A PRIMER ON HOMOTOPY COLIMITS

DANIEL DUGGER

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1. Introduction

This is an expository paper on homotopy colimits and homotopy limits. These are constructions which should arguably be in the toolkit of every modern algebraic topologist, yet there does not seem to be a place in the literature where a graduate student can easily read about them. Certainly there are many fine sources: [BK], [DwS], [H], [HV], [V1], [V2], [CS], [S], among others. Of these my favorites are [DS] and [H], the first as a general introduction and the second as an excellent reference work. Yet [H] demands that the student absorb quite a bit before reaching homotopy colimits, and [DwS] does not delve deeply into the topic. The remaining sources mentioned above present other difficulties to readers encountering these ideas for the first time.

What I found myself wanting was a relatively short paper that would start with the basic ideas and then proceed to give students a 'crash course' in homotopy colimits—a paper which would survey the basic techniques for working with them and show some examples, but not weigh the reader down with too many details. That is the aim of the present document. Like most such documents, it probably fails to truly meet its goals—as one example, it is not very short!

Many proofs are avoided, or perhaps just sketched, and the reader is encouraged to seek out the complete proofs in the above sources.

1.1. **Prerequisites.** The reader is assumed to be familiar with basic category theory, in particular with colimits and limits. [ML] is a fine reference. Some experience with simplicial sets will be helpful, as well as some experience with model categories. For the former we recommend [C], and for the latter [DwS].

Almost no model category theory is used in the first eight sections, where we keep the focus mostly on topological spaces. Readers will only have to know that a cellular inclusion is the main example of a cofibration, and that a CW-complex is the main example of a cofibrant object. "Weak equivalence" means weak homotopy equivalence—that is to say, a map inducing isomorphisms on all homotopy groups.

In Sections 7–10 model category theory is much more prevalent. Although one can state the basic properties of homotopy colimits and limits without using model categories, the most elegant proofs all use model category techniques. So it is very useful to become proficient in this way of thinking about things.

What we have just outlined is something like the 'minimum basic requirements' assumed in the paper. In reality we have assumed more, because we assume throughout that the reader has a certain amount of experience with many basic homotopy-theoretic constructions (classifying spaces, spectral sequences, etc.) Hopefully students with just one or two years experience past their first algebraic topology course will find the paper accessible, though.

1.2. **Organization.** Part 1 of the paper (Sections 2–6) develops the basic definition of homotopy colimits and limits, as well as some foundational properties. Everything is done in the context of topological spaces, although the entire discussion adapts more or less verbatim to other simplicial model categories.

Parts 2 and 3 of the paper (Sections 7–12) concern more advanced perspectives on homotopy colimits and limits. We develop spectral sequences for computing some of their invariants, explain how to adapt the constructions to arbitrary model categories, and in Part 2 we intensively discuss the connection with the theory of derived functors.

To conclude the paper we have Part 4, concerning examples. Most of the material here only depends on Part 1, but every once in a while we need to use something more advanced. Most readers will be able to understand the basic ideas without having read Parts 2 and 3 first, but will occasionally have to flip back for complete details.

1.3. **Notation.** If $\mathcal{C}$ is a category and X and Y are objects, then we will write $\mathcal{C}(X,Y)$ instead of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$. The overcategory $(\mathcal{C}\downarrow X)$ is the category whose objects are pairs $[A, A\to X]$ consisting of an object A in $\mathcal{C}$ and a map $A\to X$. A map $[A, A\to X]\to[B,B\to X]$ consists of a map $A\to B$ making the evident triangle commute. Occasionally we will denote an object of $(\mathcal{C}\downarrow X)$ as $[A, X\leftarrow A]$, depending on the circumstance.

1.4. **Acknowledgments.** I am grateful to Jesper Grodal, Robert Lipshitz, and Don Stanley for alerting me to errors in an early version, and to Owen Gwilliam for encouraging me to actually finish this manuscript. I would especially like to acknowledge an intellectual debt to Phil Hirschhorn and Dan Kan. Much of my understanding of homotopy colimits was passed down from them, and learning from [H] was one of the great pleasures in my early education.

Part 1. Getting started

2. First examples

The theory of homotopy colimits arises because of the following basic difficulty. Let I be a small category, and consider two diagrams $D, D' \colon I \to \mathcal{T}op$. If one has a natural transformation $f \colon D \to D'$, then there is an induced map $\operatorname{colim} D \to \operatorname{colim} D'$. If f is a natural weak equivalence—i.e., if $D(i) \to D'(i)$ is a weak equivalence for all $i \in I$—it unfortunately does not follow that $\operatorname{colim} D \to \operatorname{colim} D'$ is also a weak equivalence. Here is an example:

Example 2.1. Let I be the 'pushout category' with three objects and two non-identity maps, depicted as follows: $1 \longleftarrow 0 \longrightarrow 2$. Let D be the diagram

$$* \longleftarrow S^n \longrightarrow D^{n+1}$$

and let D' be the diagram

$$* \longleftarrow S^n \longrightarrow *.$$

Let $f \colon D \to D'$ be the natural weak equivalence which is the identity on S^n and collapses all of D^{n+1} to a point. Then $\operatorname{colim} D \cong S^{n+1}$ and $\operatorname{colim} D' = *$, so the induced map $\operatorname{colim} D \to \operatorname{colim} D'$ is certainly not a weak equivalence.

So the colimit functor does not preserve weak equivalences (one sometimes says that the colimit functor is not "homotopy invariant", and it means the same thing). The *homotopy* colimit functor may be thought of as a correction to the colimit, modifying it so that the result *is* homotopy invariant.

There is one simple example of a homotopy colimit which nearly everyone has seen: the mapping cone. We generalize this slightly in the following example, which concerns homotopy pushouts.

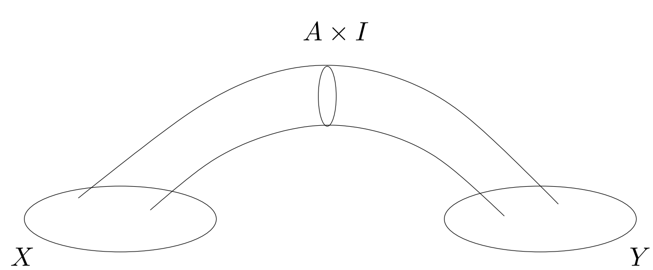
Example 2.2. Consider a pushout diagram of spaces $X \xleftarrow{f} A \xrightarrow{g} Y$. Call this diagram D. The pushout of D is obtained by gluing X and Y together along the images of the space A: that is, $f(a)$ is glued to $g(a)$ for every $a \in A$. The homotopy pushout, on the other hand, is constructed by gluing together X and Y 'up to homotopy'. Specifically, we form the following quotient space:

$$\operatorname{hocolim} D = \big[X \amalg (A \times I) \amalg Y\big]/\sim$$

where the equivalence relation has

$$(a, 0) \sim f(a) \qquad \text{and} \qquad (a, 1) \sim g(a), \qquad \text{for all } a \in A.$$

We can depict this space by the following picture:



Consider the open cover $\{U, V\}$ of $\operatorname{hocolim} D$ where U is the union of X with the image of $A \times [0, \frac{3}{4})$, and V is the union of Y with the image of $A \times (\frac{1}{4}, 1]$. Note that U deformation retracts down to X, V deformation retracts down to Y, and that the map $A \to U \cap V$ given by $a \mapsto (a, \frac{1}{2})$ is a homotopy equivalence. The Mayer-Vietoris sequence then gives a long exact sequence relating the homology of $\operatorname{hocolim} D$ with $H_*(X)$, $H_*(Y)$, and $H_*(A)$. Similarly, the Van Kampen theorem shows (assuming X, Y, and A are path-connected, for simplicity) that $\pi_1(\operatorname{hocolim} D)$ is the pushout of the diagram of groups $\pi_1(X) \longleftarrow \pi_1(A) \longrightarrow \pi_1(Y)$. The moral is that the space $\operatorname{hocolim} D$ is pretty easy to study using the standard tools of algebraic topology—in contrast to $\operatorname{colim} D$, which is much harder.

It is now easy to prove that our construction of $\operatorname{hocolim} D$ preserves weak equivalences. Suppose D' is another pushout diagram $X' \longleftarrow A' \longrightarrow Y'$, and that $D \to D'$ is a natural weak equivalence. Let $\{U', V'\}$ be the cover of $\operatorname{hocolim} D'$ defined analogously to $\{U, V\}$. Note that the map $\operatorname{hocolim} D \to \operatorname{hocolim} D'$ restricts to maps $U \to U'$, $V \to V'$, and $U \cap V \to U' \cap V'$, and these restrictions are all weak equivalences (because U and U' deformation retract down to X and X', and so forth). It then follows from the naturality of the Van Kampen theorem, and of the Mayer-Vietoris sequence, that $\operatorname{hocolim} D \to \operatorname{hocolim} D'$ induces isomorphisms on π_1 and on all homology groups with local coefficients. So it is a weak equivalence by the Whitehead Theorem [DaK, Theorem 6.71??]. (A better proof, that avoids the Whitehead Theorem and gets more to the heart of the matter, follows directly from the little-known but foundational result [Gr, 16.24]).

Before leaving this example we should relate it to mapping cones. If $f\colon A \to X$ is a map, then the quotient $X/f(A)$ is the pushout of $* \longleftarrow A \longrightarrow X$. The homotopy pushout of $* \longleftarrow A \longrightarrow X$, as defined above, is nothing other than the mapping cone of f.

There are several things to be learned from the above example, and we will return to it often as we develop the general theory. For now, here are four basic things to notice right away:

(1) Whereas the colimit of a diagram is obtained by taking the spaces in the diagram and gluing them together, the homotopy colimit will be constructed by "gluing them up to homotopy". Sometimes one says that the homotopy colimit is a "fattened up" version of the colimit. The above example is perhaps misleadingly simple, because the indexing category I is so simple—for general categories quite a bit more will be involved in encoding the necessary homotopies. Still, this basic idea of 'gluing up to homotopy' is the important one.

(2) Note that in the above example one has a map $\operatorname{hocolim} D \to \operatorname{colim} D$ obtained by collapsing the homotopy. Specifically, one defines a map

$$X \amalg (A \times I) \amalg Y \to X \amalg_A Y$$

by letting it be the natural maps on the X and Y factors, and on the $A \times I$ factor it is the projection $A \times I \to A$ followed by the evident map into $X \amalg_A Y$. This respects the identifications in the definition of $\operatorname{hocolim} D$, so we get our map $\operatorname{hocolim} D \to X \amalg_A Y$.

This situation is typical. When we finally define $\operatorname{hocolim} D$ for general diagrams we will find that there is a natural map $\operatorname{hocolim} D \to \operatorname{colim} D$ obtained by 'collapsing homotopies'.

(3) Many algebraic-topological invariants of the space $\operatorname{hocolim} D$ should be computable in terms of the invariants for the D_i's. We will see, for instance, that this is true for any cohomology theory $E^*(-)$ and any homology theory $E_*(-)$. This is one of the main ways in which homotopy colimits are better than colimits—they interact in predictable ways with the standard machinery of algebraic topology.

(4) It is not completely obvious, but it turns out that in our construction of $\operatorname{hocolim} D$ we could have replaced the interval I by any contractible space Z admitting a cofibration $\{0,1\} \rightarrowtail Z$. So we could have defined $\operatorname{hocolim} D$ as $[X \amalg (A \times Z) \amalg Y]/\sim$ where $(a,0) \sim f(a)$ and $(a,1) \sim g(a)$. This gives a space which is weakly equivalent to the definition we used above. (Even more, we could have replaced $A \times Z$ with any space B admitting a cofibration $A \amalg A \rightarrowtail B$ and a weak equivalence $B \to A$ coequalizing these two maps $A \to B$). What this is telling us is that there is not really a *single* homotopy colimit of a diagram; rather, there are lots of different *models* for the homotopy colimit, all weakly equivalent to each other. The model where we used the interval I is in some sense more natural than the others, but we don't always want to be tied down to one model.

2.3. **The million-dollar question.** Why should one learn about homotopy colimits? How are they useful? These are the kind of questions every student should ask their professors before learning about something. It is often hard to give a simple answer, but here are my attempts:

(a) As remarked above, it is relatively easy to compute the homology or cohomology of a homotopy colimit ("easy" in the sense that there is a spectral sequence). So if one is studying a space X and can identify it as being a certain homotopy colimit (or more precisely, weakly equivalent to a certain homotopy colimit), then one has a good chance of computing the homology and cohomology groups of X.

(b) Many things that happen in algebraic topology come down, in the end, to showing that two spaces X and Y are weakly equivalent. As we will see, there are many techniques for showing that different homotopy colimits are weakly equivalent. So if one can first identify X and Y as certain homotopy colimits, there are suddenly a number of tools available for proving that $X \simeq Y$.

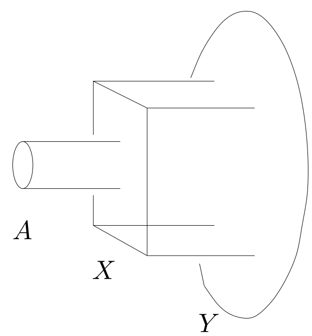
(c) Algebraic topology is full of *machinery*. This word can mean lots of things, but what I mean at the moment is a method for starting with some input data and producing a space or a sequence of spaces. For instance, one can start with a category and produce its classifying space; or start with a symmetric monoidal category and produce a Γ-space, and from the Γ-space get a spectrum. In algebraic K-theory one starts with a ring, considers the exact category of R-modules, and from this data constructs a K-theory space $K(R)$. These are only the most obvious examples—a complete list of such 'machines' would probably fill hundreds of pages.

    Anyway, the point I want to make is that homotopy colimits (and limits) play an important role in the construction of the output spaces for many of these machines. If you are a student of homotopy theory and haven't yet encountered homotopy colimits, it is only a matter of time.

2.4. **One more example.** Before ending this section we examine another brief example. Consider a diagram of spaces

$$A \xrightarrow{f} X \xrightarrow{g} Y.$$

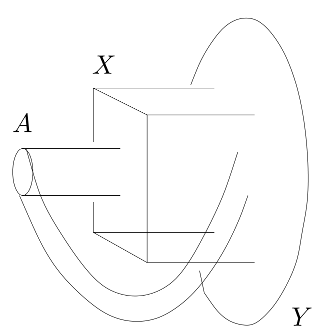
One way to construct the homotopy colimit in this case is as the double mapping cylinder shown below



This is the space $[(A \times I) \amalg (X \times I) \amalg Y]/\sim$ in which we have identified $(a, 1) \sim (f(a), 0)$ and $(x, 1) \sim g(x)$, for all $a \in A$ and $x \in X$. Note that this space deformation retracts down to Y.

Now consider the following. For the colimit of a diagram D, every map $f \colon D_i \to D_j$ in the diagram tells us to glue $a \in D_i$ to $f(a) \in D_j$. In the homotopy colimit we are supposed to glue up to homotopy, and this is what we tried to do in the double mapping cylinder above. But note that we have only done this for f and g, whereas there is a third map in our diagram—namely, the composite gf! Maybe we should glue in a homotopy for that map, too.

This suggests that we should do the following. Start with $A \amalg X \amalg Y$ and glue in a cylinder for f, g, and gf. This gives us the following space, which we'll call W:

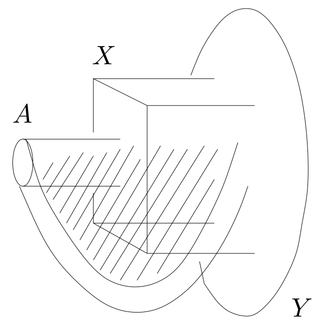


Unfortunately W is clearly not homotopy equivalent to Y, and therefore not homotopy equivalent to our double mapping cylinder above. But we can fix this as follows.

There is an evident map $A \times \partial\Delta^2$ into W: we have an $A \times I$ occuring in the mapping cylinders for f and gf, forming two of the 'sides' of $A \times \partial\Delta^2$. The third side comes from the composite $A \times I \xrightarrow{f \times id} X \times I \to W$, where the second map is the cylinder part of the mapping cylinder for g. What we will do is take W and attach a copy of $A \times \Delta^2$ along the image of $A \times \partial\Delta^2$; that is, we form the pushout

$$\begin{tikzcd} A \times \partial\Delta^2 \arrow[r] \arrow[d, tail] & W \arrow[d, dotted] \\ A \times \Delta^2 \arrow[r, dotted] & W'. \end{tikzcd}$$

It is hard to draw a picture for W', but maybe we can try something like this:



This new space W' is homotopy equivalent to the double mapping cylinder we started with: the cylinder corresponding to gf can be squeezed down into the double mapping cylinder, via the $A \times \Delta^2$ piece we just attached. So W' is another model for the homotopy colimit of our diagram

2.5. **Summary.** The previous example suggests the following. Suppose given a small category I and a diagram $D\colon I \to \mathcal{T}op$. To construct $\operatorname{hocolim} D$ we should start with $\amalg_i D(i)$, and then for every map $f\colon i \to j$ in I we should glue in a cylinder $D(i) \times \Delta^1$ corresponding to f. Then for every pair of composable maps

$$i \xrightarrow{f} j \xrightarrow{g} k$$

in I we should glue in a copy of $D(i) \times \Delta^2$. Continuing the evident pattern, for every sequence of n composable maps

$$i_0 \to i_1 \to i_2 \to \cdots \to i_n$$

we should glue in a copy of $D(i_0) \times \Delta^n$. The problem is to figure out how to keep track of all this gluing in an efficent way! We'll begin developing the techniques for this in the next section.

3. Simplicial spaces

Before giving a general construction of homotopy colimits we need some preliminary machinery.

Let Δ be the cosimplicial indexing category: the objects are the finite ordered sets $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$, and the maps are the monotone increasing functions. Note that there is an inclusion $\Delta \hookrightarrow \mathcal{T}op$ which sends $[n]$ to Δ^n and sends a map $\sigma \colon [n] \to [k]$ to the corresponding linear map $\Delta^n \to \Delta^k$ which coincides with σ on the vertices of Δ^n. Sometimes we will blur the distinction between Δ and this subcategory of $\mathcal{T}op$ which is its image; in fact, historically the category Δ first arose *as this subcategory*—the description in terms of ordered sets is really just a modern convenience.

If $\mathcal{C}$ is any category, a simplicial object in $\mathcal{C}$ is a functor $X \colon \Delta^{op} \to \mathcal{C}$. This is commonly drawn as a diagram consisting of spaces $X_n = X([n])$ together with 'face' and 'degeneracy' maps between them:

$$\cdots \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; X_2 \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; X_1 \;\substack{\longrightarrow\\[-2pt]\longrightarrow}\; X_0\ .$$

The face maps decrease dimension, and the degeneracies increase dimension; we will usually not draw the degeneracies, for typographical reasons. A cosimplicial object in $\mathcal{C}$ is a functor $Z \colon \Delta \to \mathcal{C}$, which is a similar diagram with all the arrows going in the other direction.

3.1. **Geometric realization.** Suppose $X \colon \Delta^{op} \to \mathcal{T}op$ is a simplicial space. The *geometric realization* of X is the space

$$|X| = \operatorname{coeq}\left[\coprod_{[n]\to[k]} X_k \times \Delta^n \rightrightarrows \coprod_n X_n \times \Delta^n\right]. \tag{3.2}$$

This is a 'coequalizer', which is just another name for a colimit of a diagram consisting of two parallel arrows: so the coequalizer of two arrows $f, g \colon S \rightrightarrows T$ is the quotient space $T/\sim$ in which one identifies $f(s) \sim g(s)$ for all $s \in S$.

To finish explaining the formula in (3.2), we should mention that the first coproduct in the coequalizer is taken over all maps in Δ. If $\sigma \colon [n] \to [k]$ is a map in Δ then there are two evident maps from $X_k \times \Delta^n$ into $\coprod_i X_i \times \Delta^i$. The first sends $X_k \times \Delta^n$ to $X_n \times \Delta^n$ via the map $\sigma^* \colon X_k \to X_n$, and the second sends $X_k \times \Delta^n$ to $X_k \times \Delta^k$ via the map $\sigma_* \colon \Delta^n \to \Delta^k$. This gives the two parallel maps in the coequalizer diagram.

A little thought shows that the above formula for $|X|$ can also be written as

$$|X| = \left(\coprod_n X_n \times \Delta^n\right) \Big/ \sim$$

where the equivalence relation has

$$(d_i x, t) \sim (x, d^i t) \qquad \text{and} \qquad (s_i x, t) \sim (x, s^i t).$$

Here the d_i and s_i are the face and degeneracy maps in X, whereas the d^i and the s^i are the coface and codegeneracy maps in the cosimplicial object $\Delta \to \mathcal{T}op$ sending $[n] \mapsto \Delta^n$.

Remark 3.3. Note that if each X_n is a discrete space then we can regard X as a functor $\Delta^{op} \to \mathcal{S}et$ and the above construction is the same as the usual geometric realization of a simplicial set.

3.4. **Homotopy invariance of geometric realization.** By a map of simplicial spaces $X \to Y$ we mean a natural transformation of functors. Such a map is said to be an **objectwise weak equivalence** if $X_n \to Y_n$ is a weak equivalence of spaces, for all n. It is not quite true that if $X \to Y$ is an objectwise weak equivalence of simplicial spaces then $|X| \to |Y|$ is a weak equivalence of spaces. At about the same time, Segal [Se] and May [M] independently developed conditions under which this *is* true. We will describe a modern version of such conditions next.

If $s_i\colon X_{n-1} \to X_n$ is a degeneracy map, $0 \leq i \leq n-1$, then note that one of the simplicial identities is $d_i s_i = id$; so X_{n-1} is a retract of X_n. We then have that s_i is injective, and a point-set-topology argument shows that the topology on X_{n-1} coincides with the subspace topology on its image. So s_i is an inclusion. If X_n is Hausdorff (which is necessarily true if X_n is cofibrant), more point-set topology shows that s_i is in fact a closed inclusion.

Define the $\boldsymbol{n}$**th latching object** of X to be the subspace

$$L_n X = \bigcup_{i=0}^{n-1} s_i(X_{n-1}) \subseteq X_n.$$

The inclusion $L_n X \hookrightarrow X_n$ is called the $\boldsymbol{n}$**th latching map**.

The first few latching spaces are easy to picture: $L_0 X = \emptyset$, $L_1 X \cong X_0$, and $L_2 X \cong X_1 \amalg_{X_0} X_1$. These spaces get more complicated as n grows. For instance, $L_3 X$ consists of three copies of X_2 glued together along three copies of X_1, all containing a single copy of X_0.

A simplicial space X is called **Reedy cofibrant** if the latching maps $L_n X \to X_n$ are cofibrations, for all n. If X is Reedy cofibrant then each X_n is cofibrant, by an induction starting with the fact that the 0th latching map is $\emptyset \to X_0$.

Theorem 3.5. *Suppose $X \to Y$ is an objectwise weak equivalence between two simplicial spaces, both of which are Reedy cofibrant. Then $|X| \to |Y|$ is also a weak equivalence.*

Sketch of proof. Let $\mathrm{Sk}_n |X|$ denote the subspace of $|X|$ defined by

$$\mathrm{Sk}_n |X| = \mathrm{coeq}\left[\coprod_{\substack{[k]\to[l]\\ k,l\leq n}} X_l \times \Delta^k \rightrightarrows \coprod_{k \leq n} X_k \times \Delta^k \right].$$

Then there is a sequence of closed inclusions

$$\mathrm{Sk}_0 |X| \hookrightarrow \mathrm{Sk}_1 |X| \hookrightarrow \mathrm{Sk}_2 |X| \hookrightarrow \cdots$$

and the colimit is $|X|$. One shows that there are pushout squares

$$\begin{tikzcd} (L_n X \times \Delta^n) \amalg_{(L_n X \times \partial\Delta^n)} (X_n \times \partial\Delta^n) \arrow[r] \arrow[d] & \mathrm{Sk}_{n-1} |X| \arrow[d] \\ X_n \times \Delta^n \arrow[r] & \mathrm{Sk}_n |X| \end{tikzcd}$$

for each n, and our assumption that X is Reedy cofibrant implies that the left vertical map is a cofibration.

Using that $X \to Y$ is an objectwise weak equivalence, one shows inductively that each $L_nX \to L_nY$ is a weak equivalence, and then that each $\mathrm{Sk}_n|X| \to \mathrm{Sk}_n|Y|$ is a weak equivalence. It then follows that $|X| \to |Y|$ is also a weak equivalence. □

Remark 3.6 (The fat realization)**.** Let X be a simplicial space. Define

$$||X|| = \mathrm{coeq}\left[\coprod_{[n]\hookrightarrow[k]} X_k \times \Delta^n \rightrightarrows \coprod_n X_n \times \Delta^n. \right]$$

where the left coproduct runs over all *injections* in Δ. Note that this definition completely ignores the degeneracy maps in the simplicial space X. The space $||X||$ is called the *fat realization* of X.

The disadvantage of $||X||$ over $|X|$ is that the former space is always much bigger and more complicated—in fact, it is always infinite-dimensional! For instance, suppose X is the simplicial space consisting of one point in every dimension. Then $|X|$ is just a point, but $||X||$ is a space consisting of one 0-cell, one 1-cell, one 2-cell, etc. This is because the degenerate stuff in X hasn't been collapsed, as it was in $|X|$.

The advantage of $||X||$ over $|X|$ is that this fat construction preserves weak equivalences under much weaker hypotheses. If $X \to Y$ is an objectwise weak equivalence between simplicial spaces which are cofibrant in each dimension, then $||X|| \to ||Y||$ is a weak equivalence. We will see a proof of this in Example 9.15 below.

3.7. **Collapsing the geometric realization.** One often thinks of the $X_n \times \Delta^n$ pieces in $|X|$ as 'higher homotopies'. Consider the process of collapsing them, in which one shrinks every Δ^n to a point. Thus, we consider the diagram

$$\begin{tikzcd} \coprod_{[n]\to[k]} X_k \times \Delta^n \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \coprod_{[n]} X_n \times \Delta^n \arrow[d] \\ \coprod_{[n]\to[k]} X_k \arrow[r, shift left] \arrow[r, shift right] & \coprod_{[n]} X_n \end{tikzcd}$$

where the vertical maps come from the projections $X_k \times \Delta^n \to X_k$ and $X_n \times \Delta^n \to X_n$. The coequalizer of the bottom two arrows is precisely $\mathrm{colim}_{\Delta^{op}} X$. Thus, we have a natural map

$$|X| \to \mathrm{colim}\, X.$$

Now, $\mathrm{colim}\, X$ can be identified with the coequalizer of the first two face maps $d_0, d_1 \colon X_1 \to X_0$. This is an exercise for the reader; clearly there is a map $\mathrm{coeq}(X_1 \rightrightarrows X_0) \to \mathrm{colim}\, X$, and one can prove using the simplicial identities that any map $X_0 \to Z$ which coequalizes $d_0, d_1 \colon X_1 \to X_0$ actually induces a map $\mathrm{colim}\, X \to Z$. Thus, one gets a map $\mathrm{colim}\, X \to \mathrm{coeq}(X_1 \rightrightarrows X_0)$, and one readily sees that the two compositions are the identities. (See also Example 21.1 below).

Putting everything together, we have shown that there is a natural map

$$|X| \to \mathrm{coeq}\Big[X_1 \rightrightarrows X_0\Big].$$

Remark 3.8. Note that if X is a simplicial *set* then this coequalizer is just $\pi_0(X)$, the set of path components. In this case our map is just the usual one from $|X|$ to its set of path components (equipped with the discrete topology).

3.9. **Degenerate simplicial spaces.** A simplicial space X is **degenerate in dimension q and above** if the maps $L_kX \to X_k$ are homeomorphisms for all $k \geq q$. It follows that the spaces X_k, $k \geq q$, all get collapsed inside of $|X|$. The reason is that if $x \in X_k$ then $x = s_{i_1}s_{i_2}\dots s_{i_r}y$ for some $y \in X_{q-1}$ (where $r = k-q+1$). So for any $t \in \Delta^k$ we have

$$(x,t) = (s_{i_1}\dots s_{i_r}y, t) \sim (y, s^{i_r}\cdots s^{i_1}t)$$

in $|X|$. A little thought shows that in this case we can write

$$|X| = \mathrm{Sk}_q\,|X| = \mathrm{coeq}\left[\coprod_{\substack{[n]\to[k]\\ n,k\leq q}} X_k \times \Delta^n \rightrightarrows \coprod_{n\leq q} X_n \times \Delta^n\right].$$

This observation simplifies the process of computing $|X|$ in many cases, and we will use it in the next sections when faced with some specific examples.

3.10. **Contracting homotopies.** Suppose X_* is a simplicial *set* and $*$ is a 0-simplex of X. A *contracting homotopy* for X is a collection of combinatorial data which will guarantee that $|X|$ deformation-retracts down to $*$. So we need to deform each n-simplex of X down to a point, and the deformations for different simplices need to be compatible. The easiest way to accomplish this is to specify the following data:

- For each 0-simplex a of X, a 1-simplex $S(a)$ connecting a to $*$;
- For each 1-simplex b of X, a 2-simplex $S(b)$ whose base is b, whose remaining vertex is $*$, and whose 'sides' are the 1-simplices previously specified;
- And so on—for each n-simplex c of X we will need an $(n+1)$-simplex whose base is c, whose remaining vertex is $*$, and whose sides coincide with previously specified data.

A contracting homotopy for X will therefore be a collection of maps $S\colon X_n \to X_{n+1}$ which are required to satisfy some identities. These identities will take a different form depending on whether we want the simplices $S(a)$ to point *towards* the simplex $*$ or *away from* the simplex $*$. We will differentiate these cases by calling them "sinklike" and "sourcelike" contracting homotopies, respectively (reflecting whether the vertex $*$ acts like a sink or source).

Before giving the formal definition it will be useful to generalize somewhat. By an **augmented simplicial set** we mean a simplicial set X together with a set W and a map $X_0 \to W$ which coequalizes the two maps $X_1 \rightrightarrows X_0$. This is the same as having a map of simplicial sets $X \to cW$, where cW is the constant simplicial set having W in every dimension. A contracting homotopy for an augmented simplicial set $X_* \to W$ will be a map $W \to X_0$ such that $W \to X_0 \to W$ is the identity together with a way of deformation-retracting X_* down to the image of W in X_0.

Finally, we wish to generalize our discussion from simplicial *sets* to simplicial *spaces*. The basic formalism is the same, and in particular the definition of augmented simplicial *space* is the same.

Definition 3.11. *Let $X_* \to W$ be an augmented simplicial space. It will be convenient to define X_{-1} to be W, and to have the map $X_0 \to W$ be denoted by d_0. Then a* ***sinklike contracting homotopy*** *is a collection of maps $S\colon X_n \to X_{n+1}$*

for $n \geq -1$ such that for each $a \in X_n$ one has

$$d_i(Sa) = \begin{cases} S(d_i a) & \text{if } 0 \leq i \leq n \\ a & \text{if } i = n+1 \end{cases} \qquad \text{and} \qquad S(s_i a) = s_i(Sa) \text{ for } 0 \leq i \leq n.$$

*Likewise, a **sourcelike contracting homotopy** for X is a collection of maps $S\colon X_n \to X_{n+1}$ for $n \geq -1$ such that for each $a \in X_n$ one has*

$$d_i(Sa) = \begin{cases} a & \text{if } i = 0 \\ S(d_{i-1} a) & \text{if } 0 < i \leq n+1 \end{cases} \qquad \text{and} \qquad S(s_i a) = s_{i+1}(Sa) \text{ for } 0 \leq i \leq n.$$

Proposition 3.12. *Let $X_* \to W$ be an augmented simplicial space which admits either a sinklike or sourcelike contracting homotopy. Then $|X| \to W$ is a homotopy equivalence.*

Proof. An easy exercise, or see Appendix A. □

Example 3.13. Let X be the simplicial set Δ^n. The k-simplices of X are all the monotone increasing sequences of length $k+1$ taking values in $\{0, 1, \ldots, n\}$. We regard X as augmented by the one-point space, so we set $X_{-1} = \{*\}$; it is useful to think of the element of X_{-1} as the "empty sequence".

One can define a sourcelike contracting homotopy for X by having the contraction operator $S\colon X_n \to X_{n+1}$ send a sequence $a_0 \ldots a_n$ to the sequence $0a_0 \ldots a_n$. In other words, the contracting homotopy inserts a 0 at the beginning of every sequence. One can also define a sinklike contracting homotopy for X, by inserting an n at the *end* of every sequence.

Example 3.14. Let $f\colon X \to Y$ be a map of topological spaces, and consider the simplicial space $\check{C}(f)$ defined by

$$[n] \mapsto X \times_Y X \times_Y \cdots \times_Y X \qquad ((n+1) \text{ factors}).$$

If $(x_0, \ldots, x_n)$ is an element of $\check{C}(f)_n$, then the ith face map omits x_i and the jth degeneracy repeats x_j. This simplicial space is called the **Čech complex** of f. If we forget the topological structure then this is the nerve of a category, where there is one object for every element of X and a unique map between any two objects which have the same image under f.

We may regard $\check{C}(f)$ as being augmented by Y, via the map f. Suppose $s\colon Y \to X$ is a section of f. Define a sourcelike contracting homotopy for $\check{C}(X)$ by sending the point $(x_0, \ldots, x_n)$ to $(s(f(x_0)), x_0, \ldots, x_n)$. Note that one can also obtain a sinklike contracting homotopy by appending $s(f(x_n))$ to the *end* of the tuple. So if f admits a section then $|\check{C}(f)| \to Y$ is a homotopy equivalence.

Example 3.15. This example will not be needed until Part 2, but we include it here as a titillating exercise. Let $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ be adjoint functors between two categories. Recall that such a pair is equipped with natural transformations $LR(X) \to X$ and $Z \to RL(Z)$, which we'll refer to as 'contraction' and 'expansion'. These natural transformations have the property that the two composites $RX \to RLR(X) \to RX$ and $LZ \to LRLZ \to LZ$ (both obtained by first expanding and then contracting in the evident way) are the identities.

For each $X \in \mathcal{D}$ one can construct a simplicial object $B_{LR}(X)$ over $\mathcal{C}$ having the form

$$[n] \mapsto (LR)^{n+1}(X).$$

If the LR pairs in $B_{LR}(X)_n$ are labelled as 0 through n (left to right), then the face map d_i applies contraction to the ith LR pair; the jth degeneracy s_j applies an expansion between the L and R of the jth LR pair. Using only the facts stated in the previous paragraph, one may check that these face and degeneracy maps indeed satisfy the axioms for a simplcial object.

Note that the contraction map $LR(X) \to X$ provides an augmentation for $B_{LR}(X)$. The simplicial object $B_{LR}(X)$ is called the **bar construction** on X associated to the adjoint pair (L, R). The name comes from a historical precedent described in Example 3.17 below.

Now apply R levelwise to $B_{LR}(X)$ to obtain a simplicial object over $\mathcal{C}$. One can check that $RB_{LR}(X) \to RX$ admits a sourcelike contracting homotopy, where the map $S\colon R[B_{LR}(X)]_n \to R[B_{BL}(X)]_{n+1}$ is simply an expansion before the first R—that is, S is the map $Z \to RL(Z)$ where $Z = B_{LR}(X)_n$. It is routine to check that the necessary identities are satisfied.

Likewise, consider the case where $X = LA$. The augmented simplicial object $B_{LR}(LA) \to LA$ admits a sinklike contracting homotopy, where the map $B_{LR}(LA)_n \to B_{LR}(LA)_{n+1}$ inserts an expansion between the L and the A.

Exercise 3.16. Given a map of topological spaces $f\colon X \to Y$, there are adjoint functors

$$L\colon (\mathcal{T}op \downarrow X) \rightleftarrows (\mathcal{T}op \downarrow Y)\colon R$$

where L is composition with f and R is pullback along f. Check that the bar construction for LR, applied to the terminal object of $(\mathcal{T}op \downarrow Y)$, is $\check{C}(f)$. How do the contracting homotopies of Example 3.14 relate to the ones in Example 3.15?

