology on M/tM.

Let $A = \overline{A}$ and $\mathcal{V}$ be an ideal in A. Given an r, suppose that the relation $\mathfrak{a}^r \subset \mathcal{V} + \mathfrak{a}^m$, $m \gg 0$, holds. Then $\mathfrak{a}^r \subset \mathcal{V}$, for our relation
93 implies that $\mathfrak{a}^r(A/\mathcal{V}) \subset \bigcap\limits_m \mathfrak{a}^m(A/\mathcal{V})$. Since $A/\mathcal{V}$ is Hausdorff for the $\mathfrak{a}$-adic topology, it follows that $\mathfrak{a}^r A/\mathcal{V} = (0)$, which implies $\mathfrak{a}^r \subset \mathcal{V}$.

3 The case of a henselian pair $(A, \mathfrak{a})$

Theorem 3.1. *With the same notations for A, B as in the pages preceding Theorem 2.1, suppose further that* $(A, \mathfrak{a})$ *is a* henselian pair[1] *(in*

[1]**Definition.** By a *henselian pair* we mean a ring A and an ideal $\mathfrak{a} \subset \operatorname{Rad} A$ (= Jacobson radical of A) such that given $F = (f_1, \dots, f_N), N$ elements of $A[X_1, \dots, X_N]$

particular $\mathfrak{a} \subset \operatorname{Rad} A$), and that $\overline{A}$ is the $\mathfrak{a}$-adic completion of A. Suppose we are given $\overline{a} \in \overline{A}^N$ such that $F(\overline{a}) = 0$ (i.e., a formal solution) and $\mathfrak{a}^r \cdot \overline{A}$ (or briefly $\mathfrak{a}^r$) $\subset H(\overline{a})$ for some r. (This means that the section of $\operatorname{Spec} \overline{B}$ over $\operatorname{Spec} \overline{A}$ where $\overline{B} = \overline{A}|X|/(f_i)$ represented by $\overline{a}$ passes through smooth points of the morphism $\operatorname{Spec} \overline{B} \to \operatorname{Spec} \overline{A}$ except over $V(\mathfrak{a})$.) Then for all $n \geq 1$ (or equivalently for all n sufficiently large) there is an $a \in A^N$ such that $F(a) = 0$, and $a \equiv \overline{a} (\operatorname{mod} \mathfrak{a}^n)$.

Proof. (1) *Reduction to the case $\mathfrak{a}$ principal.*

Let $\mathfrak{a} = (t_1, \ldots, t_k)$. Let us try to prove the theorem by induction on k. If $k = 0$, then the theorem is trivial. So assume the theorem proved for $(k-1)$. We observe that the couple $(A/t_k^\ell, (t_1, \ldots, t_{k-1}))$ is again a henselian pair $\forall \ell \geq 1$ (here $t_1, \ldots, t_{k-1}$ denote the canonical images of t_i in A/t_k^ℓ). Set $t = t_k$ and $A_1 = A/(t^\ell)$. We note also that the $\mathfrak{a}$-adic 94
topology on A_1 is the same as the $(t_1, \ldots, t_{k-1}) = \mathfrak{a}_1$-adic topology. If $\overline{A}$, $\overline{A}_1$ denote the corresponding completions, we get a canonical surjective homomorphism $\overline{A} \to \overline{A}_1$ whose kernel is $t^\ell \cdot \overline{A}$. Let $\overline{b}$ be the canonical image of $\overline{a}$ in $\overline{A}_1$. Then we have $F(\overline{b}) = 0$. Besides, we see that $\mathfrak{a}_1^s \subset H(\overline{b})$ for some s (this follows from the fact that $V(H) \cap \operatorname{Spec} B =$ locus of nonsmooth points for $\operatorname{Spec} B \to \operatorname{Spec} A$ and the set of smooth points behaves well by base change and for us the base change is $\overline{A} \to \overline{A}_1$. We canonot say that $s = r$, for the ideal H (or rather $H(\operatorname{mod} B)$) which we have defined using the base ring A does not behave well with respect to base change. The ideal $\mathscr{H}$ does behave well with respect to base change, and if we had used this ideal we could have got the same integers). Hence by induction hypothesis, for all $m \geq t$, there exists a $b \in A_1^N$ such that $F(b) = 0$, and $b \equiv \overline{b} (\operatorname{mod} \mathfrak{a}_1^M)$.

Lift b to an element $a_1 \in A^N$ and choose ℓ so that $\ell \geq m$. Then we see that

$$a_1 \equiv \overline{a} (\operatorname{mod} \mathfrak{a}^m), \quad \text{and}$$
$$F(a_1) \equiv 0 (\operatorname{mod} (t^\ell)).$$

---

and $x^0 \in A^n$, $x^0 = (x_1^0, \ldots, x_N^0)$ such that $F(x^0) \equiv 0 \operatorname{mod} (\mathfrak{a})$ and such that $\det \left(\dfrac{\partial f_i}{\partial X_j} \right)|_{x^0}$ is invertible $\operatorname{mod} \mathfrak{a}$, then $\exists x \in A^N$, $x \equiv x^0 \operatorname{mod} (\mathfrak{a})$ with $F(x) = 0$.

(The fact that $b \equiv \overline{b}(\mathrm{mod}\, \mathfrak{a}_1^m)$ implies $(a_1 - \overline{a}) + x \in \mathfrak{a}^m$ with $x \in \mathrm{Ker}(A \to A_1) = (t^\ell)$. Now $t \in \mathfrak{a}$, and if $\ell \geq m$, $(t) \in \mathfrak{a}^m$.) We claim that if $m \gg 0$ (and consequently $\ell \gg 0$), we have $\mathfrak{a}^r \subset H(\mathfrak{a}_1)$. By hypothesis we have $\mathfrak{a}^r \subset H(\overline{a})$, and from the relation $a_1 \equiv \overline{a}(\mathrm{mod}\, \mathfrak{a}^m)$, we get

$$\mathfrak{a}^r \subset H(a_1) + \mathfrak{a}^m$$

(as ideals in $\overline{A}$; to deduce this we use the Taylor expansion). Then, as we
95 remarked before the theorem for $m \gg 0$, this implies that $\mathfrak{a}^r \subset H(a_1)$. Since $t \in \mathfrak{a}$ it follows that $t^r \in H(a_1)$.

Let A_t denote the t-adic completion. Then the following relations in A,

$$\begin{aligned}
&\text{for all } \ell \geq 0,\ F(a_1) \equiv 0(\mathrm{mod}\,(t^I))\\
&\text{there exists } a_1 \in A^N \text{ such that } t^r \in H(a_1)
\end{aligned}$$

hold *a fortiori* in A_t, and hence by Theorem 2.1 we can find $a' \in A_t^N$ such that

$$\begin{aligned}
F(a') &= 0, \quad \text{and}\\
a' &\equiv a_1(\mathrm{mod}\,(t^{\ell-r})).
\end{aligned}$$

Note that the pair $(A,(t))$ is also henselian. Hence if the theorem were true for $k = 1$, we would have

$$\begin{aligned}
\forall n, \exists\, a \in A^N \text{ such that } F(a) = 0,\ \text{and}&\\
a \equiv a'(\mathrm{mod}\, t^n A).&
\end{aligned}$$

But we have

$$\begin{aligned}
&a' \equiv a_1(\mathrm{mod}\,(t^s)A_t),\ \text{for } s \text{ sufficiently large, and}\\
&a_1 \equiv \overline{a}(\mathfrak{a}^m \cdot \overline{A}),\ m \text{ sufficiently large.}
\end{aligned}$$

These imply $a \equiv \overline{a}(\mathrm{mod}\, \mathfrak{a}^n)$ (since $t \in \mathfrak{a}$ and $A_t \subset \overline{A}$), which implies the theorem. Hence we have only to prove the theorem in the case $k = 1$, i.e., $\mathfrak{a}$ principal. □

(2) *A general lemma:*

Lemma 3.2. *Let A, B be as in the pages preceding the theorem, i.e.,* $B = A[X_1, \ldots, X_N]/(f_1, \ldots, f_q)$, $F = (f_1, \ldots, f_0)$. *Let C be the symmetric algebra on* F/F^2 *over B, so that* Spec C *is the conormal bundle over* **96**
Spec B *(*F/F^2 *as a B-module is the conormal sheaf over* Spec B*). Let* f, g, h *denote the canonical morphisms*

$$\operatorname{Spec} C \underset{f}{\to} \operatorname{Spec} B \underset{h}{\to} \operatorname{Spec} A, \; g = h \circ f.$$

Let V be the open subschemes of Spec B *where* Spec $B \to$ Spec A *is smooth and* $V' = f^{-1}(V)$. *Then we have the following:*

(a) $g : \operatorname{Spec} C \to \operatorname{Spec} A$ *is smooth and of relative dimension N (over A) on* V' *and*