Example 3.17. Let G be a finite group, and let $G\mathcal{T}op$ denote the category of G-spaces and equivariant maps. There are adjoint functors

$$\mathcal{T}op \underset{U}{\overset{F}{\rightleftarrows}} G\mathcal{T}op$$

where U is the functor that forgets the G-action and F is the free functor $F(Y) = G \times Y$. Note that the counit $FU(X) \to X$ of the adjunction is the action map $G \times X \to X$, and the unit $Y \to UF(Y)$ is the map $Y \to G \times Y$ given by $y \mapsto (e, y)$.

If X is a G-space then consider the simplicial space $B_{FU}(X)$ from Example 3.15. A little thought reveals that this is the simplicial space

$$\cdots \, G \times G \times G \times X \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; G \times G \times X \;\substack{\longrightarrow\\[-1ex]\longrightarrow}\; G \times X$$

where the face and degeneracy maps are described as follows. Write a tuple $(g_0, g_1, \ldots, g_n, x) \in G^{n+1} \times X$ as $g_0|g_1|g_2|\cdots|g_n|x$. If the vertical bars are indexed left to right, with the first bar having index 0, then d_i removes bar i and s_j inserts "$e|$" after bar j. The use of bars in the above notation is why this simplicial space is called the "bar construction". The element $g_0|g_1|g_2|\cdots|g_n|x$ is in some contexts denoted $[g_0|g_1|\cdots|g_n|x]$, $[g_0|g_1|\cdots|g_n]x$, or $g_0[g_1|\cdots|g_n]x$.

Write $E_\bullet(G, X) = B_{FU}(X)$ and $E(G, X) = |E_\bullet(G, X)|$. The latter is a G-space, and in fact the action is free: this follows immediately from the fact that the G-action on each level of $E_\bullet(G, X)$ is free. The augmentation $E_\bullet(G, X) \to X$ induces a natural G-equivariant map $E(G, X) \to X$. If we forget the G-action then the simplicial space $E_\bullet(G, X)$ has a contracting homotopy (as in Example 3.15) and so $E(G, X) \to X$ is a weak equivalence.

When $X = *$, the space $E(G,*)$ is usually just written as EG. It is a contractible space with a free G-action.

There are other models for the space EG. For any set S let $\pi_S\colon S \to *$ be the projection, and consider the simplicial set $\check{C}(\pi_S)$. This simplicial space depends functorially on S, and the realization $|\check{C}(\pi_S)|$ is contractible by Example 3.14. In particular, if S is a G-set then G acts on $\check{C}(pi_S)$ (diagonally in each level) and hence on $|\check{C}(pi_S)|$. When $S = G$ then the action is free in every level, and so $|\check{C}(\pi_G)|$ is a contractible space with a free G-action.

The simplicial spaces $\check{C}(\pi_G)$ and $E(G,*)$ are different, as one can readily check. But they are isomorphic: verify that the maps $G^n \to G^n$ given by

$$(g_0, g_1, \dots, g_{n-1}) \mapsto (g_0, g_0^{-1}g_1, g_1^{-1}g_2, \dots, g_{n-2}^{-1}g_{n-1})$$

give an isomorphism $\check{C}(\pi_G) \to E(G,*)$ of simplicial G-spaces (recall tha the G-action on $E(G,*)$ is via the leftmost G, whereas G acts diagonally on $\check{C}(\pi_G)$).

Finally, let us turn back to $E(G,X)$ for general G-spaces X. This is a simplicial G-space, free in every degree, whose realization is naturally equivalent to X. The space $E(G,*) \times X$ (with diagonal G-action on the product) is another such space: and of course they turn out to be isomorphic. Check that the maps $G^n \times X \to G^n \times X$ given by

$$(g_0, \dots, g_{n-1}, x) \to (g_0, \dots, g_{n-1}, g_0g_1 \cdots g_{n-1}x)$$

give an isomorphism $E(G,X) \to E(G,*) \times X$ of simplicial G-spaces.

The quotient $(EG \times X)/G$ is called the **Borel construction** on X, and it appears often in algebraic topology (for more about why, see Section 7). It is often written as $EG \times_G X$, and of course it is also $E(G,X)/G$. When X is a point the Borel construction is EG/G, and this is usually denoted BG. Note that there is a principal G-bundle $G \to E(G,X) \to E(G,X)/G$, and $E(G,X) \simeq X$.

4. Construction of homotopy colimits

Let I be a small category, and let $D\colon I \to \mathcal{T}op$ be a diagram. We will now explain how to construct the homotopy colimit of D (really we should say, "*a* homotopy colimit of D").

The **simplicial replacement** of D is the simplicial space

$$\coprod_{i_0} D(i_0) \Lleftarrow \coprod_{i_0 \leftarrow i_1} D(i_1) \Lleftarrow \coprod_{i_0 \leftarrow i_1 \leftarrow i_2} D(i_2) \Lleftarrow \cdots$$

We will denote this $\operatorname{srep}(D)$. So we have

$$\operatorname{srep}(D)_n = \coprod_{i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n} D(i_n)$$

where the coproduct ranges over chains of composable maps in I. We must define the face and degeneracy maps. If $\sigma = [i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n]$ is a chain and $0 \leq j \leq n$, then we can 'cover up' i_j and obtain a chain of $n-1$ composable maps—call this new chain $\sigma(j)$. When $j < n$, the map $d_j\colon \operatorname{srep}(D)_n \to \operatorname{srep}(D)_{n-1}$ sends the summand $D(i_n)$ corresponding to σ to the identical copy of $D(i_n)$ in $\operatorname{srep}(D)_{n-1}$ indexed by $\sigma(j)$. When $j = n$ we must modify this slightly, as covering up i_n now yields a chain that ends with i_{n-1}. So $d_n\colon \operatorname{srep}(D)_n \to \operatorname{srep}(D)_{n-1}$ sends the summand $D(i_n)$ corresponding to the chain σ to the summand $D(i_{n-1})$ corresponding to $\sigma(n)$, and the map we use here is the map $D(i_n) \to D(i_{n-1})$ induced by the last map in σ.

The degeneracy maps $s_j\colon \operatorname{srep}(D)_n \to \operatorname{srep}(D)_{n+1}$, $0 \leq j \leq n$, are a bit easier to describe. Each s_j sends the summand $D(i_n)$ corresponding to the chain $\sigma = [i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n]$ to the identical summand $D(i_n)$ corresponding to the chain in which one has inserted the identity map $i_j \leftarrow i_j$.

Example 4.1. The **nerve** of a small category I is the simplicial set NI which in dimension n consists of all strings $[i_0 \to i_1 \to \cdots \to i_n]$ of n composable arrows. The face map d_j corresponds to 'covering up' the object i_j, as above. The **classifying space** of I is the geometric realization of the nerve; it will be denoted BI.

The nerve of the opposite category I^{op} may be identified with the simplicial set which in dimension n consists of all strings $[i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n]$ of n composable arrows, where the face map d_j again corresponds to covering up the object i_j. This is very similar to the nerve of I, but not identical—the order of the faces and degeneracies have been reversed. These simplicial sets are not isomorphic, but they *are* naturally weakly equivalent.

Suppose $D\colon I \to \mathcal{T}op$ is the diagram for which $D(i) = *$ for all $i \in I$. Then $\operatorname{srep}(D)$ is just the nerve of the category I^{op}.

Remark 4.2. Note that we have made a choice when defining the simplicial replacement. We could have defined the nth object to be

$$\coprod_{i_0 \to i_1 \to \cdots \to i_n} D(i_0) \tag{4.3}$$

and again defined the degeneracy d_j to be the map associated to 'covering up' i_j. This is related to the distinction between the nerve of a category I and the nerve of its opposite. The simplicial space from (4.3) is not isomorphic to $\operatorname{srep}(D)$, although their geometric realizations are homeomorphic.

So there are two natural definitions of the simplicial replacement (as well as for the nerve of a category), and one is forced to choose. Our choices were made to agree with the conventions in [H].

It turns out to be useful to have *both* definitions around at the same time. They are brought together in the *two-sided bar construction* which we will talk about in Section 11.

Remark 4.4. Note that if each $D(i)$ is a cofibrant space, then the simplicial replacement is automatically Reedy cofibrant (cf. Section 3.4). This is because the nth latching object of $\mathrm{srep}(D)$ is just the subspace of $\mathrm{srep}(D)_n$ consisting of all summands corresponding to chains which have identity maps in them. So the latching object is just a summand inside the whole space, and the complementary summand is cofibrant (being a disjoint union of cofibrant spaces). Thus, $L_n(\mathrm{srep}(D)) \to \mathrm{srep}(D)_n$ is a cofibration.

Definition 4.5. *The* ***homotopy colimit*** *of a diagram $D\colon I \to \mathcal{T}op$ is the geometric realization of its simplicial replacement. That is,*

$$\operatorname{hocolim} D = |\,\mathrm{srep}(D)|.$$

Sometimes we will write $\operatorname{hocolim}_I D$ *to remind us of the indexing category.*

4.6. **Homotopy invariance of the homotopy colimit.**

Proposition 4.7. *If $D, D'\colon I \to \mathcal{T}op$ are two diagrams consisting of cofibrant objects and $\alpha\colon D \to D'$ is a natural weak equivalence, then the induced map* $\operatorname{hocolim} D \to \operatorname{hocolim} D'$ *is a weak equivalence.*

Proof. We get a map of simplicial spaces $\mathrm{srep}(D) \to \mathrm{srep}(D')$, and this is an objectwise weak equivalence. Since $\mathrm{srep}(D)$ and $\mathrm{srep}(D')$ are both Reedy cofibrant, it follows from Theorem 3.5 that the induced map of realizations is also a weak equivalence. □

Remark 4.8. Note that we could have instead defined $\operatorname{hocolim} D$ to be $\|\,\mathrm{srep}(D)\|$. That is, we could have used the fat realization instead of the usual geometric realization. This would still give a homotopy invariant construction, and would be weakly equivalent to the definition of $\operatorname{hocolim} D$ adopted above. This is further demonstration that there is not really a *single* homotopy colimit construction; there are many such constructions, all weakly equivalent to each other.

Remark 4.9 (Cofibrancy assumptions)**.** Proposition 4.7 is perhaps weaker than one would hope for, because of the cofibrancy conditions on the objects of D and D'. There are two things to say about this. In a general model category, to get the 'correct' homotopy colimit of a diagram D one should first arrange things so that all the objects are cofibrant—for instance, by applying a cofibrant-replacement functor to all the objects of D. Then one can apply specific formulas for the hocolim, such as the one above.

In the category $\mathcal{T}op$, though, an 'accident' occurs, in that the cofibrancy conditions on the objects are not necessary at all! That is to say, Proposition 4.7 is true even without these conditions. A proof can be found in [DI, Appendix]. We will tend to ignore this, however, and continue to state results with the objectwise cofibrant hypotheses in them. This is because we want to state the results so that they generalize to other model categories.

4.10. **The natural map from the homotopy colimit to the colimit.** Note that $\operatorname{colim} D$ is the coequalizer of d_0 and d_1 in $\operatorname{srep}(D)$: that is, it is the quotient space $[\amalg_i D(i)]/\sim$ where for every map $\sigma\colon i \to j$ in I we identify points $x \in D(i)$ with $\sigma_*(x) \in D(j)$. The canonical map

$$|\operatorname{srep}(D)| \to \operatorname{coeq}\Big[\operatorname{srep}(D)_1 \rightrightarrows \operatorname{srep}(D)_0\Big]$$

from Section 3.7 therefore can be written as a map $\operatorname{hocolim} D \to \operatorname{colim} D$.

Example 4.11. Let us return to our most basic example, where I is the pushout category and D is a diagram $X \xleftarrow{f} A \xrightarrow{g} Y$. The simplicial replacement has $X \amalg A \amalg Y$ in dimension 0, and $X \amalg A \amalg A \amalg Y$ in dimension 1; everything in dimensions 2 and higher is degenerate. So by the discussion in Section 3.9, when forming $|\operatorname{srep}(D)|$ we only have to pay attention to the spaces in dimensions 0 and 1.

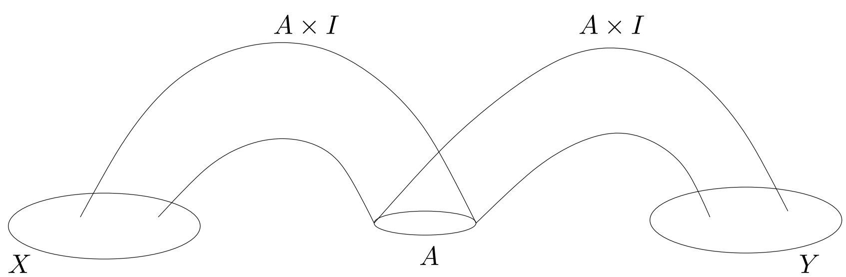
It is perhaps better to write $\operatorname{srep}(D)_1 = X_{id} \amalg A_f \amalg A_g \amalg Y_{id}$, where we are now keeping track of the maps in I indexing the summands (thus, "A_f" is the copy of A indexed by the map f). We see that the X and Y are degenerate, and a little thought shows that $|\operatorname{srep}(D)|$ is the quotient space

$$[X \amalg A \amalg Y \amalg (A_f \times \Delta^1) \amalg (A_g \times \Delta^1)]/\sim$$

in which the following identifications are made:

(1) $(a,0) \in A_f \times \Delta^1$ is identified with $f(a) \in X$, whereas $(a,1) \in A_f \times \Delta^1$ is identified with $a \in A$.
(2) $(a,0) \in A_g \times \Delta^1$ is identified with $g(a) \in Y$, whereas $(a,1) \in A_f \times \Delta^1$ is identified with $a \in A$.

We thus get something like the following picture (but where the two cylinders do not really intersect except at their ends):



Note that this is homeomorphic to the space from Example 2.2.

Exercise 4.12. Work through the definition of $\operatorname{hocolim} D$ when D is the diagram $A \to X \to Y$, and check that it is homeomorphic to the space W' from our example in Section 2.4.

4.13. **A different formula.** Here is another formula for the homotopy colimit. Although it looks quite different at first, the space it describes is homeomorphic to that of our previous definition (we will explain why below). The new formula is:

$$\operatorname*{hocolim}_I D = \operatorname{coeq}\Bigg[\coprod_{i\to j} D_i \times B(j\downarrow I)^{op} \rightrightarrows \coprod_i D_i \times B(i\downarrow I)^{op}\Bigg]. \tag{4.14}$$

There are a few things to say about this formula. If $\mathcal{C}$ is a category, then $B\mathcal{C}$ is its classifying space—the geometric realization of its nerve. And $\mathcal{C}^{op}$ denotes the opposite category. The *op*'s are needed in the above formula only to make it conform with the choices we made in defining the simplicial replacement. The category $(i \downarrow I)$ is the undercategory of i, defined dually to the overcategories described in Section 1.3. Finally, if $i \to j$ is a map in I then there is an evident induced map of categories $(j \downarrow I) \to (i \downarrow I)$, and this is being used in one of the maps from our coequalizer diagram.

The formula in (4.14) gives a more direct comparison between the homotopy colimit and the ordinary colimit. The colimit is, after all, the coequalizer

$$\operatorname{colim}_I D = \operatorname{coeq}\left[\coprod_{i\to j} X_i \rightrightarrows \coprod_i X_i\right].$$

One finds a map from the previous coequalizer diagram to this one simply by collapsing the spaces $B(i \downarrow I)^{op}$ to a point; thus, one gets the map $\operatorname{hocolim} D \to \operatorname{colim} D$.

Below we will prove rigorously that the space defined in (4.14) is homeomorphic to the space $|\operatorname{srep}(D)|$, but let us pause to explain the general idea. In constructing $|\operatorname{srep}(D)|$, for every chain $i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n$ we have added a copy of $D_{i_n} \times \Delta^n$. So if we fix a particular spot D_i of the diagram, this means that we are adding a copy of $D_i \times \Delta^n$ for every string $i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_{n-1} \leftarrow i$. Such a string gives an n-simplex in $B(i \downarrow I)^{op}$, corresponding to the chain

$$[i, i_0 \leftarrow i] \leftarrow [i, i_1 \leftarrow i] \leftarrow \cdots \leftarrow [i, i_{n-1} \leftarrow i] \leftarrow [i, i \leftarrow i]$$

(which is a chain in $(i \downarrow I)$). In the formula (4.14) we are simply grouping all these $D_i \times \Delta^n$'s together—fixing i and letting n vary—into the space $D_i \times B(i \downarrow I)^{op}$. In other words, the space $B(i \downarrow I)^{op}$ is parameterizing all the 'D_i-homotopies' that are being added into the homotopy colimit.

Here is a simple example:

Example 4.15. Consider again the case where I is the pushout category $1 \leftarrow 0 \to 2$ and D is a diagram $X \leftarrow A \to Y$. Then $(1 \downarrow I)$ and $(2 \downarrow I)$ are both the trivial category with one object, whereas $(0 \downarrow I)$ is the category $a \leftarrow b \to c$ (isomorphic to I again). So $B(0 \downarrow I)$ is the space consisting of two intervals joined at one endpoint:

The above formula says

$$\operatorname{hocolim}_I D = \Big[X \amalg \Big(A \times B(0 \downarrow I)^{op}\Big) \amalg Y\Big] \Big/ \sim$$

and one checks that the quotient relations give the same space we saw in Example 4.11.

If one is willing to learn some more machinery, there is a very slick proof that our two formulas for hocolim D are naturally homeomorphic. We give this in Section 11. For the moment we will be content with an argument which is more longwinded, but requires less background.

Consider the following big diagram:

$$\begin{tikzcd}
\cdots \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \cdots \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \cdots \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
\coprod_{i,k_0\leftarrow k_1\leftarrow j\leftarrow i} X_i \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \coprod_{i,j_0\leftarrow j_1\leftarrow i} X_i \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \coprod_{j_0\leftarrow j_1} X_{j_1} \arrow[d, shift left] \arrow[d, shift right] \\
\coprod_{i,k_0\leftarrow j\leftarrow i} X_i \arrow[r, shift left] \arrow[r, shift right] & \coprod_{i,j_0\leftarrow i} X_i \arrow[r] & \coprod_{j_0} X_{j_0}
\end{tikzcd}$$

Each column is a simplicial space. The rightmost column is $\operatorname{srep}(X)$, the middle column is $\coprod_i (X_i \times N(i\downarrow I)^{op})$, and the leftmost column is $\coprod_{i\to j}(X_i \times N(j\downarrow I)^{op})$.

We have a map of simplicial spaces from the middle column to the right column. In degree n this sends the summand X_i indexed by the string $[j_0 \leftarrow j_1 \leftarrow \cdots j_n \leftarrow i]$ to the summand X_{j_n} indexed by $[j_0 \leftarrow \cdots \leftarrow j_n]$, via the map $X_i \to X_{j_n}$ induced by $i \to j_n$. This is clearly compatible with face and degeneracies.

We have *two* maps of simplicial spaces from the left column to the middle column. In simplicial degree n, one map sends the summand X_i indexed by the string $[i, k_0 \leftarrow k_1 \leftarrow \cdots \leftarrow k_n \leftarrow j \leftarrow i]$ to the summand X_i indexed by the string $[i, k_0 \leftarrow \cdots \leftarrow k_n \leftarrow i]$ (forget about j). The other map sends our summand X_i to the summand X_j indexed by $[j, k_0 \leftarrow \cdots \leftarrow k_n \leftarrow j]$ (forget about i).

Now, it is easy to check that each horizontal level of our diagram is a coequalizer diagram; that is to say, the objects in the right column are the coequalizers of the objects in the other two columns. Geometric realization is a left adjoint, and therefore will commute with coequalizers. So this identifies $|\operatorname{srep}(D)|$ with the coequalizer of

$$\coprod_{i\to j} |X_i \times N(j\downarrow I)^{op}| \rightrightarrows \coprod_i |X_i \times N(i\downarrow I)^{op}|.$$

Finally, observe that if K is a simplicial set than $X \times |K|$ can be identified with the geometric realization of the simplicial space

$$[n] \mapsto X \times K_n = \coprod_{K_n} X$$

(use the fact that both $|-|$ and $X \times (-)$ are left adjoints, therefore they commute). So the above coequalizer can instead be written as

$$\coprod_{i\to j} X_i \times |N(j\downarrow I)^{op}| \rightrightarrows \coprod_i X_i \times |N(i\downarrow I)^{op}|$$

and this completes the argument.

5. Homotopy limits and some useful adjunctions

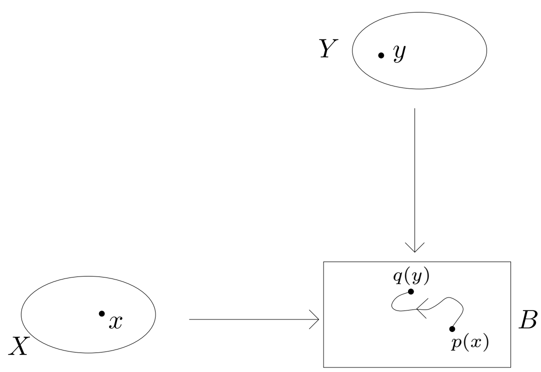
We have not yet talked about homotopy *limits*. The story is completely dual to that for homotopy colimits, the main difference being that the pictures are not quite as easy to draw. We will just outline the basic constructions, accentuating the small differences.

Example 5.1. We again start with the most basic example, generalizing slightly the notion of a homotopy fiber. Let I be the pullback category $1 \to 0 \leftarrow 2$, and let $D\colon I \to \mathcal{T}op$ be a diagram $X \xrightarrow{p} B \xleftarrow{q} Y$. A point in the pullback $X \times_B Y$ consists of a point $x \in X$ and a point $y \in Y$ such that $p(x) = q(y)$. A point in the homotopy pullback will consist of a point $x \in X$, a point $y \in Y$, and a *path* from $p(x)$ to $q(y)$.

Formally, we define $\operatorname{holim} D$ to be the pullback of the diagram

$$\begin{tikzcd} & B^I \arrow[d] \\ X \times Y \arrow[r, "p\times q"] & B \times B \end{tikzcd}$$

where B^I is the space of maps $\gamma\colon I \to B$ and $B^I \to B$ sends γ to $(\gamma(0), \gamma(1))$. It is sometimes useful to depict a point in $\operatorname{holim} D$ via a picture like the following:



Note that if $X \xrightarrow{p} B$ is a map and $* \in B$ is a basepoint, then the homotopy fiber of p, as classicaly defined, is just the homotopy pullback of the diagram

$$X \longrightarrow B \longleftarrow *.$$

Generally speaking, if I is any indexing category and $D\colon I \to \mathcal{T}op$ is a diagram, then a point in $\lim D$ consists of points in each $D(i)$ which 'match up' as you move around the diagram. A point in $\operatorname{holim} D$ will consist of points in each $D(i)$, *together with* paths connecting their images as you move around the diagram, as well as 'higher homotopies' connecting the paths, and paths of paths, etc. It is a bit hard to describe, but here is one more example.

Example 5.2. Consider a diagram D of the form $A \xrightarrow{f} X \xrightarrow{g} Y$. A point in $\operatorname{holim} D$ will consist of points $a \in A$, $x \in X$, $y \in Y$, together with the following extra data. First, we need a path α from $f(a)$ to x, a path β from $g(x)$ to y, and a path γ from $g(f(a))$ to y. Applying g to α gives a path from $g(f(a))$ to $g(x)$, and so now we have a map $\partial\Delta^2 \to Y$ consisting of the three paths $g(\alpha)$, β, and γ.

Finally, we also require a map $\Delta^2 \to Y$ extending our map $\partial\Delta^2 \to Y$. This is a 'higher homotopy'.

5.3. **Tot of a cosimplicial space.** A cosimplicial space is a functor $X\colon \Delta \to \mathcal{T}op$, drawn as follows:

$$X_0 \rightrightarrows X_1 \substack{\rightarrow\\\rightarrow\\\rightarrow} X_2 \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots$$

(and here we are omitting the codegeneracy maps for typographical reasons). Let Δ^* denote the cosimplicial space corresponding to the standard inclusion $\Delta \hookrightarrow \mathcal{T}op$. As a cosimplicial space, Δ^* is

$$\Delta^0 \rightrightarrows \Delta^1 \substack{\rightarrow\\\rightarrow\\\rightarrow} \Delta^2 \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots$$

If X is any cosimplicial space we can talk about the space of maps from Δ^* to X: the points are the natural transformations $\Delta^* \to X$, and they are topologized as a subspace of $\prod_n X_n^{\Delta^n}$. This space of maps is sometimes denoted $\mathrm{Map}(\Delta^*, X)$, but is more commonly denoted $\mathrm{Tot}\, X$. It is called the **totalization** of X, or usually just "Tot of X", for short. We can also describe it as an equalizer:

$$\mathrm{Tot}\, X = \mathrm{eq}\Bigg[\prod_n X_n^{\Delta^n} \rightrightarrows \prod_{[n]\to[k]} X_k^{\Delta^n}\Bigg].$$

The two maps in the equalizer can be defined as follows, using that any map $\sigma\colon [n] \to [k]$ induces a corresponding map $\sigma_*\colon \Delta^n \to \Delta^k$. Given a sequence of elements $s_n \in X_n^{\Delta^n}$, one of our maps sends this to the collection $\sigma \mapsto s_k \circ \sigma_* \in X_k^{\Delta^n}$. The other map sends the sequence s_n to the collection $\sigma \mapsto X(\sigma) \circ s_n \in X_k^{\Delta^n}$, where $X(\sigma)$ is the induced map $X_n \to X_k$.

In words, a point in $\mathrm{Tot}\, X$ consists of a point $x_0 \in X_0$, an edge x_1 in X_1, a 2-simplex x_2 in X_2, and so on, which are compatible in the following two ways:

(1) The two images of x_0 under $X_0 \rightrightarrows X_1$ are the two endpoints of x_1; the three images of x_1 under the maps $d^0, d^1, d^2\colon X_1 \to X_2$ are the three faces of the 2-simplex x_2; and so on.
(2) The image of x_1 under the codegeneracy $X_1 \to X_0$ is the map $\Delta^1 \to X_0$ collapsing everything to x_0; the image of x_2 under the two codegeneracies $X_2 \rightrightarrows X_1$ are the two maps $\Delta^2 \rightrightarrows \Delta^1 \xrightarrow{x_1} X_1$, etc.

There doesn't seem to be a particularly simple way to think about all this! Usually I think of a point in $\mathrm{Tot}\, X$ as being a point $x_0 \in X_0$ plus an edge connecting its two images in X_1, plus a 2-simplex connecting the three images of this edge in X_2, and so on, with the proviso that all this data must be compatible under the codegeneracies.

Note that there is an evident map $\mathrm{eq}(X_0 \rightrightarrows X_1) \to \mathrm{Tot}\, X$ defined as follows. If $x_0 \in X_0$ is equalized by the two maps to X_1, then we can choose our 1-simplex x_1 in X_1 to be constant. Then we can also choose our 2-simplex in X_2 to be constant, and so on down the line. All of these choices are automatically compatible under codegeneracies, so we get a point in $\mathrm{Tot}\, X$.

5.4. **Reedy fibrancy.** It is not true that if $X \to Y$ is an objectwise weak equivalence between cosimplicial spaces then $\mathrm{Tot}\, X \to \mathrm{Tot}\, Y$ is a weak equivalence. It *is* true if X and Y satisfy some conditions, which we now explain.

Let X be a cosimplicial space and let $a \in X_n$. Applying the codegeneracy maps to a gives an n-tuple $(s^0a, s^1a, \ldots, s^{n-1}a) \in (X_{n-1})^n$. This is not an arbitrary

n-tuple, as the cosimplicial identities give us some relations among the coordinates. If we relabel this n-tuple as $(x_0, \dots, x_{n-1})$, we find that $s^i x_i = s^i x_{i+1}$ for each i in the range $0 \le i \le n-2$. The ***n*th matching object** of X is the subspace of all n-tuples satisfying these relations; that is,

$$M_n X = \{(y_0, y_1, \dots, y_{n-1}) \in (X_{n-1})^n \mid s^i y_i = s^i y_{i+1} \text{ for } 0 \le i \le n-2\}.$$

The map $X_n \to M_n X$ sending a to $(s^0 a, \dots, s^{n-1} a)$ is called the ***n*th matching map**.

Definition 5.5. *A cosimplicial space is* ***Reedy fibrant*** *if the associated matching maps $X_n \to M_n X$ are fibrations, for all $n \ge 0$.*

Proposition 5.6. *Let $X \to Y$ be an objectwise weak equivalence between cosimplicial spaces, each of which is Reedy fibrant. Then $\operatorname{Tot} X \to \operatorname{Tot} Y$ is a weak equivalence of spaces.*

Proof. See [BK, ????]. □

5.7. **Construction of homotopy limits.** Let I be a small category and let $D\colon I \to \mathcal{T}op$ be a diagram. The **cosimplicial replacement** of D is the cosimplicial space $\operatorname{crep}(D)$ defined as

$$\operatorname{crep}(D)_n = \prod_{i_0 \to i_1 \to \cdots \to i_n} D(i_n).$$

The cofaces and codegeneracies are the evident ones, defined analogously to the case of simplicial replacements.

The cosimplicial replacement of a diagram is always Reedy fibrant, provided that the diagram was objectwise fibrant (which is always true in $\mathcal{T}op$, since all spaces are fibrant). So one defines the homotopy limit of D by

$$\operatorname{holim} D = \operatorname{Tot}[\operatorname{crep}(D)].$$

It readily follows from Proposition 5.6 that this construction is homotopy invariant.

The equalizer of $\operatorname{crep}(D)_0 \rightrightarrows \operatorname{crep}(D)_1$ is just $\lim D$; a point in this equalizer consists of choices of points in each D_i which are compatible as one moves around the diagram. The natural map from this equalizer into $\operatorname{Tot}(\operatorname{crep}(D))$ gives us a natural map $\lim D \to \operatorname{holim} D$.

Just as for homotopy colimits, we can describe $\operatorname{holim} D$ via another formula—this time an equalizer formula:

$$\operatorname{holim} D \cong \operatorname{eq}\left[\prod_i X_i^{B(I\downarrow i)} \rightrightarrows \prod_{i \to j} X_j^{B(I\downarrow i)}\right].$$

5.8. **Adjunctions.** If $D\colon I \to \mathcal{T}op$ and $X \in \mathcal{T}op$, there is a useful adjunction formula

$$\mathcal{T}op(\operatorname*{colim}_I D, X) \cong \lim_I \mathcal{T}op(D(i), X).$$

Here $\mathcal{T}op(A, B)$ denotes the set of maps from A to B in the category $\mathcal{T}op$. The formula just says that giving a map $\operatorname{colim} D \to X$ is the same as giving a bunch of maps $D(i) \to X$ which are compatible as i changes. There is a similar formula

$$\mathcal{T}op(A, \lim_I D) \cong \lim_I \mathcal{T}op(A, D(i))$$

which has an analogous interpretation.

When generalizing to homotopy limits and colimits, the difference is that one replaces the set of maps in $\mathcal{T}op$ with the mapping space $\operatorname{Map}(X,Y)$ (also denoted X^Y) . For this we need to assume we are working in a 'good' category of spaces where the mapping space is a true right adjoint (like the category of compactly-generated spaces). We then have natural isomorphisms

$$\operatorname{Map}(\operatorname{hocolim}_I D, X) \to \operatorname{holim}_{I^{op}} \operatorname{Map}(D(i), X) \tag{5.9}$$

and

$$\operatorname{Map}(A, \operatorname{holim}_I D) \to \operatorname{holim}_I \operatorname{Map}(A, D(i)). \tag{5.10}$$

We will only explain the map in (5.9), as the other one is similar. Using the description of $\operatorname{hocolim} D$ from (4.14), we have maps

$$\begin{array}{c}
\operatorname{Map}(\operatorname{hocolim} D, Z) \\
\big\downarrow \cong \\
\operatorname{eq}\Big[\operatorname{Map}\Big(\coprod_i D_i \times B(i\downarrow I)^{op}, Z\Big) \rightrightarrows \operatorname{Map}\Big(\coprod_{i\to j} D_i \times B(j\downarrow I)^{op}, Z\Big)\Big] \\
\big\downarrow \cong \\
\operatorname{eq}\Big[\prod_i \operatorname{Map}\Big(D_i \times B(i\downarrow I)^{op}, Z\Big) \rightrightarrows \prod_{i\to j} \operatorname{Map}\Big(D_i \times B(j\downarrow I)^{op}, Z\Big)\Big] \\
\big\downarrow \cong \\
\operatorname{eq}\Big[\prod_i \operatorname{Map}\Big(D_i, Z\Big)^{B(i\downarrow I)^{op}} \rightrightarrows \prod_{i\to j} \operatorname{Map}\Big(D_i, Z\Big)^{B(j\downarrow I)^{op}}\Big] \\
\big\downarrow \cong \\
\operatorname{eq}\Big[\prod_i \operatorname{Map}\Big(D_i, Z\Big)^{B(I^{op}\downarrow i)} \rightrightarrows \prod_{i\to j} \operatorname{Map}\Big(D_i, Z\Big)^{B(I^{op}\downarrow j)}\Big] \\
\Big\| \\
\operatorname{holim}_{I^{op}} \operatorname{Map}(D(i), Z).
\end{array}$$

In the first two maps we are using that $\operatorname{Map}(-,Z)$ takes colimits to limits, which follows from the adjointness properties. The third map just uses the adjunction, and in the fourth map we have used the identification $(i\downarrow I)^{op} = (I^{op}\downarrow i)$.

6. Changing the indexing category

As mentioned briefly in Section 2.3, one is often in the situation of wanting to prove that the homotopy colimits of two different diagrams are weakly equivalent. There are a variety of techniques for this, and we will describe a few in this section. Unfortunately, the proofs of these results require more technology than is yet at our disposal—so we will defer the proofs until Section 10.

Let $\alpha\colon I \to J$ be any functor between small categories. Then given any diagram $X\colon J \to \mathcal{T}op$, one obtains a new diagram $\alpha^* X\colon I \to \mathcal{T}op$ by $\alpha^* X = X \circ \alpha$. We wish to compare $\operatorname{hocolim}_J X$ with $\operatorname{hocolim}_I(\alpha^* X)$. In particular, under what conditions will they be weakly equivalent?

6.1. **The classical problem for colimits.** The corresponding problem in the case of ordinary colimits is probably familiar. There is a canonical map

$$\operatorname*{colim}_I(\alpha^* X) \to \operatorname*{colim}_J X$$

and one wants to know when this is an isomorphism. A common situation is that I is a subcategory of J, and one usual definition for I to be 'cofinal' in J is something like:

(1) For each $j \in J$, there is an $i \in I$ and a map $j \to i$.
(2) For any two parallel maps $j \rightrightarrows i$ where $i \in I$, there is a map $i \to i'$ in I such that the two composites $j \to i'$ are the same.

This is actually a special case of a much more general definition. Recall that for any $j \in J$, the undercategory $(j \downarrow \alpha)$ is the category whose objects are pairs $[i, f\colon j \to \alpha(i)]$ consisting of an object $i \in I$ and a map $f\colon j \to \alpha(i)$ in J. A map from $[i, f]$ to $[i', f']$ consists of a map $i \to i'$ in I making the diagram

$$\begin{tikzcd} j \arrow[r, "f"] \arrow[dr, "f'"'] & \alpha(i) \arrow[d] \\ & \alpha(i') \end{tikzcd}$$

commute.

Definition 6.2. *The functor $\alpha\colon I \to J$ is **terminal** (or **final**, or **left cofinal**) if for each $j \in J$ the undercategory $(j \downarrow \alpha)$ is non-empty and connected.*

Theorem 6.3. *If $\alpha\colon I \to J$ is terminal then for every diagram $X\colon J \to \mathcal{T}op$, the map $\operatorname{colim}_I(\alpha^* X) \to \operatorname{colim}_J X$ is an isomorphism.*

Proof. See [ML, Thm. IX.3.1]. □

Remark 6.4. There is a nice way to remember the above definition and theorem. One particularly simple case is when J has a terminal object w, and $I = \{w\}$ is the subcategory consisting of this single object. In this case it is clear that $\operatorname{colim}_J X$ should just be $X(w)$, which is $\operatorname{colim}_I(\alpha^* X)$.