(b) $\exists$ *an imbedding* $\operatorname{Spec} C \hookrightarrow \mathbb{A}_A^{2N+q}$ *(A-morphism) such that the restriction of the normal sheaf (for this imbedding) to every affine open subset* $U \hookrightarrow V'$ *is trivial.*

Proof of Lemma 3.2. We set

$$C = B[Y_1, \ldots, Y_q]/I$$
$$A[X, Y]/(F, I), \; K = (F, I).$$

On V we have the exact sequence

(i) $$0 \to F/F^2 \to \Omega_{A[X]/A} \otimes_{A[X]} B \to \Omega_{B/A} \to 0.$$

(This is an abuse of notation; strictly speaking we have to write $(F/F^2)|_V \ldots$, etc.) On V we have the exact sequence

(ii) $$0 \to f^*(\Omega_{B/A}) \to \Omega_{C/A} \to \Omega_{C/B} \to 0, \quad \text{and}$$
$$\Omega_{C/B} \approx f^*(F/F^2).$$

Taking f^* of the first sequence, we get sequences **97**

$$0 \to f^*(F/F^2) \to (\text{Free}) \to f^*(\Omega_{B/A}) \to 0$$

$$0 \to f^*(\Omega_{B/A}) \to \Omega_{C/A} \to f^*(F/F^2) \to 0$$

which are exact on any affine open set U in V'. These exact sequences are split on U, so that we conclude

$$\Omega_{C/A} \quad \text{is free on} \quad U.$$

On V' we have the exact sequence

$$0 \to K/K^2 \to \Omega_{A[X,Y]/A} \otimes_{A[X,Y]} C \to \Omega_{C/A} \to 0.$$

Thus it follows that on U

$$K/K^2 \oplus \underbrace{(\text{Free})}_{\text{of rank } N} = (\text{Free})$$

(From the exact sequence (ii) it follows that $\Omega_{C/A}$ is of rank N over C and this implies the assertion (a).) If we introduce N more indeterminates $Z_1, \ldots, Z_N$, then

(*) $$C = A[X, Y, Z_1, \ldots, Z_n]/(K, Z_1, \ldots, Z_n).$$

Let $K' = (K, Z_1, \ldots, Z_N)$. Then we see easily that

$$K'/K'^2 = K/K^2 \oplus (\text{Free of } rk\ N).$$

It follows that K'/K'^2 is free on U. Thus for the embedding $\operatorname{Spec} C \hookrightarrow \mathbb{A}_A^{2N+q}$, the restriction of the normal bundle K'/K'^2 to U is trivial. This completes the proof of the lemma.

98 (3) We saw in (1) above, that for the theorem it suffices to prove it in the case $\mathfrak{a} = (t)$. The condition $\mathfrak{a}^r\overline{A} \subset H(\overline{a})$ becomes $t^r \in H(\overline{a})$. (Note that $H(\overline{a})$ is the ideal in $\overline{A}$ generated by evaluating at $\overline{a}$ elements of H and H *is an ideal in* $A[X_1, \ldots, X_N]$ not in $\overline{A}[X_1, \ldots, X_N]$.)

We claim now that there is an $h \in H$ *such that*

$$h(\overline{a}) = (\text{unit}) \cdot t^r.$$

A priori it is clear there is an $\overline{h} \in H \cdot \overline{A}[X]$ such that $\overline{h}(\overline{a}) = t^r$. Since $\overline{A}$ is the t-adic completion of A, we can find an $h \in H$ such that

$$h \equiv \overline{h} (\operatorname{mod} (t^{r+1}))$$

(i.e., the coefficients of h and $\overline{h}$ differ respectively by an element of (t^{r+1})). This implies that

$$\begin{aligned} h(\overline{a}) - \overline{h}(\overline{a}) &= *t^{r+1}, \quad \text{hence} \\ h(\overline{a}) &= t^r(1 + *t). \end{aligned}$$

Now $(1 + *t)$ is a unit in $\overline{A}$. This proves the required claim.

(4) *The final step.*

Since $\operatorname{Spec} C$ is a vector bundle over $\operatorname{Spec} B$, we have the 0-section $\operatorname{Spec} C \overset{\curvearrowleft}{\to} \operatorname{Spec} B$. We are given $\overline{a} \in \overline{A}^N$ such that $F(\overline{a}) = 0$. Now $\overline{a}$ determines a section of $\operatorname{Spec}(B \otimes_A \overline{A})$ over $\overline{A}$, or equivalently a morphism $s : \operatorname{Spec} \overline{A} \to \operatorname{Spec} B$ forming a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \overline{A} \arrow[rr, "s"] \arrow[dr] & & \operatorname{Spec} B \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd}$$

Using the 0-section, s can be lifted to a morphism $s_1 : \operatorname{Spec} \overline{A} \to$ **99**
$\operatorname{Spec} C$:

$$\begin{tikzcd} \operatorname{Spec} \overline{A} \arrow[rr, "s_1"] \arrow[dr] & & \operatorname{Spec} C \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd}$$

Now by (3), s_1 carries $(\operatorname{Spec} A - V(t))$ to $\operatorname{Spec} C[1/h]$, where h is as in (3) (here h denotes the canonical image in C of the h in (3)). We observe that

$$\operatorname{Spec} C[1/h] \subset V',$$

$(V' = f^{-1}(V)$, V = locus of smooth points of $\operatorname{Spec} B \to \operatorname{Spec} A)$ and V' is contained in the locus of smooth points of the map $\operatorname{Spec} C \to \operatorname{Spec} A$. We have $C = A[X, Y, Z]/K'$ where $K' = (F, I, Z_1, \ldots, Z_N)$. The section s_1 defines a solution $K'(\overline{a}') = 0$, where $\overline{a}' \in \overline{A}^{2N+q}$ ($\overline{a}'$ extends the section $\overline{a}$). We have $t^k \in H(\overline{a}')$ for a suitable k. Hence the conditions, similar to those of $\operatorname{Spec} B \to \operatorname{Spec} A$, are now satisfied for the map

$\operatorname{Spec} C \to \operatorname{Spec} A$. It is immediately seen that it suffices to solve for the case $\operatorname{Spec} C \to \operatorname{Spec} A$, in fact if we get a solution $a' \in A^{2N+q}$, we have to take for a the first N coordinates of a'.

Let $U = \operatorname{Spec} C[1/h]$. Then the restriction of the normal bundle of the imbedding $\operatorname{Spec} C \hookrightarrow \mathbb{A}_A^{2N+q}$ is trivial on U and, U being smooth over A, it follows easily that U is open in a global complete intersection. By this we mean there exist $g_1, \ldots, g_{N+q} \in A[X, Y, Z]$ such that $V(g_1, \ldots, g_{N+q})$ has dimension N and we have an open immersion

$$(\operatorname{Spec} C[1/t]) = \operatorname{Spec} C[1/h] \hookrightarrow \operatorname{Spec} A[X, Y, Z]/(g_1, \ldots, g_{n+q}).$$

Let $G = (g_1, \ldots, g_{N+q})$. Then we have

$$K'_t = G_t.$$

100 (We denote the localization with respect to t by a subscript. Note that localization with respect to t is the same as localization with respect to b.) The given solution $\overline{a}'$ in $\overline{A}^{2N+q}$ is such that $K'(\overline{a}')$ gives rise to a solution $G(\overline{a}') = 0$ by changing the g_i, multiplying them by suitable powers of t. Conversely, suppose we have solved the problem for G, i.e., we have found $a' \in A^{2N+q}$ such that $G(a) = 0$ and $a' \equiv \overline{a}(\operatorname{mod} t^n)$. Then we see easily that there is a θ such that

$$t^\theta \cdot F(a') = 0.$$

Since $F(\overline{a}) = 0$ by Taylor expansion, it follows that $F(a') \equiv 0$ $(\operatorname{mod} t^n)$. Now if $n \gg 0$, by Lemma 2.1, it follows that $F(a') = 0$. Thus it suffices to solve the problem for G, i.e., for the morphism

$$\operatorname{Spec} C' \to \operatorname{Spec} A, \quad C' = A[X, Y, Z]/G.$$

We have seen that $\overline{a}'$ defines a section of this having the required properties. Further, $\operatorname{Spec} C'$ is a *global* complete intersection and smooth over A in $\mathbb{A}^{2N+q}$, i.e., we have reduced the theorem to the following lemma.