The condition for being a terminal object is that the undercategories $(j \downarrow \{w\})$ are trivial categories consisting of one object and an identity map. This is a very special case of the connectedness condition above.

6.5. **Extension to the case of homotopy colimits.** Let $\alpha\colon I \to J$ be a functor between small categories. We first note that for any diagram $X\colon J \to \mathcal{T}op$ there is a natural map of simplicial spaces

$$\phi_\alpha\colon \operatorname{srep}(\alpha^*X) \to \operatorname{srep}(X).$$

In simplicial degree n this is the map

$$\coprod_{i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n} (\alpha^*X)(i_n) \longrightarrow \coprod_{j_0 \leftarrow j_1 \leftarrow \cdots \leftarrow j_n} X(j_n)$$

which sends the summand $(\alpha^*X)(i_n)$ indexed by the chain $[i_0 \leftarrow \cdots \leftarrow i_n]$ to the summand $X(\alpha(i_n))$ corresponding to the chain $[\alpha(i_0) \leftarrow \cdots \leftarrow \alpha(i_n)]$. Note that $(\alpha^*X)(i_n) = X(\alpha(i_n))$, and the map is really just the identity on these summands. This is clearly compatible with the face and degeneracy maps, and so gives a map of simplicial spaces.

Taking realizations gives us a natural map

$$\phi_\alpha\colon \operatorname*{hocolim}_I \alpha^*X \to \operatorname*{hocolim}_J X.$$

Definition 6.6. *The functor $\alpha\colon I \to J$ is* ***homotopy terminal*** *(or* ***homotopy final****, or* ***homotopy left cofinal****) if for each $j \in J$ the undercategory $(j \downarrow \alpha)$ is non-empty and contractible (meaning that its nerve is contractible).*

See Remark 6.13 for more about the above choices in terminology.

Theorem 6.7 (Cofinality Theorem)**.** *If α is homotopy terminal then for every diagram $X\colon J \to \mathcal{T}op$, the map $\operatorname{hocolim}_I(\alpha^*X) \to \operatorname{hocolim}_J X$ is a weak equivalence.*

Proof. See Section 10.6 for a complete proof, and Section 11 for a different perspective. □

There is one special case of Theorem 6.7 which we *will* prove now, both because the proof is simple and because we will need it later.

Lemma 6.8. *Suppose that J has a terminal object z. Then for every diagram $X\colon J \to \mathcal{T}op$, the map $\operatorname{hocolim}_J X \to \operatorname{colim}_J X \to X(z)$ is a weak equivalence.*

Proof. Consider the simplicial space $\operatorname{srep}(X)$. There is an evident augmentation $\operatorname{srep}(X) \to X(z)$, and we claim that this augmented simplicial space admits a sourcelike contracting homotopy (see Definition 3.11). The contraction operator $S\colon \operatorname{srep}(X)_n \to \operatorname{srep}(X)_{n+1}$ will send the summand $X(i_n)$ labelled by $[i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n]$ to the summand $X(i_n)$ labelled by $[z \leftarrow i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n]$. It is routine to check that this satisfies the identities for a contracting homotopy, and therefore by Proposition 3.12 we find that $|\operatorname{srep}(X)| \to X(z)$ is a homotopy equivalence. □

It is often useful to know how the maps ϕ_α behave under composition. Suppose now that $I_1 \xrightarrow{\alpha} I_2 \xrightarrow{\beta} I_3$ are two functors between categories, and that $X\colon I_3 \to \mathcal{T}op$ is a diagram. We have three natural maps of simplicial spaces, forming a triangle which is readily checked to commute:

$$\begin{tikzcd}
\operatorname{srep}[(\beta\alpha)^*X] \arrow[r, "\phi_\alpha"] \arrow[dr, "\phi_{\beta\alpha}"'] & \operatorname{srep}(\beta^*X) \arrow[d, "\phi_\beta"] \\
& \operatorname{srep}(X).
\end{tikzcd}$$

This yields a commutative triangle of homotopy colimits:

$$\begin{tikzcd}
\operatorname{hocolim}_{I_1}(\beta\alpha)^*X \arrow[r, "\phi_\alpha"] \arrow[dr, "\phi_{\beta\alpha}"'] & \operatorname{hocolim}_{I_2}\alpha^*X \arrow[d, "\phi_\beta"] \\
& \operatorname{hocolim}_{I_3} X.
\end{tikzcd}$$

Here is another result about changing the indexing category. Suppose again that $\alpha\colon I \to J$ is a functor and $X\colon J \to \mathcal{T}op$. For each $j \in J$, let $u_j\colon (\alpha \downarrow j) \to J$ be the map sending $[i, \alpha(i) \to j]$ to $\alpha(i)$. Notice that there is a canonical map

$$\operatorname*{colim}_{(\alpha\downarrow j)} u_j^* X \to X_j.$$

Theorem 6.9. *Let $\alpha\colon I \to J$ be a functor, and let $X\colon J \to \mathcal{T}op$. Suppose that for each $j \in J$ the composite map*

$$\operatorname*{hocolim}_{(\alpha\downarrow j)} u_j^* X \to \operatorname*{colim}_{(\alpha\downarrow j)} u_j^* X \to X_j$$

is a weak equivalence. Then the map $\operatorname{hocolim}_I \alpha^* X \to \operatorname{hocolim}_J X$ is a weak equivalence.

Proof. See Section 10.6. □

6.10. **Dual results for homotopy limits.** Suppose $\alpha\colon I \to J$ and $X\colon J \to \mathcal{T}op$. There is a natural map of cosimplicial spaces

$$\operatorname{crep}(X) \to \operatorname{crep}(\alpha^* X),$$

and after taking Tot this gives a map $\alpha^*\colon \operatorname{holim}_J X \to \operatorname{holim}_I(\alpha^* X)$.

Definition 6.11. *The functor $\alpha\colon I \to J$ is **homotopy initial** (or **homotopy cofinal**, or **homotopy right cofinal**) if for each $j \in J$ the overcategory $(\alpha \downarrow j)$ is non-empty and contractible (meaning that its nerve is contractible).*

The following is the dual version of Theorem 6.7:

Theorem 6.12. *If α is homotopy initial then for every diagram $X\colon J \to \mathcal{T}op$, the map $\operatorname{holim}_J X \to \operatorname{holim}_I(\alpha^* X)$ is a weak equivalence.*

Remark 6.13. The terms 'final/cofinal' and—even worse—'left/right cofinal' are easily mixed up, and it is also easy to mix up which one goes with colimits and which one goes with limits. The terms 'homotopy initial' and 'homotopy terminal' are better in this regard, as they fit naturally with the notions of initial and terminal object.

If a category has a terminal object, it is easy to compute the homotopy *colimit*. The condition that a category I has a terminal object ω says something about the undercategories $(i \downarrow \omega)$ for each object i; likewise, the condition that a functor $\alpha\colon K \to I$ be homotopy terminal says something about the undercategories $(i \downarrow \alpha)$. So the adjective 'terminal' lets one remember how to connect all these concepts (and likewise for 'initial').

One also has the following analog of Theorem 6.9:

Theorem 6.14. *Let $\alpha\colon I \to J$ be a functor, and let $X\colon J \to \mathcal{T}op$. Suppose that for each $j \in J$ the composite map*

$$X_j \to \lim_{(j\downarrow\alpha)} u_j^* X \to \operatorname{holim}_{(j\downarrow\alpha)} u_j^* X$$

is a weak equivalence, where $u_j\colon (j\downarrow\alpha) \to J$ is the evident functor. Then the map $\operatorname{holim}_J X \to \operatorname{holim}_I \alpha^* X$ *is a weak equivalence.*

6.15. **Further techniques.** We give one more result related to changing the indexing category. We will only state the hocolim version; the holim version is entirely analogous.

Suppose that $\alpha, \alpha'\colon I \to J$ are two functors and $\eta\colon \alpha \to \alpha'$ is a natural transformation. If $X\colon J \to \mathcal{T}op$ then η induces a natural transformation $\eta_*\colon \alpha^* X \to (\alpha')^* X$. The following triangle commutes in the homotopy category $\operatorname{Ho}(\mathcal{T}op)$:

$$\begin{tikzcd}
\operatorname{hocolim}_I \alpha^* X \arrow[r, "\phi_\alpha"] \arrow[d, "\eta_*"'] & \operatorname{hocolim}_J X \\
\operatorname{hocolim}_I (\alpha')^* X. \arrow[ur, "\phi_{\alpha'}"'] &
\end{tikzcd}$$

Proposition 6.16. *Let $\alpha\colon I \to J$ be a functor between small categories, and let $X\colon J \to \mathcal{T}op$ be a diagram. Suppose that there is a functor $\beta\colon J \to I$ together with natural transformations $\eta\colon \alpha\beta \to id_J$ and $\theta\colon \beta\alpha \to id_I$ such that the following two conditions hold:*

(1) Applying X to the maps $\eta(j)\colon \alpha\beta(j) \to j$ yields weak equivalences, for all $j \in J$; and

(2) Applying $\alpha^* X$ to the maps $\theta(i)\colon \beta\alpha(i) \to i$ also yields weak equivalences, for all $i \in I$.

Then the induced map $\operatorname{hocolim}_I \alpha^* X \to \operatorname{hocolim}_J X$ *is a weak equivalence.*

Moreover, the same conclusion holds if there are zig-zags of natural transformations between $\beta\alpha$ and id_I, and between $\alpha\beta$ and id_J, provided each step in the zig-zags induces weak equivalences after applying X and $\alpha^* X$, respectively.

Proof. See [D, Proposition A.4]. □

7. A few examples

Part 2. A closer look

So far we have understood the homotopy colimit as a ‘fattened up’ version of the colimit. Whereas taking a colimit can be thought of as gluing objects together, taking a homotopy colimit amounts to indirectly gluing them together via homotopies and higher homotopies. We saw that this process can be described by a certain formula (the geometric realization of the simplicial replacement), which is not hard to describe but perhaps not so easy to manipulate.

In the next few sections we will take a closer look at this formulaic approach to homotopy colimits, and we will encounter several variations of the main idea. The ostensible goal will be to learn some clever techniques for manipulating these formulas, but along the way we will make discoveries which slowly take us further and further away from the formulaic perspective. In Part 3 we will then take up those discoveries from a more abstract point-of-view.

There is a central theme which drives most of what follows. Given a diagram $D\colon I \to \mathcal{T}op$, there is a way of constructing the homotopy colimit by first replacing D with an ‘equivalent’ (but nicer) diagram $QD\colon I \to \mathcal{T}op$ (having $QD_i \simeq D_i$ for each i) and then taking the ordinary colimit of QD. The diagram QD is in some sense a *resolution* of D, and this leads us to view the homotopy colimit as a derived functor of the colimit. When we first encounter this idea in Section 9 it might seem like there is not much content to it—we are just rewriting the old formula for the homotopy colimit in a different way. But the power of homological (or homotopical) algebra comes in realizing that one doesn’t have to use the same resolution every time; *any* nice enough resolution will do the job. So in the end this new way of looking at things will prove very useful.

Here is a good analogy to keep in mind (and it turns out to be more than just an analogy). In homological algebra, one could choose to define Tor and Ext groups by always using the standard resolution (also called the bar resolution) of a module. From a theoretical perspective this is perfectly reasonable, and in some ways very convenient, but it makes computations almost impossible. There are very few instances where one can get enough control over the standard resolution to successfully compute something. But one eventually realizes that Tor and Ext groups *are* computable, either by using smaller resolutions or techniques that involve patching more manageable pieces of information together. This is what remains in our journey through homotopy colimits: we haved defined them via a “standard resolution”, and this is enough to prove some basic properties, but we need different techniques for actually getting our hands on them.

This is probably belaboring the point, but I can’t resist one last analogy—again because it goes deeper than it first seems. Imagine teaching the theory of ordinary homology by writing down the singular chain complex on the first day of class. One can do this without too much trouble, and one can maybe even prove some basic results about this theory; but it is nearly impossible to make a computation based on the definition itself. The singular chain complex is huge, and the large size plays out opposing ways: good because it is easy to write down and convenient for proving basic properties, but bad in the sense of allowing for computations. One should think of our initial definition of homotopy colimits in similar terms.

8. Brief review of model categories

Model categories will weave their way in and out of the next few sections. They have proven themselves to be a valuable ally when dealing with derived functors and homotopical algebra.

We will not recall the notion of a model category here. The reader may consult [DwS], [H], or [Ho] for nice overviews. Suffice it to say that a model category $\mathcal{M}$ is a category equipped with three collections of maps—the cofibrations, fibrations, and weak equivalences—which are required to satisfy five basic axioms. A map is called a 'trivial cofibration' if it is both a cofibration and a weak equivalence, and similarly for 'trivial fibration'.

The basic examples are as follows:

(1) $\mathcal{T}op$, where the weak equivalences are weak homotopy equivalences, the fibrations are Serre fibrations, and the cofibrations are retracts of cellular inclusions.
(2) $s\mathcal{S}et$, where the weak equivalences are the maps which become weak homotopy equivalences after geometric realization. The fibrations are the Kan fibrations, and the cofibrations are the monomorphisms.
(3) $Ch_{\geq 0}(R)$, where R is a ring. This is the category of non-negatively graded chain complexes. We equip it with the so-called *projective* model category structure: the weak equivalences are the quasi-isomorphisms, the fibrations are maps which are surjective in positive dimensions, and the cofibrations are the monomorphisms which in each level are split with projective cokernel.
(4) $Ch_{\leq 0}(R)$, where R is a ring. This is the category of non-positively graded chain complexes (or cochain complexes, after re-indexing). Here we use the so-called *injective* model category structure: the weak equivalences are the quasi-isomorphisms, the cofibrations are the monomorphisms, and the fibrations are the surjections which in each level are split with injective kernel.

There are many other examples, for instance several different model categories of *spectra*.

8.1. **Quillen functors.** If $\mathcal{M}$ and $\mathcal{N}$ are two model categories, a **Quillen pair** is an adjoint pair

$$L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R$$

which satisfies the following two equivalent conditions:

(1) L preserves cofibrations and trivial cofibrations—that is so say, if f is a cofibration (resp. trivial cofibration) in $\mathcal{M}$ then $L(f)$ is a cofibration (resp. trivial cofibration) in $\mathcal{N}$.
(2) R preserves fibrations and trivial fibrations.

The most familiar example is the adjoint pair

$$|-|\colon s\mathcal{S}et \rightleftarrows \mathcal{T}op\colon \mathrm{Sing}$$

where $|-|$ is geometric realization and Sing is the functor which sends a space X to the simplicial set $[n] \mapsto \mathcal{T}op(\Delta^n, X)$.

One can prove that when (L, R) is a Quillen pair, L preserves weak equivalences between cofibrant objects and R preserves weak equivalences between fibrant objects. The 'derived functor' of L applied to an object $A \in \mathcal{M}$ is obtained by choosing a weak equivalence $QA \to A$ in which QA is cofibrant, and then applying L to QA. If $Q'A \to A$ is another weak equivalence in which $Q'A$ is cofibrant, then the model category axioms show that there is a weak equivalence $QA \to Q'A$; thus,

$L(QA) \to L(Q'A)$ is also a weak equivalence. This tells us that the derived functor of L gives a well-defined homotopy type.

Similarly, the derived functor of R applied to an object $Z \in \mathcal{N}$ is obtained by choosing a weak equivalence $Z \to FZ$ in which FZ is fibrant, and then applying R to FZ.

8.2. **Simplicial model categories.** We will need this material only rarely in the following, but this is a reasonable place to quickly review it. A model category $\mathcal{M}$ is called **simplicial** if there are bifunctors

$$\otimes\colon s\mathcal{S}et \times \mathcal{M} \to \mathcal{M}, \qquad \mathrm{Map}\colon \mathcal{M}^{op} \times \mathcal{M} \to s\mathcal{S}et, \qquad F\colon s\mathcal{S}et^{op} \times \mathcal{M} \to \mathcal{M}$$

satisfying the usual adjunctions

$$\mathrm{Map}(K \times X, Y) \cong \underline{s\mathcal{S}et}(K, \mathrm{Map}(X,Y)) \cong \mathrm{Map}(X, F(K,Y))$$

where $\underline{s\mathcal{S}et}(-,-)$ is the simplicial mapping space between two simplicial sets. Associativity and unital properties of $\otimes$ are also required, and these bifunctors need to be compatible with the model category structure in the sense that the following axiom is satisfied:

[SM7] For every cofibration $j\colon K \rightarrowtail L$ in $s\mathcal{S}et$ and every fibration $p\colon X \twoheadrightarrow Y$ in $\mathcal{M}$, the map

$$\mathrm{Map}(L,X) \to \mathrm{Map}(K,X) \times_{\mathrm{Map}(K,Y)} \mathrm{Map}(L,Y)$$

is a fibration. Moreover, it is a trivial fibration if either j or p is a weak equivalence.

The following two conditions are known to be equivalent to SM7:

(1) For any cofibrations $j\colon K \rightarrowtail L$ in $s\mathcal{S}et$ and $f\colon A \rightarrowtail B$ in $\mathcal{M}$, the induced map $(K \otimes B) \amalg_{(K\otimes A)} (L \otimes A) \to L \otimes B$ is a cofibration, and it is trivial if either j or f is so.
(2) For any cofibration $j\colon K \rightarrowtail L$ in $s\mathcal{S}et$ and fibration $p\colon X \twoheadrightarrow Y$ in $\mathcal{M}$, the map

$$F(L,X) \to F(K,X) \times_{F(K,Y)} F(L,Y)$$

is a fibration, and it is trivial if either j or p is.

Example 8.3. The model category $\mathcal{T}op$ becomes a simplicial model category via

$$K \otimes X = |K| \times X, \quad F(K,X) = X^{|K|}, \quad \text{and} \quad \mathrm{Map}(X,Y) = \mathrm{Sing}(Y^X)$$

where Sing is the usual singular complex functor $\mathrm{Sing}\colon \mathcal{T}op \to s\mathcal{S}et$.

The model category $s\mathcal{S}et$ is a simplicial model category where

$$K \otimes X = K \times X, \quad F(K,X) = \underline{s\mathcal{S}et}(K,X), \quad \text{and} \quad \mathrm{Map}(X,Y) = \underline{s\mathcal{S}et}(X,Y).$$

These are the two standard examples, and most other examples are model categories that are derived from $\mathcal{T}op$ or $s\mathcal{S}et$ in some way.

Remark 8.4. While not every model category can be given a simplicial structure, there is a certain sense in which every model category is *almost* simplicial. The Dwyer-Kan theory of framings gives a way of defining $K \otimes X$, $F(K,X)$, and $\mathrm{Map}(X,Y)$ in any model category, but things like the adjunction formula and the associativity of $\otimes$ only turn out to be true up to homotopy, not on the nose. See ??? for more information.

Remark 8.5. The theory of homotopy colimits and homotopy limits that we have developed so far adapts more or less verbatim to any simplicial model category. With only slightly more trouble, the theory of framings allows one to also extend the definitions to any model category. We will not need this for a while, but it is good to know right at the start that the theory really does work in a very great generality.

9. The derived functor perspective

In this section we explain a sense in which the homotopy colimit is the derived functor of the colimit functor. We also discuss a universal property (of sorts) enjoyed by the homotopy colimit.

Example 9.1. To motivate what follows, we return to our basic example of a pushout diagram $X \xleftarrow{f} A \xrightarrow{g} Y$. Recall that the homotopy pushout consists of a copy of X, a copy of Y, and a cylinder $A \times I$ in which the two ends of the cylinder have been glued to X and Y via the maps f and g. We can arrive at this construction in a different way, as follows.

Let $\mathrm{Cyl}(f)$ and $\mathrm{Cyl}(g)$ denote the mapping cylinders of f and g; for example, the former is the quotient space $[X \amalg (A \times I)]/\sim$ where $(a,0) \sim f(a)$. Let $i\colon A \hookrightarrow \mathrm{Cyl}(f)$ denote the inclusion $a \mapsto (a,1)$, and let $j\colon A \hookrightarrow \mathrm{Cyl}(g)$ be defined similarly. We have the new pushout diagram of the form $\mathrm{Cyl}(f) \xleftarrow{i} A \xrightarrow{j} \mathrm{Cyl}(g)$; let's call this new diagram QD. Note that there is a natural weak equivalence $QD \to D$ obtained by collapsing the cylinders, and that the colimit of QD is a model for $\mathrm{hocolim}\, D$.

To summarize, we have found the following prescription for constructing the homotopy colimit. First replace the diagram D by a new one QD in which one adds homotopies to the objects in a certain way, without affecting their homotopy type. Sometimes this is called 'fattening up' the diagram D. The homotopy colimit of D is then just the colimit of the new diagram QD.

9.2. **Construction of QX.** We will next explain how to adapt the above example to the general case. Let $X\colon I \to \mathcal{T}op$ be a diagram. Basically what we want to do is replace each object X_i with the homotopy colimit of all the objects in the diagram mapping *to* X_i. To say this precisely, for each $i \in I$ consider the overcategory $(I \downarrow i)$ and the forgetful functor $u_i\colon (I \downarrow i) \to I$ sending the pair $[j, j \to i]$ to j. Write

$$(QX)_i = \underset{(I\downarrow i)}{\mathrm{hocolim}}\, u_i^* X.$$

The category $(I \downarrow i)$ has a terminal object, namely the pair $[i, id\colon i \to i]$. So there are natural maps

$$X_i = (u_i^* X)([i, i \to i]) \to (QX)_i \to \underset{(I\downarrow i)}{\mathrm{colim}}\, u_i^* X \to X_i$$

and the composite is the identity. It follows from Lemma 6.8 that $(QX)_i \to X_i$ a weak equivalence.

Note that $(QX)_i$ is the realization of the following simplicial space:

$$\coprod_{i \leftarrow i_0} X(i_0) \Leftarrow \coprod_{i \leftarrow i_0 \leftarrow i_1} X(i_1) \Lleftarrow \coprod_{i \leftarrow i_0 \leftarrow i_1 \leftarrow i_2} X(i_2) \cdots \tag{9.3}$$

Now suppose that we have a map $f\colon i \to j$, and let u_f denote the functor $(I \downarrow i) \to (I \downarrow j)$ sending $[k, k \to i]$ to $[k, k \to j]$ (obtained by composing with f). This functor induces a map

$$(u_f)_*\colon (QX)_i = \underset{(I\downarrow i)}{\mathrm{hocolim}}\, u_i^* X \to \underset{(I\downarrow j)}{\mathrm{hocolim}}\, u_j^* X = (QX)_j.$$

At the simplicial level (9.3), this is just the map that composes with $i \to j$ in all the indexing strings. If $i \xrightarrow{f} j \xrightarrow{g} k$ are two maps in I then we have a commutative diagram

$$\begin{tikzcd} (I \downarrow i) \arrow[r, "u_f"] \arrow[dr, "u_{gf}"'] & (I \downarrow j) \arrow[d, "u_g"] \\ & (I \downarrow k) \end{tikzcd}$$

and therefore get a commutative triangle

$$\begin{tikzcd} (QX)_i \arrow[r, "(u_f)_*"] \arrow[dr, "(u_{gf})_*"'] & (QX)_j \arrow[d, "(u_g)_*"] \\ & (QX)_k. \end{tikzcd}$$

So QX is a new diagram $I \to \mathcal{T}op$.

The natural maps $\operatorname{hocolim}_{(I\downarrow i)} u_i^* X \to \operatorname{colim}_{(I\downarrow i)} u_i^* X \cong X_i$ compile to give a map of diagrams $QX \to X$. By the remarks above, this is an objectwise weak equivalence.

Remark 9.4. Note that we also have weak equivalences $X_i \to QX_i$ coming from the terminal object of $(I \downarrow i)$, but these are not compatible as i varies. That is, they do not assemble to give a map of diagrams $X \to QX$. See Example 9.1.

Our final claim is that $\operatorname{colim}_I(QX) \cong \operatorname{hocolim}_I X$. It is not so hard to just think about it and see that this must be true. We will be able to explain it better after a brief detour, though.

9.5. **Homotopy coherent maps and the universal property.** Suppose that $X, Y : I \to \mathcal{T}op$ are two diagrams. A map of diagrams $X \to Y$ consists of a collection of maps $X_i \to Y_i$ which are compatible as i varies. A **homotopy coherent** map $X \to Y$ consists of a collection of maps $X_i \to Y_i$ (which might not be compatible as i varies), *together with* the following data:

(1) For every map $i \to j$ in I, a homotopy $X_i \times \Delta^1 \to Y_j$ between the composites $X_i \to X_j \to Y_j$ and $X_i \to Y_i \to Y_j$.
(2) For every composable pair $i \to j \to k$, a map $X_i \times \Delta^2 \to Y_k$ whose restriction to $X_i \times \partial\Delta^1$ gives the three homotopies corresponding to the maps $i \to j$, $i \to k$, and $j \to k$.
(3) For every chain of n morphisms $i_0 \to i_1 \to \cdots \to i_n$, a map $X_{i_0} \times \Delta^n \to Y_{i_n}$ which extends previous data on the subspace $X_{i_0} \times \partial\Delta^n$.

Of course we have been very sloppy in writing down the third condition. We have also left out something: for any chain of maps containing an identity, the corresponding Δ^n-homotopy should be an appropriate degeneration of the Δ^{n-1}-homotopy associated to the smaller chain in which the identity map is omitted. That's quite a mouthful, and pretty intimidating to deal with.

A convenient way to be more rigorous is as follows. One can form a cosimplicial space $\operatorname{Map}(X, Y)$

$$\prod_i \operatorname{Map}(X_i, Y_i) \rightrightarrows \prod_{i_0 \to i_1} \operatorname{Map}(X_{i_0}, Y_{i_1}) \Rrightarrow \prod_{i_0 \to i_1 \to i_2} \operatorname{Map}(X_{i_0}, Y_{i_2}) \cdots$$

with the evident coface and codegeneracy maps, and a homotopy coherent map $X \to Y$ is precisely a point in Tot of this cosimplicial space. The maps from Δ^n

into the nth level are all the n-fold homotopies, compatibility with coface maps is condition (3) from above, and compatibility with codegeneracies is the wordy condition that we initially left out.

The cosimplicial space that appears above is called the **two-sided cobar construction**. We will explore it in much more detail in Section 12.

Remark 9.6. Let Z be a space. To give a map $\operatorname{colim} X \to Z$ is equivalent to giving a map of diagrams $X \to cZ$, where cZ is the constant diagram containing Z at every spot (and all identity maps). The reader may check that to give a map $\operatorname{hocolim} X \to Z$ is the same as giving a *homotopy coherent* map of diagrams $X \to cZ$. This can be thought of as the 'universal property' for homotopy colimits.

Let $\mathcal{T}op^I$ denote the category whose objects are functors $I \to \mathcal{T}op$, and where the maps are natural transformations. To distinguish these maps from the homotopy coherent maps, we will occasionally refer to them as "honest maps". If X and Y are I-diagrams and we write $X \to Y$, this will always denote an honest map.

Let $\operatorname{hc}(X, Y)$ denote the set of homotopy coherent maps from X to Y. Note that honest maps of diagrams $X' \to X$ and $Y \to Y'$ give maps $\operatorname{hc}(X, Y) \to \operatorname{hc}(X', Y)$ and $\operatorname{hc}(X, Y) \to \operatorname{hc}(X, Y')$ in the evident way.

Let QX denote the diagram constructed in the previous section. We claim that to give a map of diagrams $QX \to Y$ is the same as giving a homotopy coherent map from X to Y:

Proposition 9.7. *There is a natural bijection between* $\mathcal{T}op^I(QX, Y)$ *and* $\operatorname{hc}(X, Y)$.

Proof. This is just a matter of chasing through the definitions. If you are willing to wait for more machinery, a slick proof is given in Section 12 below. □

Corollary 9.8. *There is a natural isomorphism* $\operatorname{colim}_I QX \to \operatorname{hocolim}_I X$.

Proof. We first give a handwavy argument that nevertheless offers a bit of intuition. To give a map $\operatorname{colim}_I QX \to Z$ is to give, for each $i \in I$, maps $\operatorname{hocolim}_{(I\downarrow i)} X \to Z$ which are compatible as i varies. This is the same as giving, for each $i \in I$, a homotopy coherent map $X|_{(I\downarrow i)} \to (cZ)|_{(I\downarrow i)}$, which are again compatible as i varies. But clearly this is the same thing as just giving a homotopy coherent map $X \to cZ$! Since this is in turn the same as giving a map $\operatorname{hocolim}_I X \to Z$, it follows that $\operatorname{colim}_I QX \cong \operatorname{hocolim}_I X$.

A cleaner argument goes like this. For any space Z we have a sequence of natural bijections

$$\mathcal{T}op(\operatorname{colim} QX, Z) \cong \mathcal{T}op^I(QX, cZ) \cong \operatorname{hc}(X, cZ) \cong \mathcal{T}op(\operatorname{hocolim} X, Z),$$

where the middle bijection is from Proposition 9.7. This implies that $\operatorname{colim} QX \cong \operatorname{hocolim} X$. □

For future reference we make the following observation.

Proposition 9.9. *Let $E \to E'$ be an objectwise trivial fibration of I-diagrams in $\mathcal{T}op$, and let D be an objectwise cofibrant I-diagram. Then* $\operatorname{hc}(D, E) \to \operatorname{hc}(D, E')$ *is surjective.*

Proof. The proof is a straightforward induction. Suppose given a homotopy coherent map from D to E'. The maps $D_i \to E'_i$ lift to maps $D_i \to E_i$. Then for each map $i \to j$ in I we have a diagram

$$\begin{tikzcd} D_i \times \partial\Delta^1 \arrow[r] \arrow[d, tail] & E_j \arrow[d, two heads, "\sim"] \\ D_i \times \Delta^1 \arrow[r] & E'_j \end{tikzcd}$$

where the top horizontal map consists of the two composites $D_i \to E_i \to E_j$ and $D_i \to D_j \to E_j$. We get a lifting because the left vertical map is a cofibration and the right vertical map is a trivial fibration. When $i \to j$ is an identity map we choose the particular lift that is the constant homotopy.

For each composable pair $i \to j \to k$ in I we now have a diagram

$$\begin{tikzcd} D_i \times \partial\Delta^2 \arrow[r] \arrow[d, tail] & E_k \arrow[d, two heads, "\sim"] \\ D_i \times \Delta^2 \arrow[r] & E'_k \end{tikzcd}$$

and again we choose a lifting. If either $i \to j$ or $j \to k$ is an identity map then we choose the explicit lifting given by the constant homotopy.

Proceeding in this way, one inductively constructs a point in $\mathrm{hc}(D, E)$ that lifts the original point in $\mathrm{hc}(D, E')$. $\square$

9.10. **Model categories of diagrams.** Model categories provide a very useful way for understanding the derived functor perspective on homotopy colimits. It turns out that $\mathcal{T}op^I$ has a model category structure in which a map $X \to Y$ is a

(1) weak equivalence if and only if each $X_i \to Y_i$ is a weak equivalence, and
(2) a fibration if and only if each $X_i \to Y_i$ is a fibration.

The cofibrations are a bit awkward to describe, but they are the maps with the left-lifting-property with respect to the trivial fibrations. We will talk more about the cofibrant objects in Section ?????.

There are adjoint functors

$$\mathrm{colim}\colon \mathcal{T}op^I \rightleftarrows \mathcal{T}op\colon c$$

where c is the constant diagram functor. Clearly c preserves fibrations and trivial fibrations, so this is a Quillen pair. To compute the derived functor of colim applied to a diagram X, one first chooses a weak equivalence $\hat{X} \to X$ where $\hat{X}$ is cofibrant, and then $\mathbf{L}\,\mathrm{colim}(X)$ is just $\mathrm{colim}\,\hat{X}$. For this to be useful, we need to be able to find the cofibrant model $\hat{X}$:

Proposition 9.11. *If $X\colon I \to \mathcal{T}op$ is objectwise cofibrant then $QX \to X$ is a cofibrant-replacement in $\mathcal{T}op^I$.*

Proof. We have already remarked that $QX \to X$ is an objectwise weak equivalence, so we just need to prove that QX is cofibrant. Let $W \to Z$ be an objectwise trivial fibration. Then the map $\mathcal{T}op^I(QX, W) \to \mathcal{T}op^I(QX, Z)$ is isomorphic to $\mathrm{hc}(X, W) \to \mathrm{hc}(X, Z)$. By Proposition 9.9, this is surjective; so we have a lift $QX \to W$ as desired. We summarize this: $\square$

Note that since $\mathrm{colim}\, QX \cong \mathrm{hocolim}\, X$, we have now identified hocolim with the derived functor of colim.

Remark 9.12. Below it will be useful to have a name for a construction parallel to QX. Namely, if $X\colon I \to \mathcal{T}op$ is a diagram let

$$\operatorname*{hocolim'}_I X = \operatorname{coeq}\left[\coprod_{i\to j} X_i \times B(j\downarrow I) \rightrightarrows \coprod_i X_i \times B(i\downarrow I)\right].$$

This is the same formula as (4.14), but without the "op" symbols on the overcategories; it can also be described as the realization of the simplicial space

$$\coprod_{i_0} X(i_0) \Lleftarrow \coprod_{i_0\to i_1} X(i_0) \Lleftarrow \coprod_{i_0\to i_1\to i_2} X(i_0) \Lleftarrow \cdots$$

Clearly $\operatorname{hocolim'}_I X$ and $\operatorname{hocolim}_I X$ are isomorphic, but this hinges on the fact that reversing the order of the face and degeneracy maps in a simplicial spaces yields an isomorphic geometric realization. At the simplicial level, pre-realization, the two constructions are somewhat different, although they are clearly "doing the same thing".

Let us now define $Q'X$ to be the diagram

$$i \mapsto \operatorname*{hocolim'}_{(I\downarrow i)}(u_i^* X).$$

Repeating the arguments from above, one finds when X is objectwise cofibrant that $Q'X$ is also a cofibrant-replacement for X in $\mathcal{T}op^I$. Note that if $*$ denotes the constant diagram $I \to \mathcal{T}op$ whose value is a single point, then $Q'(*)$ is the diagram $i \mapsto B(I\downarrow i)$ whereas $Q(*)$ is the diagram $i \mapsto B(I\downarrow i)^{op}$.

9.13. **Tensor products of diagrams.** Suppose $X\colon I \to \mathcal{T}op$ and $\Omega\colon I^{op} \to \mathcal{T}op$ are given diagrams. The tensor product $X \otimes \Omega$ is defined to be

$$X \otimes \Omega = \operatorname{coeq}\left[\coprod_{i\to j} X_i \times \Omega_j \rightrightarrows \coprod_i X_i \times \Omega_i\right].$$

This kind of construction is called a **coend**, and we have seen it several times already.

Example 9.14.

(a) A simplicial space is a functor $X\colon \Delta^{op} \to \mathcal{T}op$. If $j\colon \Delta \hookrightarrow \mathcal{T}op$ is the canonical functor, then $|X|$ is just $X \otimes j$.

(b) Recall that $\Delta_f \hookrightarrow \Delta$ is the subcategory consisting of the face inclusions. If $X'\colon \Delta_f^{op} \to \mathcal{T}op$ denotes the restriction of X to Δ_f^{op}, and $j'\colon \Delta_f \to \mathcal{T}op$ is the restriction of j, then $||X||$ is $X' \otimes j'$.

(c) Let $X\colon I \to \mathcal{T}op$ be a diagram, and let $\Omega\colon I^{op} \to \mathcal{T}op$ be the diagram such that $\Omega(i) = *$ for each i. Then $X \otimes \Omega \cong \operatorname{colim} X$.

(d) Let $X\colon I \to \mathcal{T}op$ be a diagram. Let $B(-\downarrow I)^{op}\colon I^{op} \to \mathcal{T}op$ denote the functor $i \mapsto B(i\downarrow I)^{op}$. Then $\operatorname{hocolim}_I X \cong X \otimes B(-\downarrow I)^{op}$.

If we fix X, the functor $X \otimes (-)$ has a nice adjointness property. Namely, it is the left adjoint to the functor $\mathcal{T}op \to \mathcal{T}op^{I^{op}}$ which sends Z to the diagram $i \mapsto \mathcal{T}op(X_i, Z)$. We will call this functor $\operatorname{Hom}(X,-)$. Our adjoint pair is therefore

$$X \otimes (-)\colon \mathcal{T}op^{I^{op}} \rightleftarrows \mathcal{T}op\colon \operatorname{Hom}(X,-).$$

Assuming X is objectwise cofibrant, then $\operatorname{Hom}(X,-)$ takes fibrations to objectwise fibrations, and trivial fibrations to objectwise trivial fibrations. So the above is a Quillen pair. One useful consequence is that the left adjoint preserves weak equivalences between cofibrant objects.

Let B_I denote the I^{op}-diagram $i \mapsto B(i \downarrow I)^{op}$. There is of course a map $B_I \to *$, and this is an objectwise weak equivalence because each category $(i \downarrow I)$ has an initial object and is therefore contractible. What's more, B_I is actually cofibrant in $\mathcal{T}op^{I^{op}}$. This is because B_I is none other than the diagram $Q'(*)$, where $*$ denotes the constant I^{op}-diagram consisting of a point in every spot (see Remark 9.12 for Q'). That is to say, for each $i \in I$ one has

$$Q'(*)_i = B(I^{op} \downarrow i) \cong B(i \downarrow I)^{op}.$$

The second isomorphism is canonical, and so gives an isomorphism of diagrams $B_I \cong Q'(*)$. The latter is cofibrant by Proposition 9.11 (the version that applies to Q' rather than Q).

So now we understand the formula for the homotopy colimit from another perspective: it came from taking a cofibrant approximation to $*$ in $\mathcal{T}op^{I^{op}}$, and then tensoring this with our given diagram. But model category theory now tells us that we could have used *any* cofibrant approximation to $*$, and we would have gotten something weakly equivalent (since any two cofibrant approximations are weakly equivalent, and $X \otimes (-)$ preserves weak equivalences between cofibrant objects). This is useful for obtaining other models for homotopy colimits.

Example 9.15. Recall that Δ_f denotes the subcategory of Δ consisting only of inclusions. Let $D\colon \Delta_f \to \mathcal{T}op$ denote the diagram $[n] \mapsto \Delta^n$, obtained by restricting the canonical diagram $\Delta \to \mathcal{T}op$. The map $D \to *$ is obviously an objectwise weak equivalence, and we claim additionally that D is cofibrant in $\mathcal{T}op^{\Delta_f}$. This is easy to see because if $X \xrightarrow{\sim} Y$ is an objectwise trivial fibration and $D \to Y$ is a map, then one can inductively product a lifting $D \to X$.

So D and B_{Δ_f} are both cofibrant replacements for the constant diagram $*$ in $\mathcal{T}op^{\Delta_f}$. They are therefore weakly equivalent. If $X\colon \Delta_f^{op} \to \mathcal{T}op$ is any objectwise cofibrant diagram then $X \otimes (-)$ is a left Quillen functor, and so we conclude that $X \otimes D$ is weakly equivalent to $X \otimes B_{\Delta_f}$. That is to say, $||X||$ is weakly equivalent to $\operatorname{hocolim}_{\Delta_f} X$.

This is one way to justify our claim—from way back in Remark 3.6—that if $X \to Y$ is an objectwise weak equivalence between objectwise cofibrant simplicial spaces, then $||X|| \to ||Y||$ is necessarily a weak equivalence.

10. More on changing the indexing category

We discuss relative homotopy colimits (also called homotopy left Kan extensions), and use these to revisit the problem of changing the indexing category. Combining these new ideas with the techniques from the last section, we will be able to give proofs of two results we skipped in Part 1.

10.1. **Relative homotopy colimits.** Let $\alpha\colon I \to J$ be a functor. Denote by $\alpha^*\colon \mathcal{T}op^J \to \mathcal{T}op^I$ the functor sending a diagram $X\colon J \to \mathcal{T}op$ to the composition $I \to J \to \mathcal{T}op$. We call the functor α^* 'restriction along α'. It has a left adjoint called the **relative colimit** or **left Kan extension**, denoted $\operatorname{colim}_{I\to J}$ or $\operatorname{colim}_\alpha$. In the case where $J = *$, the trivial category, this is the usual colimit functor.

There is a simple formula for $\operatorname{colim}_{I\to J}(A)$, where A is in $\mathcal{T}op^I$. Namely, it is the diagram in $\mathcal{T}op^J$ given by

$$j \mapsto \operatorname*{colim}_{(\alpha\downarrow j)}(u_j^* A)$$

where $u_j\colon (\alpha \downarrow j) \to I$ is the forgetful functor. This is a simple exercise using the universal property of colimits.

The adjoint pair

$$\operatorname{colim}_{I\to J}\colon \mathcal{T}op^I \rightleftarrows \mathcal{T}op^J : \alpha^*$$

is a Quillen pair, as the right adjoint α^* clearly preserves fibrations and trivial fibrations. Given $A\colon I \to \mathcal{T}op$ one defines the **relative homotopy colimit** (or homotopy left Kan extension) to be the J-diagram given by

$$\operatorname*{hocolim}_{I\to J} A = \operatorname*{colim}_{I\to J} QA.$$

Observe that this is the derived functor of $\operatorname{colim}_{I\to J}$.

We can also give a more explicit description of the relative homotopy colimit:

Proposition 10.2. *Fix $\alpha\colon I \to J$. Then for $A\colon I \to \mathcal{T}op$, $\operatorname{hocolim}_{I\to J} A$ is the J-diagram*

$$j \mapsto \operatorname*{hocolim}_{(\alpha\downarrow j)}(u_j^* A).$$

Proof. Let j be an object in J. Notice that

$$[\operatorname*{hocolim}_{I\to J} A]_j = [\operatorname*{colim}_{I\to J} QA]_j = \operatorname*{colim}_{(\alpha\downarrow j)} u_j^*(QA) \quad \text{and} \quad \operatorname*{hocolim}_{(\alpha\downarrow j)} u_j^* A = \operatorname*{colim}_{(\alpha\downarrow j)} Q(u_j^* A).$$

So it suffices to prove that $Q(u_j^* A) = u_j^*(QA)$. The former is the $(\alpha \downarrow j)$-diagram sending

$$[i, \alpha(i) \to j] \mapsto \operatorname*{hocolim}_{(\alpha\downarrow j)\downarrow[i,\alpha(i)\to j]} u_j^* A.$$

One readily checks that the category $\big((\alpha \downarrow j) \downarrow [i, \alpha(i) \to j]\big)$ may be identified with $(I \downarrow i)$, and so we are looking at the diagram

$$[i, \alpha(i) \to j] \mapsto \operatorname*{hocolim}_{(I\downarrow i)} u_i^* A.$$

But this is just $u_j^*(QA)$, so we are done. $\square$

10.3. **Changing the indexing category.** Let $\alpha\colon I\to J$ be a functor. Our next goal will be to relate $\operatorname{hocolim}_I \alpha^*X$ to relative homotopy colimits; this will then allow us to prove the Cofinality Theorem. Let $\alpha^{op}\colon I^{op}\to J^{op}$ be the associated functor of opposite categories, and let $B(-\downarrow\alpha)^{op}$ denote the diagram $J^{op}\to \mathcal{T}op$ sending $j\mapsto B(j\downarrow\alpha)^{op}$.

The following proposition uses the functor Q' discussed in Remark 9.12.

Proposition 10.4. *For any $\alpha\colon I\to J$ one has:*

(a) $B(-\downarrow\alpha)^{op}\cong \operatorname*{colim}_{I^{op}\to J^{op}} Q'(*)$, where $*$ is the constant I^{op}-diagram whose value is $*$.

(b) $B(-\downarrow\alpha)^{op}$ is cofibrant in $\mathcal{T}op^{J^{op}}$.

(c) For any $X\colon I\to\mathcal{M}$, there is a natural isomorphism $\operatorname{hocolim}_I X\cong X\otimes_{I^{op}} Q'(*)$.

(d) There is a natural isomorphism

$$X\otimes B(-\downarrow\alpha)^{op}\cong \operatorname*{hocolim}_I \alpha^*X.$$

Proof. For part (a) we begin by applying Proposition 10.2—or more precisely, the analogous result where every hocolim is relaced with hocolim'. This tells us that for every object j in J,

$$\Big[\operatorname*{colim}_{I^{op}\to J^{op}} Q'(*)\Big]_j\cong \operatorname*{hocolim'}_{(\alpha^{op}\downarrow j)} *\cong B(\alpha^{op}\downarrow j)=B(j\downarrow\alpha)^{op}.$$

Part (b) is an immediate consequence of (a), since $Q'(*)$ is cofibrant in $\mathcal{T}op^{I^{op}}$ (see Remark 9.12) and $\operatorname{colim}_{I^{op}\to J^{op}}$ is a left Quillen functor.

Part (c) is just a restatement of things we have seen before: (4.14) says that $\operatorname{hocolim}_I X\cong X\otimes_{I^{op}} B(-\downarrow I)$, and $B(-\downarrow I)\cong Q'(*)$ by Remark 9.12.

Finally, part (d) is an argument with adjunctions. For all spaces Z we have

$$\begin{aligned}\mathcal{T}op\Big(X\otimes B(-\downarrow\alpha)^{op},Z\Big)&\cong \mathcal{T}op\Big(X\otimes \operatorname*{colim}_{\alpha^{op}} Q'(*),\,Z\Big)\\&\cong \mathcal{T}op^{J^{op}}\Big(\operatorname*{colim}_{\alpha^{op}} Q'(*),\,\operatorname{Hom}(X,Z)\Big)\\&\cong \mathcal{T}op^{I^{op}}\Big(Q'(*),\,\alpha^*\operatorname{Hom}(X,Z)\Big)\\&= \mathcal{T}op^{I^{op}}\Big(Q'(*),\,\operatorname{Hom}(\alpha^*X,Z)\Big)\\&= \mathcal{T}op\big(\alpha^*X\otimes Q'(*),\,Z\big).\end{aligned}$$

Since these isomorphisms are natural and hold for all spaces Z, it follows that

$$X\otimes B(-\downarrow\alpha)^{op}\cong\alpha^*X\otimes Q'(*).$$

But by (c) the object on the right is precisely $\operatorname{hocolim}_I\alpha^*X$. ◻

Remark 10.5. Note that in part (a) we could also have written

$$B(-\downarrow\alpha)^{op}\cong \operatorname*{hocolim'}_{I^{op}\to J^{op}} *.$$

10.6. **Proof of the cofinality theorem.** We can now give two of the proofs we skipped over in Part 1.

Proof of Theorem 6.7. First note that if $\alpha\colon I\to J$ is a functor then there is a map of J^{op}-diagrams

$$B(-\downarrow\alpha)^{op}\to B(-\downarrow J)^{op}.$$

So for any diagram $X\colon J \to \mathcal{T}op$ there is an induced map

$$X \otimes B(- \downarrow \alpha)^{op} \to X \otimes B(- \downarrow J)^{op}.$$

The right object is $\operatorname{hocolim}_J X$, and by Proposition 10.4(d) the left object is $\operatorname{hocolim}_I \alpha^* X$. One checks that the above is the natural map $\operatorname{hocolim}_I \alpha^* X \to \operatorname{hocolim}_J X$.

Suppose now that $\alpha\colon I \to J$ is homotopy terminal. This means that for all objects j in J, the space $B(j \downarrow \alpha)$ is contractible. So the map of J^{op}-diagrams

$$B(- \downarrow \alpha)^{op} \to B(- \downarrow J)^{op}$$

is an objectwise weak equivalence, since both diagrams are objectwise contractible. As $X \otimes (-)$ is a left Quillen functor, it necessarily preserves weak equivalences between cofibrant objects. So

$$X \otimes B(- \downarrow \alpha)^{op} \to X \otimes B(- \downarrow J)^{op}$$

is a weak equivalence of spaces, which is what we wanted. □

Proof of Theorem 6.9. Recall that $\alpha\colon I \to J$, $X\colon J \to \mathcal{T}op$, and we assume that for each object j in J the composite

$$\operatorname*{hocolim}_{(\alpha\downarrow j)} u_j^* X \to \operatorname*{colim}_{(\alpha\downarrow j)} u_j^* X \to X_j \tag{10.7}$$

is a weak equivalence. Consider the two adjoint pairs

$$\mathcal{T}op^I \underset{\alpha^*}{\overset{\operatorname{colim}_\alpha}{\rightleftarrows}} \mathcal{T}op^J \underset{c}{\overset{\operatorname{colim}}{\rightleftarrows}} \mathcal{T}op.$$

The composite of the right adjoints is the constant diagram functor, so the composite of the left adjoints is the colimit functor.

Start by observing that there is a natural diagram

$$\begin{tikzcd} Q(\alpha^* X) \arrow[rr, "\phi"] \arrow[dr, "\simeq"'] & & \alpha^*(QX) \arrow[dl, "\simeq"] \\ & \alpha^* X. & \end{tikzcd}$$

The two diagonal maps are familiar, and are both objectwise equivalences. At a particular object i in I, the horizontal morphism ϕ is the evident map

$$\operatorname*{hocolim}_{(I\downarrow i)} u_i^*(\alpha^* X) \to \operatorname*{hocolim}_{(J\downarrow \alpha(i))} u_{\alpha(i)}^* X$$

(induced by α). Note that ϕ must also be an objectwise weak equivalence, since the other two maps in the triangle are so.

Apply $\operatorname{colim}_\alpha$ to the above triangle to get

$$\begin{tikzcd} \operatorname{colim}_\alpha Q(\alpha^* X) \arrow[rr, "\phi"] \arrow[dr, "\simeq"'] & & \operatorname{colim}_\alpha \alpha^*(QX) \arrow[r] \arrow[dl, "\simeq"] & QX \arrow[dl, "\simeq"] \\ & \operatorname{colim}_\alpha \alpha^* X \arrow[r] & X. & \end{tikzcd}$$

Consider the composite

$$\operatorname*{colim}_{\alpha} Q(\alpha^* X) \to \operatorname*{colim}_{\alpha}(\alpha^* X) \to X.$$

This is a map of J-diagrams, and in spot j it is precisely the map from (10.7). So our hypothesis is precisely that this map is an objectwise weak equivalence. It then follows from the above diagram that

$$\operatorname*{colim}_{\alpha} Q(\alpha^* X) \to QX \tag{10.8}$$

is also an objectwise weak equivalence.

The diagram $Q(\alpha^* X)$ is cofibrant in $\mathcal{T}op^I$, by Proposition 9.11. So $\operatorname{colim}_\alpha Q(\alpha^* X)$ is cofibrant in $\mathcal{T}op^J$, as $\operatorname{colim}_\alpha$ is a left Quillen functor. Therefore (10.8) is an objectwise weak equivalence between cofibrant diagrams. Applying colim_J to this yields

$$\operatorname*{hocolim}_{I} \alpha^* X \to \operatorname*{hocolim}_{J} X,$$

where on the left side we are using that $\operatorname{colim}_J \circ \operatorname{colim}_\alpha = \operatorname{colim}_I$. Since left Quillen functors preserve weak equivalences between cofibrant objects, this map is a weak equivalence and we are done. $\square$

11. The two-sided bar construction

The material in this section is from the beautiful paper [HV]. We will see that there is a single construction which unifies almost everything we have talked about so far. Using this, one obtains very slick proofs of most of the main theorems.

11.1. **Basic definitions.** Let $\mathcal{M}$ be a simplicial model category (see Section 8.2). The reader is free to assume $\mathcal{M} = \mathcal{T}op$, but we have reason for the extra generality.

Let I be a small category and let $X: I \to \mathcal{M}$ and $W: I^{op} \to \mathcal{M}$. Define $B_\bullet(W, I, X)$ to be the simplicial object

$$[n] \mapsto \coprod_{i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n} W(i_0) \times X(i_n). \tag{11.2}$$

The face map d_j corresponds to 'covering up i_j', with two provisos. In $d_n: B_n(W, I, X) \to B_{n-1}(W, I, X)$ one must use the map $X(i_n) \to X(i_{n-1})$, whereas in d_0 one must use the map $W(i_0) \to W(i_1)$. The degeneracies correspond to insertion of identity maps, as we are used to. The simplicial object $B_\bullet(W, I, X)$ is called the **two-sided bar construction**.

Example 11.3. For the case $W = *$ (the constant diagram) one has $B_\bullet(*, I, X) = \operatorname{srep}(X)$. We can also regard X as a functor $(I^{op})^{op} \to \mathcal{M}$ and thereby consider the object $B_\bullet(X, I^{op}, *)$. This is not $\operatorname{srep}(X)$ but rather the *other* simplicial replacement that was defined in Remark 4.2.

Remark 11.4. Note that if S is a set and $X \in \mathcal{M}$ then the notation $S \times X$ makes sense: it means the coproduct of copies of X, one for each element $s \in S$. Given a map of sets $S \to T$ and a map $X \to Y$ in $\mathcal{M}$, there are natural maps $S \times X \to T \times X$ and $S \times X \to S \times Y$. Using this observation, the construction $B_\bullet(W, I, X)$ makes sense if $X: I \to \mathcal{M}$ and $W: I^{op} \to \mathcal{S}et$, or if $X: I \to \mathcal{S}et$ and $W: I^{op} \to \mathcal{M}$. It even makes sense if $X: I \to \mathcal{S}et$ and $W: I^{op} \to \mathcal{S}et$, in which case it produces a simplicial set.

The two-sided bar construction also makes sense if $X: I \to \mathcal{M}$, $Y: I^{op} \to s\mathcal{S}et$, and we replace $\times$ with $\otimes$ in the definition (this is the tensor that is part of the simplicial structure on $\mathcal{M}$). We will still write $B_\bullet(W, I, X)$ in this case, as it should always be clear from context what exactly is meant by this notation.

Example 11.5. $B_\bullet(*, I, *)$ is the nerve of I^{op}.

Assume again that $X: I \to \mathcal{M}$ and $W: I^{op} \to \mathcal{M}$, but keep in mind that all of our remarks apply to other settings as well. Let $B(W, I, X) = |B_\bullet(W, I, X)|$. Note that one has a natural map

$$B(W, I, X) \to \operatorname{coeq}\Big[B_1(W, I, X) \rightrightarrows B_0(W, I, X)\Big] = W \otimes_I X$$

(see Section 9.13 for the tensor product). One thinks of $B(W, I, X)$ as a fattened up version of the tensor product; or sometimes as the 'homotopy tensor product'. Note that if $f: X \to X'$ and $g: W \to W'$ are maps of diagrams then there are induced maps

$$B(W, I, X) \to B(W', I, X) \qquad \text{and} \qquad B(W, I, X) \to B(W, I, X').$$

We would like to claim that if f and g are weak equivalences then both these maps are also weak equivalences, but this requires some additional assumptions.

Proposition 11.6. *Suppose that for any cofibrant object Z in $\mathcal{M}$, the functor $(-)\times Z$ preserves weak equivalences between cofibrant objects. Then for any objectwise weak equivalences $f\colon X \to X'$ and $g\colon W \to W'$ between objectwise-cofibrant diagrams, the two maps $B(W,I,X) \to B(W',I,X)$ and $B(W,I,X) \to B(W,I,X')$ are weak equivalences.*

Proof. The main step is to check that if W and X are objectwise-cofibrant diagrams then $B_\bullet(W,I,X)$ is a Reedy cofibrant simplicial object. This is true for basically the same reason as in the case of the simplicial replacement: the nth latching object sits inside $B_n(W,I,X)$ as a summand of the coproduct. However, there is one extra hitch: we need to know that the summands $W(a)\times X(b)$ of these coproducts are themselves cofibrant. This is taken care of by the hypothesis on $\mathcal{M}$.

Once the Reedy cofibrancy is established, the result follows immediately from Theorem 3.5. □

As explained in [HV], it is useful to think of the theory of diagrams as being a generalization of the theory of modules. One should think of a diagram $X\colon I \to \mathcal{M}$ as a left I-module, and a diagram $W\colon I^{op} \to \mathcal{M}$ as a right I-module. This is particularly satisfying if the objects of $\mathcal{M}$ are sets with extra structure: for an $x \in X(i)$ and a map $f\colon i \to j$, write $f.x$ for the image of x under $X(i) \to X(j)$; for $w \in W(j)$ write $w.f$ for the image of w under $W(j) \to W(i)$.

If I is a small category then write $\mathcal{M}od{-}I$ for the category of right I-modules, i.e. the category of functors $I^{op} \to \mathcal{M}$. Likewise, write $I{-}\mathcal{M}od$ for the category of left I-modules, i.e. the category of functors $I \to \mathcal{M}$. Note that we are leaving out $\mathcal{M}$ from the notation, though it should be understood from context.

If I and J are small categories, then an $\boldsymbol{I-J}$ **bimodule** is a diagram $I\times J^{op} \to \mathcal{M}$. If W is an $I-J$ bimodule and X is a $J-K$ bimodule, then by $B(W,J,X)$ we mean the $I-K$ bimodule defined as

$$(i,k) \mapsto B(W_i, J, X_k).$$

Here W_i is the J^{op}-diagram $j \mapsto W(i,j)$, and X_k is the J-diagram $j \mapsto X(j,k)$. Note that the construction of $B(W,J,X)$ makes sense even if the target of X is $\mathcal{S}et$, or if the target of W is $\mathcal{S}et$.

Example 11.7. If I is a category then for each $i \in I$ we obtain a left I-module $I(i,-)$ and a right I-module $I(-,i)$. These are free modules, in the following sense. For any object $Z \in \mathcal{M}$, consider the right I-module $I(-,i)\otimes Z$ sending $j \mapsto I(j,i)\otimes Z$. Then for any right I-module X there is a bijection

$$\mathrm{Hom}_{\mathcal{M}od\text{–}I}(I(-,i)\otimes Z, X) \cong \mathrm{Hom}_{\mathcal{M}}(Z, X_i) \tag{11.8}$$

obtained by restricting to the canonical copy of Z in the ith spot of the diagram. Similarly, there are bijections

$$\mathrm{Hom}_{I\text{–}\mathcal{M}od}(I(i,-)\otimes Z, W) \cong \mathrm{Hom}_{\mathcal{M}}(Z, W_i)$$

for each left I-module W.

An easy adjointness argument now shows that there are natural isomorphisms $I(-,i)\otimes_I W \cong W_i$ and $X\otimes I(i,-) \cong X_i$. For example, the former follows from the fact that for any object U in $\mathcal{M}$ we have the natural bijections

$$\begin{aligned}\mathrm{Hom}_{\mathcal{M}}(I(-,i)\otimes W, U) \cong \mathrm{Hom}_{\mathcal{M}od\text{–}I}(I(-,i), \mathrm{Hom}(W,U)) &\cong \mathrm{Hom}(W,U)_i \\ &\cong \mathrm{Hom}_{\mathcal{M}}(W_i, U).\end{aligned}$$

Note that at the second stage $\mathcal{M}od–I$ refers to the category of right I-modules with values in $\mathcal{S}et$, and the second isomorphism is an instance of (11.8) where $Z = *$.

Example 11.9. Putting the left and right modules $I(j, -)$ and $I(-, i)$ together, we have an $I - I$ bimodule given by the functor $I \times I^{op} \to \mathcal{S}et$ sending $(i, j) \mapsto I(j, i)$. We will call this functor I, by abuse. [The switching in the order of i and j is annoying, but seems unavoidable; the problem is that the notation in mathematics always wants to be right to left, so that to talk about maps from a to b we should really write "$\mathrm{Hom}(b, a)$"; but we don't.]

By the above observations, for any left I-module X (that is to say, for any diagram $X: I \to \mathcal{M}$) we get a left I-module $B(I, I, X)$. Notice that the I in the first slot refers to the bimodule from the preceeding paragraph, whereas the I in the second slot is the category. Similarly, for any right I-module W we get another right I-module $B(W, I, I)$. We will see in a moment that these are precisely the diagrams QX and $Q'W$ defined in Section 9.

Exercise 11.10. If $X: I \to \mathcal{S}et$, then $B_\bullet(I, I, X)$ is an I-diagram of simplicial sets. Check that $B_\bullet(I, I, *)$ is the diagram $i \mapsto N(I \downarrow i)^{op}$. Similarly, check that $B_\bullet(*, I, I)$ is the diagram $i \mapsto N(i \downarrow I)^{op}$.

Exercise 11.11. Let $\alpha: I \to J$ be a functor. There there is a functor $J \times I^{op} \to \mathcal{S}et$ given by $(j, i) \mapsto J(\alpha(i), j)$. This is really obtained by starting with the $J - J$ bimodule J and restricting the right action along α. We will still call this bimodule J, but now regard it as a $J - I$ bimodule.

Check that $B_\bullet(J, I, *)$ is the left J-module given by $j \mapsto N(\alpha \downarrow j)^{op}$. Similarly, $B_\bullet(*, I, J)$ is the right J-module given by $j \mapsto N(j \downarrow \alpha)^{op}$.

Exercise 11.12. Let $\alpha: I \to J$ be a functor. For any right I-module X, the tensor product $X \otimes_I J$ is a right J-module. Make sense of this and prove that $X \otimes_I J = \mathrm{colim}_{\alpha^{op}} X$.

11.13. **Main properties and applications.** The central result of [HV] is the following:

Theorem 11.14. *Let I, J, K, and L be small categories. Suppose given an $I - J$ bimodule X, a $J - K$ bimodule Y, and a $K - L$ bimodule Z. Then there is a canonical isomorphism*

$$B(X, J, Y) \otimes_K Z \cong B(X, J, Y \otimes_K Z)$$

of $I - L$ bimodules.

Similarly, if W is an $H - I$ bimodule then there is a canonical isomorphism

$$W \otimes_I B(X, J, Y) \cong B(W \otimes_I X, J, Y)$$

of $H - K$ bimodules.

Remark 11.15. The above theorem has an open-ended interpretation, as we have not specified the target categories for the bimodules X, Y, and Z. For instance, X and Y could take their values in $\mathcal{S}et$ and Z could take its values in $\mathcal{M}$; or X and Z could take their values in $\mathcal{S}et$ and Y could take its values in $\mathcal{M}$; or all three functors could take their values in $\mathcal{S}et$. The isomorphism of the theorem is valid in all these cases.

The proof of Theorem 11.14 is a simple exercise in adjoint functors. We will give it at the end of the section. What we will do now is point out that the theorem allows one to give very slick proofs of many of our results about homotopy colimits.

Example 11.16 (The two formulas for hocolim)**.** Recall that if $X\colon I \to \mathcal{T}op$ then $B(*, I, X) = |\operatorname{srep}(X)| = \operatorname{hocolim}_I X$. By Theorem 11.14 we can also write

$$B(*, I, X) \cong B(*, I, I \otimes_I X) \cong B(*, I, I) \otimes_I X.$$

But $B(*, I, I)$ is the diagram $i \mapsto N(i \downarrow I)^{op}$, and so the right-most object is the formula from (4.14). This seems to be the slickest proof that the two formulas for $\operatorname{hocolim} X$ are isomorphic.

Example 11.17 (The diagrams QX)**.** Let $X\colon I \to \mathcal{M}$ and consider the left I-module $B(I, I, X)$. This is the diagram

$$i \mapsto B(I(-, i), I, X) = B(I(-, i), I, I) \otimes_I X.$$

But it is easy to check that $B_\bullet(I(-, i), I, I) = B_\bullet(*, I \downarrow i, I)$. So we are really looking at the diagram

$$i \mapsto B(*, I \downarrow i, I) \otimes_I X = B(*, I \downarrow i, X) = \operatorname*{hocolim}_{I\downarrow i} u_i^* X.$$

Therefore $B(I, I, X)$ is the I-diagram QX defined in Section 9.

Recall that we have a natural map of I-diagrams $B(I, I, X) \to I \otimes_I X = X$. This is our map $QX \to X$. Finally, note that one has

$$\operatorname*{colim}_I QX = \operatorname*{colim}_I B(I, I, X) = * \otimes_I B(I, I, X) \cong B(*, I, X) \cong \operatorname*{hocolim}_I X.$$

This gives a very elegant proof of Corollary 9.8 that doesn't go through Proposition 9.7.

Example 11.18 (The diagrams $Q'X$)**.** We again start with $X\colon I \to \mathcal{M}$, but now we regard X as a right I^{op}-module. It is easy to see that $B_\bullet(X, I^{op}, *)$ is the 'other' simplicial replacement for X considered in Remark 4.2; and so $B(X, I^{op}, *)$ is what we called hocolim' X in Remark 9.12.

The object $B(X, I^{op}, I^{op})$ is a right I^{op}-module, or equivalently a left I-module; in other words, it is a diagram $I \to \mathcal{M}$. An analysis similar to the one in the previous example shows that this is precisely the diagram $Q'X$ defined in (9.12).

Just as in the previous example, we find that

$$\operatorname*{colim}_I Q'X = B(X, I^{op}, I^{op}) \otimes_I * \cong B(X, I^{op}, *) = \operatorname*{hocolim'}_I X.$$

Example 11.19 (Changing the indexing category)**.** Suppose $\alpha\colon I \to J$ is a functor and $X\colon J \to \mathcal{M}$. Then we can write

$$\operatorname*{hocolim}_I \alpha^* X = B(*, I, X) \cong B(*, I, J \otimes_J X) \cong B(*, I, J) \otimes_J X.$$

Note that the natural map $\operatorname{hocolim}_I \alpha^* X \to \operatorname{hocolim}_J X$ is the map

$$\alpha_*\colon B(*, I, J) \otimes_J X \to B(*, J, J) \otimes_J X.$$

Observe that $B(*, I, J)$ is the J^{op}-diagram given by $j \mapsto N(j \downarrow \alpha)^{op}$ (see Example 11.11). So the above formula for $\operatorname{hocolim}_I \alpha^* X$ recovers Proposition 10.4(d).

We can also recover the other parts of Proposition 10.4. For instance, let us consider part (a). For any diagram $X\colon I \to \mathcal{M}$, we have already remarked that

$Q'X = B(X, I^{op}, I^{op})$. So if we want to apply Q' to the constant I^{op}-diagram whose value is a point, then we have

$$Q'(*) = B(*, I, I).$$

It follows that

$$\operatorname*{colim}_{I^{op}\to J^{op}} Q'(*) = Q'(*) \otimes_I J = B(*, I, I) \otimes_I J = B(*, I, I \otimes_I J) = B(*, I, J),$$

where in the first equality we are using Exercise 11.12. But we have already remarked that $B(*, I, J)$ is the J^{op}-diagram $j \mapsto N(j \downarrow \alpha)^{op}$, so this proves Proposition 10.4(a).

This completes our examples. Hopefully they demonstrate the power of learning to manipulate the two-sided bar construction. After proving just a few basic results, many significant corollaries come along almost for free.

For ease of future reference, we now summarize the relations between the two-sided bar construction and other objects considered in this paper. In the following, $I \to J$ is a map of small categories and X is a diagram $I \to \mathcal{M}$.

$$\begin{aligned}
B(*, I, *) &= BI^{op}\\
B(I, I, X) &= QX = B(I, I, I) \otimes_I X\\
B(X, I^{op}, I^{op}) &= Q'X = X \otimes_{I^{op}} B(I^{op}, I^{op}, I^{op})\\
B_\bullet(*, I, X) &= \operatorname{srep}(X)\\
B(*, I, X) &= \operatorname*{hocolim}_I X = B(*, I, I) \otimes_I X\\
B(J, I, X) &= \operatorname*{hocolim}_{I\to J} X = B(J, I, I) \otimes_I X\\
B(X, I^{op}, *) &= \operatorname{hocolim}'_I X = X \otimes_{I^{op}} B(I^{op}, I^{op}, *)\\
B(X, I^{op}, J^{op}) &= \operatorname{hocolim}'_{I\to J} X = X \otimes_{I^{op}} B(I^{op}, I^{op}, J^{op})
\end{aligned}$$

11.20. **Proof of the Hollender-Vogt theorem.** We have one last thing to wrap up before closing this section:

Proof of Theorem 11.14. ??? □

12. Function spaces and the two-sided cobar construction

This section is a companion to the last one. We saw that the two-sided bar construction is a homotopical version of the tensor product of diagrams. Dually, there is a two-sided "cobar construction" that serves as a homotopical version of the function space between two diagrams.

As in the last section, fix a simplicial model category $\mathcal{M}$. Let I be a small category, and let X and Y be left I-modules with values in $\mathcal{M}$. One defines the **function space** from X to Y to be

$$F_I(X,Y) = \mathrm{eq}\left(\prod_i \mathrm{Map}(X(i),Y(i)) \rightrightarrows \prod_{i\to j} \mathrm{Map}(X(i),Y(j))\right).$$

Note that this is a simplicial set.

If X is an $I-K$ bimodule, then the natural extension of this definition gives a left K-module $F_I(X,Y)$ (taking values in $s\mathcal{S}et$). Likewise, if Y is an $I-L$ bimodule then $F_I(X,Y)$ is a right L-module.

Proposition 12.1. *Let I and K be small categories. Let Z be a left K-module, X an $I-K$ bimodule, and Y a left I-module. Then there are natural adjunction isomorphisms*
(a) $\mathrm{Hom}_{K-\mathcal{M}od}(Z, F_I(X,Y)) \cong \mathrm{Hom}_{I-\mathcal{M}od}(X \otimes_K Z, Y)$*, and*
(b) $F_K(Z, F_I(X,Y)) \cong F_I(X \otimes_K Z, Y)$.

Proof. An easy argument, left to the reader. □

Remark 12.2. Recall that if I is a small category we have the I-I bimodule $i,j \mapsto I(j,i)$, and this is also denoted by the symbol I. If i is an object of I, an easy adjointness argument using Proposition 12.1(a) and the fact that $I(i,-)\otimes_I Y \cong Y_i$ shows that $F_I(I(i,-),Y) \cong Y_i$. This isomorphism is natural in i, and so can be interpreted as an isomorphism of left I-modules

$$F_I(I,Y) \cong Y, \tag{12.3}$$

where the left I-module structure on $F_I(I,Y)$ is induced by the right I-module structure on I.

Just as the tensor product $(-)\otimes_I(-)$ can be expanded to a homotopical version $B(-,I,-)$, its adjoint $F_I(-,-)$ also has a homotopical version which we denote $\Omega_I(X,Y)$. We define $\Omega_I^\bullet(X,Y)$ to be the cosimplicial object

$$[n] \mapsto \prod_{i_0\to i_1\to\cdots\to i_n} \mathrm{Map}(X(i_0),Y(i_n)),$$

with the evident coface and codegeneracy operators, and we define $\Omega_I(X,Y) = \mathrm{Tot}\,\Omega_I^\bullet(X,Y)$. Note that there is a natural map $F_I(X,Y) \to \Omega_I(X,Y)$.

Recall that we have seen the construction $\Omega_I^\bullet(X,Y)$ before, in Section 9.5 when we discussed homotopy coherent maps between diagrams. For $X,Y\colon I \to \mathcal{T}op$ we defined $\mathrm{hc}(X,Y)$ to be the underlying set of $\Omega_I(X,Y)$.

The following results are dual versions of Theorem 11.14:

Theorem 12.4. *Let I and K be small categories. Let Z be a left K-module, let X be an I-K bimodule, and let Y be a left I-module.* **Hypothesis on $\mathcal{M}$?** *There are natural isomorphisms*

(a) $F_K(Z, \Omega_I(X,Y)) \cong \Omega_I(X \otimes_K Z, Y)$,
(b) $\Omega_K(Z, F_I(X,Y)) \cong F_I(B(X,K,Z),Y)$, *and*
(c) $\Omega_K(Z, \Omega_I(X,Y)) \cong \Omega_I(B(X,K,Z),Y)$.