4 Tougeron's lemma

Lemma 4.1. *Let $(A, \mathfrak{a})$ be a henselian pair and $f_i \in A[Y_1, \ldots, Y_N]$, $1 \leq i \leq m$. Let $J = \left(\dfrac{\partial f_i}{\partial Y_j}\right)$ be the Jacobian matrix, $1 \leq i \leq m$, $1 \leq j \leq N$. Suppose we are given $y^0 = (y_1^0, \ldots, y_N^0) \in A^N$ such that*

$$f(y^0) \equiv 0 (mod\, \Delta^2 \mathcal{V})$$

where $V(\mathcal{V}) = V(\mathfrak{a})$ (or $\Leftrightarrow$ $\mathcal{V}$ is also a defining ideal for $(A, \mathfrak{a})$) and Δ is the annihilator of the A-module C presented by the relation matrix **101**
(i.e., C is the cokernel of the homomorphism $A^N \to A^m$ whose matrix is $J(y_0)$). Then there is a $y \in A^N$ such that

$$f(y) = 0 \quad and \quad y \equiv y^0 (\mathrm{mod}\, \Delta \mathcal{V}).$$

Proof. The henselian property of $(A, \mathfrak{a})$ is used in the following manner: Let $F = (F_1, \ldots, F_N)$ be N elements of $A[Y_1, \ldots, Y_N]$ and $y^0 = (y_1^0, \ldots, y_N^0) \in A^N$ such that

$$\text{(P)} \qquad \begin{aligned} &\text{(i) } F(y^0) \equiv 0 (\mathrm{mod}\, \mathfrak{a}) \\ &\text{(ii) } \det\left(\frac{\partial F_i}{\partial Y_j}\right)_{y=y^0} \quad \text{is a } \textit{unit} \quad (\mathrm{mod}\, \mathfrak{a}). \end{aligned}$$

Then there is a $y \in A^N$ such that $F(y) = 0$ and $y \equiv y^0 (\mathrm{mod}\, \mathfrak{a})$.

Let $\delta_1, \ldots, \delta_r$ generate the annihilator of Δ. This implies that there exist $N \times m$ matrices such that

$$JN_i = \delta_i I, \quad J = J(y^0), \quad I = \mathrm{Id}_{(m \times m)}.$$

Write

$$f(y^0) = \sum_{i,j} \delta_i \delta_j \epsilon_{ij},$$

$$\epsilon_{ij} = (\epsilon_{ij1}, \ldots, \epsilon_{ij\nu}, \ldots, \epsilon_{ijm}), \ \epsilon_{ij} \in \mathcal{V}$$

$\searrow$

m components.

We try to solve the equations

$$f\left(y^0 + \sum_{i=1}^{r} \delta_i U_i\right) = 0$$

for elements $U_i = (U_{il}, \ldots, U_{iN}) \in A^N$ (we consider vectors in A^n to
102 be column matrices). Expansion by Taylor's formula in vector notation gives

$$0 = f(y^0) + J \cdot (\sum \delta_i U_i) + \sum_{i,j} \delta_i \delta_j Q_{ij},$$

where

J is an $(m \times N)$ matrix,

$f(y^0)$ is an $(m \times 1)$ matrix,

U_i is an $(N \times 1)$ matrix (not $(1 \times N)$ matrix as it is written),

and

$$Q_{ij}, \epsilon_{ij} \quad \text{are} \quad (m \times 1) \quad \text{matrices.}$$

Expanding, we get

$$\begin{aligned}
0 &= J \cdot \underbrace{(\sum_{i=1}^{r} \delta_i U_i)}_{(m\times N)(N\times 1)\text{ matrix}} + \sum_{i,j} \delta_i \delta_j \underbrace{(Q_{ij} + \epsilon_{ij})}_{(m\times 1)\text{ matrix}} \\
&= \sum_{i=1}^{r} \delta_i (JU_i) + \sum_{i,j} \delta_i \cdot JN_j \cdot (Q_{ij} + \epsilon_{ij}) \quad (\delta_j \mathrm{Id} = JN_j) \\
&= \sum_{i=1}^{r} (\delta_i J) \cdot U_i + \sum_{i} \delta_i J (\sum_{J} N_j (Q_{ij} + \epsilon_{ij})) \quad (\delta_i \text{ are scalars}).
\end{aligned}$$

Thus it suffices to solve the r equations

$$(*) \qquad 0 = U_i + \sum_{j} N_j (Q_{ij} + \epsilon_{ij}), \quad 1 \le i \le r.$$

This is an equation for an $(N\times 1)$ matrix. Thus $(*)$ gives *Nr equations in the Nr unknowns $U_{i\nu}$, $1 \le i \le r$, $1 \le \nu \le N$.*

We note that Q_{ij} are vectors of polynomials in $U_{i\nu}$ all of whose terms are of degree ≥ 2. Let $F = F_1, \ldots, F_{NR} \in A[U_{i\nu}]$ represent the right hand side of $(*)$. Write $Z_1, \ldots, Z_{Nr}$ for the indeterminates $U_{i\nu}$. Then

$$\left(\frac{\partial F_k}{\partial Z_\ell}\right) = \mathrm{Id} + M, \quad M = (m_{\alpha\beta}), \quad (Nr \times Nr) \text{ matrix}$$

where M is an $Nr \times Nr$ matrix of polynomials in Z_ℓ and every $m_{\alpha\beta}$ has no constant term. **103**

Let $x^0 \in A^{Nr}$ represent the vector $(0, \ldots, 0)$; then we have

$$F(z^0) \equiv 0 (\mathrm{mod}\, \mathcal{V}) \quad \text{since} \quad \epsilon_{ij\nu} \in \mathcal{V}.$$

Without loss of generality we can suppose $\mathcal{V} = \mathfrak{a}$ since $\mathcal{V}$ is also a defining ideal for $(A, \mathfrak{a})$

$$F(Z^0) \equiv 0 (\mathrm{mod}\, \mathfrak{a}).$$

Further, we have $\left(\frac{\partial F_k}{\partial Z_1}\right)_{Z=z^0}$ is a unit in $A/\mathfrak{a}$. Hence by the henselian property of $(A, \mathfrak{a})$, we have a solution z of $(*)$ in A such that $z \equiv z^0$ $(\mathrm{mod}\, \mathfrak{a})$, i.e., $z \equiv 0 (\mathrm{mod}\, \mathfrak{a})$ since $z^0 \equiv (0)$. Set

$$y = y^0 + z.$$

Then we have

$$y \equiv y^0 (\mathrm{mod}\, \Delta\mathfrak{a}) \quad \text{and} \quad f(y) = 0,$$

which proves the lemma. ◻

Remark 4.1. It is possible to take $\mathcal{V}$ such that $\mathcal{V} \subset \mathfrak{a}^r$, for if $(A, \mathfrak{a})$ is a henselian pair, the henselian property is true for $\mathcal{V}$, $\mathcal{V} \subset \mathfrak{a}$. Then the above proof also goes through for this case.

Corollary. *Let* $(A, \mathfrak{a})$ *be a henselian pair and*

$$f_1, \ldots, f_m \in A[y_1, \ldots, y_N]$$

and φ the canonical morphism

$$\begin{array}{c}\varphi : \operatorname{Spec} A[y_1, \ldots, y_N/(f_1, \ldots, f_m) \to \operatorname{Spec} A \\ \| \\ B\end{array}$$

104 *Suppose that φ* is smooth and a relative complete intersection. *Then given $\overline{y} \in \hat{A}$ ($\mathfrak{a}$-adic completion of A) such that*

$$f(\overline{y}) = 0,$$

there is a $y \in A$ such that $f(y) = 0$ *and*

$$y \equiv y(mod\, \mathfrak{a}^c)$$

for any given $c \geq 1$.

Proof. We can take $c = 1$ for $\mathfrak{a}^c$ is also a defining ideal for $(A, \mathfrak{a})$. We can *a fortiori* find $y^0 \in A^N$ such that

$$f(y^0) \equiv 0 (\operatorname{mod} \mathfrak{a}) (\Leftarrow f(\overline{y}) = 0).$$

y^0 then defines B a section of $\operatorname{Spec}(B)$ over $\operatorname{Spec} A/\mathfrak{a}$. The morphism is a complete intersection and smooth at points of this section. This implies that the Ideal generated by canonical images in $A/\mathfrak{a}$ of the determinants of the $(m \times m)$ minors of $J(y_0)$ is the unit ideal in $A/\mathfrak{a}$, i.e., the canonical image of Δ in $A/\mathfrak{a}$ is the unit ideal. Since $\mathfrak{a}$ is in the Jacobson radical, it follows that Δ is itself the unit ideal. Indeed Δ being the unit ideal in $A/\mathfrak{a}$ implies there exists a $u \in \Delta$ such that $u \equiv 1 (\operatorname{mod} \mathfrak{a})$, hence $u - 1 \in \mathfrak{a}$, hence $u = 1 + r$, $r \in \mathfrak{a}$. Since $\mathfrak{a} \subset \operatorname{Rad} A$, u is a unit in A. $\square$

5 Existence of algebraic deformations of isolated singularities

Definition 5.1. A family of isolated singularities is a scheme $X \xrightarrow{\pi} S$ over $S = \operatorname{Spec} A$, A a k-algebra such that (i) π is flat, of finite presentation and X is affine, $X = \operatorname{Spec} \mathscr{O}$ and (ii) if Γ is the closed subset of X where π is not smooth, then $\Gamma \to S$ is a *finite morphism* (for this let us say that Γ is endowed with the canonical structure of a reduced scheme).