Proof. ??? □

To demonstrate the power of this machinery we give a very quick proof of a result we encountered back in Section 9.5:

Alternate Proof of Proposition 9.7. Recall that $X, Y : I \to \mathcal{T}op$ are two diagrams. Observe that the set $\mathcal{T}op^I(X,Y)$ is just the underlying set of $F_I(X,Y)$. We simply write

$$F_I(QX,Y) = F_I(B(I,I,X),Y) = \Omega_I(X, F_I(I,Y)) = \Omega_I(X,Y).$$

The second equality is by Theorem 12.4(b), and the third equality uses (12.3). Taking underlying sets gives us $\mathcal{T}op^I(QX,Y) = \mathrm{hc}(X,Y)$, as desired. □

Notice that $\Omega_I^\bullet(*,Y) = \mathrm{crep}(Y)$ and $\Omega_I(*,Y) = \mathrm{holim}\, Y$. As further practice with the machinery we will develop the dual of the construction $X \mapsto QX$. To this end, if Y is a left I-module define $RY = \Omega_I(I,Y)$. This is also a left I-module, and it comes with a natural map

$$Y = F_I(I,Y) \to \Omega_I(I,Y) = RY.$$

Note that

$$\lim_I RY = F_I(*, RY) = F_I(*, \Omega_I(I,Y)) = \Omega_I(I \otimes_I *, Y) = \Omega_I(*,Y) = \operatorname*{holim}_I Y.$$

We claim that the natural map $Y \to RY$ is an objectwise weak equivalence. This is simply because for every object i in I the co-augmented cosimplicial space

$$Y_i \longrightarrow \prod_{i_0} \mathrm{Map}(I(i,i_0), Y_{i_0}) \rightrightarrows \prod_{i_0 \to i_1} \mathrm{Map}(I(i,i_0), Y_{i_1}) \Rrightarrow \cdots$$

has a contracting homotopy. To write this down, notice that in level n we have the product, indexed over all strings $i \to i_0 \to \cdots \to i_n$, of Y_{i_n}. It is convenient, though not technically correct, so imagine the set-theoretic situation and write such an element as a tuple $(s_{[i \to i_0 \to \cdots \to i_n]} \in Y_{i_n})$. The contraction (i.e., the extra codegeneracy) sends such a tuple to the assignment

$$[i \to i_0 \to \cdots \to i_{n-1}] \mapsto s_{[i \to i \to i_0 \to \cdots \to i_{n-1}]}$$

where of course the map $i \to i$ is the identity.

Remark 12.5. There is another way to see the above contracting homotopy. Let $\mathrm{ob}(I)$ be the category with the same objects as I, but where the only maps are the identities. There is a forgetful functor $U : \mathcal{T}op^I \to \mathcal{T}op^{\mathrm{ob}(I)}$, and this has both a left and right adjoint. The right adjoint G may be readily checked to be the map that sends $W \in \mathcal{T}op^{\mathrm{ob}(I)}$ to the diagram

$$i \mapsto \prod_{i \to i_0} W_{i_0}.$$

The adjoint pair (U, G) allows us to write down the (coaugmented) cosimplicial object

$$Y \longrightarrow GU(Y) \rightrightarrows GUGU(Y) \Rrightarrow \cdots$$

We leave the reader to check that Tot of this cosimplicial object is precisely RY. Applying U everywhere, we pick up a contracting homotopy by the dual of the situation in Exercise 3.15. This shows that $Y \to RY$ is an objectwise weak equivalence.

By analogy with what we saw for QX, one would expect RY to be a fibrant replacement for Y in the model category $\mathcal{M}^I$ and for our conclusion to be that $\operatorname{holim}_I(-)$ is the derived functor of $\lim_I(-)$. This does not quite fit with what we have done so far, though. Notice that the adjoint functors

$$c\colon \mathcal{M} \rightleftarrows \mathcal{M}^I\colon \lim$$

are not usally Quillen functors. For example, the constant functor c need not preserve cofibrant objects; likewise, the right adjoint lim typically doesn't preserve fibrations or trivial fibrations.

The trouble here is that the model category structure on $\mathcal{M}^I$ that was the right one to use for the colimit story is not the right one to use for the limit story. We will talk more about this in the next section (????).

12.6. **Relative limits and homotopy limits.** Let $\alpha\colon I \to J$ be a functor between small categories. The restriction functor $\alpha^*\colon \mathcal{M}^J \to \mathcal{M}^I$ has a left adjoint that we have used before (the relative colimit), but it also has a right adjoint. Write $\lim_\alpha$ or $\lim_{I\to J}$ for this right adjoint. If $X\colon I \to \mathcal{M}$ then this right adjoint is given by

$$\big[\lim_{I\to J} X\big]_j = \lim_{(j\downarrow\alpha)} u_j^* X$$

for each object j in J.

Another formula for the relative limit is

$$\lim_{I\to J} X = F_I(J, X)$$

where J is the usual J-J bimodule, regarded as an I-J bimodule via restriction along α. We can see this through the adjunctions

$$J\text{–}\mathcal{M}od(A, F_I(J,X)) \cong I\text{–}\mathcal{M}od(J \otimes_J A, X) \cong I\text{–}\mathcal{M}od(\alpha^* A, X) \cong J\text{–}\mathcal{M}od(A, \lim_{I\to J} X)$$

where we have used $J \otimes_J A = A$ (but it equals $\alpha^* A$ when we restrict and think of it as a left I-module).

We define the relative homotopy limit via

$$\operatorname*{holim}_{I\to J} X = \lim_{I\to J}(RX) = F_I(J, \Omega_I(I,X)) = \Omega_I(I \otimes_I J, X) = \Omega_I(J,X).$$

12.7. **Wrap-up.** We close by listing relations between the cobar construction Ω and other things we have considered in this paper. Here $I \to J$ is a functor between small categories, and $Y\colon I \to \mathcal{M}$ is a diagram.

$$\begin{aligned}
\Omega_I^\bullet(*, Y) &= \operatorname{crep}(Y)\\
\Omega_I(*, Y) &= \operatorname{holim} Y\\
\Omega_I(X, Y) &= \operatorname{hc}(X,Y)\\
\Omega_I(I, Y) &= RY\\
\Omega_I(J, Y) &= \operatorname*{holim}_{I\to J} Y.
\end{aligned}$$

Part 3. The homotopy theory of diagrams

Let I be a small category and consider the category of diagrams $\mathcal{T}op^I$. A map of diagrams $X \to Y$ is defined to be an objectwise weak equivalence if $X_i \to Y_i$ is a weak equivalence for every i in I. Two diagrams are said to be weakly equivalent if they can be connected by a (possibly very long) zig-zag of objectwise weak equivalences. The homotopy category of diagrams, denoted $\mathcal{H}o(\mathcal{T}op^I)$, is the localization of $\mathcal{T}op^I$ with respect to these objectwise weak equivalences. (In general, there are set-theoretic problems in forming such localizations: they can be handled by adopting the axioms of Grothendieck universes if necessary, but they can also be handled by showing that $\mathcal{T}op^I$ has a model category structure with the aforementioned weak equivalences.)

In general, when $\mathcal{M}$ is a model category then $\mathcal{H}o(\mathcal{M})$ only encodes a fraction of the homotopical information in $\mathcal{M}$. Dwyer and Kan showed that for any two objects X and Y in $\mathcal{M}$ one can associate a *homotopical mapping space* $\operatorname{Map}(X,Y)$ (a simplicial set defined up to weak equivalence), where π_0 of this mapping space is $\mathcal{H}o(\mathcal{M})(X,Y)$. Appropriate models of these mapping spaces can even be bundled together to give a simplicially enriched category. It gradually became clear that it was somehow this simplicially enriched category, rather than the model category structure, that contained the truly homotopical information from $\mathcal{M}$. Dwyer and Kan gave a construction of this category that only depended on $\mathcal{M}$ and the class of weak equivalences, making it clear that the classes of cofibrations and fibrations were not essential—rather, they are just tools to help get at the homotopical information. Kan often advocated for the perspective that a "homotopy theory" was really just a pair $(\mathcal{C},\mathcal{W})$ consisting of a category $\mathcal{C}$ and a subclass $\mathcal{W}$ of the morphisms (the "weak equivalences"); but in practice one often accesses this homotopy theory via a model category structure (Kan likened it to accesing a manifold via a choice of coordinate system).

In the last several sections we have already found ourselves working with the homotopy theory of diagrams. We gave ourselves a certain model category structure on $\mathcal{T}op^I$ and observed that $\operatorname{colim}\colon \mathcal{T}op^I \rightleftarrows \mathcal{T}op\colon c$ was a Quillen pair (where c is the constant diagram functor), leading to an adjunction

$$\mathbb{L}\operatorname{colim}\colon \mathcal{H}o(\mathcal{T}op^I) \rightleftarrows \mathcal{H}o(\mathcal{T}op)\colon \mathbb{R}c$$

(note that $\mathbb{R}c$ really is just the evident 'constant diagram' functor). We saw that our hocolim functor was just a particular model for the left derived functor $\mathbb{L}\operatorname{colim}$, and we began to understand that by "homotopy colimit" one should really mean *any* model for that derived functor. We also saw the bar and cobar constructions, which can also be interpreted as models for derived functors. For example, if X is in $\mathcal{T}op^I$ then $B(X,I,-)$ is a model for the derived functor of $X\otimes(-)\colon \mathcal{T}op^{I^{op}} \to \mathcal{T}op$.

In the next few sections we want to...

13. Model structures on diagram categories

Let $\mathcal{M}$ be a model category, and let I be a small category. If we are lucky then the diagram category $\mathcal{M}^I$ has a model category structure where the weak equivalences are the objectwise ones induced from $\mathcal{M}$. In some cases it actually has several such model category structures. We give a brief guide to these results:

(1) If $\mathcal{M}$ is cofibrantly-generated, then $\mathcal{M}^I$ has the so-called **projective model structure** where the weak equivalences are objectwise, the fibrations are objectwise, and the cofibrations are forced to be the maps with the left-lifting-property with respect to the trivial fibrations.
(2) If $\mathcal{M}$ is combinatorial, then $\mathcal{M}^I$ has the so-called **injective model structure** where the weak equivalences are objectwise, the cofibrations are objectwise, and the fibrations are forced to be the maps with the right-lifting-property with respect to the trivial cofibrations.
(3) If I is a Reedy category then $\mathcal{M}^I$ has the so-called **Reedy model structure** in which the weak equivalences are objectwise, the cofibrations are maps that induced $\mathcal{M}$-cofibrations on all latching maps, and the fibrations are maps that induced $\mathcal{M}$-fibrations on all matching maps. See ???

For the notion of cofibrantly-generated model category, see [H, Chapter 11]. A model category is called combinatorial if it is both cofibrantly-generated and its underlying category is locally presentable. Ref??

In the cases where these model structures simultaneously exist, every projective-cofibration is a Reedy-cofibration and every Reedy cofibration is an injective-cofibration. Dually, every injective-fibration is a Reedy fibration and every Reedy fibration is a projective-fibration. This says that the identity maps give Quillen equivalences

$$\mathcal{M}^I_{prog} \xrightarrow{\sim} \mathcal{M}^I_{Reedy} \xrightarrow{\sim} \mathcal{M}^I_{inj}$$

where the arrows indicate the left adjoints in the Quillen pairs.

13.1. **The projective model structure.** Recall that if $L\colon \mathcal{A} \rightleftarrows \mathcal{B}\colon R$ is an adjoint pair and $\mathcal{A}$ has a model category structure, then in many cases one can lift this model structure to $\mathcal{B}$ by defining a map $b_1 \to b_2$ in $\mathcal{B}$ to be a weak equivalence (respectively, fibration) if and only if $Rb_1 \to Rb_2$ is a weak equivalence (respectively, fibration) in $\mathcal{A}$. The cofibrations in $\mathcal{B}$ are defined to be the maps having the left-lifting-property with respect to the trivial fibrations. This produces a model structure when $\mathcal{A}$ is cofibrantly-generated and one has enough control in $\mathcal{B}$ to be able to perform the small object argument. This is Kan's *Recognition Theorem*, see [H, Theorem 11.3.2]. We will apply this principle to obtain the projective model structure on $\mathcal{M}^I$, using an appropriate adjoint pair.

Let $\mathcal{M}$ be a category, and let I be a small category. For any object i in I, consider the functor $\mathrm{ev}_i\colon \mathcal{M}^I \to \mathcal{M}$ given by $\mathrm{ev}_i(X) = X_i$. If $\mathcal{M}$ has small coproducts then ev_i has a left adjoint $A \to F_i(A)$ where $F_i(A)$ is the I-diagram given by

$$[F_i(A)]_j = I(i,j) \otimes A.$$

Recall that if S is a set and A is an object of $\mathcal{M}$ then $S \otimes A$ denotes a coproduct of copies of A indexed by the set S. We call $F_i(A)$ the **free diagram generated by A at spot i**, or sometimes the **i-free diagram generated by** A. The idea is simple enough to remember: if one imagines putting a copy of A at spot i and writing down the most general possible diagram derived from this, then at spot j

we must add one copy of A for every map $i \to j$. Note that this free diagram will typically have *more than* just a single copy of A at spot i: indeed, there should be one copy for every element of $I(i,i)$.

Similarly, if $\mathcal{M}$ has small products then ev_i has a right adjoint $A \to CF_i(A)$ given by

$$[CF_i(A)]_j = A^{I(j,i)}$$

where if S is a set then A^S denotes the product of copies of A indexed by the set S. We call $CF_i(A)$ the **co-free diagram generated by A at spot i**. We will not need co-free diagrams in the present section, but they will reappear later.

Of course $\mathrm{ob}(I)$ denotes the set of objects of I, but we will also use this to denote the category having this set of objects and only identity maps. Consider the adjoint pair

$$\mathcal{M}^{\mathrm{ob}(I)} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{M}^I$$

where $U(X)$ is the assignment $i \mapsto X_i$ (forgetting all the diagram maps $X_i \to X_j$) and $F(\{A_i\}) = \amalg_i F_i(A_i)$. If $\mathcal{M}$ is a model category then $\mathcal{M}^{\mathrm{ob}(I)}$ inherits a model structure in the evident way, and the objectwise weak equivalences (resp., objectwise fibrations) in $\mathcal{M}^I$ are precisely the maps that become weak equivalences (resp., fibrations) upon applying U.

If $\mathcal{M}$ is cofibrantly-generated with generating cofibrations $\{A_\alpha \rightarrowtail B_\alpha\}$ and generating trivial cofibrations $\{I_\alpha \overset{\sim}{\rightarrowtail} J_\alpha\}$, then the sets

$$\{F_i(A_\alpha) \to F_i(B_\alpha) \,|\, i \in \mathrm{ob}(I)\} \quad \text{and} \quad \{F_i(I_\alpha) \to F_i(I_\alpha) \,|\, i \in \mathrm{ob}(I)\}$$

detect the trivial fibrations and fibrations in $\mathcal{M}^I$, respectively. That is to say, a map in $\mathcal{M}^I$ is a trivial fibration if and only if it has the right-lifting-property with respect to all the maps $F_i(A_\alpha) \to F_i(B_\alpha)$ for all i and α, and similary for the fibrations. This is a triviality coming out of the adjointness properties. The existence of the projective model structure on $\mathcal{M}^I$ is a direct application of Kan's Recognition Theorem.

The following properties are trivial, but worth noting:

Proposition 13.2.

(a) *If $f\colon A \to B$ is a projective-cofibration then f is an objectwise cofibration.*

(b) *For any cofibration $A \rightarrowtail B$ in $\mathcal{M}$ and any object i in I, the map $F_i(A) \to F_i(B)$ is a projective-cofibration. In particular, if A is cofibrant in $\mathcal{M}$ then $F_i(A)$ is projective-cofibrant.*

Proof. Let i be an object in I. Let $X \to Y$ be any trivial fibraion in $\mathcal{M}$, and suppose given a lifting diagram

$$\begin{tikzcd} A_i \arrow[r] \arrow[d, tail] & X \arrow[d] \\ B_i \arrow[r] & Y. \end{tikzcd}$$

By adjointness we get

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & CF_i(X) \arrow[d] \\ B \arrow[r] & CF_i(Y), \end{tikzcd}$$

and the construction of $CF_i(-)$ shows immediately that $CF_i(X) \to CF_i(Y)$ is an objectwise trivial fibration. So a lifting $B \to CF_i(X)$ exists, and by adjointness this gives a lifting $B_i \to X$ in the original diagram. Since $A_i \to B_i$ has the left-lifting-property with respect to all trivial fibrations, it must be a cofibration in $\mathcal{M}$.

Part (b) is also an easy exercise using adjointness, or it comes for free from the fact that (F, U) is a Quillen pair. □

Note that the adjoint pair

$$\mathcal{M}^I \underset{c}{\overset{\mathrm{colim}}{\rightleftarrows}} \mathcal{M}$$

is a Quillen pair, as c preserves fibrations and trivial fibrations. So if X is a projective-cofibrant diagram then the natural map $\operatorname{hocolim} X \to \operatorname{colim} X$ is a weak equivalence.

13.3. **The injective structure.** The projective model structure on $\mathcal{M}^I$ is a very reasonable construction that fit in naturally with the historical development of model categories. The injective structure, though similar in some ways, is also rather different. The dual theory all works fine, except for the problem that model categories are rarely fibrantly-generated: so at first there seem to be very few examples where the theory applies at all.

The first constructions of the injective model structure on $s\mathcal{S}et^I$ were given by Heller [He] and Jardine [J1] (Jardine was really concerned with a whole class of model structures on $s\mathcal{S}et^I$, of which the injective model structure is only one example). The proof for the existence of this model structure is very different than for the projective version: it uses large cardinals and the transfinite version of the small object argument in a crucial way. Jeff Smith later showed that this proof goes through in the general context of combinatorial model categories: the "locally presentable" hypothesis on the underlying category is exactly what one needs to make the transfinite small object argument work out. Some of Smith's ideas were written up by Beke in [Be], but the existence of the injective model structure on $\mathcal{M}^I$ is not explicitly stated there. It can readily be deduced from [Be, Theorem 1.7 and Propositions 1.15, 1.18] using the accessible functor $U\colon \mathcal{M}^I \to \mathcal{M}^{\mathrm{ob}(I)}$, and there is also some general discussion in [Be, Section 3]. Additionally, there are some recent sources for the existence of this model structure: it follows easily from [BHKKRS, Theorem 2.23], and it is addressed explicitly in [L, Proposition A.2.8.2].

Once one has constructed the injective model structure, it is trivial to check that

$$\mathcal{M}^I \underset{CF}{\overset{U}{\rightleftarrows}} \mathcal{M}^{\mathrm{ob}(I)}$$

is a Quillen pair, where $CF(\{X_i\}) = \prod_i CF_i(X_i)$ (the left adjoint U clearly preserves cofibrations and trivial cofibrations). The following properties of the injective model structure are automatic, with proofs that are completely dual to the projective version we already saw:

Proposition 13.4.

(a) If $f\colon X \to Y$ is an injective-fibration then f is an objectwise fibration.

(b) For any fibration $X \rightarrowtail Y$ in $\mathcal{M}$ and any object i in I, the map $CF_i(X) \to CF_i(Y)$ is an injective-fibration. In particular, if X is fibrant in $\mathcal{M}$ then $CF_i(X)$ is injective-fibrant.

Also note that the adjoint pair

$$\mathcal{M} \underset{\lim}{\overset{c}{\rightleftarrows}} \mathcal{M}^I$$

is a Quillen pair, as c preserves cofibrations and trivial cofibrations. So if X is an injective-fibrant diagram then $\lim X$ is equivalent to the homotopy limit of X.

13.5. **The Reedy structure.** If $X, Y \colon \Delta \to \mathcal{M}$ then maps of diagrams $X \to Y$ can be produced inductively: at each level there is an extension problem, where one must produce a map $X_n \to Y_n$ that is compatible with the previous maps $X_i \to Y_i$ for $i < n$. This compatibility question breaks up into two distinct prices: compatibility with the coface maps and compatibility with the codegeneracies. Specifically, there is a "latching object" $L_n X$ obtained by taking a colimit of all the coface maps landing in degree n, and a dual "matching object" $M_n X$ obtained by taking a limit of all the codegeneracies emanating from degree n. The maps $X_i \to Y_i$ for $i < n$ induce maps $L_n X \to L_n Y$ and $M_n X \to M_n Y$, and the extension problem is indicated in the diagram below:

$$\begin{tikzcd}
L_n X \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d, dotted] & M_n X \arrow[d] \\
L_n Y \arrow[r] & Y_n \arrow[r] & M_n Y.
\end{tikzcd}$$

Bousfield-Kan [BK] figured out how to put a model category structure on cosimplicial spaces so that the notions of cofibration/fibration interact well with the above extension problem. Later Reedy, in an unpublished preprint, determined how to do something similar for simplicial (rather than cosimplicial) objects, and he did this over any model category. Many years later Kan isolated what makes this work and generalized it into the notion of a *Reedy category*. Very briefly, a category is Reedy if the objects can be assigned an $\mathbb{N}$-grading and the maps can all be decomposed into "special up-maps" and "special down-maps" that are subject to certain conditions. The conditions are precisely those that allow for the inductive arguments to go through. This material first appeared in [H, Chapter 15].

We are going to give an extremely quick overview of Reedy model categories, since there already exists a very nice and thorough discussion in [H].

Definition 13.6. *A* ***Reedy category*** *is a small category I together with two subcategories I^{up} and I^{dn}, both of which are required to contain all the objects of I, such that there exists a function* $\deg\colon \mathrm{ob}(I) \to \mathbb{N}$ *such that*

(i) Every non-identity map in I^{up} raises degree;
(ii) Every non-identity map in I^{dn} lowers degree;
(iii) Every map g in I has a unique factorization $g = g^{up} \circ g^{dn}$ where g^{up} is in I^{up} and g^{dn} is in I^{dn}.

A ***directed Reedy category*** *is a Reedy category in which either I^{up} or I^{dn} consists only of identity maps; in the former case we call it an* ***upwards-directed Reedy category****, and in the latter we call it a* ***downwards-directed Reedy category****.*

Example 13.7.

(a) Let Δ^{up} denote the subcategory of injections and Δ^{dn} denote the subcategory of surjections. The degree function $\deg([n]) = n$ shows that $(\Delta, \Delta^{up}, \Delta^{dn})$ is a Reedy category.

(b) Let $(\Delta^{op})^{up}$ denote the subcategory consisting of maps that are opposites of surjections, and let $(\Delta^{op})^{dn}$ denote the subcategory whose maps are the opposites of injections. The degree function $\deg([n]) = n$ again shows that $(\Delta^{op}, (\Delta^{op})^{up}, (\Delta^{op})^{dn})$ is a Reedy category.

(c) Generalizing the previous example, it is easy to check that if (I, I^{up}, I^{dn}) is a Reedy category then so is $(I^{op}, (I^{dn})^{op}, (I^{up})^{op})$.

(d) Let I be a category for which there exists a function $\deg\colon \mathrm{ob}(I) \to \mathbb{N}$ having the property that every non-identity morphism raises degree. Then $(I, I, \mathrm{ob}(I))$ is an upwards-directed Reedy category. Similarly, if there exists a function $\deg\colon \mathrm{ob}(I) \to \mathbb{N}$ such that every non-identity morphism lowers degree then $(I, \mathrm{ob}(I), I)$ is a downwards-directed Reedy category.

For the rest of this section we assume that (I, I^{up}, I^{dn}) is a Reedy category and that $\mathcal{M}$ is a given model category.

Definition 13.8. *Let $X\colon I \to \mathcal{M}$ be a diagram, and let i be an object in I.*

*(a) Let $\partial(I^{up} \downarrow i)$ be the full subcategory of $(I^{up} \downarrow i)$ containing all objects except the identity map at i. This is called the **latching category** of I and i.*

*(b) Let $\partial(i \downarrow I^{dn})$ be the full subcategory of $(i \downarrow I^{dn})$ containing all objects except the identity map at i. This is called the **matching category** of I at i.*

*(c) Define the **latching object** of X at i to be*

$$L_i X = \operatorname*{colim}_{\partial(I^{up}\downarrow i)} X.$$

*The natural map $L_i X \to X_i$ is called the **latching map** of X at i.*

*(d) Define the **matching object** of X at i to be*

$$M_i X = \lim_{\partial(i\downarrow I^{dn})} X.$$

*The natural map $X_i \to M_i X$ is called the **matching map** of X at i.*

Let deg be a chosen degree function for the Reedy category I. Let $A, B, X, Y\colon I \to \mathcal{M}$, and consider a lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dotted, "\lambda"] & Y. \end{tikzcd}$$

Imagine a situation in which a partial lifting λ has been produced up through degree $n-1$, and we wish to extend to degree n. We leave it to the reader to check that this is equivalent to producing, for each $i \in \mathrm{ob}(I)$ of degree n, a lifting in the $\mathcal{M}$-diagram

$$\begin{tikzcd} A_i \amalg_{L_i A} L_i B \arrow[r] \arrow[d] & X_i \arrow[d] \\ B_i \arrow[r] \arrow[ur, dotted, "\lambda_i"] & Y_n \times_{M_i Y} M_i X \end{tikzcd}$$

(where all the solid-arrow maps are the evident ones induced by those in the original lifting diagram). This is the key observation behind the following result:

Theorem 13.9 (Kan)**.** *Let I be a Reedy category, and let $\mathcal{M}$ be a model category. Then there is a model category structure on $\mathcal{M}^I$ where*

(a) The weak equivalences are the objectwise weak equivalences;
(b) A map $A \to B$ is a cofibration if and only if the maps $A_i \amalg_{L_i A} L_i B \to B_i$ are $\mathcal{M}$-cofibrations, for all objects i in I.
(c) A map $X \to Y$ is a fibration if and only if the maps $X_i \to Y_i \times_{M_i Y} M_i X$ are $\mathcal{M}$-fibrations for all objects i in I.

Proof. See [H, Theorem 15.3.4] or do it yourself as an easy exercise. □

The following proposition shows, in particular, that Reedy cofibrations are objectwise cofibrations with some extra conditions; and likewise Reedy fibrations are objectwise fibrations with some extra conditions.

Proposition 13.10. *Let I be a Reedy category and $\mathcal{M}$ be a model category. Then*
(a) If $A \to B$ is a Reedy cofibration in $\mathcal{M}^I$ then both $A_i \to B_i$ and $L_i A \to L_i B$ are $\mathcal{M}$-cofibrations, for every object i in I.
(b) If $X \to Y$ is a Reedy fibration in $\mathcal{M}^I$ then both $X_i \to Y_i$ and $M_i X \to M_i Y$ are $\mathcal{M}$-fibrations, for every object i in I.

Proof. See [H, Proposition 15.3.11]. □

Corollary 13.11. *Let I be a Reedy category.*
(a) If $\mathcal{M}$ is a cofibrantly-generated model category then the identity maps give a Quillen pair $\mathcal{M}^I_{proj} \rightleftarrows \mathcal{M}^I_{Reedy}$, where the top map is the left adjoint.
(b) If $\mathcal{M}$ is a combinatorial model category then the identity maps given a Quillen pair $\mathcal{M}^I_{Reedy} \rightleftarrows \mathcal{M}^I_{inj}$, where the top map is the left adjoint.
In both parts, the given Quillen pairs are in fact Quillen equivalences.

Proof. In (a), the right adjoint preserves fibrations and trivial fibrations. In (b), the left adjoint preserves cofibrations and trivial cofibrations. In both parts, the given Quillen functors induced the identity maps on the homotopy categories. But any Quillen pair that induces an equivalence on homotopy categories is a Quillen equivalence [Ho, Proposition 1.3.13]. □

In some cases the Reedy model category structure coincides with either the projective or injective structure:

Proposition 13.12. *Let I be a small category and let $\mathcal{M}$ be a model category.*
(a) If I is an upwards-directed Reedy category then the Reedy and projective model structures on $\mathcal{M}^I$ are equal.
(b) If I is a downards-directed Reedy category then the Reedy and injective model structures on $\mathcal{M}^I$ are equal.

Proof. In (a), all the matching categories are empty and so the matching objects for any I-diagram are all equal to the terminal object. Consequently, $X \to Y$ is a Reedy fibration if and only if $X_i \to Y_i$ is an $\mathcal{M}$-fibration for every i in I. The weak equivalences and fibrations in $\mathcal{M}^I_{Reedy}$ and $\mathcal{M}^I_{proj}$ are therefore equal, hence the cofibrations are also equal. The proof for (b) is the same. □

Note that the adjoint functors $\operatorname{colim}\colon \mathcal{M}^I \rightleftarrows \mathcal{M}\colon c$ do not, in general, give a Quillen pair when $\mathcal{M}^I$ is equipped with the Reedy model structure. It is not true in general that if X is a fibrant object of $\mathcal{M}$ then cX is Reedy-fibrant. Similar remarks hold for the dual pair $c\colon \mathcal{M} \rightleftarrows \mathcal{M}^I\colon \lim$. But this does happen in some situations, and they were completely identified by Hirschhorn:

Theorem 13.13. *Let I be a Reedy category, let $\mathcal{M}$ be a model category, and equip $\mathcal{M}^I$ with the Reedy model structure.*

(a) $\operatorname{colim}\colon \mathcal{M}^I \rightleftarrows \mathcal{M}\colon c$ *is a Quillen pair (with c the right adjoint) if and only if for every object i of I, the matching category $\partial(i \downarrow I^{dn})$ is empty or connected.*

(b) $c\colon \mathcal{M} \rightleftarrows \mathcal{M}^I\colon \lim$ *is a Quillen pair (with c the left adjoint) if and only if for every object i of I, the latching category $\partial(I^{up} \downarrow i)$ is empty or connected.*

Proof. See [H, Section 15.10]. □

Note that part (a) includes all upwards-directed Reedy categories, and part (b) includes all downwards-directed Reedy categories (as we already knew based on Proposition 13.12).

14. Cofibrant diagrams

Suppose $\mathcal{M}$ is a model category, I is a small category, and $X: I \to \mathcal{M}$ is a diagram. If $\mathcal{M}^I$ has a model category structure where $\operatorname{colim}: \mathcal{M}^I \rightleftarrows \mathcal{M}: c$ is a Quillen pair, then the homotopy colimit of X can be computed by finding a cofibrant-replacement $QX \to X$ in $\mathcal{M}^I$ and taking the ordinary colimit of QX. We have seen that our standard formula for homotopy colimits comes from a certain "standard" cofibrant-replacement functor, but the fact that *any* cofibrant-replacement will do sometimes allows one to find a model for the homotopy colimit that is easier to understand. For this reason, it is useful to have some experience determining what cofibrant diagrams look like in different model structures on $\mathcal{M}^I$. We will explore several examples in this section.

14.1. **Upwards-directed Reedy diagrams.** Let I be an upwards-directed Reedy category, and let $X: I \to \mathcal{T}op$ be a diagram. Recall that for each object i in I we have a latching object $L_iX = \operatorname{colim}_{\partial(I\downarrow i)} X$, and this comes equipped with a latching map $L_iX \to X_i$.

Proposition 14.2. *If the latching map $L_iX \to X_i$ is a cofibration for each object i, then X is projective-cofibrant and so $\operatorname{hocolim}_I X \to \operatorname{colim}_I X$ is a weak equivalence.*

Proof. The assumed conditions on the latching maps say precisely that X is Reedy cofibrant. But by Proposition 13.12(a), the Reedy and projective model structure coincide; so X is in fact projective-cofibrant. It is then immediately that $\operatorname{hocolim}_I X \to \operatorname{colim}_I X$ is a weak equivalence. □

We will see several explicit examples of upwards-directed Reedy categories in the further examples below.

14.3. **Pushout diagrams.** Let I denote the pushout category $1 \leftarrow 0 \to 2$. It is easy to see that if $A_1 \leftarrow A_0 \to A_2$ is a diagram in $\mathcal{M}$ then it is projective-cofibrant if and only if A_0 is cofibrant and both $A_0 \to A_1$, $A_0 \to A_2$ are cofibrations. To see this directly, note that a lifting problem

$$\begin{tikzcd} & {[X_1 \leftarrow X_0 \to X_2]} \arrow[d, "\simeq"] \\ {[A_1 \leftarrow A_0 \to A_2]} \arrow[r] & {[Y_1 \leftarrow Y_0 \to Y_2]} \end{tikzcd}$$

can be solved in three steps, by sequentially lifting in the following three diagrams:

$$\begin{tikzcd} & X_0 \arrow[d, "\simeq"] \\ A_0 \arrow[r] \arrow[ur, dotted] & Y_0 \end{tikzcd} \qquad \begin{tikzcd} A_0 \arrow[r] \arrow[d] & X_1 \arrow[d, "\simeq"] \\ A_1 \arrow[r] \arrow[ur, dotted] & Y_1 \end{tikzcd} \qquad \begin{tikzcd} A_0 \arrow[r] \arrow[d] & X_2 \arrow[d, "\simeq"] \\ A_2 \arrow[r] \arrow[ur, dotted] & Y_2, \end{tikzcd}$$

where $A_0 \to X_1$ is the composite $A_0 \to X_0 \to X_1$, and likewise for $A_0 \to X_2$.

Alternatively, we can make I into a Reedy category by having both maps raise degree:

$$\begin{tikzcd} 1 & & 2 \\ & 0. \arrow[ul] \arrow[ur] \end{tikzcd}$$

We know from Proposition 13.12 that the Reedy and projective model structures agree in this case. Given a diagram $A_1 \leftarrow A_0 \rightarrow A_2$, the latching objects are $L_0 A = \emptyset$ and $L_1 A = L_2 A = A_0$; so the condition for Reedy cofibrancy are precisely that $\emptyset \to A_0$, $A_0 \to A_1$, and $A_1 \to A_2$ are all cofibrations.

Since $\operatorname{colim}\colon \mathcal{M}^I_{proj} \to \mathcal{M}$ is a left Quillen functor, we deduce the following:

Proposition 14.4. *If $A_1 \leftarrow A_0 \rightarrow A_2$ is a pushout diagram in a model category $\mathcal{M}$ such that A_0 is cofibrant and $A_0 \to A_1$, $A_1 \to A_2$ are cofibrations, then the natural map* $\operatorname{hocolim} A \to \operatorname{colim} A$ *is a weak equivalence.*

We can improve on the above proposition slightly. Observe that I can be made into a Reedy category in a different way by arranging the arrows as follows:

$$\begin{tikzcd} 2 \\ 0 \arrow[u] \arrow[d] \\ 1 \end{tikzcd}$$

For a diagram $A_1 \leftarrow A_0 \rightarrow A_2$ we then have $L_1 A = \emptyset$, $L_0 A = \emptyset$, and $L_2 A = A_0$. So A is Reedy cofibrant if and only if A_0 and A_1 are cofibrant, and $A_0 \to A_2$ is a cofibration. The matching categories are empty for the objects 0 and 1, and the trivial category with one object for 2; so by Theorem 13.13 $\operatorname{colim}\colon \mathcal{M}^I_{Reedy} \to \mathcal{M}$ is again a Quillen functor. We therefore deduce:

Proposition 14.5. *Let $A_1 \leftarrow A_0 \rightarrow A_2$ be a diagram in a model category $\mathcal{M}$. Then if A_0 and A_1 are cofibrant and $A_0 \to A_2$ is a cofibration, the natural map* $\operatorname{hocolim} A \to \operatorname{colim} A$ *is a weak equivalence.*

Example 14.6. If $A \to X$ is a cofibration between cofibrant objects in $\mathcal{T}op$, then the homotopy pushout of $* \leftarrow A \rightarrow X$ is weakly equivalent to X/A.