Definition 5.2. We say that two families $X \to S$ and $X' \to S$ of isolated **105**
singularities are *equivalent* or *isomorphic* if there is a family of isolated singularities $X'' \to S$ (note that S is the same) and étale morphisms $X'' \to X'$, $X'' \to X$ such that

(i) the following diagram is commutative

$$\begin{tikzcd} & X'' \arrow[dl, "\text{étale}"'] \arrow[dd] \arrow[dr, "\text{étale}"] & \\ X \arrow[dr] & & X' \arrow[dl] \\ & S & \end{tikzcd}$$

and

(ii) these maps induce isomorphisms

$$\begin{tikzcd} & \Gamma'' \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ \Gamma & & \Gamma'. \end{tikzcd}$$

An equivalence class of isolated singularities represented by $X \to S$ is therefore the *henselization of* X *along* Γ.

Consider in particular a one point "*family*" $X_0 \to \operatorname{Spec} k$ *with an isolated singularity.* (We could also take a finite number of isolated singularities.) We see easily that the formal deformation space of X_0 (in the sense of Schlessinger defined before) depends only on the equivalence class of X_0. Let A be the *formal versal deformation space* associated to X_0 (we can speak of the versal deformation space of X_0 by taking Zariski tangent space of $A = \dim T^1_{X_0}$), i.e., A is a complete local ring, and we are given a sequence $\{X_n\}$ of deformations over A_n

$$X_n = \operatorname{Spec} \mathscr{O}_n, \quad A_n = A/m_A^{n+1}, \quad \mathscr{O}_A \otimes A_{n-1} \cong \mathscr{O}_{n-1}$$

satisfying the versal property mentioned before. Note that by a versal deformation it is not meant that there is a deformation of X_0 over A. The **106** following theorem, proved by Elkik, asserts that in fact a deformation of X_0 over A does exist (i.e., in the case of isolated singularities). It is the crucial step for the existence of "algebraic" versal deformations for X_0.

Theorem 5.1. *Let $X_0 = \operatorname{Spec} \mathscr{O}_0$, $\mathscr{O}_0 = k[X_1, \ldots, X_m]/(f)$, $(f) = (f_1, \ldots, f_r)$. Let $X_n = \operatorname{Spec} \mathscr{O}_n$ be as above, but we suppose moreover that X_0 is equidimensional of dimension d. Then there is a deformation X' over A such that $X' \otimes A_n \approx \mathscr{O}_n$, and if $X' = \operatorname{Spec} \mathscr{O}'$ then $\mathscr{O}'$ is an A-algebra of finite type. (We do not claim that $\mathscr{O}'$ has the same presentation as $\mathscr{O}_0$.)*

Proof. Let $\overline{\mathscr{O}} = \varprojlim \mathscr{O}_n$, $\mathscr{O}_n = A_n[X_1, \ldots, X_m]/(f^{(n)})$, where $f_i^{(n)} \in A_n[X]$ is a lifting of $f_i \in k[X]$. Let $A[X]^\wedge$ denote the m-adic ($m = m_A$) completion of $A[X]$. We see that $A[X]$ is the set of formal power series $\sum a_i X^{(i)}$ such that $a_i \to 0$ in the m-adic topology of A. Then $\overline{f}_i = \lim f_i^{(n)}$ is in $A[X]$ and we see easily that

$$\overline{\mathscr{O}} = A[X]^\wedge/(\overline{f}).$$

(For, we see that we have a canonical homomorphism

$$\alpha : A[X]^\wedge/(\overline{f}) \to \overline{\mathscr{O}}$$

obtained from the canonical homomorphism $A[X]^\wedge/(\overline{f}) \to A_n[X]/(f^{(n)})$. It is easy to see that α is an isomorphism.)

The proof of the theorem is divided into the following steps:

(1) *It is enough to find a (flat) deformation X'' over A of X_0 such that*
107 *$\mathscr{O}'' \otimes_A A_1 \approx \mathscr{O}_1$, where $X'' = \operatorname{Spec} \mathscr{O}''$ (recall that $A_1 = A/m^2$).*

For, given a (flat) deformation X'' over A we get deformations $\{X''_n\} = X'' \otimes A_n$ over A_n. For each n, we get then by the versal property of A, a homomorphism

$$\alpha_n : A \to A_n.$$

Note that α_n is defined by

$$(\alpha_n)_m : A_m \to A_n, \quad m \gg 0,$$

so that $X_m \otimes_{A_m} A_n \approx X_n''$. These $\{\alpha_n\}$ are consistent and hence $\{a_n\}$ patch up to define a homomorphism of rings

$$\alpha : A \to A.$$

The hypothesis that $\mathscr{O}'' \otimes_A A_1 \approx \mathscr{O}_1$ implies that

$$\alpha \equiv \mathrm{Id}(\mathrm{mod}\, m_A^2).$$

This condition on α implies that α is an isomorphism; for it follows that α induces an isomorphism on the Zariski tangent spaces, so that $\mathrm{Im}\,\alpha$ contains a set of generators of m_A, hence α is *surjective*; further, this condition implies that the induced homomorphisms $\alpha_n : A/m_A^n \to A/m_A^n$ are surjective, and these vector spaces being finite-dimensional, it follows that α_n is an isomorphism for all n (in particular injective). It follows easily that $\mathrm{Ker}\,\alpha \subset \bigcap_n m_A^n = (0)$, i.e., α is injective. Hence α is an isomorphism.

Now define the deformation X' over A as the pull back of X'' over A by the isomorphism $\alpha - 1$. It is easily checked that $X' \otimes_A A_n \approx X_n$, and this proves (1).

Let us set $\overline{X} = \mathrm{Spec}\,\overline{\mathscr{O}}$. Consider the Jacobian matrix $J = \left(\dfrac{\partial \overline{f}_i}{\partial X_j}\right)$ $1 \leq i \leq r$, $1 \leq j \leq m$ $((\overline{f}) = (\overline{f}_1, \ldots, \overline{f}_r))$. Let $\overline{\Gamma}$ denote the locus 108
of points in $\overline{X} = \mathrm{Spec}\,\overline{\mathscr{O}}$ where $rk\ J < (m-d)$. Then $\overline{\Gamma}$ is a *closed subscheme* in $\overline{X}$. (We note that $\overline{f}_i \in A[X]^\wedge$ and $\dfrac{\partial \overline{f}_i}{\partial X_j} \in A[X]^\wedge$ so that the Jacobian matrix J is a matrix of elements is $A[X]^\wedge$.) Hence if $x \in \mathrm{Spec}\,A[X]^\wedge$ (in particular if $x \in \mathrm{Spec}\,\overline{\mathscr{O}} = \overline{X} \hookrightarrow A[X]$), we can talk of the rank of J at x, i.e., the matrix $J(x)$ whose elements are the canonical images in $k(x)$ (residue field at x) of the elements of J. It follows then easily that the locus of points x of $\overline{X}$ where $J(x)$ is of rank $< (m-d)$ is closed in $\overline{X}$; in fact we see that $\overline{\Gamma} = V(I)$ where I is the ideal generated by the determinants of all the $(m-d) \times (m-d)$ minors of J' where J' is J with elements replaced by their canonical images in $\overline{\mathscr{O}}$. It is clear that $\overline{\Gamma} \cap X_0$ is precisely the set of singular points of X_0, which is by our hypothesis a finite subset of X_0. It can then be seen without

much difficulty that $\overline{\Gamma}$ *is finite over* A ($\overline{\Gamma}$ is endowed with the canonical structure of a reduced scheme or a scheme structure from the ideal I introduced above). The proof of this is similar to the fact: quasifinite implies finite in the "formal case", i.e., in the situation $A \to B$ where A, B are complete local rings and B is the completion of a local ring of an A-algebra of finite type.

Let $\overline{\Gamma} = \operatorname{Spec} \overline{\mathscr{O}}/\Delta$ and let Δ_0 be the ideal in $\mathscr{O}_0$ defined by Δ (i.e., the canonical image of $\Delta \otimes k$ in $\mathscr{O}_0$). By the Noether Normalization lemma we can find $y_1^0, \ldots, y_d^0$ in Δ_0 such that $\mathscr{O}_0$ is a finite $k[y_1^0, \ldots, y^0 d]$ module such that the set of common zeros of y_i^0 is precisely the set of singular points of X_0. Lift y_i^0 to elements $y_1, \ldots, y_d$ in Δ so that $y_1, \ldots, y_d$ vanish on $\overline{\Gamma}$. Then we have

109 (2) $\overline{\mathscr{O}}$ *is a finite* $A[y]^\wedge$ *module* (and $\overline{\Gamma}$ is precisely the locus of $y_i = 0$).

This is again obtained by an argument generalizing "quasifinite implies finite in the "formal" cases."

(3) *The open subscheme* $\overline{X} - \overline{\Gamma}$ *of* $\overline{X}$ *is regular (over A) (i.e.,* $\overline{X} - \overline{\Gamma}$ *is flat over A and the fibres are regular).*

This is a generalization of the Jacobian criterion of regularity to the adic and formal case.

(4) *Outside the set* $\{y = 0\}$ *in* $\operatorname{Spec} A[y]^\wedge$ *(this is a section of* $\operatorname{Spec} A[y]^\wedge$ *over* $\operatorname{Spec} A$ *and* $\overline{\Gamma}$ *lies over this set),* $\overline{\mathscr{O}}$ *is locally free over* $A[y]^\wedge$ *say of rank r, i.e.,* $p_*(\mathscr{O}_{\overline{X}-\overline{\Gamma}})$ *is a locally free sheaf of* $\mathscr{O}_{\operatorname{Spec} A[y]^\wedge - \{y_i=0\}}$ *modules (*$p : \overline{X} \to \operatorname{Spec} A[y]^\wedge$ *canonical morphism).*

For, $p_*(\mathscr{O}_{\overline{X}})$ is finite over $\operatorname{Spec} A[y]^\wedge$. Now $\operatorname{Spec} A[y]^\wedge$ is regular over $\operatorname{Spec} A$. We have seen that $\overline{X} - \overline{\Gamma}$ is regular over $\operatorname{Spec} A$, so that it is in particular Cohen-Macaulay over $\operatorname{Spec} A$. Now a Cohen-Macaulay module M (of finite type) over a regular local ring B is free (cf. Serre's "Algèbre locale") and from this (4) follows. (We can use this property only for the corresponding fibres, but then the required property is an easy consequence of this.)

(5) *Set $\hat{P} = A[y]^\wedge$. Then for $\overline{\mathscr{O}}$ considered as a $\hat{P}$ module we have a representation of the form*

$$\hat{P}^m \xrightarrow{(a_{ij})} \hat{P}^n \to \overline{\mathscr{O}} \to 0 \quad \text{(exact as } \hat{P} \text{ modules)}$$

with $rk(a_{ij}) \leq (n-r)$ (i.e., determinants of all minors of $rk(n-r+1)$ of (a_{ij}) are zero).

More generally, let us try to describe an *R-algebra B* having a representation of the form 110

$$(*) \qquad \begin{cases} R^m \xrightarrow{(a_{ij})} R^n \to B \to 0 & \text{exact sequence} \\ rk(a_{ij}) \leq (n-r). & \text{of } R\text{-modules} \end{cases}$$

The we have ◻

Lemma 5.1. $\exists$ *an affine scheme V (of finite type) over* $\operatorname{Spec}\mathbb{Z}$ *such that every R-algebra of the form* $(*)$ *is induced by an R-valued point of V.*

Proof of Lemma. To each R we consider the functor $\mathfrak{F}(R)$

$$\mathfrak{F}(R) = \left\{ \begin{array}{c} \text{set of all commutative } R\text{-algebras } B \\ \text{with a representation of the form } (*) \end{array} \right\}.$$

One would like to represent the functor $\mathfrak{F}$ by an affine scheme, etc. We don't succeed in doing this, but we will represent a functor G such that we have a *surjective* morphism $G \to \mathfrak{F}$.

An algebra structure on B is given by an R-homomorphism

$$B \otimes_R B \to B.$$

Then in the diagram

$$\begin{array}{ccc} R^n \otimes RR^n & \longrightarrow & B \otimes_R B \\ {\scriptstyle\exists}\downarrow & & \downarrow \\ R^n & \longrightarrow & B \end{array}$$

the homomorphism $R^n \otimes_R R^n \to R^n$ factors via a homomorphism $R^m \otimes R^n \to R^n$ (of course not uniquely determined). Now the following sequence

$$(R^m \otimes_R R^n) \oplus (R^n \otimes R^m) \xrightarrow{\psi} R^n \otimes R^n \to B \otimes B \to 0$$

111 is exact where $\psi = (a_{ij}) \otimes \mathrm{Id} \oplus \mathrm{Id} \otimes (a_{ij})$. [We have

$$(\mathrm{Ker}(R^n \to B) \otimes R^n) \oplus (R^n \otimes (\mathrm{Ker}\, R^n \to B)) \xrightarrow{\text{can hom}} \mathrm{Ker}(R^n \otimes R^n \to B \otimes B) \to 0.$$

This implies exactness of the given diagram.] Again there exists a lifting of the above commutative diagram

$$(I_0) \qquad \begin{tikzcd} (R^m \otimes R^n) \oplus (R^n \otimes R^m) \arrow[r, "\psi"] \arrow[d, "\exists"', "c"] & R^n \otimes R^n \arrow[r] \arrow[d, "\exists"', "b"] & B \otimes_R B \arrow[r] & 0 \\ R^m \arrow[r, "(a_{ij})=a"] & R^n \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

On the other hand, giving a commutative diagram

$$(I) \qquad \begin{tikzcd} (R^m \otimes R^n) \oplus (R^n \otimes R^m) \arrow[r, "\psi"] \arrow[d, "c"'] & R^n \otimes R^n \arrow[d, "b"'] \\ R^m \arrow[r, "(a_{ij})=(a)"] & R^n \end{tikzcd}$$

where $\psi = (a_{ij}) \otimes \mathrm{Id} \oplus \mathrm{Id} \otimes (a_{ij})$ determines the commutative diagram (I_0).

The algebra structure on B induced by (I_0) is associative if the diagram

$$\begin{tikzcd} R^n \otimes R^n \otimes R^n \arrow[rr, "\mathfrak{b}\cdot(\mathrm{Id}\otimes b)"] \arrow[dr, "\mathrm{Id}\otimes b"'] & & R^n \\ & R^n \otimes R^n \arrow[ur, "b"'] & \end{tikzcd}, \quad \begin{tikzcd} R^n \otimes R^n \otimes R^n \arrow[rr, "\mathfrak{b}\cdot(b\otimes\mathrm{Id})"] \arrow[dr, "b\otimes\mathrm{Id}"'] & & R^n \\ & R^n \otimes R^n \arrow[ur, "b"'] & \end{tikzcd},$$

and the map $R^n \otimes R^n \otimes R^{n \mathfrak{b}\cdot(\mathrm{Id}\otimes b) - \mathfrak{b}\cdot(b\otimes \mathrm{Id})} \to R^n$ factorizes through $R^m \to$

R^n, i.e.,

(II)
$$\begin{tikzcd} R^n \otimes R^n \otimes R^n \arrow[rr, "b\cdot(\mathrm{Id}\otimes b)-b\cdot(b\otimes \mathrm{Id})"] \arrow[dr, "\exists\alpha"'] & & R^n \\ & R^m \arrow[ur, "(a_{ij})=a"'] & \end{tikzcd}$$

or, $b\cdot(\mathrm{Id}\otimes b) - b\cdot(b\otimes \mathrm{Id})$ is zero in B.

Let $\widetilde{b} : R^n \times R^n \to R^n$ be the homomorphism obtained by changing **112**
$b : R^n \times R^n \to R^n$ by the involution defined by $x\otimes y \mapsto y\otimes x$ in $R\otimes R$. Then the algebra structure is commutative if there is a homomorphism $\delta : R^n \otimes_R R^n \to R^m$ such that

(III)
$$\begin{tikzcd} R^n \otimes_R R^n \arrow[rr, "b-b"] \arrow[dr, "\delta"'] & & R^n \\ & R^m \arrow[ur, "(a_{ij})=a"'] & \end{tikzcd}$$

commutes.

Finally the identity element $1 \in B$ can be lifted to an element $e \in R^n$ (determines a homomorphism $R \to R^n$) and there is a map $\epsilon : R^n \to R^m$ such

(IV)
$$\begin{tikzcd} R^{ne} \otimes b - \mathrm{Id} \arrow[rr] \arrow[dr, "\epsilon"'] & & R^n \\ & R^m \arrow[ur, "a=(a_{ij})"'] & \end{tikzcd}$$

commutes, i.e., $e\otimes b - \mathrm{Id} = 0$ in B.