14.7. **Coequalizer diagrams.** Using exactly the same kind of analysis as for pushout categories, one can prove the following:

Proposition 14.8. *Let $A_0 \rightrightarrows A_1$ be a coequalizer diagram in a model category $\mathcal{M}$. Then if A_0 is cofibrant and the evident map $A_0 \amalg A_0 \to A_1$ is a cofibration, then A is a projective-cofibrant diagram and the map* $\operatorname{hocolim} A \to \operatorname{colim} A$ *is a weak equivalence.*

Proof. Make the coequalizer category into an upwards-directed Reedy category, so that the Reedy and projective model structures coincide. The latching objects for A are $L_0 A = \emptyset$ and $L_1 A = A_0 \amalg A_0$. □

14.9. **Sequential colimit diagrams.** Let ω denote the category

$$0 \to 1 \to 2 \to \cdots$$

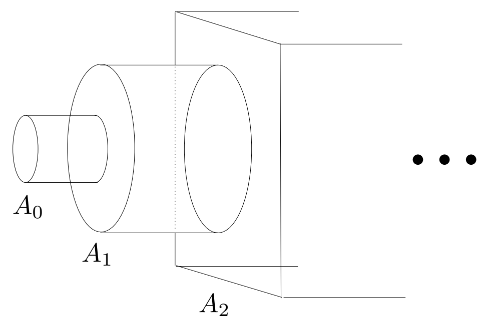
We can again make this into an upwards-directed Reedy category, so that the projective and Reedy model structures coincide. The latching object for a diagram A at level n is just the object A_{n-1}, so the projective-cofibrant objects are the diagrams such that A_0 is cofibrant and $A_n \to A_{n+1}$ is a cofibration, for all n. We deduce the following:

Proposition 14.10. *If $A_0 \to A_1 \to \cdots$ is a diagram in a model category $\mathcal{M}$ such that A_0 is cofibrant and each $A_n \to A_{n+1}$ is a cofibration, then* $\operatorname{hocolim} A \to \operatorname{colim} A$ *is a weak equivalence.*

In the category of topological spaces one often models sequential homotopy colimits by "mapping telescopes": if $A_0 \to A_1 \to \cdots$ is a diagram of topological spaces where all the maps are denoted f, then the telescope is

$$\mathrm{Tel}(A) = \Big(\coprod_n A_n \times I\Big)\Big/\!\sim$$

where the equivalence relation has $(x,1) \sim (f(x),0)$ for all $x \in A_n$ and all $n \geq 0$. There is a picture that goes with this:



Corollary 14.11. *Let $A_0 \to A_1 \to \cdots$ be a diagram of cofibrant spaces in $\mathcal{T}op$. Then the mapping telescope* $\mathrm{Tel}(A)$ *is a model for the homotopy colimit.*

Proof. Define $\mathrm{Tel}_n(A) = \big(\coprod_{i=0}^n A_i\big)/\sim$ where the equivalence relation is the same as for $\mathrm{Tel}(A)$. Then we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Tel}_0(A) \arrow[r, tail] \arrow[d, "\simeq"'] & \mathrm{Tel}_1(A) \arrow[r, tail] \arrow[d, "\simeq"'] & \mathrm{Tel}_2(A) \arrow[r, tail] \arrow[d, "\simeq"'] & \cdots \\
A_0 \arrow[r] & A_1 \arrow[r] & A_2 \arrow[r] & \cdots
\end{tikzcd}$$

where the horizontal maps are the evident inclusions and the vertical maps are the evident projections. This diagram shows

$$\operatorname*{hocolim}_n \mathrm{Tel}_n(A) \to \operatorname*{hocolim}_n A_n$$

is a weak equivalence. But the maps $\mathrm{Tel}_n(A) \to \mathrm{Tel}_{n+1}(A)$ are cofibrations, and so $\mathrm{hocolim}_n \mathrm{Tel}_n(A) \to \mathrm{colim}_n \mathrm{Tel}_n(A)$ is a weak equivalence by Proposition 14.10. Finish by observing that $\mathrm{Tel}(A) = \mathrm{colim}_n \mathrm{Tel}_n(A)$. □

14.12. **Group actions.** Let G be a discrete group and let $\mathbb{B}G$ denote the category with one object and endormorphism monoid G. A $\mathbb{B}G$-diagram in $\mathcal{M}$ is simply an object X of $\mathcal{M}$ with a map of monoids $G \to \mathcal{M}(X,X)$, which is one way of saying that we have a left G-action on X

Here we do not have a Reedy structure on $\mathcal{M}^{\mathbb{B}G}$, so we will assume that $\mathcal{M}$ is cofibrantly-generated with generating cofibrations $\{A_\alpha \rightarrowtail B_\alpha\}$ and will use the projective model structure on $\mathcal{M}^{\mathbb{B}G}$. For an object X in $\mathcal{M}$ the free $\mathbb{B}G$-diagram is $F(X) = G \otimes X$, where recall that this denotes a coproduct of copies of X indexed by the elements of G. The G-action is the evident one, where G acts by permutation of the copies. Then the maps $\{F(A_\alpha) \to F(B_\alpha)\}$ are generating cofibrations for $\mathcal{M}^{\mathbb{B}G}$.

Let us specialize a bit further by taking $\mathcal{M} = \mathcal{T}op$. Then our generating cofibrations are $\{S^{n-1} \hookrightarrow D^n\}$, and the generating cofibrations of $\mathcal{T}op^{\mathbb{B}G}$ are $\{G \times S^{n-1} \hookrightarrow G \times D^n\}$. We conclude:

Proposition 14.13. *Every free G-cell complex X is cofibrant in $\mathcal{T}op^{\mathbb{B}G}$, and so has the property that* $\operatorname{hocolim}_{\mathbb{B}G} X \to X/G$ *is a weak equivalence.*

Let EG be a free G-cell complex whose underlying space is contractible. Define a G-cell complex to be any G-space built up from cell attachments via maps $G/H \times S^{n-1} \hookrightarrow G/H \times D^n$, where H ranges over all subgroups of G. If X is any G-cell complex, then $X \times EG$ has an induced G-cell structure where all the G-cells are free (the basic point here is that $G \times (G/H)$ with its diagonal G-action is isomorphic, as a G-space, to a disjoint union of copies of G). So $X \times EG$ is cofibrant as a $\mathbb{B}G$-diagram, and the projection $X \times EG \to X$ is a weak equivalence of $\mathbb{B}G$-diagrams. We conclude that $\operatorname{colim}_{\mathbb{B}G}(X \times EG)$ is a model for $\operatorname{hocolim}_{\mathbb{B}G} X$. That is:

Proposition 14.14. *If X is any G-cell complex then $(X \times EG)/G$ is a model for $hocolim_{\mathbb{B}G} X$.*

14.15. **Cubical diagrams.** For any set S, let $\mathcal{P}(S)$ be the poset of finite subsets of S, ordered by inclusion. When S is itself finite, $\mathcal{P}(S)$ has a terminal object and often we will want to omit that: so let $i\mathcal{P}(S)$ denote the full subcategory of $\mathcal{P}(S)$ consisting of all *proper* finite subsets. The 'i' stands for 'initial'. We will write $i\mathcal{P}_n$ for $i\mathcal{P}(\{1, \dots, n\})$.

Note that $i\mathcal{P}_2$ is just the pushout category. The category $i\mathcal{P}_3$ is depicted by the diagram

$$\begin{tikzcd}
 & \{3\} \arrow[rr] \arrow[dd] & & \{1,3\} \\
\emptyset \arrow[ur] \arrow[rr] \arrow[ddd] & & \{1\} \arrow[ur] \arrow[dd] & \\
 & \{2,3\} & & \\
\{2\} \arrow[ur] \arrow[rr] & & \{1,2\}, &
\end{tikzcd}$$

and in general the diagram for $i\mathcal{P}_n$ can be depicted by a "punctured n-cube".

There are different ways to regard $i\mathcal{P}_n$ as a Reedy category, but in some sense the most obvious is to make it an upward-directed Reedy category by defining the degree of a subset of be its number of elements. If $A \colon P_n \to \mathcal{T}op$ is a diagram then for any $S \subseteq \{1, 2, \dots, n\}$ we can identify the latching object as

$$L_S(X) = \Big[\coprod_{T \subset S} A_T \Big] \Big/ \sim$$

where the quotient relation says that for any two proper subsets $T, T' \subset S$ and any $x \in A_{T \cap T'}$, the images of x under $A_{T\cap T'} \to A_T$ and $A_{T \cap T'} \to A_{T'}$ are identified. Note that we could also describe $L_S(A)$ as a quotient space $[\coprod A_U]/\sim$ where U runs over the proper subsets of S with $|U| = |S| - 1$.

The following result is just a restatement of the fact that for an upwards-directed Reedy category the Reedy and projective model structures are identical:

Proposition 14.16. *A diagram $A \colon i\mathcal{P}_n \to \mathcal{T}op$ is projective-cofibrant if and only if for every finite $S \subsetneq \{1, 2, \dots, n\}$ the latching map $L_S(A) \to A_S$ is a cofibration.*

Here is a simple application of what we have learned so far. Let X be a topological space, and let $\{A_1, \dots, A_n\}$ be a collection of closed sets which cover X. For any $S \subseteq \{1, 2, \dots, n\}$, set

$$\hat{A}_S = \bigcup_{j \notin S} A_j.$$

For example, $\hat{A}_\emptyset = A_1 \cap \dots \cap A_n$. Notice that this defines a diagram $\hat{A} \colon i\mathcal{P}_n \to \mathcal{T}op$ by having the maps in the diagram be the evident inclusions. The induced map $\operatorname{colim}_{i\mathcal{P}_n} \hat{A} \to X$ is clearly a bijection, and it is easy to check that it is actually a homeomorphism (but note that this uses that we have a *finite* cover). We obtain the following:

Corollary 14.17. *Let $\{A_1, \dots, A_n\}$ be a closed cover for a space X, and let $\hat{A} \colon i\mathcal{P}_n \to \mathcal{T}op$ be as defined above. Assume that for every proper subset $S \subset \{1, \dots, n\}$, the inclusion $\bigcup_{T \subset S} \hat{A}_T \hookrightarrow \hat{A}_S$ is a cofibration. Then* $\operatorname{hocolim}_{i\mathcal{P}_n} \hat{A} \to X$ *is a weak equivalence.*

Proof. One simply checks that the latching maps are the maps $\bigcup_{T \subset S} \hat{A}_T \to \hat{A}_S$. $\square$

Let $A \colon i\mathcal{P}_n \to \mathcal{T}op$ be a diagram. Just as we saw for the pushout category $i\mathcal{P}_2$, one can obtain different criteria for $\operatorname{hocolim} A \to \operatorname{colim} A$ to be a weak equivalence by noticing that $i\mathcal{P}_n$ can be given the structure of a Reedy category in several ways. For example, the following diagram shows a Reedy structure on $i\mathcal{P}_3$:

$$\begin{tikzcd}
 & \{1,2\} & \\
\{1\} \arrow[ur] \arrow[d] & & \{2\} \arrow[ul] \arrow[ddd] \\
\{1,3\} & \emptyset \arrow[ul] \arrow[ur] \arrow[d] & \\
 & \{3\} \arrow[ul] \arrow[dr] & \\
 & & \{2,3\}.
\end{tikzcd}$$

More precisely, the degrees of objects are determined by their vertical position on the page (with the lowest object having degree zero, for example). The category $i\mathcal{P}_3{}^{up}$ is the subcategory generated by all of the drawn maps that raise degree, and $i\mathcal{P}_3{}^{dn}$ is the subcategory generated by the drawn maps that lower degree.

For this Reedy structure to apply to our problem we need to check two things:

(1) Every map in the diagram has a unique factorization as a down-map followed by an up-map (the condition for being a Reedy category);
(2) For each spot in the diagram, the matching category is either empty or connected (the condition for $\operatorname{colim} \colon \mathcal{M}^I \rightleftarrows \mathcal{M} \colon c$ to be a Quillen pair, cf. Theorem 13.13).

In this case both are easy and left to the reader. Consequently, we obtain the following:

Proposition 14.18. *Suppose given a diagram $A \colon i\mathcal{P}_3 \to \mathcal{T}op$ such that*

(1) all objects are cofibrant,

(2) $A_3 \to A_{13}$ *is a cofibration,*
(3) $A_\emptyset \to A_1$ *and* $A_\emptyset \to A_2$ *are cofibrations, and*
(4) The induced map $A_1 \amalg_{A_\emptyset} A_2 \to A_{12}$ *is a cofibration.*

Then the natural map $\operatorname{hocolim} A \to \operatorname{colim} A$ *is a weak equivalence.*

Proof. The listed conditions are just the various latching-map conditions for a diagram to be Reedy cofibrant. □

Exercise 14.19. Consider the following two pictures showing proposed Reedy structures on $i\mathcal{P}_3$:

$$\begin{tikzcd}
 & \{1,2\} & \\
\{1\} \arrow[ur] \arrow[d] & & \\
\{1,3\} & \emptyset \arrow[ul] \arrow[d] \arrow[dr] & \\
 & \{3\} \arrow[ul] \arrow[dr] & \{2\} \arrow[uuul] \arrow[d] \\
 & & \{2,3\}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \{3\} \arrow[dl] \arrow[dr] & \\
\{1,3\} & & \{2,3\} \\
 & \emptyset \arrow[uu] \arrow[dl] \arrow[dr] & \\
\{1\} \arrow[uu] \arrow[dr] & & \{2\} \arrow[uu] \arrow[dl] \\
 & \{1,2\} &
\end{tikzcd}$$

One of these is not, in fact, a Reedy structure. The other one *is* a Reedy structure but does not satisfy the property about matching categories needed for Hirschhorn's Theorem (13.13). Which is which?

15. Diagrams in the homotopy category

The canonical functor $\mathcal{M} \to \mathcal{H}o(\mathcal{M})$ induces $\mathcal{M}^I \to \mathcal{H}o(\mathcal{M})^I$, and clearly this latter map sends objectwise weak equivalences to isomorphisms. So there is an induced functor $\mathcal{H}o(\mathcal{M}^I) \to \mathcal{H}o(\mathcal{M})^I$. This is typically far from being an equivalence, so let us investigate some examples demonstrating the differences. We will use the model category $\mathcal{T}op_*$ of pointed topological spaces; the basepoints are used here for convenience, rather than for any essential reason.

Example 15.1 (Differences in morphisms)**.** Let I be the category $0 \to 1$, so that $\mathcal{T}op_*^I$ is the category of maps in $\mathcal{T}op_*$. Morphisms in $\mathcal{H}o(\mathcal{T}op_*)^I$ from $[S^n \to *]$ to $[j\colon A \to X]$ are in bijective correspondence with $\{f \in \pi_n(A) \,|\, j \circ f \simeq *\}$. To compute morphisms in $\mathcal{H}o(\mathcal{T}op_*^I)$ we first replace $[S^n \to *]$ with the cofibrant model $[S^n \hookrightarrow D^{n+1}]$, then we compute homotopy classes of maps from this model into $[A \to X]$. One readily checks that this coincides with the classical definition of the relative homotopy group $\pi_{n+1}(X, A)$.

Note that when $A = *$ then

$$\mathcal{H}o(\mathcal{T}op_*)^I\Big([S^n \to *], [* \to X]\Big) = \{*\}$$

whereas

$$\mathcal{H}o(\mathcal{T}op_*^I)\Big([S^n \to *], [* \to X]\Big) = \pi_{n+1}(X).$$

So clearly these can be different.

The map $\mathcal{H}o(\mathcal{T}op_*^I)([S^n \to *], [A \to X]) \to \mathcal{H}o(\mathcal{T}op_*)^I([S^n \to *], [A \to X])$ takes the form

$$\pi_{n+1}(X, A) \to \{f \in \pi_n(A) \,|\, j \circ f \simeq *\}$$

and is readily checked to be the boundary map in the long exact sequence for relative homotopy groups. Exactness precisely says that the above map is surjective.

It is a general fact that for I equal to $0 \to 1$ that $\mathcal{H}o(\mathcal{T}op_*^I) \to \mathcal{H}o(\mathcal{T}op_*)^I$ always induces surjections on sets or morphisms (we leave this as an exercise for the reader). It is possible to given categories I for which this is not true, but coming up with concrete examples is difficult.

Example 15.2 (Differences in objects)**.** We next give an example where $\mathcal{H}o(\mathcal{T}op_*^I) \to \mathcal{H}o(\mathcal{T}op_*)^I$ is not surjective on isomorphism classes. Starting with a diagram $X\colon I \to \mathcal{H}o(\mathcal{T}op_*)$, a lifting to a diagram $\tilde{X}\colon I \to \mathcal{T}op_*$ is called a **rigidification** of X (or sometimes a *realization* of X). So our example will show that rigidifications do not always exist.

The following example is due to Cooke [Co]. Let I be the category with one object and endomorphism monoid $\mathbb{Z}/2$. An object of $\mathcal{H}o(\mathcal{T}op_*)^I$ can be described as a homotopy $\mathbb{Z}/2$-action on a space X. Cooke constructed such a homotopy action that could not be rigidified into an "honest" action. To describe this it will be useful for us to work stably, which means replacing $\mathcal{T}op_*$ with the category of spectra. We will do this, but at the same time be casual in our notation and write things like S^k instead of $\Sigma^\infty(S^k)$. If one does not want to use spectra, one can make all of the arguments below in $\mathcal{T}op_*$ by simply suspending enough times.

Before giving Cooke's construction let us describe his obstruction to the existence of rigidifications. If X is a space and $f\colon X \to X$ is such that $f^2 \simeq id$, choose a homotopy $H\colon X \times I \to X$ such that $H_0 = id$ and $H_1 = f^2$. Then $H \circ (f \times id)$ and $f \circ H$ are both homotopies from f to f^3. We regard these as paths in the mapping space $\mathrm{Map}(X,X)$. Since these two paths have the same beginning and ending points, we can use them to make a loop: set

$$J_{f,H} = H(f \times id) * \overline{fH} \in \pi_1(\mathrm{Map}(X,X), f).$$

Here the overline indicates that the path fH is run in reverse.

The loop $J_{f,H}$ depends on the choice of homotopy H, so we set

$$S_f = \{J_{f,H} \mid H \text{ is a homotopy from } id \text{ to } f^2\} \subseteq \pi_1(\mathrm{Map}(X,X), f).$$

This set is an invariant of the pair (X, f). One can readily check that if (X, f) has a rigidification then S_f contains the constant path. Cooke constructed a pair (X, f) where he could prove that S_f does *not* contain the constant path, and so concluded that the pair did not have a rigidication.

A similar invariant is the Toda bracket $\langle f-1, f+1, f-1\rangle \subseteq [\Sigma X, X]_*$. If (X, f) is rigidifiable then this Toda bracket must contain the null map.

Let $\alpha \in \pi_{n-1}(S^k)$ be any element whose order is a multiple of 4 (the first such example is $\nu \in \pi_7(S^4)$). Let $A = S^k \cup_{2\alpha} e^n$, and let $X = A \vee S^{n-1} \vee S^{n-1}$. Let $i\colon S^k \hookrightarrow A$ be the inclusion of the bottom cell, and let $\pi\colon A \to S^n$ be the map which squashes the bottom cell to a point. Since the stable homotopy category is additive we can describe maps $X \to X$ via 3×3 matrices. Let f be the element of $[X, X]_*$ represented by the matrix

$$\begin{bmatrix} I & i\alpha & 0 \\ 0 & I & 0 \\ \eta\pi & 0 & I. \end{bmatrix}$$

Matrix multiplication shows that f^2 is represented by

$$\begin{bmatrix} I & 2i\alpha & 0 \\ 0 & I & 0 \\ 2\eta\pi & \eta\pi i\alpha & I \end{bmatrix}$$

and since $2i\alpha\colon S^{n-1} \to A$, $2\eta\colon S^n \to S^{n-1}$, and πi are all null-homotopic we find that $f^2 \simeq id$. So (X, f) is a homotopy $\mathbb{Z}/2$-action.

Example 15.3 (Colimits in the homotopy category)**.** We give an example showing that $\mathcal{H}o(\mathcal{T}op_*)^I$ generally does not have colimits. Let I be the pushout category $1 \leftarrow 0 \rightarrow 2$, and let X be the diagram $* \longleftarrow S^1 \xrightarrow{\times 2} S^1$ in $\mathcal{H}o(\mathcal{T}op_*)^I$. It is easy to see that for any pointed space Z one has a natural bijection

$$\mathcal{H}o(\mathcal{T}op_*)^I(X, cZ) \cong \{a \in \pi_1(Z) \mid a^2 = 0\}.$$

Let us write $\Theta(Z)$ for the group on the right. If W is a colimit for X in $\mathcal{H}o(\mathcal{T}op_*)$ then there is a natural bijection $\Theta(Z) \cong \mathcal{H}o(\mathcal{T}op_*)(W, Z)$. Consequently, if $Z_1 \to Z_2 \to Z_3$ is a fiber sequence in $\mathcal{H}o(\mathcal{T}op_*)$ then the sequence

$$\Theta(Z_1) \to \Theta(Z_2) \to \Theta(Z_3)$$

is exact (as a sequence of pointed sets) in the middle spot. But it is easy to find counterexamples to this. For example, consider the fiber sequence

$$K(\mathbb{Z}/4, 1) \to K(\mathbb{Z}/2, 1) \xrightarrow{\mathrm{Sq}^1} K(\mathbb{Z}/2, 2).$$

Applying Θ gives the sequence

$$\{0, 2\} \to \{0, 1\} \to \{0\}$$

but where the first map sends both elements to 0; clearly this is not exact. Another example is the fiber sequence $S^1 \to \mathbb{R}P^\infty \to \mathbb{C}P^\infty$. In any case, we have a contradiction and so conclude that X does not have a colimit in $\mathcal{H}o(\mathcal{T}op_*)$.

16. Homotopy coherent diagrams

16.1. **Introduction.** Let I be a small category. A diagram $X\colon I \to \mathcal{T}op$ is a pair of assignments $i \mapsto X_i$ and $[f\colon i \longrightarrow j] \mapsto [X_f\colon X_i \to X_j]$ such that $X_{id} = id$ and for every pair of composable morphisms $[i \xrightarrow{f} j \xrightarrow{g} k]$ the diagram

$$\begin{tikzcd} X_i \arrow[r, "X_f"] \arrow[dr, "X_{gf}"'] & X_j \arrow[d, "X_g"] \\ & X_k \end{tikzcd}$$

is commutative. A *homotopy commutative* diagram is a similar pair of assignments but where the triangle is only required to commute up to homotopy: that is, there must exist homotopies $H_{g,f}\colon X_i \times I \to X_k$ such that $H_0 = X_g \circ X_f$ and $H_1 = X_{gf}$, but no specific homotopy is assumed to be chosen.

Suppose that we actually choose homotopies $H_{g,f}$, for every composable pair. Then for every composable triple $h \circ g \circ f$ we get a sequence of homotopies

$$X_h \circ (X_g \circ X_f) \text{ --- } X_h \circ X_{gf} \text{ --- } X_{h(gf)} = X_{(hg)f} \text{ --- } X_{hg} \circ X_f \text{ --- } (X_h \circ X_g) \circ X_f.$$

That is, we get an explicit set of homotopies $H_{h,g,f}$ demonstrating that composition of the X-maps is homotopy-associative. Well, for any composable 4-tuple

$$i_0 \xrightarrow{f} i_1 \xrightarrow{g} i_2 \xrightarrow{h} i_3 \xrightarrow{j} i_4$$

we then get the classical Stasheff pentagon

$$\begin{tikzcd}[column sep=small] & & X_j(X_h(X_gX_f)) \arrow[dll, dash] \arrow[drr, dash] & & \\ (X_jX_h)(X_gX_f) \arrow[dr, dash] & & & & X_j((X_hX_g)X_f) \arrow[dl, dash] \\ & ((X_jX_h)X_g)X_f \arrow[rr, dash] & & (X_j(X_hX_g))X_f. & \end{tikzcd}$$

These five homotopies assemble to give a loop in the mapping space $\mathrm{Map}(X_{i_0}, X_{i_4})$, and we can ask that these loops all be null-homotopic. This is a kind of "higher homotopy" condition on our diagram. Moreover, explicit choices of null-homotopies for these Stasheff pentagons then lead to even *higher* homotopy elements in various mapping spaces, which we can again require to be null-homotopic.

There is some bookkeeping required to make sense of all this, but very briefly a *homotopy coherent* diagram is a homotopy commutative diagram together with an explicit choice of higher and higher homotopies demonstrating that various homotopy elements of mapping spaces are actually null.

Note that every "honest" diagram $X\colon I \to \mathcal{T}op$ will yield a homotopy coherent diagram, by simply taking all the required homotopies to be constant. Likewise, evey homotopy coherent diagram yields a homotopy commutative diagram, simply by forgetting the choices of all the higher homotopies.

The theory of homotopy coherent diagrams goes back to Vogt, Dwyer-Kan, and others ?????. Some of the main points are:

(1) There are notions of homotopy colimit and homotopy limit that make sense for homotopy coherent diagrams.
(2) Every homotopy coherent diagram can be "rigidified" into an honest diagram. Moreover, the homotopy theories of honest diagrams and homotopy coherent diagrams are equivalent.
(3) Given a homotopy commutative diagram, there is an obstruction theory for giving it the structure of a homotopy coherent diagram. In light of (2), this is an obstruction theory for "rigidifying" the given homotopy commutative diagram into an honest diagram.

16.2. **Simplicial diagams.** Let $\mathcal{C}at$ denote the category of small categories. By a **simplicial category** we mean a simplicial object in $\mathcal{C}at$ where the categories in each level have the same object set. Alternatively, we may regard such a thing as a category enriched over $s\mathcal{S}et$. To be concrete, a (small) simplicial category I consists of

(1) A set of objects (denoted I by abuse);
(2) For each $i, j \in I$, a simplicial set $I(i,j)$;
(3) For each $i \in I$, a distinguished 0-simplex $id_i \in I(i,i)$;
(4) For each $i, j, k \in I$, composition maps $I(j,k) \times I(i,j) \to I(i,k)$ which satisfy associativity and unital axioms.

One defines functors between simplicial categories in the evident manner.

If $\mathcal{C}$ is another simplicial category, then an I-diagram in $\mathcal{C}$ is just a functor $X\colon I \to \mathcal{C}$. Concretely, this consists of a collection of objects $X_i \in \mathcal{C}$ together with maps of simplicial sets $I(i,j) \to \mathcal{C}(X_i, X_j)$ for each $i, j \in I$ such that

(1) id_i maps to id_{X_i}, and
(2) for each $i, j, k \in I$, the diagram

$$\begin{tikzcd} I(j,k) \times I(i,j) \arrow[r] \arrow[d] & I(i,k) \arrow[d] \\ \mathcal{C}(X_j, X_k) \times \mathcal{C}(X_i, X_j) \arrow[r] & \mathcal{C}(X_i, X_k) \end{tikzcd}$$

commutes.

If $X, Y\colon I \to \mathcal{C}$ are two functors then a natural transformation from X to Y is a collection of maps $f_i\colon X_i \to Y_i$ such that for any objects i and j, the diagram

$$\begin{tikzcd} I(i,j) \arrow[r] \arrow[d] & \mathcal{C}(Y_i, Y_j) \arrow[d, "(-)\circ f_i"] \\ \mathcal{C}(X_i, X_j) \arrow[r, "f_j \circ (-)"] & \mathcal{C}(X_i, Y_j) \end{tikzcd}$$

commutes. Here the bottom horizontal map is the composite

$$\mathcal{C}(X_i, X_j) = \{f_j\} \times \mathcal{C}(X_i, X_j) \longrightarrow \mathcal{C}(X_j, Y_j) \times \mathcal{C}(X_i, X_j) \longrightarrow \mathcal{C}(X_i, Y_j),$$

and similary for the right vertical map.

Write $\mathcal{C}^I$ for the category of all I-diagrams in $\mathcal{C}$. This is itself a simplicial category, under the following definition. Given $X, Y\colon I \to \mathcal{C}$, define an n-simplex of $\mathrm{Map}(X,Y)$ to be a collection of maps $\Delta^n \to \mathcal{C}(X_i, Y_i)$ (for every object i) such that for every two objects i and j the following diagram commutes:

$$\begin{tikzcd}
\Delta^n \times I(i,j) \arrow[rr, "\cong"] \arrow[d] & & I(i,j) \times \Delta^n \arrow[d] \\
\mathcal{C}(X_j, Y_j) \times \mathcal{C}(X_i, X_j) \arrow[dr] & & \mathcal{C}(Y_i, Y_j) \times \mathcal{C}(X_i, Y_i) \arrow[dl] \\
& \mathcal{C}(X_i, Y_j). &
\end{tikzcd}$$

It takes a moment to verify that there are evident maps $\mathrm{Map}(Y,Z)\times\mathrm{Map}(X,Y) \to \mathrm{Map}(X,Z)$ satisfying the necessary associativity and unital axioms.

Definition 16.3. *A **simplicially-powered category** $\mathcal{C}$ is a simplicial category together with functors $\otimes\colon s\mathcal{S}et \times \mathcal{C}_0 \to \mathcal{C}_0$ and $F\colon s\mathcal{S}et^{op} \times \mathcal{C}_0 \to \mathcal{C}_0$ such that the following axioms are satisfied:*
(1) $\mathcal{C}(X \otimes K, Y) \cong \mathrm{Map}(K, \mathcal{C}(X,Y)) \cong \mathcal{C}(X, F(K,Y))$.
(2) ???

Note that this actually has the structure of a simplicial category, and is simplicially tensored and cotensored if $\mathcal{C}$ is so. ?????

Remark 16.4. Now let $\mathcal{M}$ be a simplicial model category, and let $X\colon I \to \mathcal{M}$ be a diagram. Note that the maps $I(i,j) \to \underline{\mathcal{M}}(X_i, X_j)$ yield maps $I(i,j)\otimes X_i \to X_j$ via adjointness. So an I-diagram in $\mathcal{M}$ can be thought of as a collection of objects X_i and a collection of 'action' maps $I(i,j)\otimes X_i \to X_j$ satisfying the evident associativity and unital conditions. Just as we did for diagrams indexed by ordinary categories, we will think of diagrams $I \to \mathcal{M}$ as 'left I-modules'.

Let $\mathcal{C}$ be simplicially tensored. Then for any i in I, the evaluation functor $\mathrm{ev}_i\colon \mathcal{C}^I \to \mathcal{C}$ has a left adjoint F_i given by

$$[F_i(A)]_j = I(i,j) \otimes A.$$

The structure maps $I(j,k)\otimes [F_i(A)]_j \to [F_i(A)]_k$ are the evident ones that use the composition pairings in I. We leave it to the reader to verify that this is indeed a left adjoint to ev_i.

Theorem 16.5 (???). *Let $\mathcal{M}$ be a cofibrantly-generated, simplicial model category. Then $\mathcal{M}^I$ has a model category structure where the weak equivalences and fibrations are determined objectwise. This is called the **projective model structure** on $\mathcal{M}^I$. Moreover, the canonical simplicial structure on $\mathcal{M}^I$ makes this into a simplicial model category.*

Let $\alpha\colon I \to J$ be a map of simplicial categories. Then there is a restriction functor $\alpha^*\colon \mathcal{M}^J \to \mathcal{M}^I$, and this functor has both left and right adjoints, denoted L_α and R_α. If $X\colon I \to \mathcal{M}$ then $L_\alpha X$ is given by

$$[L_\alpha X]_j = \mathrm{coeq}\left[\coprod_i J(\alpha i, j)\otimes X_i \leftleftarrows \coprod_{i_0,i_1} J(\alpha i_1, j)\otimes I(i_0,i_1)\otimes X_{i_0}\right]$$

where the two maps in the coequalizer come from the pairings

$$I(i_0,i_1)\otimes X_{i_0} \longrightarrow X_{i_1},$$
$$J(\alpha i_1, j)\otimes I(i_0,i_1) \longrightarrow J(\alpha i_1, j)\otimes J(\alpha i_0, \alpha i_1) \longrightarrow J(\alpha i_0, j).$$

We leave it to the reader to check that there are evident pairings

$$J(j,j')\otimes[L_\alpha X]_j \to [L_\alpha X]_{j'}$$

making $L_\alpha X$ into a J-diagram, and to verify that this is indeed left adjoint to α^*.

Note that α^* preserves objectwise weak equivalences and objectwise fibrations, and so we have a Quillen pair

$$L_\alpha\colon \mathcal{M}^I \rightleftarrows \mathcal{M}^J : \alpha^*.$$

Theorem 16.6 (???)**.** *Let $\alpha\colon I \to J$ be a map of simplicial categories having the same object set (and assume that α is the identity on objects). Suppose that $I(i,j)\to J(i,j)$ is a weak equivalence for all objects i and j. Then the adjoint pair $L_\alpha\colon \mathcal{M}^I \rightleftarrows \mathcal{M}^J : \alpha^*$ is a Quillen equivalence.*

Before proving this result let us introduce the version of the bar construction that is relevant to simplicial diagrams. We will be concerned with the object $B_\bullet(J,I,X)$, which to every j in J associates the simplicial object

$$[n] \mapsto \coprod_{i_0,i_1,\dots,i_n}\Big[J(\alpha i_0,j)\otimes I(i_1,i_0)\otimes\cdots\otimes I(i_n,i_{n-1})\otimes X_{i_n}\Big].$$

Note the similarities to (11.2). The tensor product of the I-factors generalizes the strings $i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n$ that we used in the version for non-simplicial categories.

?????

Proof of Theorem 16.6. Let X be a J-diagram. We must show that $[L_\alpha Q(\alpha^*X)] \to X$ is an objectwise weak equivalence, so let j be an object in J. The key step is to examine the following map of simplicial objects:

$$\begin{array}{ccc}
\coprod_{i_0} J(\alpha i_0,j)\otimes X_{i_0} & \Lleftarrow & \coprod_{i_0,i_1} J(\alpha i_1,j)\otimes I(i_0,i_1)\otimes X_{i_1} \Lleftarrow \cdots \\
\downarrow & & \downarrow \\
\coprod_{j_0} J(j_0,j)\otimes X_{j_0} & \Lleftarrow & \coprod_{j_0,j_1} J(j_1,j)\otimes J(j_0,j_1)\otimes X_{j_1} \Lleftarrow \cdots
\end{array}$$

where in level n the vertical map is induced by the maps $\alpha\colon I(a,b)\to J(a,b)$ and on indices can be described by the "substitution" $i_r \mapsto j_r$ for $r<n$ and $\alpha i_n \to j_n$. The hypotheses on α imply that the above is a levelwise weak equivalence, and so it induces a weak equivalence on realizations. The bottom simplicial object has a contracting homotopy induced by the identity map on j (it is an instructive exercise to write this down). This completes the argument. For Y an I-diagram, the argument for showing that $Y \to \alpha^*[L_\alpha(QY)]$ is an objectwise weak equivalence is very similar.

Now we give the slick version. The map we are trying to analyze is the composite

$$B(J,I,X) \longrightarrow B(J,J,X) \longrightarrow X.$$

The first map is an objectwise equivalence because $I \to J$ is so, and the second map is always an objectwise equivalence. For the other direction we are looking at

$$Y \longrightarrow B(I,I,Y) \longrightarrow B(J,I,Y).$$

Again, the first map is always an objectwise equivalence and the second map is so because of the hypothesis on α. $\square$

Finally, we note that the theory of homotopy colimits for simplicial diagrams works just as for ordinary diagrams, via the evident modifications. For example, if $X\colon I \to \mathcal{C}$ is a simplicial diagram then its homotopy colimit is defined to be the realization of the simplicial replacement

$$\coprod_{i_0} X(i_0) \Leftarrow \coprod_{i_0,i_1} I(i_1,i_0)\otimes X_{i_1} \Lleftarrow \coprod_{i_0,i_1,i_2} I(i_1,i_0)\otimes I(i_2,i_1)\otimes X_{i_2} \Lleftarrow \cdots$$

We leave the reader to check to develop the details of the theory for him- or herself.

16.7. **Resolutions of categories.** Let I be an ordinary category. We can regard I as a simplicial category by regarding all its mappings sets as discrete simplicial sets. By a **resolution of** I we mean a simplicial category $\tilde{I}$ with the same set of objects as I, together with a map of simplicial categories $\tilde{I} \to I$ with the property that for every $i, j \in I$ the map $\tilde{I}(i,j) \to I(i,j)$ is a weak equivalence.

If S is a set, define a **graph** $\mathcal{G}$ with object set S to be an assignment $(s,t) \mapsto \mathcal{G}(s,t)$ for $s,t \in S$. Morphisms of graphs are the evident ones, and $\mathcal{G}raph(S)$ will denote the category of all graphs with object set S. Likewise, let $\mathcal{C}at(S)$ denote the category of small categories with object set S.