Let us then define the functor G : (Rings) $\to$ (Sets) as follows:

$G(R)$ =(i) *Set of homomorphisms* $a : R^m \to R^n$ such that $rk\ a \leq (n-r)$ (i.e., determinants of all minors of a of $rk(n-r+1)$ vanish), *together with* (0)

(ii) *Set of homomorphisms* b, c

$$\begin{array}{ccc} R^n \otimes R^n & , & (R^m \otimes R^n) \oplus (R^n \otimes R^m) \\ \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle c} \\ R^n & & R^m \end{array}$$

113 such that the following diagram is commutative:

(I)
$$\begin{tikzcd}
(R^m \otimes R^n) \oplus (R^n \otimes R^m) \arrow[r, "a\otimes \mathrm{Id} \oplus \mathrm{Id} \otimes a"] \arrow[d, "c"'] & R^n \otimes R^n \arrow[d, "b"] \\
R^m \arrow[r, "a"] & R^n
\end{tikzcd}$$

together with

(iii) an R-homomorphism $\alpha : R^n \otimes R^n \otimes R^n \to R^m$ such that the following is commutative

(II)
$$\begin{tikzcd}
R^n \otimes R^n \otimes R^n \arrow[rr, "bo(\mathrm{Id}\otimes b) - bo(b\otimes \mathrm{Id})"] \arrow[dr] & & R^n \\
& R^m \arrow[ur, "a"'] &
\end{tikzcd},$$

and

(iv) *an R-homomorphism $\delta : R^n \otimes R^n \to R^m$ such that*

(III)
$$\begin{tikzcd}
R^n \otimes R^n \arrow[rr, "b - \tilde{b}"] \arrow[dr, "\delta"'] & & R^n \\
& R^m \arrow[ur, "a"'] &
\end{tikzcd}$$

commutes, and

(v) an R-homomorphism $e : R \to R^n$ and $\epsilon : R^n \to R^m$ such that

(IV)
$$\begin{tikzcd}
R^n \arrow[rr, "e\otimes b - \mathrm{Id}"] \arrow[dr, "\epsilon"'] & & R^n \\
& R^m \arrow[ur, "a"'] &
\end{tikzcd}$$

commutes.

It is now easily seen that $G(R)$ can be identified with a subset $S \hookrightarrow R^P$ such that there exist polynomials $F_i(X_1, \ldots, X_p)$ over $\mathbb{Z}$ such that

$s = (s_1, \ldots, s_p) \in S$ iff $F_i(s_1, \ldots, s_p) = 0$, and the set $\{F_i\}$ and p are
114 independent of R. From this it is clear that $G(R)$ is represented by a scheme V of finite type over $\mathbb{Z}$ and since $G(R) \to F(R)$ is surjective, Lemma 5.1 follows immediately.

It follows in particular that $\overline{\mathscr{O}}$ is represented by *a homomorphism* $\varphi : \operatorname{Spec} \hat{P} \to V(\hat{P} = A[y_1, \ldots, y_m]^\wedge)$.

(6) *The image of* $\operatorname{Spec} \hat{P} - \{y = 0\}$ *in V lies in the smooth locus of V over* $\operatorname{Spec} \mathbb{Z}$*.*

We have a representation

$$(*) \qquad \hat{P}[z_1, \ldots, z_s] \to \overline{\mathscr{O}} \to 0$$

(homomorphisms of rings and homomorphisms as P modules). Now $\hat{P}[z_1, \ldots, z_s]$ is *regular* over A and $\overline{X} - \overline{\Gamma}$ is *regular* over $\operatorname{Spec} A - \{y = 0\}$. Hence the immersion $\overline{X} \hookrightarrow \operatorname{Spec} \hat{P}[z_1, \ldots, z_s] \simeq \mathbb{A}^s_{\hat{P}}$, being an A-morphisms, is a local complete intersection at every point of $\overline{X} - \overline{\Gamma}$ (we use the fact that a regular local ring which is the quotient of another regular local ring is a complete intersection in the latter; we use this fact for the corresponding local rings of the fibres and then lifting the m-sequence, etc.). Now codim $\overline{X}$ in $\mathbb{A}^s_{\hat{P}}$ is s. Now take a closed point $x_0 \in \operatorname{Spec} \hat{P} - \{y = 0\}$. Then tensoring $(*)$ by $k(x_0)$ we get

$$k(x_0)[z_1, \ldots, z_s] \to \mathscr{O} \otimes_{\hat{P}} k(x_0) \to 0 \quad \text{exact.}$$

Now $\operatorname{Spec} \hat{\mathscr{O}} \otimes_{\hat{P}} k(x_0)$ is precisely the fibre of $\overline{X}$ over x_0 for the morphism $\overline{X} \hookrightarrow \operatorname{Spec} \hat{P}$. We *claim* that $\operatorname{Spec} \overline{\mathscr{O}} \otimes_{\hat{P}} k(x_0)$ is also a local complete intersection in $k(x_0)[z_1, \ldots, z_s]$ wherever $x_0 \in \operatorname{Spec} P - \{y = 0\}$ (it is a 0-dimensional subscheme of $\operatorname{Spec} k(x_0)[z_1, \ldots, z_s] \hookrightarrow \mathbb{A}^s_{k(x_0)}$) 115
and in fact that $\operatorname{Spec} \overline{\mathscr{O}} \otimes_{\hat{P}} R_0 \hookrightarrow \operatorname{Spec} R_0[z_1, \ldots, z_s]$ is a *morphism of local complete intersection* over R_0 for any base change $\operatorname{Spec} R \to \operatorname{Spec} \hat{P} - \{y = 0\}$ (i.e., flat and the fibres of the morphism over $\operatorname{Spec} R$ is a local complete intersection). To prove this we note that $\overline{\mathscr{O}} \otimes_{\hat{P}} R_0$ is locally free (of rank r) ($\operatorname{Spec} R_0 \to \operatorname{Spec} \hat{P} - \{y = 0\}$) and $\operatorname{Spec} R_0[z_1, \ldots, z_s] \to \operatorname{Spec} R_0$ is a regular morphism. In particular a Cohen-Macaulay morphism. Now the claim is an immediate consequence of

Lemma 5.2. *Let B, C, R be local rings such that B, C are R-algebras flat over R and B is Cohen-Macaulay over R. Let $f : B \to C$ be a surjective (local) homomorphism of R-algebras such that C is a complete intersection in B. Then for all $R \to R_0 \to 0$, the surjective homomorphism*

$$B \otimes R_0 \to C \otimes R_0 \to 0$$

(the morphism $\operatorname{Spec}(C \otimes R_0) \hookrightarrow \operatorname{Spec}(B \otimes R_0)$*) is a morphism of complete intersection over* $\operatorname{Spec} R_0$.

Proof. Since the flatness hypothesis is satisfied, it suffices to prove that $C \otimes k$ is a complete intersection in $B \otimes k (k = R/m_R)$. Now we have

$$0 \to I \to B \to C \to 0 \quad \text{exact,}$$

$I = \ker B \to C$, and $I = (f_1, \ldots, f_s)$, f_i is an m-sequence in B. Now the codimension of C, $\operatorname{Spec}(C \otimes k)$ in $\operatorname{Spec}(B \otimes k)$, is equal to the codimension of $\operatorname{Spec} C$ in $\operatorname{Spec} B$, which is s. (This follows by flatness of B, C
116 over R and the fact that flat implies equidimensional.) Let $\overline{f}_i$ denote the canonical images of f_i in $B \otimes k$. We have then $(B \otimes k)/(\overline{f}_1, \ldots, \overline{f}_s) = C \otimes k$. Now $(B \otimes k)$ is Cohen-Macaulay and the codimension of $\operatorname{Spec}(C \otimes k)$ in $\operatorname{Spec}(B \otimes k)$ is s. It follows by Macaulay's theorem that $\overline{f}_1, \ldots, \overline{f}_s$ is an m-sequence in $\overline{B} \otimes k$. This implies that $\overline{C} \otimes k$ is a complete intersection in $\overline{B} \otimes k$, and proves Lemma 5.2.

The complete intersection trick. We go back to the proof of (5). Let $\lambda : \operatorname{Spec} R_0 \to \operatorname{Spec} \hat{P}$ be a morphism such that R_0 is Artin local and $\lambda(\operatorname{Spec} R_0) \subset \operatorname{Spec} \hat{P} - \{y = 0\}$. Consider the morphism $(\varphi \circ \lambda) : \operatorname{Spec} R_0 \to V$. Then $(\varphi \circ \lambda)$ defines an R_0 algebra B which is a free R_0-module of rank r (in particular flat over R_0), and B is a morphism of local complete intersection over R_0 (B is of relative dimension 0 over R_0). Let $R \to R_0 \to 0$ be such that $\operatorname{Spec} R$ is an infinitesimal neighborhood of $\operatorname{Spec} R_0$. Then the assertion (5) follows if we show that $(\varphi \circ \lambda) : \operatorname{Spec} R_0 \to V$ factors through $\operatorname{Spec} R \to V$.