There is a forgetful functor $\tilde{U}\colon \mathcal{C}at(S) \to \mathcal{G}raph(S)$, and this has a left adjoint $\tilde{F}$: if $\mathcal{G}$ is a graph, then the morphisms in $\tilde{F}(\mathcal{G})$ are formal compositions of the elements in the appropriate $\mathcal{G}(s,t)$ sets. If I is an object in $\mathcal{C}at(S)$ we can look at the (augmented) bar resolution

$$\cdots \Rrightarrow \tilde{F}\tilde{U}\tilde{F}\tilde{U}\tilde{F}\tilde{U}(I) \Rrightarrow \tilde{F}\tilde{U}\tilde{F}\tilde{U}(I) \rightrightarrows \tilde{F}\tilde{U}(I) \longrightarrow I$$

from Example 3.15. Write $(\tilde{F}\tilde{U})_\bullet(I)$ for the simplicial category (without the augmentation). Applying $\tilde{U}$, we pick up a contracting homotopy; this shows that $[(\tilde{F}\tilde{U})_\bullet(I)](i,j) \to I(i,j)$ is a weak equivalence for all objects i and j. In other words, $(\tilde{F}\tilde{U})_\bullet(I) \to I$ is a resolution of categories.

There is a variant of this construction that is also useful. Define the **trivial pointed graph** on a set S to be graph $[S]$ for which $[S](s,t) = \emptyset$ if $s \neq t$ and $[S](s,s) = \{*\}$ for every $s \in S$. Define a **pointed graph** on a set S to a graph $\mathcal{G}$ together with a morphism $[S] \to \mathcal{G}$. Let $\mathcal{G}raph_*(S)$ be the category of pointed graphs on S. There is then an evident forgetful functor $U\colon \mathcal{C}at(S) \to \mathcal{G}raph_*(S)$, where the extra "pointed" structure consists of the identity maps. This has a left adjoint $F\colon \mathcal{G}raph_*(S) \to \mathcal{C}at(S)$ which can be described as "take the category freely generated by the non-identity edges of the graph". One still has that $(FU)_\bullet(I) \to I$ is a resolution of I by free categories.

While both constructions are large, we prefer $(FU)_\bullet(I)$ because it is noticeably smaller than $(\tilde{F}\tilde{U})_\bullet(I)$. It is convenient to not have to add formal generators correponding to identity maps when one forms $(FU)(I)$.

Definition 16.8. *Let I be a small category and let $\mathcal{M}$ be a simplicial model category. A* ***homotopy coherent diagram*** *in $\mathcal{M}$ is a simplicial functor $(FU)_\bullet(I) \to \mathcal{M}$. The category of homotopy coherent diagrams is $\mathcal{M}^{(FU)_\bullet(I)}$.*

We aim to investigate the simplicial categories $(FU)_\bullet(I)$ in some examples. Let $[n]$ denote the category $0 \to 1 \to \cdots \to n$. That is, the object set is $\{0, 1, \ldots, n\}$ and there is a unique map $i \to j$ when $i \leq j$. We will investigate the simplicial categories $(FU)_\bullet([n])$.

It is immediate that $(FU)([0]) = [0]$ (note that this would not be true if we used $(\tilde{F}\tilde{U})$), and also that $(FU)([1]) = [1]$. So we have $(FU)_\bullet([0]) = c[0]$ and $(FU)_\bullet([1]) = c[1]$, where the $c(-)$ means regard the argument as a constant simplicial category.

The first place things are interesting is with $(FU)([2])$. Let the maps in $[2]$ be denoted $0 \xrightarrow{g} 1 \xrightarrow{f} 2$. If u is a morphism in a category I, write $[u]$ for the corresponding "free generator" in the category $FU(I)$. Then $(FU)([2])$ is the category

$$\begin{tikzcd} & 1 \arrow[dr, "{[f]}"] & \\ 0 \arrow[ur, "{[g]}"] \arrow[rr, shift left, "{[f][g]}"] \arrow[rr, shift right, "{[fg]}"'] & & 2 \end{tikzcd}$$

and $(FUFU)([2])$ is the category

$$\begin{tikzcd} & 1 \arrow[dr, "{[[f]]}"] & \\ 0 \arrow[ur, "{[[g]]}"] \arrow[rr, shift left, "{[[f][g]]}"] \arrow[rr, shift right, "{[[fg]]}"'] \arrow[rr, bend right=40, "{[[f]][[g]]}"'] & & 2. \end{tikzcd}$$

In general, $(FU)^n([2])$ has exactly one map from 0 to 1, one map from 1 to 2, and exactly $n+1$ maps from 0 to 2. A little work verifies that the morphism simplicial sets are

$$\mathrm{Hom}_{(FU)_\bullet([2])}(i,j) = \begin{cases} * & \text{if } i<j \text{ and } (i,j) \neq (0,2) \\ \Delta^1 & \text{if } (i,j) = (0,2). \end{cases}$$

For $(FU)_\bullet([3])$, the full subcategory consisting of objects 0, 1, and 2 is readily identified with what we just described. Similarly for the full subcategory consisting of objects 1, 2, and 3. The only thing new is the space of morphisms from 0 to 3. We leave the reader to verify that there are four morphisms from 0 to 3 in $(FU)([3])$, and nine in $(FUFU)([3])$.

Definition 16.9. *Fix $n \geq 1$. For $0 \leq i \leq j \leq n$ let $P_{i,j}$ denote the poset of subsets of $\{i, i+1, \ldots, j\}$ that contain i and j. Let $\mu\colon P_{j,k} \times P_{i,j} \to P_{i,k}$ be the unique functor given on objects by union of subsets.*

Let NP_n be the simplicial category whose object set is $\{0, 1, \ldots, n\}$ and whose simplicial set of morphisms from i to j is the nerve $NP_{i,j}$, with composition induced by the maps μ.

Remark 16.10. Note that if $j > i$ then $P_{i,j}$ is isomorphic to the poset consisting of all subsets of $\{i+1, \ldots, j-1\}$, and so $NP_{i,j} \cong (\Delta^1)^{j-i-1}$.

Proposition 16.11. *There is an isomorphism of simplicial categories $(FU)_\bullet([n]) \cong NP_n$.*

Proof. The proof we give is adopted from [DS, Section A.7]. Let m_i denote the unique map in $[n]$ from $i-1$ to i. Then morphisms from i to j in $FU([n])$ are in bijective correspondence with bracketings of the expression $m_j m_{j-1} \cdots m_{i+1}$ having the property that every m_r is inside exactly one set of brackets. For example, the maps from [0] to [3] are

$$[m_3m_2m_1],\quad [m_3m_2][m_1],\quad [m_3][m_2m_1],\quad [m_3][m_2][m_1].$$

Such bracketings can be parameterized by subsets of $\{i, i+1, \ldots, j\}$ containing i and j: namely, send an expression to the union of $\{i\}$ and the set of indices that occur immediately after a left bracket. In order, the bracketed expressions listed above correspond to $\{0,3\}$, $\{0,1,3\}$, $\{0,2,3\}$, $\{0,1,2,3\}$.

If one thinks of maps in $FU([n])$ as formal compositions of maps in $[n]$, the associated subset can be thought of as the set of "intermediate stops" in the formal composition.

Generalizing the above analysis, maps from i to j in $(FU)^r([n])$ correspond to bracketings of the expression $m_jm_{j-1}\cdots m_{i+1}$ having the property that every m_s is inside exactly t sets of brackets. For example, when $r=3$ here are some maps from 0 to 3:

$$[[[m_3m_2m_1]]],\quad [[m_3][m_2]][[m_1]],\quad [[m_3m_2][m_1]],\ldots$$

In such a bracketed expression, rank the brackets by "interiority": outermost brackets have rank 0, and innermost brackets have rank $r-1$. The face map d_j corresponds to removing all brackets of rank j, whereas the degeneracy s_j amounts to doubling all brackets of rank j. For example,

$$d_0\Big([[m_3][m_2m_1]]\Big) = [m_3][m_2m_1],\quad d_1\Big([[m_3][m_2m_1]]\Big) = [m_3m_2m_1],$$

$$s_0\Big([[m_3][m_2m_1]]\Big) = [[[m_3][m_2m_1]]],\quad s_1\Big([[m_3][m_2m_1]]\Big) = [[[m_3]][[m_2m_1]].$$

Given a bracketed expression ω, define the sth vertex $v_s(\omega)$ to be the expression obtained by removing all brackets except those of rank s. Using the bijection between simple bracketed expressions and subsets that we discussed already, each $v_s(\omega)$ corresponds to a subset S_s of $\{i, i+1, \ldots, j\}$. The fact that the rank s brackets are inside the rank $s-1$ brackets implies that $S_{s-1} \subseteq S_s$.

A bracketed expression with s layers therefore corresponds to a chain $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_{s-1} \subseteq \{i, i+1, \ldots, j\}$, and it is easy to see that this indeed gives a bijection. It is immediate to check that this is compatible with face and degeneracy operators, giving an isomorphism of simplicial sets $(FU)_\bullet([n])(i,j) \cong NP_{i,j}$.

Composition in $(FU)_\bullet([n])$ corresponds to concatenation of bracketed expressions, which in turn clearly corresponds to the union of subsets. A little thought completes the proof. $\square$

Part 4. Other useful tools

17. Homology and cohomology of categories

This section is really a prelude to the following one. If $D\colon I \to \mathcal{T}op$ is a diagram and $E_*(-)$ is a homology theory, it turns out that there is a certain spectral sequence that starts with the groups $E_*(D_i)$ and computes the groups $E_*(\operatorname{hocolim} D)$. The starting page for this spectral sequences is a collection of groups written $H_p(I; E_q(D))$ that are called the "homology of I with coefficients in the functor $E_q(D)$". Here $E_q(D)$ denotes the diagram $I \to \mathcal{A}b$ given by $i \mapsto E_q(D_i)$.

There is a similar spectral sequence for computing $E^*(\operatorname{hocolim} D)$, starting with *cohomology* groups $H^p(I^{op}; E^q(D))$. Likewise, there is a spectral sequence for computing $\pi_*(\operatorname{holim} D)$ starting with the groups $H^p(I; \pi_q(D))$ (assuming appropriate connectivity hypotheses on the spaces D_i).

Section 18 will describe all of these spectral sequences in detail. In the present section we develop the algebraic constructions $H_*(I; F)$ and $H^*(I; F)$, where F is a functor $I \to \mathcal{A}b$.

17.1. **Homology and cohomology of a category with coefficients in a functor.** Fix an abelian category $\mathcal{A}$. In our applications below this will always be the category of abelian groups.

Let $\mathcal{I}$ be a small category, and let $D\colon \mathcal{I} \to \mathcal{A}$ be a functor. We will define objects $H^p(\mathcal{I}; D)$ and $H_p(\mathcal{I}; D)$ in $\mathcal{A}$, for each $p \geq 0$. One approach starts by writing down the cosimplicial replacement for F:

$$\prod_i D(i) \rightrightarrows \prod_{i_0 \to i_1} D(i_1) \rightrightarrows \prod_{i_0 \to i_1 \to i_2} D(i_2) \rightrightarrows \cdots$$

This is a cosimplicial object over $\mathcal{A}$. Taking the alternating sum of the coface maps gives a cochain complex over $\mathcal{A}$, and we define $H^p(\mathcal{I}; D)$ to the the pth cohomology group of this complex.

Note that $H^0(\mathcal{I}; F)$ is just the equalizer of the first two arrows in our cosimplicial object, which is precisely $\lim F$. So in some sense the groups $H^p(\mathcal{I}; F)$ are 'higher limit functors'. One somtimes writes

$$H^p(\mathcal{I}; F) = \lim{}^p D.$$

We will make the connection with derived functors more precise in a moment.

Similarly, the homology group $H_p(\mathcal{I}; D)$ is defined to be the pth homology group of the chain complex associated to the simplicial replacement of D:

$$\coprod_{i_0} D(i_0) \leftleftarrows \coprod_{i_0 \leftarrow i_1} D(i_1) \leftleftarrows \coprod_{i_0 \leftarrow i_1 \leftarrow i_2} D(i_2) \leftleftarrows \cdots$$

Here we have $H_0(\mathcal{I}; D) \cong \operatorname{colim}_{\mathcal{I}} D$, and one sometimes writes

$$H_p(\mathcal{I}; D) = \operatorname{colim}_p D.$$

The connection with derived functors is given by *relative* homological algebra, and we will take a brief digression to describe this general theory.

Definition 17.2. *Let $\mathcal{C}$ be an abelian category. A* ***projective class*** *in $\mathcal{C}$ consists of a pair $(\mathcal{P}, \mathcal{E})$ where $\mathcal{P}$ is a class of objects, $\mathcal{E}$ is a class of morphisms, and the following axioms are satisfied:*

(1) An object U is in $\mathcal{P}$ if and only if $\mathcal{C}(U, X) \to \mathcal{C}(U, Y)$ is surjective for every $X \to Y$ in $\mathcal{E}$;

(2) A map $X \to Y$ lies in $\mathcal{E}$ if and only if $\mathcal{C}(U,X) \to \mathcal{C}(U,Y)$ is surjective for every U in $\mathcal{P}$;

(3) For every X in $\mathcal{C}$, there is a morphism $P \to X$ in $\mathcal{E}$ such that P is in $\mathcal{P}$.

*Objects of $\mathcal{P}$ are called **$\mathcal{P}$-projectives** or **relative projectives**. Elements of $\mathcal{E}$ are called **$\mathcal{P}$-epimorphisms** or **relative epimorphisms**.*

Example 17.3. Here are three standard examples of projective classes:

(a) R is a ring and $\mathcal{C}$ is the category of left R-modules. Let $\mathcal{E}$ consist of the surjective maps, and and $\mathcal{P}$ consist of the direct sums of free modules (the "categorical" projectives).

(b) Let $\mathcal{A}$ be any abelian category, let $\mathcal{P}$ consist of all objects of $\mathcal{A}$, and let $\mathcal{E}$ consist of the split-epimorphisms. This is called the **trivial projective class**.

(c) Let $\mathcal{A}$ be an abelian category, let $\mathcal{I}$ be a small category, and let $\mathcal{C} = \mathcal{A}^I$. Define $\mathcal{P}$ to consist of all retracts of coproducts of the free diagrams $F_i(A)$ where A is in $\mathcal{A}$ and i is in I. Define $\mathcal{E}$ to be the class of objectwise split-epimorphisms: the maps of diagrams $D_1 \to D_2$ such that $D_1(A) \to D_2(A)$ is a split-epimorphism for every A in $\mathcal{A}$. A little thought shows that $(\mathcal{P}, \mathcal{E})$ is a projective class; we will call it the **standard projective class** on $\mathcal{A}^{\mathcal{I}}$.

Remark 17.4 (Pullbacks of projective classes)**.** If $L\colon \mathcal{C}_1 \rightleftarrows \mathcal{C}_2\colon R$ are an adjoint pair between abelian categories and $(\mathcal{P}_1, \mathcal{E}_1)$ is a projective class on $\mathcal{C}_1$, then one can lift this to a projective class on $\mathcal{C}_2$. Define $\mathcal{P}_2$ to be the collection of all retracts of objects $L(P)$ for P in $\mathcal{P}_1$, and define $\mathcal{E}_2$ to be $R^{-1}(\mathcal{E}_1)$. One can readily theck that $(\mathcal{P}_2, \mathcal{E}_2)$ is indeed a projective class. Example 17.3(c) is an example of this, using the usual adjoint pair $\mathcal{A}^{\mathrm{ob}(\mathcal{I})} \rightleftarrows \mathcal{A}^{\mathcal{I}}$ and lifting the trivial projective class from $\mathcal{A}^{\mathrm{ob}(\mathcal{I})}$.

Given a projective class $(\mathcal{P}, \mathcal{E})$ in $\mathcal{C}$, a sequence $A \to B \to C$ in $\mathcal{C}$ is defined to be **$\mathcal{P}$-exact** if the composite is zero and if for every P in $\mathcal{P}$ the sequence

$$\mathcal{C}(P,A) \to \mathcal{C}(P,B) \to \mathcal{C}(P,C)$$

is an exact sequence of abelian groups. Homological algebra then goes through in the context of projective classes with little change from the usual story. If F is an additive functor on $\mathcal{C}$ then one gets derived functors $L_k^{\mathcal{P}}F$ as follows: for X in $\mathcal{C}$ one builds a $\mathcal{P}$-projective resolution $\Omega_* \to X$, and then $(L_k^{rel}F)(X) = H_k(F(\Omega))$. The usual arguments show that this is independent of the choice of resolution, up to unique isomorphism.

Note that there is a completely dual notion of **injective class**, and that one can use these to define right derived functors.

17.5. **(Co)homology of categories as a derived functor.** Let $\mathcal{A}$ be an abelian category, $\mathcal{I}$ be a small category, and let $(\mathcal{P}, \mathcal{E})$ be the standard projective class on $\mathcal{A}^{\mathcal{I}}$. Recall that for each i in $\mathcal{I}$ we have the adjoint pair $F_i\colon \mathcal{A} \rightleftarrows \mathcal{A}^{\mathcal{I}}\colon \mathrm{ev}_i$, and by definition the objects $F_i(A)$ are all $\mathcal{P}$-projectives for any A in $\mathcal{A}$. For j in $\mathcal{I}$ one has $F_i(A) = \mathcal{I}(i,j) \otimes A$. Note that if $f\colon i \to j$ is a map in $\mathcal{I}$ then there is a canonical map $F_j(A) \to F_i(A)$ which is adjoint to the map $A \to [F_i(A)]_j$ which includes the copy of A indexed by the map f. So in fact we have a diagram $F_{(-)}(A)\colon \mathcal{I}^{op} \to \mathcal{A}^{\mathcal{I}}$.

For a given diagram $D\colon \mathcal{I} \to \mathcal{A}$ consider the following augmented simplicial object in $\mathcal{A}^{\mathcal{I}}$:

$$D \longleftarrow \coprod_{i_0} F_{i_0}(D_{i_0}) \Lleftarrow \coprod_{i_0 \leftarrow i_1} F_{i_0}(D_{i_1}) \Lleftarrow \coprod_{i_0 \leftarrow i_1 \leftarrow i_2} F_{i_0}(D_{i_2}) \Lleftarrow \cdots$$

This is the bar resolution associated to the adjoint pair $F\colon \mathcal{A}^{\mathrm{ob}(\mathcal{I})} \rightleftarrows \mathcal{A}^{\mathcal{I}}\colon U$, and so the usual arguments (???) show that upon taking alternating sums of face maps we get a resolution of D by $\mathcal{P}$-projectives.

The next step is to apply the colimit functor to every stage of this resolution. Since colim is a left adjoint, it commutes with coproducts. Also, the pair of adjoint functors

$$\mathcal{A} \underset{\mathrm{ev}_i}{\overset{F_i}{\rightleftarrows}} \mathcal{A}^{\mathcal{I}} \underset{c}{\overset{\mathrm{colim}}{\rightleftarrows}} \mathcal{I}$$

gives that $\mathrm{colim} \circ F_i$ is left adjoint to $\mathrm{ev}_i \circ c$. But since the latter is the identity this implies that there exists natural isomorphisms $\mathrm{colim}(F_i(A)) \cong A$. So applying colim to the above resolution exactly reproduces the simplicial replacement of D, and we have therefore proven that

$$H_k(\mathcal{I}; D) \cong [L_k^{rel}\,\mathrm{colim}](D).$$

The situation for cohomology is completely similar. Here one uses the coaugmented cosimplicial object

$$D \longrightarrow \prod_{i_0} CF_{i_0}(D_{i_0}) \Rrightarrow \prod_{i_0 \to i_1} CF_{i_0}(D_{i_1}) \Rrightarrow \prod_{i_0 \to i_1 \to i_2} CF_{i_0}(D_{i_2}) \Rrightarrow \cdots$$

which gives a relative injective resolution of D with respect to the standard injective class on $\mathcal{A}^{\mathcal{I}}$. The natural isomorphisms $\lim CF_i(A) \cong A$ then lead to the conclusion that

$$H^k(\mathcal{I}; D) \cong [R^k_{rel} \lim](D).$$

17.6. **One more persepective on (co)homology of categories.**

Let K be a simplicial set. The **category of simplices** of K is the Grothendieck construction of $K\colon \Delta^{op} \to \mathcal{S}et$. That is, an object of ΔK is a pair $([n], \alpha\colon \Delta^n \to K)$ and a morphism $([n], \alpha) \to ([k], \beta)$ is a map $\sigma\colon [k] \to [n]$ in Δ (it goes the other way in Δ^{op}) such that $\beta = \alpha \circ \sigma$.

Define a **homological coefficient system** on K to be a functor $F\colon \Delta K \to \mathcal{A}b$. A coefficient system is called **locally constant** if it takes every map in ΔK to an isomorphism in $\mathcal{A}b$.

Given a coefficient system F on K, we can produce the following simplicial abelian group:

$$\coprod_{a_0 \in K_0} F([0], a_0) \Lleftarrow \coprod_{a_1 \in K_1} F([1], a_1) \Lleftarrow \coprod_{a_2 \in K_2} F([2], a_2) \Lleftarrow \cdots$$

The face and degeneracy operators are induced, in an evident way, from the ones on K and the functor structure of F. We then define $C_*(K; F)$ to be the associated chain complex, and $H_*(K; F)$ as the homology.

Analogously, define a cohomological coefficient system on K to be a functor $F\colon (\Delta K)^{op} \to \mathcal{A}b$. Given such a system, we can construct the cosimplicial abelian group

$$\prod_{a_0\in K_0} F([0],a_0) \rightrightarrows \prod_{a_1\in K_1} F([1],a_1) \overrightarrow{\rightrightarrows} \prod_{a_2\in K_2} F([2],a_2) \overrightarrow{\overrightarrow{\rightrightarrows}} \cdots$$

and the associated cochain complex $C^*(K;F)$. Then we write $H^*(K;F)$ for the cohomology groups.

Note that these constructions are quite general: given a simplicial set K and a (co)homological coefficient system F, we get (co)homology groups $H_*(K;F)$ or $H^*(K;F)$.

Now let I be a category and $F\colon I\to \mathcal{A}b$ be a functor. This induces a coefficient system $\mathcal{F}$ on the nerve NI by

$$\big([n],[i_0\to i_1\to\cdots\to i_n]\big)\mapsto F(i_0).$$

One readily checks that our definitions of $H_*(I;F)$ and $H_*(NI;\mathcal{F})$.

17.7. **Examples of homology and cohomology groups of categories.**

Example 17.8 (Homology of groups)**.** Let $\mathcal{A}$ be the category of vector spaces over a field k. Let G be a group, and let $\mathbb{B}G$ be the category with one object whose endomorphism group is G. Then a functor $D\colon \mathbb{B}G\to\mathcal{A}$ is just a representation of G over k, and $H^*(\mathbb{B}G;D)$ is just classical group cohomology. In particular, note that the cohomology groups are often nontrivial when $*>0$, even though $\mathcal{A}$ is "just" the category of vector spaces.

As one specific example, let X be an object in $\mathcal{A}$ with a $\mathbb{Z}/2$-action. Let σ denote the generator of $\mathbb{Z}/2$. The free diagram $F(X)$ is $X\oplus X$ where σ acts by swapping the two factors. We surject $F(X)$ onto X using the equivariant map $1+\sigma\colon X\oplus X\to X$, and then kernel is $\{(x,-\sigma x)\,|\,x\in X\}$. We can equivariantly surject $F(X)$ onto this kernel in the evident way, and continuing like this we object the resolution

$$\cdots\longrightarrow X\oplus X\xrightarrow{h} X\oplus X\xrightarrow{g} X\oplus X\xrightarrow{h} X\oplus X\xrightarrow{1+\sigma} X\longrightarrow 0$$

where h sends $(x,0)\mapsto(x,-\sigma x)$ and g sends $(x,0)\mapsto(x,\sigma x)$ (the behavior of h and g on $(0,x)$ is forced by equivariance). In this case the colimit functor quotients by the $\mathbb{Z}/2$-action, which identifies the two copies of X in each appearance of $F(X)$. So $H_k(\mathbb{BZ}/2;X)$ is the kth homology group of

$$\cdots X\xrightarrow{1-\sigma} X\xrightarrow{1+\sigma} X\xrightarrow{1-\sigma} X\longrightarrow 0.$$

This is, of course, the usual computation of the group homology $H_*(\mathbb{Z}/2;X)$. We leave the reader to go through the similar process to check that $H^*(\mathbb{BZ}/2;X)\cong H^*(\mathbb{Z}/2;X)$, using that the co-free object is $X\times X$ where σ again acts by swapping the two factors. (Note that $X\times X=X\oplus X$, since we are in an additive category).

Example 17.9 (The pushout category)**.** Let $\mathcal{I}$ be the category $1\leftarrow 0\to 2$, and let $A=[A_1\leftarrow A_0\to A_2]$ be a diagram in some abelian category $\mathcal{A}$. The three types of free diagrams are readily checked to be

$$F_0(X)=[X\xleftarrow{id} X\xrightarrow{id} X],\qquad F_1(X)=[X\longleftarrow 0\longrightarrow 0]$$
$$F_2(X)=[0\longleftarrow 0\longrightarrow X].$$

Using these, we surject onto A in an evident way and readily compute the kernel, which turns out to be free. The resolution so obtained is:

$$\begin{array}{c} 0 \\ \downarrow \\ [A_0 \leftarrow 0 \to 0] \oplus [0 \leftarrow 0 \to A_0] \\ \downarrow \\ [A_0 \leftarrow A_0 \to A_0] \oplus [A_1 \leftarrow 0 \to 0] \oplus [0 \leftarrow 0 \to A_2] \\ \downarrow \\ [A_1 \xleftarrow{f} A_0 \xrightarrow{g} A_2] \\ \downarrow \\ 0, \end{array}$$

where the maps are evident ones that we will leave the reader to work out. Applying the colimit functor, we find that the groups $H_*(\mathcal{I}; A)$ are the homology groups of the chain complex

$$0 \longrightarrow A_0 \oplus A_0 \longrightarrow A_0 \oplus A_1 \oplus A_2 \longrightarrow 0$$

where the middle map sends $(x, y) \mapsto (x + y, -f(x), -g(y))$. One readily finds that

$$H_*(\mathcal{I}; A) = \begin{cases} \operatorname{colim}_{\mathcal{I}} A & \text{when } * = 0, \\ \ker f \cap \ker g & \text{when } * = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This is a good example of how we can effectively compute by using resolutions that are not the standard resolution.

Example 17.10 (The pullback category)**.** Let $\mathcal{I}$ be the category $1 \to 0 \leftarrow 2$. The resolution we used in the previous example dualizes to give a co-free resolution in this case. We leave the reader to work out that

$$H^*(\mathcal{I}; A) = \begin{cases} \lim_{\mathcal{I}} A & \text{when } * = 0, \\ A/(\operatorname{im} f + \operatorname{im} g) & \text{when } * = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This can also be deduced from the previous example by an appropriate use of opposite categories.

Example 17.11 (The indexing category for sequential colimits)**.** Let $\mathcal{I}$ denote the category $0 \to 1 \to 2 \to \cdots$ and let $A\colon \mathcal{I} \to \mathcal{A}$ be a diagram. Let us write $f\colon A_i \to A_{i+1}$ to denote all the maps in the diagram.

The free diagram $F_i(X)$ looks like

$$0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow X \xrightarrow{id} X \xrightarrow{id} \cdots$$

where the first X occurs in spot i. To surject onto A we need to use $\oplus_i F_i(A_i)$, but it is easy to see that the kernel of $\oplus_i F_i(A_i) \to A$ is then free. That is to say, we have a short free resolution of the form

$$0 \longrightarrow \oplus_i F_{i+1}(A_i) \xrightarrow{g} \oplus_i F_i(A_i) \longrightarrow A \longrightarrow 0$$

where the map g can be described as follows: on $F_{i+1}(A_i)$ it is the adjoint of the map

$$\begin{tikzcd} A_i \arrow[r] \arrow[rr, bend right, "id\oplus f"'] & {[F_i(A_i)]_{i+1} \oplus [F_{i+1}(A_{i+1})]_{i+1}} \arrow[r, equal] & A_i \oplus A_{i+1} \end{tikzcd}$$

(where the equality sign denotes the canonical isomorphism). This is a mouthful, but this is a case where it is really easier to work it out oneself than to read an explanation.

Applying the colimit functor to each stage of the resolution, we find that the groups $H_*(\mathcal{I};A)$ are the homology groups of the complex

$$0 \longrightarrow \oplus_i A_i \xrightarrow{\tilde{g}} \oplus_i A_i \longrightarrow 0$$

where $\tilde{g}$ restricts to the map $id \oplus f$ on A_i. It is easy to check that $H_0(\mathcal{I};A) = \operatorname{colim}_{\mathcal{I}} A$ here, but an analysis of H_1 is difficult in this generality. So let us now assume that $\mathcal{A}$ is the category of left R-modules, for some ring R. Then $\tilde{g}$ acts on elements as

$$\tilde{g}(a_0, a_1, a_2, \ldots) = (a_0, a_1 - f(a_0), a_2 - f(a_1), \ldots).$$

Note that the tuples must be eventually zero, being in the direct sum and not the direct product of the A_i. Now one readily finds that

$$H_*(\mathcal{I};A) = \begin{cases} \operatorname{colim}_{\mathcal{I}} A & \text{if } * = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Be warned that it is *not* true that $H_1(\mathcal{I};A) = 0$ in all abelian categories, we have only proven this for categories of modules over a ring. We will shortly see a counterexample to the general statement.

Example 17.12 (The indexing category for sequential limits)**.** Here let $\mathcal{J} = \mathcal{I}^{op}$, where $\mathcal{I}$ is from the previous example. So $\mathcal{J}$ indexes sequential *limit* diagrams. The co-free diagram $CF_i(X)$ looks like

$$\cdots \longrightarrow X \xrightarrow{id} X \xrightarrow{id} X \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0$$

where the rightmost X occurs in spot i. A diagram A can be resolved by co-free diagrams via

$$0 \longrightarrow A \longrightarrow \prod_i F_i(A_i) \xrightarrow{g} \prod_i F_{i+1}(A_i) \longrightarrow 0$$

where g is defined dually to what we saw for $\mathcal{I}$-diagrams. At spot j this is the exact sequence

$$0 \longrightarrow A_j \xrightarrow{(id,f,\ldots,f^j)} A_j \times A_{j-1} \times \cdots \times A_0 \xrightarrow{g} A_{j-1} \times \cdots \times A_0 \longrightarrow 0$$

where the component of g mapping into A_k is the difference $\pi_k - f\pi_{k+1}$ and π_i is the evident projection onto A_i.

Applying the limit functor to each stage of our resolution gives that $H^*(\mathcal{J};A)$ is calculated as the cohomology groups of

$$0 \longrightarrow \prod_i A_i \xrightarrow{\tilde{g}} \prod_i A_i \longrightarrow 0$$

where the component of $\tilde{g}$ mapping into A_i is again $\pi_i - f\pi_{i+1}$. The kernel of $\tilde{g}$ is clearly $H^0(\mathcal{J};A)$, but the cokernel is hard to analyze. Even in the case where the abelian category $\mathcal{A}$ is R-modules, this cokernel is not easy to describe. It is usually denoted $\lim^1 A$. So we have computed

$$H^*(\mathcal{J};A) = \begin{cases} \lim_{\mathcal{J}} A & \text{if } * = 0, \\ \lim^1 A = \operatorname{coker}(\tilde{g}) & \text{if } * = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 17.13 (A non-vanishing $\lim^1$)**.** Let $\mathcal{A}$ be the category of abelian groups, and consider the diagram

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z}.$$

Then $\lim^1$ of this diagram is the cokernel of

$$\prod_i \mathbb{Z} \longrightarrow \prod_i \mathbb{Z}, \quad (a_0, a_1, \ldots) \mapsto (a_0 - 2a_1, a_1 - 2a_2, \ldots).$$

If a sequence $(u_0, u_1, \ldots)$ is in the image of this map, then notice that

$$u_0 \equiv a_0 \pmod 2, \quad u_0 + 2u_1 \equiv a_0 \pmod 2, \quad u_0 + 2u_1 + 4u_2 \equiv a_0 \pmod 2$$

and so forth. So the series $u_0 + 2u_1 + 4u_2 + \cdots$ converges to an integer in the 2-adic topology on $\mathbb{Z}$. It is easy to give examples of sequences where this fails, for example $1 = u_i$ for all i. This proves that $\lim^1$ is non-vanishing in this case.

We can actually compute $\lim^1$ explicitly for this example. Consider the exact sequence of diagams

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "8"] \arrow[d, "2"'] & \mathbb{Z} \arrow[r] \arrow[d, "1"] & \mathbb{Z}/8 \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "4"] \arrow[d, "2"'] & \mathbb{Z} \arrow[r] \arrow[d, "1"] & \mathbb{Z}/4 \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[d, "2"'] & \mathbb{Z} \arrow[r] \arrow[d, "1"] & \mathbb{Z}/2 \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "1"] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

Call the columns A, B and C, from left to right. The long exact sequence for $\lim^*$ gives

$$\begin{tikzcd}
0 \arrow[r] & \lim A \arrow[r] \arrow[d, equal] & \lim B \arrow[r] \arrow[d, equal] & \lim C \arrow[r] \arrow[d, equal] & \lim^1 A \arrow[r] \arrow[d, equal] & \lim^1 B \arrow[r] \arrow[d, equal] & \lim^1 C \arrow[r] \arrow[d, equal] & 0 \\
 & 0 & \mathbb{Z} & \mathbb{Z}_2^{\wedge} & ?? & 0 & \lim^1 C &
\end{tikzcd}$$

where $\mathbb{Z}_2^{\wedge}$ denotes the 2-adic completion of $\mathbb{Z}$. The fact that $\lim^1 B = 0$ follows directly from the definition. Consequently, the above exact sequence shows that $\lim^1 C = 0$ and also $\lim^1 A = \mathbb{Z}_2^{\wedge}/\mathbb{Z}$. Note that $\mathbb{Z}_2^{\wedge}$ is uncountable, and the same is therefore true of $\lim^1 A$.

Remark 17.14. Let us now return to our analysis of $H_1(\mathcal{I};A)$ in Example 17.11. We saw that this vanished when the abelian category is the category of R-modules. To see a case where it does not vanish, take $\mathcal{A} = \mathcal{A}b^{op}$ (the opposite category of abelian groups). The computations for $H_1(\mathcal{I};-)$ are then the same as for $H^1(\mathcal{J};-)$ in $\mathcal{A}b$, and we have just seen that these groups can be nonzero.

This example illustrates an important point regarding abelian categories. It is sometimes said that the Freyd-Mitchell Embedding Theorem shows that one can always pretend that any abelian category is a category of modules, and can therefore prove theorems by assuming that the objects in the abelian category have underlying sets that behave just as in the module case. This is fine up to a point, but our analysis of $H_1(\mathcal{I};-)$ shows that it only goes so far.

Specifically, the Freyd-Mitchell Embedding Theorem says that if $\mathcal{A}$ is a small abelian category then there is a fully faithful and exact functor $E\colon \mathcal{A} \to R-\mathcal{M}od$ for some ring R. Here "exact" means that the functor preserves finite limits and colimits. It is the "finite" word that is important for our discussion, as our analysis of $H_1(\mathcal{I};-)$ required us to work with an infinite coproduct. As the Freyd-Mitchell theorem does not guarantee that E preserves such coproducts, one cannot reduce the $H_1(\mathcal{I};-)$ computation to the case where $\mathcal{A}$ is a subcategory of $R-\mathcal{M}od$.

18. Spectral sequences for holims and hocolims

If $D\colon I \to \mathcal{T}op$ is a diagram, there is a spectral sequence for computing $\pi_*(\operatorname{holim} D)$ from knowledge of $\pi_*(D_i)$ for each i. Actually, this is not always a true spectral sequence due to the fact that π_0 may not be a group, and π_1 may not be an abelian group. So one has these problems on the 'fringe'. There are ways to deal with these problems, but very often one is in a situation where they actually aren't there. One way this can happen is if one is really dealing with spectra rather than spaces. Another way is if one is dealing with spaces which are all connected with abelian fundamental groups. We will develop things in these two special cases.