Now B is defined by

$$R_0^m \xrightarrow{a^0} R_0^n \to B \to 0,$$

where $a^0 = (a^0_{ij})$. Now $\operatorname{Spec} B \to \operatorname{Spec} R_0$ is a morphism of local complete intersection for a suitable imbedding and $\operatorname{Spec} B$. It is clear that $\operatorname{Spec} B \to \operatorname{Spec} R_0$ is in fact a morphism of global complete intersection for the corresponding imbedding, for it is easily seen that a 0-dimensional closed subscheme of $\mathbb{A}^n_k$ which is a local complete intersection, is in fact a global intersection. We have seen in § 4, Part 1, That the *functor of global deformations of a complete intersection is unobstructed i.e., formally smooth.* Hence there is a *flat* R-algebra B' such 117
that $B' \otimes_R R_0 \approx B$. Hence the sequence $R_0^m \xrightarrow{a_0} R_0^n \to 0$ can be lifted to an exact sequence $R^m \xrightarrow{a} R^n \to B' \to 0$; the proof of this is in spirit analogous to imbedding a deformation (infinitesimal) of $X \hookrightarrow \mathbb{A}^n$ in the same affine space. A homomorphism $(R_0^n \to B)$ over R_0 is given by n elements in B. Hence this homomorphism can be lifted to $R^n \to B'$ and it becomes surjective. Now it is seen easily that $R_0^m \to R_0^n$ can be lifted to $R^m \to R^n$. It follows that determinants of all minors of a of rank $(n - r + 1)$ vanish. Thus the relations (0) above can be lifted to R.

Consider the relations (I). We are ginve b_0, c_0 such that the following diagram is commutative:

$$\begin{tikzcd}
(R_0^m \otimes R_0^n) \oplus (R_0^n \otimes R_0^m) \arrow[r, "a_0 \otimes \mathrm{Id} \oplus \mathrm{Id} \otimes a_0 = \psi_0"] \arrow[d, "c_0"'] & R_0^n \otimes R_0^n \arrow[d, "b_0"] \\
R_0^m \arrow[r, "a_0"] & R_0^n.
\end{tikzcd}$$

With the above lifting of (a_0) to a we get

$$
(*) \qquad
\begin{tikzcd}
(R^m \otimes R^n) \oplus (R^n \otimes R^m) \arrow[r, "a\otimes \mathrm{Id} \oplus \mathrm{Id}\otimes a"] \arrow[d, "\exists"', "c"] \arrow[dr, phantom, "\text{(B)}"] & R^n \otimes R^n \arrow[r, "\text{can}"] \arrow[d, dashed, "b"', "\exists"] \arrow[dr, phantom, "\text{(A)}"] & B' \otimes_R B' \to 0 \arrow[r] \arrow[d, "\text{can}"] & 0 \\
R^m \arrow[r, "a"] & R^n \arrow[r] & B \arrow[r] & 0.
\end{tikzcd}
$$

The two rows are exact. We claim *that there is a b which lifts b_0, and such that the square* (A) *is commutative.* In fact this is quite easy, for it is clear that we can find $b' : R^n \otimes R^n \to R^n$ such that (A) is commutative. Let b'_0 be the reduction mod R_0 of b'. Then we see that the composition of the canonical map $R_0^n \to B$ with $b_0 - b'_0$ is zero. Hence **118**

$$b_0(z) \equiv b'_0(z) \quad (\text{mod } \operatorname{Im} a_0, \text{ or } \operatorname{Ker} R_0^n \to B).$$

Now $b_0(z) - b'_0(z)$ can be lifted to elements in $\operatorname{Im} a$, so that taking for $\{z_a\}$ a canonical basis in $R^n \to R^n$ we define $b : R^n \otimes R^n \to R^n$

$$b(z_a) = b'(z_a) + \theta_a,$$

where $\theta_a \in \operatorname{Im} a$ lifts $b_0(a) - b'_0(z)$. It is now clear that b lifts b_0 and the square (A) is commutative. This proves the claim.

By a similar argument as above there is a c such that c reduces to c_0 and the square (B) is commutative. (If necessary we can prolong to the left the exact sequence of the second row in the diagram (*).) Thus it follows that the relations in I can be lifted to R.

A similar argument shows that a_0, δ_0, e_0 and ϵ_0 which are given representing the point $\operatorname{Spec} R_0 \to V$ can be lifted to R so that the diagrams (II), (III) and (IV) are still commutative. This means that the $\operatorname{Spec} R_0 \to V$ can be lifted to an R-valued point of V. As we remarked before, the assertion (5) now follows.

We go back to the usual notations in the theorem. Then: (7) *Let $\mathfrak{a} = m_A(y_1, \ldots, y_d)$ ideal in $A[y]$. Then $\hat{P} = A[y]^\wedge$ is also the $\mathfrak{a}$-adic completion of $A[y]$ (of course $\hat{P}$ is also the $m_A \cdot (A[y])$-adic completion of $A[y]$).*

The proof of this assertion is immediate for a convergent series $\sum_i f_i$, $f_i \in A[y]$ in the $\mathfrak{a}$-adic topology is precisely one such that the coefficients of f_i tend to zero in the m_A-adic topology and the degree of the monomials $\to \infty$. This implies that a convergent series is precisely a **119**
formal power series in $\{y_i\}$ such that the coefficients (in A) tend to 0 in the m_A-adic topology. This is the description of $\hat{P}$ we had and (7) is proved. (8) Now for the morphism $\varphi : \operatorname{Spec} \hat{P} \to V$ we have that the image of $\operatorname{Spec} \hat{P} - \{y = 0\}$ is in the smooth locus of V over $\mathbb{Z}$. *We note*

that $V(\mathfrak{a}) = (\operatorname{Spec} k[y]) \cup \{y = 0\}$ (k *residue field of* A), *i.e.,* $V(\mathfrak{a})$ *contains* $\{y = 0\}$ so that the image of $\operatorname{Spec} \hat{P} - V(\mathfrak{a})$ (by φ) is also in the smooth locus of V over $\mathbb{Z}$. Now the $\mathfrak{a}$-adic completion of $A[y]$ is $\hat{P}$. Let $\widetilde{P}$ denote the henselization of $(P, \mathfrak{a})$, $P = A[y]$. Now apply Theorem 2 proved above. Hence we can find an étale map $\operatorname{Spec} R \to \operatorname{Spec} P$ which is trivial over $V(\mathfrak{a})$ and morphism $\varphi' : \operatorname{Spec} R \to V$ (note that $\hat{P}$ is also the $\mathfrak{a}$-adic completion of $\widetilde{P}$) such that

$$\varphi' \equiv \varphi(\bmod\ ^N), \quad \text{for any given } N.$$

(Note that the $\mathfrak{a}$-adic (i.e., $\mathfrak{a}R$-adic, $\mathfrak{a}$ is not an ideal in R) completion of R is also $A|y|^\wedge$). Let O' be the R-algebra defined by φ'. Then we have $\mathscr{O} \equiv \overline{\mathscr{O}}(\bmod\ \mathfrak{a}^N)$ (i.e., $\mathscr{O}'/\mathfrak{a}^N = \overline{\mathscr{O}}/\mathfrak{a}^N$). Now $\mathscr{O}'$ becomes an A-algebra and then we have

$$\mathscr{O}' \equiv \overline{\mathscr{O}}(\bmod\ m_A^N)$$

for $(m_A R)^N \supset \mathfrak{a}^N$. Take in particular $N = 2$. Thus we can find an R-algebra $\mathscr{O}'$ of finite type and consequently of *finite type over* A such that

$$\mathscr{O}' \equiv \overline{\mathscr{O}}(\bmod\ m_A^2).$$

Thus to conclude the proof of the theore, it suffices to prove that $\mathscr{O}'$ is *flat* over A.

120 (9) *Choice of* $\mathscr{O}'$ *such that* $\mathscr{O}'$ *is flat/A.*

We had a presentation of $\overline{\mathscr{O}}$ as follows:

$$\hat{P}^m \xrightarrow{a} \hat{P}^n \to \overline{\mathscr{O}} \to 0.$$

We claim that we have a presentation such that

(*)
$$\left\{\begin{aligned}
&\hat{P}^\ell \xrightarrow{\theta} \hat{P}^m \xrightarrow{a} \hat{P}^n \to \overline{\mathscr{O}} \to 0,\\
&\text{where} \quad a \cdot \theta = 0 \quad \text{and} \quad \hat{P}^m \xrightarrow{a} \hat{P}^n \to \overline{\mathscr{O}} \to 0 \text{ is exact, and}\\
&\hat{P}^\ell \otimes_A k_A \xrightarrow{\theta \otimes k_A} \hat{P}^m \otimes k_A \xrightarrow{a \otimes k_A} \hat{P}^n \otimes k_A \to \overline{\mathscr{O}} \otimes k_A \to 0 \text{ is exact,}
\end{aligned}\right.$$

where $k_A = A/m_A$.

This follows if we prove that $\overline{\mathscr{O}}$ is flat over A (cf., § 3, Part 1). However, (*) can be established directly as follows: We can find an exact sequence of the form

$$\hat{P}^{\ell} \otimes k_A \xrightarrow{\overline{\theta}_0} \hat{P}^m \otimes k_A \xrightarrow{\overline{a}_0} \hat{P}^n \otimes k_A \to \overline{\mathscr{O}} \otimes k_A \to 0.$$

(Note that $\operatorname{Spec} \overline{\mathscr{O}} \otimes k_A = X_0 = \operatorname{Spec} \mathscr{O}_0$ is the scheme whose deformation we are considering and that $X_n = \overline{\mathscr{O}} \otimes A/m_A^{n+1} = \operatorname{Spec} \mathscr{O}_n$ are infinitesimal deformations of X_0.) The above exact sequence can be lifted to an *exact sequence*

$$\hat{P}^{\ell} \otimes A/m_A^{n+1} \xrightarrow{\overline{\theta}_n} \hat{P}^m \otimes A/m_A^{n+1} \xrightarrow{\overline{a}_n} \hat{P}^n \otimes A/m_A^{n+1} \to \mathscr{O}_n \to 0$$

since $\mathscr{O}_n$ is flat over A/m_A^{n+1}. Passing to the limit, we have an exact sequence

$$\hat{P}^{\ell} \otimes A/m_A^{n+1} \xrightarrow{\overline{\theta}_n} \hat{P}^m \otimes A/m_A^{n+1} \xrightarrow{\overline{a}_n} \hat{P}^n \otimes A/m_A^{n+1} \to \mathscr{O}_n \to 0$$

since $\mathscr{O}_n$ is flat over A/m_A^{n+1}. Passing to the limit, we have an exact sequence

$$\hat{P}^{\ell} \xrightarrow{\overline{\theta}} P^m \xrightarrow{\overline{a}} P^n \to \overline{\mathscr{O}} \to 0$$

such that $\overline{a} \cdot \overline{\theta} = 0$, and $\hat{P}^m \xrightarrow{\overline{a}} P^n \to \overline{\mathscr{O}} \to 0$ is exact. This proves the **121**
existence of (*).

Now define a functor G' which is a modification of G as follows: $G'(R) =$ Set of $\{\theta, a, b, c, \alpha, \delta, e, \epsilon$ where $a, b, c, \alpha, \delta, e, \epsilon$ are as in definition of $G(R)$; and θ is defined by $R^{\ell} \xrightarrow{\theta} R^m \xrightarrow{a} R^n$ with $a \cdot \theta = 0\}$.

Then as in the case of $G(R)$, we see that G' is represented by a scheme V' of finite type over $\mathbb{Z}$. The given representation for $\overline{\mathscr{O}}$ as in $(*)$ above gives rise to a morphism $\psi : \operatorname{Spec} P \to V'$. We claim that as in the case of V, the image of $\operatorname{Spec} P - \{y = 0\}$ lies in the smooth locus of V'. With the same notations for R, R_0 as in the proof of the statement for the case V, it suffices to prove the following: Given

$$R_0^{\ell} \xrightarrow{\theta_0} R_0^m \xrightarrow{a_0} R_0^n \to B \to 0$$

such that $a_0 \cdot \theta_0 = 0$ and $R_0^m \xrightarrow{a_0} R_0^n \to B \to 0$ is exact, and a flat lifting B' over R (hence free over R), we have to lift this sequence to R (the proof of the lifting of the quantities involved is the same as for $G(R)$). As we have seen before, for $G(R)$ we have a lifting

$$R^m \xrightarrow{a} R^n \to B' \to 0.$$

Now $\operatorname{Im}\theta_0$ are a set of relations. Since B' is flat over R these relations can also be lifted, i.e., we have a lifting

$$R^\ell \xrightarrow{\theta} R^m \xrightarrow{a} R^n \to B' \to 0$$

such that $a \cdot \theta = 0$ and $R^m \xrightarrow{a} R^n \to B' \to$ is exact. This proves
122 the required claim and hence it follows $\psi(\operatorname{Spec} P - \{y = 0\})$ lies in the smooth locus (over $\mathbb{Z}$) of V'.

Applying Theorem 2, we can find an étale $\operatorname{Spec} R \to \operatorname{Spec} P$, $P = A[y]$ and a morphism $\psi' : \operatorname{Spec} R \to V'$ such that

$$\psi' \equiv \psi (\operatorname{mod} \mathfrak{a}^2).$$

Let $\mathscr{O}'$ be the R-algebra defined by ψ'. Then as we have seen before, we have

$$\mathscr{O}' \equiv \overline{\mathscr{O}} (\operatorname{mod} m_A^2).$$

We claim that $\mathscr{O}'$ is flat over A. For this we observe that we have *a fortiori*

$$\mathscr{O}' \equiv \overline{\mathscr{O}} (\operatorname{mod} m_A).$$

This implies that $\mathscr{O}'/m_A \cdot \mathscr{O}' = \overline{\mathscr{O}}/m_A\overline{\mathscr{O}} \approx \mathscr{O}_0$. Let

$$(\mathrm{I}') \qquad \begin{gathered} \hat{P}^\ell \xrightarrow{\theta'} \hat{P}^m \xrightarrow{a'} \hat{P}^n \to \mathscr{O}' \to 0 \\ a' \circ \theta' = 0,\ \hat{P}^m \to \hat{P}^n \to \hat{P}^n \to \mathscr{O}' \to 0 \quad \text{exact} \end{gathered}$$

be a representation of $\mathscr{O}'$. Recall we have the representation for $\overline{\mathscr{O}}$

$$(\mathrm{I}) \qquad \hat{P}^\ell \xrightarrow{\theta} \underbrace{\hat{P}^m \xrightarrow{a} \hat{P}^n \to \overline{\mathscr{O}} \to 0}_{\text{exact}}, \quad \text{and} \quad a \circ \theta = 0.$$

Now tensoring (I′) with A/m_A^n yields

$$\hat{P}^\ell \otimes A/m_A^n \xrightarrow{\theta' \otimes A/m_A^n} \underbrace{\hat{P}^m \otimes A/m_A^n \xrightarrow{a' \otimes A/m_A^n} \hat{P}^n \otimes A/m_A^n \to \mathscr{O}' \otimes Am_A^n \to 0}_{\text{exact}},$$

and

$$(a' \otimes A/m_A^n) \cdot (\theta' \otimes A/m_A^n) = 0.$$

We have $(I') \otimes A/m_A = (I) \otimes A/m_A$, as a consequence of the fact that **123**
$\mathscr{O}' \equiv \overline{\mathscr{O}} (\operatorname{mod} m_A)$. By (I) $I \otimes A/m_A$ is *exact*. This implies that $(I') \otimes A/m_A$ is exact. So we have that $(I') \otimes A/m_A$ is *exact*, and $(I') \otimes A/m_A^n$ has the property, $(a' \otimes A/m_A^n) \cdot (\theta' \otimes A/m_A^n) = 0$ and

$$\hat{P}^m \otimes A/m_A^n \xrightarrow{a' \otimes A/m_A^n} \hat{P}^n \otimes A/m_A^n \to \mathscr{O}' \otimes A/m_A^n \to 0,$$

for all n. This implies, as we saw in the first few lectures, that $\mathscr{O}' \otimes A/m_A^n$ is flat over A/m_A^n for every n.

Now $\mathscr{O}'$ is an A-algebra of finite type and so $\mathscr{O}' \otimes A/m_A^n$ flat over A/m_A^n for all n implies that $\mathscr{O}'$ is flat over A (cf. SGA exposes on flatness).

The proof of the theorem is now complete. $\square$

Remark 5.1. The fact that $\mathscr{O}'$ is flat over A can also be shown in a different manner. This can be done using only the functor G (i.e., V), but a better approximation (i.e., better than $N = 2$) for $\mathscr{O}'$ may be needed. This uses the following result of Hironaka: Let B be a complete local ring, $b \subset m$ an ideal in B and M a finite B-module locally free (of rank r) outside $V(b)$. Then there is an N such that whenever M' is a finite B-module locally free of rank r outside $V(b)$ and $M' = M (\operatorname{mod} b^N)$, then $M' \approx M$. Take in our present case $B = A[[y]]$ so that we have

$$\begin{tikzcd}
A \arrow[rr, hook] \arrow[dr, dash] & & A[y]^\wedge \arrow[r, hook] & A[[y]] \\
& A[y] \arrow[r, hook] & R \arrow[u, hook] &
\end{tikzcd}$$

R étale over $A[y]$ such that $A[y]/\mathfrak{a} \approx R/\mathfrak{a}$ and that all the extensions are faithfully flat. Take a coherent sheaf on $\operatorname{Spec} R$; to verify that it is flat
124 over A it suffices to verify that its lifting to $\operatorname{Spec} A[[y]]$ is flat over A. Take $\underline{b}$ to be $\mathfrak{a} \cdot A[[y]]$. Then by taking a suitable approximation for $\mathscr{O}'$ it follows that the liftings to $A[[y]]$ of $\mathscr{O}'$ and $\overline{\mathscr{O}}$ are isomorphic. This implies that $\overline{\mathscr{O}} \otimes A/m^n \approx \mathscr{O}' \otimes A/m^n$, hence that $\mathscr{O}'$ is flat over A, since $\operatorname{Spec}(\overline{\mathscr{O}} \otimes A/m^n) = X_n$ is flat over A.