If E is a cohomology theory, then there is a related spectral sequence for computing $E^*(\operatorname{hocolim} D)$ from knowledge of $E^*(D_i)$, for all i. In fact this can be obtained as a special case of the above, using the adjunctions in the category of spectra

$$E^n(\operatorname{hocolim} D) = \pi_{-n} \operatorname{Map}(\operatorname{hocolim} D, E) = \pi_{-n}\Big[\operatorname*{holim}_I \operatorname{Map}(D(i), E)\Big].$$

Here we are writing E also for some spectrum representing our given cohomology theory. In this section we will explain these two spectral sequences.

18.1. **A motivating example.** Before tackling the general case, we pause to consider one special example. Given a pushout diagram $X = [B \leftarrow A \to C]$ of cofibrant objects, we have seen that one model for $\operatorname{hocolim} X$ is the double mapping cylinder obtained as a pushout of

$$B \amalg C \longleftarrow A \amalg A \longrightarrow A \times I.$$

As discussed in Example 2.2, there is an evident open cover $\{U, V\}$ of this space obtained by roughly cutting the cylinder in half (but allowing a little overlap between the two pieces). The intersection $U \cap V$ is then homotopy equivalent to A, and $U \simeq B$ and $V \simeq C$. Mayer-Vietoris gives a long exact sequence

$$\cdots \to E^*(\operatorname{hocolim} X) \to E^*(B) \oplus E^*(C) \to E^*(A) \to E^{*+1}(\operatorname{hocolim} X) \to \cdots$$

This long exact sequence is a degenerate case of the spectral sequence we are after.

There is another way to get this long exact sequence that works if A is well-pointed. Choose a basepoint $*$ in A such that $* \hookrightarrow A$ is a cofibration, and consider the subspace $W = B \cup (* \times I) \cup C$ inside of $\operatorname{hocolim} X$. The quotient $(\operatorname{hocolim} X)/W$ is isomorphic to ΣA and $W \simeq B \vee C$, so the long exact sequence for the pair $(\operatorname{hocolim} X, W)$ can be written as

$$\cdots \to E^*(\Sigma A, *) \to E^*(\operatorname{hocolim} X) \to E^*(B \vee C) \to E^{*+1}(\Sigma A, *) \to \cdots$$

One can check that this is "the same" as the previous long exact sequence, after appropriate use of the suspension isomorphism.

This example indicates the basic idea: for a general diagram X, the construction of $\operatorname{hocolim} X$ should yield a certain decomposition of this space into simpler components—either via an open covering or otherwise. The usual tools of cohomology theories will take such a decomposition and churn out a spectral sequence. In fact, the description of $\operatorname{hocolim} X$ as a geometric realization immediately suggests the "skeletal" filtration where one decomposes the space by the dimension of the attached simplices (this is basically what we did in the second approach above). The problem to be solved is really just one of bookkeeping: how does one describe the filtration quotients and compute the initial input for the spectral sequence.

The situation for homotopy limits is completely dual. If $Y = [B \to A \leftarrow C]$ is a diagram of pointed spaces then one model for the homotopy limit is the pullback of $B \times C \longrightarrow A \times A \xleftarrow{\pi} A^I$. The map $A^I \to A \times A$ is a fibration whose fiber over $(*, *)$ is ΩA, and so pulling back gives a fibration sequence

$$\Omega A \to \operatorname{holim} Y \to B \times C.$$

The long exact sequence in homotopy groups becomes

$$\cdots \to \pi_{i+1}(A) \to \pi_i(\operatorname{holim} Y) \to \pi_i(B) \oplus \pi_i(C) \to \pi_i(A) \to \cdots$$

and again this can be regarded as a certain degenerate example of a spectral sequence.

18.2. **The spectral sequences.** Let $D\colon I \to \mathcal{T}op$ be a diagram, and let $\mathcal{E}_*$ be a homology theory. Write $\mathcal{E}_k(D)$ for the diagram $I \to \mathcal{A}b$ given by $i \mapsto \mathcal{E}_k(D_i)$, and write $\mathcal{E}^k(D)$ for the analogous diagram $I^{op} \to \mathcal{A}b$.

Theorem 18.3 (Spectral sequences for homotopy colimits)**.**

(a) There is a spectral sequence $E^2_{p,q} = H_p(I; \mathcal{E}_q(D)) \Rightarrow \mathcal{E}_{p+q}(\operatorname{hocolim} D)$. *The differentials have the form* $d^r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$.

(b) There is a spectral sequence $E_2^{p,q} = H^p(I^{op}; \mathcal{E}^q(D)) \Rightarrow \mathcal{E}^{p+q}(\operatorname{hocolim} D)$. *The differentials have the form* $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$.

The dual version for homotopy limits is a bit more dicey to state, because of the problem that the homotopy groups π_i of a space depend on basepoints, are nonabelian when $i = 1$, and are not groups at all when $i = 0$. There are ways to deal with these issues and talk about a "fringed spectral sequence", but we will not tackle this and instead assume we are in a situation where these problems are not present.

Theorem 18.4 (Spectral sequences for homotopy limits)**.** *Assume either that*

(1) Each space D_i is path-connected with abelian fundamental group, or

(2) D is actually a diagram in some category of spectra.

Then there is a spectral sequence $E_2^{p,q} = H^p(I; \pi_q D) \Rightarrow \pi_{q-p}(\operatorname{holim} D)$. *The differentials have the form* $d_r\colon E_r^{p,q} \to E_r^{p+r,q+r-1}$.

We will later discuss where these spectral sequences come from and how to remember the bigrading of the differentials.

18.5. **Examples.**

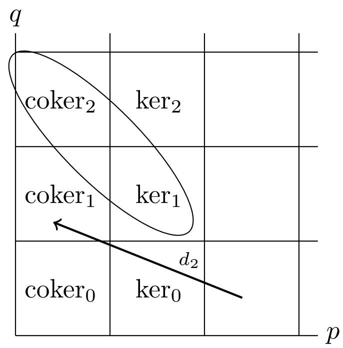
Example 18.6 (Homology of a homotopy pushout)**.** Let I be the category $1 \leftarrow 0 \to 2$, and let $D = [B \leftarrow A \to C]$ be a I-diagram in $\mathcal{T}op$. Let $\mathcal{E}_*(-)$ be a homology theory. Recall that $H_p(I; -)$ is nonzero only for $p = 0, 1$. Since the d_r-differential maps the $p = k$ line to the $p = k - r$ line, the spectral sequence collapses at the E_2-page.

Also, recall that we computed

$$H_0(I; \mathcal{E}_q(D)) = \operatorname{coker}\big[\mathcal{E}_q(A) \to \mathcal{E}_q(B) \oplus \mathcal{E}_q(C)\big] = \operatorname{coker}_q$$
$$H_1(I; \mathcal{E}_q(D)) = \ker\big[\mathcal{E}_q(A) \to \mathcal{E}_q(B)\big] \cap \ker\big[\mathcal{E}_q(A) \to \mathcal{E}_q(C)\big] = \ker_q.$$

Note that coker_q and $\ker_q$ are simply abbreviations that will be useful below.

We have the following picture:



The spectral sequence tells us there is a very short filtration $\mathcal{E}^n(\operatorname{hocolim} D) = F_0 \subseteq F_1$, and the groups $(F_0/F_1) \oplus F_1$ are the ones appearing in the nth diagonal in the E_∞-term (where the second"diagonal" is the one circled in the picture). It remains to remember the *order* in which these two terms appear along the diagonal: to do this, move along the diagonal in the vague direction of the differentials. In our picture this takes us from lower to higher in the diagonal. The quotient group of $\mathcal{E}^n(\operatorname{hocolim} D)$ (F_0/F_1 in our notation) is always the one near the tail of the differential. So in our case this says

$$F_0/F_1 \cong \ker_{n-1} \qquad \text{and} \qquad F_1 \cong \operatorname{coker}_n .$$

Putting everything together, the spectral sequence is giving us short exact sequences

$$0 \longrightarrow \operatorname{coker}_n \longrightarrow \mathcal{E}_n(\operatorname{hocolim} D) \longrightarrow \ker_{n-1} \longrightarrow 0.$$

Note that this is the same information as in the long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & E_n(A) \arrow[r] & E_n(B) \oplus E_n(C) \arrow[r] & E_n(\operatorname{hocolim} D) \arrow[dll, out=0, in=180] & \\
& & E_{n-1}(A) \arrow[r] & E_{n-1}(B) \oplus E_{n-1}(C) \arrow[r] & \cdots
\end{tikzcd}$$

Exercise 18.7. Check that the spectral sequence for computing the cohomology of a homotopy pushout, and for computing the homotopy groups of a homotopy pullback, represent the same information that we saw in the motivating examples from Section 18.1.

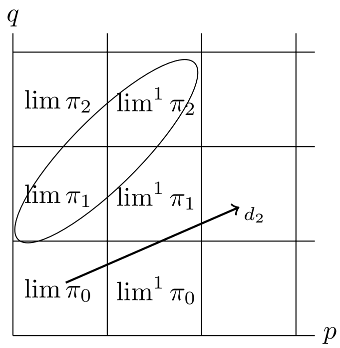
Example 18.8 (Homotopy of a sequential homotopy limit)**.** Let $\cdots \to X_2 \to X_1 \to X_0$ be an inverse limit system of pointed topological spaces (connected with abelian fundamental groups). Let $\mathcal{J}$ denote the indexing category $\cdots \to 2 \to 1 \to 0$. The spectral sequence

$$H^p(\mathcal{J}; \pi_q(X)) \Rightarrow \pi_{q-p}(\operatorname{holim} X)$$

is again concentrated along the lines $p = 0, 1$, and the differentials leave this range and so are all zero. Recall that

$$H^0(\mathcal{J}; \pi_q(X)) = \lim \pi_q(X), \qquad H^1(\mathcal{J}; \pi_q(X)) = \lim{}^1 \pi_q(X).$$

This time our spectral sequence looks like



Reasoning as in the previous example, the spectral sequence gives us short exact sequences

$$0 \longrightarrow \lim^1 \pi_{n+1}(D) \longrightarrow \pi_n(\operatorname{holim} D) \longrightarrow \lim \pi_n(D) \longrightarrow 0.$$

Exercise 18.9. Given a sequence of spaces $X_0 \to X_1 \to \cdots$, use the spectral sequence for the cohomology of a sequential colimit to derive the short exact sequences

$$0 \longrightarrow \lim^1 \mathcal{E}^{n-1}(X) \longrightarrow \mathcal{E}^n(\operatorname{hocolim} X_n) \longrightarrow \lim \mathcal{E}^n(X) \longrightarrow 0.$$

This is usually called the **Milnor exact sequence**.

18.10. **Spectral sequences for simplicial and cosimplicial spaces.** ?????

This is an immediate consequence of the following result about cosimplicial spaces. If X is a cosimplicial pointed space, one may form a cosimplicial abelian group by applying $\pi_n(-,*)$ to each level (assuming that $n \geq 2$, or that the spaces X_i are connected with abelian fundamental group). After taking the alternating sum of the coface maps, the cosimplicial abelian group becomes a cochain complex. Let $H^p(\pi_q(X))$ denote the pth cohomology group.

Theorem 18.11. *Let X be a Reedy fibrant simplicial space, such that each X_n is connected with abelian fundamental group. Then there is a spectral sequence of the form $E_2^{p,q} = H^p(\pi_q(X)) \Rightarrow \pi_{q-p}(\operatorname{Tot} X)$, where the differentials have the form $d_r \colon E_r^{p,q} \to E_r^{p+r,q+r-1}$.*

Remark 18.12. There is an easy way to remember how the differentials work in the above spectral sequence, at least if one understands the two spectral sequences associated to a double chain complex. Suppose that, instead of X being a cosimplicial space, X were a cosimplicial chain complex. That is, suppose that instead of working in the model category $\mathcal{T}op$ we were working in the model category $Ch(\mathbb{Z})$.

Now each X_n is a chain complex, which we draw vertically with the differentials going down. We are now looking at a cosimplicial chain complex, which after taking the alternating sum of coface maps becomes a double complex. In this case $\operatorname{Tot} X$ "is" the totalization of this double complex, and the spectral sequence "is" the spectral sequence obtained by first taking homology groups in the vertical direction and then in the horizontal direction. So if one knows how the indexing works in the latter spectral sequence, one also knows how it works in the former.

18.13. **Spectral sequences for homotopy colimits.** Suppose $D \colon I \to \mathcal{T}op$, and that $\mathcal{E}^*$ is a cohomology theory represented by a spectrum $\mathcal{E}$. Note that for each

n one obtains an I^{op}-diagram of abelian groups by $i \mapsto \mathcal{E}^n(D(i))$. We'll call this diagram $\mathcal{E}^n(D)$, for short.

Proposition 18.14. *There is a spectral sequence $E_2^{p,q} = H^p(I^{op}; \mathcal{E}^q(D)) \Rightarrow \mathcal{E}^{p-q}(\operatorname{hocolim} D)$. The differentials have the form $d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$.*

Proof. This is obtained by dualizing the spectral sequence for a homotopy limit. □

One can also derive a spectral sequence for computing the E-homology of a homotopy colimit. This is based on the following spectral sequence for the homotopy groups of a geometric realization of spectra:

Proposition 18.15. *Let $[n] \mapsto G_n$ be a simplicial spectrum. Then there is a spectral sequence*

$$E^1_{p,q} = \pi_p G_q \Rightarrow \pi_{p+q}|G|$$

where the differentials have the form $d^r \colon E^r_{p,q} \to E^r_{p+r-1,q-r}$. The differential d_1 is the alternating sum of the face maps in the cosimplicial abelian group $[n] \mapsto \pi_* G_n$.

Proof. This is the homotopy spectral sequence associated to the tower of homotopy cofiber sequences

$$\begin{tikzcd} * \arrow[r] & {|\operatorname{Sk}_0 G|} \arrow[r] \arrow[d] & {|\operatorname{Sk}_1 G|} \arrow[r] \arrow[d] & {|\operatorname{Sk}_2 G|} \arrow[r] \arrow[d] & \cdots \\ & G_0 & \Sigma G_1 & \Sigma^2 G_2 & \cdots \end{tikzcd}$$

In spectra, homotopy cofiber sequences are also homotopy fiber sequences—so each layer in the tower gives a long exact sequence in homotopy groups, resulting in an exact couple. □

Remark 18.16. Again, there is a nice way to remember how the differentials go in the above spectral sequence. Imagine the parallel situation in which the G_i are chain complexes rather than spectra. Then what we really have is a double complex, and we are looking at the spectral sequence whose G_2-term is obtained by first taking the homology of the G_i's and then taking homology in the other direction. Provided one can remember how the differentials work in the spectral sequence of a double complex, one also knows how they work in the spectral sequence of Proposition 18.15.

Proposition 18.17. *Let $\mathcal{E}$ be a spectrum and let $X \colon I \to \mathcal{T}op$ be a diagram of spaces. Then for each p one gets a diagram of abelian group $i \mapsto \mathcal{E}_p(X_i)$; call this diagram $\mathcal{E}_p X$.*

There is a spectral sequence

$$E^2_{p,q} = H_q(I; \mathcal{E}_p X) \Rightarrow \mathcal{E}_{p+q}(\operatorname*{hocolim}_I X).$$

The differentials have the form $d^r \colon E^r_{p,q} \to E^r_{p+r-1,q-r}$.

Proof. Consider the simplicial spectrum $[n] \mapsto \mathcal{E} \wedge \Sigma^\infty(\operatorname{srep}(X)_n)$. The geometric realization of this simplicial spectrum is $\operatorname{hocolim}_I(\mathcal{E} \wedge \Sigma^\infty(X_i)_+)$, which is the same as

$$\mathcal{E} \wedge \Sigma^\infty(\operatorname*{hocolim}_I X_i).$$

The spectral sequence of Proposition 18.15 converges to the $(p+q)$th homotopy group of this geometric realization, which is therefore $\mathcal{E}_{p+q}(\operatorname{hocolim}_I X)$. □

19. Homotopy limits and colimits in other model categories

So far we have been almost exclusively working in the model category of topological spaces. In this section we will explain some of the ways in which our methods adapt to more general model categories. In many cases this takes the form, "If a model category satisfies P and Q then everything we did before works exactly the same. However, if the model category does *not* satisfy P or Q then one can still get the same basic results, but it requires harder work."

We also make some remarks particular to the case where the model category is chain complexes over an abelian category. Here, the study of homotopical algebra is really just ordinary homological algebra. So the theory of homotopy colimits can be phrased in somewhat more algebraic terms. We make some of this explicit.

19.1. **Simplicial model categories.** A model category $\mathcal{M}$ is called *simplicial* if for every $X, Y \in \mathcal{M}$ and $K \in s\mathcal{S}et$ one has functorial constructions

$$X \otimes K \in \mathcal{M}, \qquad F(K,X) \in \mathcal{M}, \qquad \text{and} \qquad \mathrm{Map}(X,Y) \in s\mathcal{S}et$$

together with adjunction isomorphisms

$$\mathrm{Map}(X \otimes K, Y) \cong \mathrm{Map}(X, F(K,Y)) \cong \underline{s\mathcal{S}et}(K, \mathrm{Map}(X,Y))$$

(note that these are isomorphisms of simplicial sets). One assumes there is a composition law $\mathrm{Map}(Y,Z) \times \mathrm{Map}(X,Y) \to \mathrm{Map}(X,Z)$ and identity maps $* \to \mathrm{Map}(X,X)$ satisfying the expected properties, and also an isomorphism $\mathrm{Map}(X,Y)_0 \cong \mathcal{M}(X,Y)$ that commutes with composition. Finally, one assumes the *pushout-product axiom* SM7; there are several equivalent versions, but we will use the one saying that if $i\colon A \rightarrowtail B$ is a cofibration in $\mathcal{M}$ and $j\colon K \hookrightarrow L$ is a cofibration in $s\mathcal{S}et$, then the map

$$i \square j\colon (A \otimes L) \amalg_{(A \otimes K)} (B \otimes K) \to B \otimes L$$

is a cofibration which is a weak equivalence if either i or j is so. A detailed treatment of simplicial model categories can be found in [H, Chapter 9].

Example 19.2. The model category $\mathcal{T}op$ is a simplicial model category, where one defines

$$X \otimes K = X \times |K|, \qquad F(K,X) = X^{|K|}$$

and where $\mathrm{Map}(X,Y)$ is the simplicial set $[n] \mapsto \mathcal{T}op(X \times \Delta^n, Y)$.

Similarly, $s\mathcal{S}et$ is a simplicial model category where one defines

$$X \otimes K = X \times K, \quad F(K,X) = \underline{s\mathcal{S}et}(K,X), \quad \text{and} \quad \mathrm{Map}(X,Y) = \underline{s\mathcal{S}et}(X,Y).$$

In a simplicial model category, one can give formulas for homotopy limits and colimits exactly like what we have described for $\mathcal{T}op$. One uses exactly the same definitions, and all the same results hold.

19.3. **The homotopy theory of diagrams.** Let $\mathcal{M}$ be any model category, and let I be a small category. Let $\mathcal{M}^I$ denote the category of I-diagrams and natural transformations.

One would like there to be a model category structure on $\mathcal{M}^I$ where the weak equivalences are the objectwise weak equivalences. Unfortunately this probably doesn't exist in general. However, it does exist if I is a so-called Reedy category, and for all I if $\mathcal{M}$ is a *cofibrantly-generated* model category.

Theorem 19.4. *Assume $\mathcal{M}$ is a cofibrantly-generated model category. Then for any small category I there is a model category structure on $\mathcal{M}^I$ where the weak equivalences and fibrations are determined objectwise. This is commonly called the* ***projective*** *model category structure on $\mathcal{M}^I$.*

Proof. See [H, Section 11.6]. □

If $\mathcal{M}$ is cofibrantly-generated, we can again consider the adjoint functors

$$\operatorname{colim}\colon \mathcal{M}^I \rightleftarrows\colon \mathcal{M}\colon c$$

and these are again a Quillen pair. One can define the homotopy colimit of a diagram as the derived functor of the colimit, just as we did in $\mathcal{T}op$. Notice that this works even if $\mathcal{M}$ is not simplicial! Relative homotopy colimits can also be defined, and the whole theory is exactly the same as for $\mathcal{T}op$.

The dual story for homotopy limits is also a little different. Here one wants a model category structure on $\mathcal{M}^I$ where the weak equivalences and *cofibrations* are defined objectwise. For the following, recall that a model category is called combinatorial if it is cofibrantly-generated and the underlying category is locally presentable.

Theorem 19.5 (J. Smith, unpublished)**.** *Assume that $\mathcal{M}$ is a combinatorial model category. Then for any small category I there is a model category structure on $\mathcal{M}^I$ in which the weak equivalences and cofibrations are determined objectwise. This is commonly called the* ***injective*** *model category structure on $\mathcal{M}^I$.*

If $\mathcal{M}$ is a combinatorial model category one can then consider the adjoint functors

$$c\colon \mathcal{M} \rightleftarrows \mathcal{M}^I\colon \lim$$

(where c is the left adjoint), and observe that c preserves cofibrations and trivial cofibrations. To this is a Quillen pair, and one can define the homotopy limit of a diagram to be the derived functor of lim.

Remark 19.6. Even if the appropriate model category structure on $\mathcal{M}^I$ does not exist, there are other techniques for making the derived functor perspective work. One can still define a homotopy category of diagrams $\operatorname{Ho}(\mathcal{M}^I)$, even though an underlying model category structure may not exist. And one can still talk about the derived functors of colim and lim. See [DHKS] for this approach.

For yet another approach to homotopy limits and colimits in general model categories, see [CS].

19.7. **Non-simplicial model categories.** Formulas for homotopy limits and colimits can also be given without assuming a simplicial structure on the model category; one just has to work a little harder. This is due to Dwyer-Kan, and it is described in detail in [H, Chapters 16, 19].

If X is a cofibrant object in a simplicial model category, then one can obtain a cylinder object for X by looking at $X \otimes \Delta^1$. One also has cylinder objects in non-simplicial model categories: they can be constructed by factoring the fold map $\nabla\colon X \amalg X \to X$ into a cofibration followed by a trivial fibration:

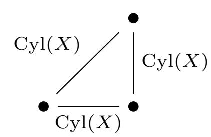
$$X \amalg X \rightarrowtail \operatorname{Cyl}(X) \xrightarrow{\sim} X.$$

These are even functorial, using that our factorizations are functorial.

In the same way, in any model category one can construct objects which "look like" $X \otimes \Delta^2$, $X \otimes \Delta^3$, etc. This is due to Dywer-Kan and is referred to as the theory of *framings*. For instance, to construct an object that looks like $X \otimes \Delta^2$ one does the following. Recall that our cylinder object $\mathrm{Cyl}(X)$ came with two maps $d^0, d^1 \colon X \to \mathrm{Cyl}(X)$. One can make an object that looks like $X \otimes \partial\Delta^1$ by forming the colimit of the diagram

$$\begin{tikzcd}
X \arrow[d, "d^0"'] \arrow[drr, "d^1"' near end] & & X \arrow[dll, "d^0"' near start] \arrow[drr, "d^1" near start] & & X \arrow[d, "d^1"] \arrow[dll, "d^0" near end] \\
\mathrm{Cyl}(X) & & \mathrm{Cyl}(X) & & \mathrm{Cyl}(X).
\end{tikzcd}$$

This corresponds to gluing three copies of $\mathrm{Cyl}(X)$ to make the picture



corresponding to $\partial\Delta^1$. Let Z denote this colimit.

Our canonical map $\mathrm{Cyl}(X) \to X$ coequalizes d^0 and d^1, and therefore induces a map $Z \to X$. Factoring this again as a cofibration followed by trivial fibration gives

$$Z \rightarrowtail X[2] \xrightarrow{\sim} X$$

and this $X[2]$ is our object which "looks like" $X \otimes \Delta^2$. Note that it is also functorial in X, due to the functoriality of our factorizations.

For an object $X \in \mathcal{M}$, let cX denote the constant cosimplicial object which is X in every dimension. Briefly, a *cosimplicial frame* on X is a cosimplicial object $\hat{X}$ in $\mathcal{M}$, together with an objectwise weak equivalence $\hat{X} \to cX$ which is an isomorphism in level 0. When X is cofibrant, one also requires that $\hat{X}$ satisfy a certain Reedy cofibrancy condition having to do with latching maps being cofibrations—we will not write this down. The nth object of $\hat{X}$ is our object which "looks like" $X \otimes \Delta^n$. Dwyer and Kan showed that cosimplicial frames exist in any model category, essentially by inductively continuing the procedure we began above.

Let I be a small category. Given a diagram $D \colon I \to \mathcal{M}$, a cosimplicial frame on D is a diagram $\hat{D} \colon I \to c\mathcal{M}$ (a diagram of cosimplicial objects on $\mathcal{M}$) together with natural weak equivalences $\hat{D}(i) \to c[D(i)]$ which make each $\hat{D}(i)$ a cosimplicial frame on $D(i)$. Again, cosimplicial frames on diagrams always exist.

Once one has a cosimplicial frame on D, one can again write down explicit formulas for the homotopy colimit. (For the homotopy limit one needs a simplicial frame on D—we have not defined this but it is completely dual). The formulas are exactly what we wrote down in the simplicial case, one just has to develop enough machinery to realize that they really do make sense.

There is no point in us describing this theory in more detail because the reader should just go read [H]. The theory of frames and homotopy limits/colimits in general model categories is wonderfully presented there.

19.8. **Abelian categories.** Let $\mathcal{A}$ be an abelian category with enough projectives and injectives. Then there are model categories on $Ch_{\geq 0}(\mathcal{A})$ and $Ch_{\leq 0}(\mathcal{A})$ which exactly parallel the two model category structures described at the beginning of this section, when $\mathcal{A}$ is the category of modules over a ring. In these categories the theory of homotopy limits and colimits becomes somewhat simpler and more familiar.

Recall that if $\mathcal{B}$ is an additive category then there is an equivalence between the category of simplicial objects in $\mathcal{B}$ and the category $Ch_{\geq 0}(B)$. In one direction one replaces a simplicial object by its normalized chain complex; up to quasi-isomorphism, this is the same as the chain complex obtained by just taking the alternating sum of face maps.

Also, recall that if $D_{*,*}$ is a double chain complex then one may form a total chain complex in two ways. One way has $\mathrm{Tot}^{\oplus}(D)_n = \bigoplus_{p+q=n} D_{p,q}$ and the other has $\mathrm{Tot}^{\otimes}(D)_n = \bigotimes_{p+q=n} D_{p,q}$. We will have need for both of these.

Suppose given a simplicial object X_* of $Ch_{\geq 0}(\mathcal{A})$. Since the category of chain complexes is additive, we may take the alternating sum of face maps…and what we get is a double complex! Let X_*^{alt} denote this double complex. The result we are after is the following:

Proposition 19.9. *The two chain complexes* $\operatorname{hocolim} X_*$ *and* $\mathrm{Tot}^{\oplus}(X_*^{alt})$ *are quasi-isomorphic.*

Similarly, suppose Z^* is a cosimplicial object in $Ch_{\leq 0}(\mathcal{A})$. Let Z^*_{alt} denote the double complex obtained by taking the alternating sum of coface maps. Then

Proposition 19.10. *The complexes* $\operatorname{holim} Z^*$ *and* $\mathrm{Tot}^{\otimes}(Z^*_{alt})$ *are quasi-isomorphic.*

What these propositions say is that the theory of homotopy colimits in $Ch_{\geq 0}(\mathcal{A})$ (and of homotopy limits in $Ch_{\leq 0}(\mathcal{A})$) can be drastically simplified by using total complexes. For instance, if $D\colon I \to Ch_{\geq 0}(\mathcal{A})$ is a diagram then to construct $\operatorname{hocolim} D$ one can form the simplicial replacement, take the alternating sum of faces, and then apply $\mathrm{Tot}^{\oplus}$. No geometric realization (or Tot) is needed.

What about homotopy *limits* in $Ch_{\geq 0}(\mathcal{A})$? Here the story is a little more complicated, but only barely. The difficulty is as follows. Suppose Z^* is a cosimplicial object in $Ch_{\geq 0}(\mathcal{A})$. Taking alternating sums of coface maps gives a double complex Z^*_{alt}. But taking the total complex now gives a complex which has terms in negative degrees, so it does not lie in $Ch_{\geq 0}(\mathcal{A})$. How does one fix this? Well, for any $\mathbb{Z}$-graded chain complex C_* one can obtain a non-negatively graded chain complex by considering the truncation $\tau_{\geq 0}(C_*)$ given by

$$Z_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \cdots$$

where Z_0 is the subobject of cycles in degree 0.

Proposition 19.11. *If Z^* is a cosimplicial object in $Ch_{\geq 0}(\mathcal{A})$, then* $\operatorname{holim} Z^*$ *is quasi-isomorphic to* $\tau_{\geq 0} \mathrm{Tot}^{\otimes}[Z^*_{alt}]$.

Similarly, we have

Proposition 19.12. *If X_* is a simplicial object in $Ch_{\leq 0}(\mathcal{A})$, then* $\operatorname{hocolim} X_*$ *is quasi-isomorphic to $\tau_{\leq 0} \mathrm{Tot}^{\oplus}[X^*_{alt}]$. Here, if C_* is a $\mathbb{Z}$-graded chain complex then $\tau_{\leq 0}(C_*)$ is the non-positively graded chain complex given by*

$$C_0/B_0 \to C_{-1} \to C_{-2} \to C_{-3} \to \cdots$$

where B_0 is the subobject of boundaries in degree 0.

Again, the above propositions show that the theory of homotopy limits and colimits in $Ch_{\geq 0}(\mathcal{A})$ and $Ch_{\leq 0}(\mathcal{A})$ can be drastically simplified by using total complexes in place of geometric realizations or Tot.

20. Various results concerning simplicial objects

This section is under construction!

Let I and J be two small categories, and let $X \to I \times J \to \mathcal{T}op$ be a diagram. Note that for any $i \in I$ we get a J-diagram by $j \mapsto X(i,j)$, and likewise for any $j \in J$ we get an I-diagram

Proposition 20.1. *There are canonical zig-zags of weak equivalences between the three objects*

$$\operatorname*{hocolim}_I [i \mapsto \operatorname*{hocolim}_J X(i,-)], \qquad \operatorname*{hocolim}_J [j \mapsto \operatorname*{hocolim}_I X(-,j)],$$

and $\operatorname{hocolim}_{I\times J} X$.

20.2. **Homotopy colimits and realizations.** Let $X\colon \Delta^{op} \to \mathcal{T}op$. We have already talked about the geometric realization $|X|$, but we can also form the homotopy colimit $\operatorname{hocolim} X$. These are both homotopy invariant constructions, but they are usually different. We can compare them, though:

Proposition 20.3. *There is a natural map* $\operatorname{hocolim} X \to |X|$ *called the* ***Bousfield-Kan map****. It is a weak equivalence when X is Reedy cofibrant.*

Similarly, if $Z\colon \Delta \to \mathcal{T}op$ is a cosimplicial space then there is a natural map $\operatorname{Tot} Z \to \operatorname{holim} Z$; this is a weak equivalence if Z is Reedy fibrant.

The proof of the above proposition requires more techniques than we have at the moment. However, we can at least describe the map. Recall that

$$\begin{aligned}\operatorname*{hocolim}_{\Delta^{op}} X &= \operatorname{coeq}\Bigg[\coprod_{[n]\to[k]} X_k \times B([n]\downarrow \Delta^{op})^{op} \rightrightarrows \coprod_n X_n \times B([n]\downarrow\Delta^{op})^{op}\Bigg] \\ &= \operatorname{coeq}\Bigg[\coprod_{[n]\to[k]} X_k \times B(\Delta\downarrow[n]) \rightrightarrows \coprod_n X_n \times B(\Delta\downarrow[n])\Bigg].\end{aligned}$$

Likewise, we have

$$|X| = \operatorname{coeq}\Bigg[\coprod_{[n]\to[k]} X_k \times \Delta^n \rightrightarrows \coprod_n X_n \times \Delta^n\Bigg].$$

We can produce a map $\operatorname{hocolim} X \to |X|$ by finding maps $\alpha_n\colon B(\Delta\downarrow[n]) \mapsto \Delta^n$ having the property that for every $\sigma\colon [n]\to[k]$ one gets a commutative square

$$\begin{tikzcd} B(\Delta\downarrow[n]) \arrow[r, "\sigma_*"] \arrow[d] & B(\Delta\downarrow[k]) \arrow[d] \\ \Delta^n \arrow[r, "\sigma_*"] & \Delta^k. \end{tikzcd}$$

We'll actually produce maps of simplicial sets $N(\Delta\downarrow[n]) \mapsto \Delta^n$. Recall that Δ^n is the simplicial set $[k] \mapsto \Delta([k],[n])$. A k-simplex in $N(\Delta\downarrow[n])$ is a string

$$[i_0] \to [i_1] \to \cdots \to [i_k] \to [n].$$

We can produce a map $[k]\to[n]$—that is, a k-simplex in Δ^n—by sending an element $j \in [k]$ to the image in $[n]$ of the last vertex of $[i_j]$ under the above composition of maps. Note that this gives a monotone increasing function $[k]\to[n]$, as desired. The resulting map $N(\Delta\downarrow[n]) \to \Delta^n$ is called the **last vertex map**. The reader may easily check that it gives the necessary commutative squares.

20.4. **Fat vs. non-fat.** Recall that $\Delta_f \subseteq \Delta$ is the subcategory consisting of all the co-face maps. A 'Δ-complex' is a functor $\Delta_f^{op} \to \mathcal{T}op$—it is a simplicial set without the degeneracy maps. If Z is a Δ-complex then the above Bousfield-Kan construction gives a natural map $\operatorname{hocolim}_{\Delta_f} Z \to ||Z||$.

So if X is a simplicial space one has the following square:

$$
\begin{array}{ccc}
\operatorname*{hocolim}_{\Delta^{op}} X & \longrightarrow & |X| \\
\uparrow & & \uparrow \\
\operatorname*{hocolim}_{\Delta_f^{op}} X & \longrightarrow & ||X||
\end{array}
$$

Proposition 20.5. *If X is objectwise cofibrant, the two maps with domain $\operatorname{hocolim}_{\Delta_f^{op}} X$ are weak equivalences. If X is also Reedy cofibrant, the other two maps are weak equivalences as well.*

Part 5. Examples

21. Homotopy initial and terminal functors

In this section we present several specific examples of functors which are homotopy initial or terminal.

Our first example is a functor which is merely initial, not homotopy initial:

Example 21.1. Let $J \hookrightarrow \Delta$ denote the subcategory consisting of the objects [0], [1], and the two maps d^0, d^1 between them. We claim that $J \hookrightarrow \Delta$ is initial; this is equivalent to saying that $J^{op} \to \Delta^{op}$ is terminal. This will justify our claim from Section 3.7 that if X is a simplicial space then $\operatorname{colim}_{\Delta^{op}} X$ is homeomorphic to the coequalizer of $d_0, d_1 \colon X_1 \rightrightarrows X_0$.

To see that $i\colon J \hookrightarrow \Delta$ is initial, we must verify that for every $n \geq 0$ the category $(i \downarrow [n])$ is connected. The objects in this category consist of all maps $[0] \to [n]$ and all maps $[1] \to [n]$. Let $e_k\colon [0] \to [n]$ denote the map whose image is $\{k\}$. Let $f_k\colon [1] \to [n]$ denote the map whose image is $\{k\}$. Finally, if $k < l$ let $g_{k,l}\colon [1] \to [n]$ be the map sending $0 \mapsto k$ and $1 \mapsto l$. These are all the objects in $(i \downarrow [n])$.

One readily checks that there are maps in $(i \downarrow [n])$ from e_k to $g_{k,l}$, e_l to $g_{k,l}$, and from e_k to f_k. This proves that $(i \downarrow [n])$ is connected.

Example 21.2. Let $\Delta_f \hookrightarrow \Delta$ be the subcategory consisting of all maps which are monomorphisms (that is, all coface maps). We claim that the inclusion functor $i\colon \Delta_f \hookrightarrow \Delta$ is homotopy initial. As a consequence, i^{op} is homotopy terminal; so the homotopy colimit of a simplicial object can be obtained by instead taking the homotopy colimit of the object obtained by forgetting all degeneracies.

We must prove that for every $n \geq 0$, the overcategory $(i \downarrow [n])$ is contractible. To do this, consider the functor

$$F\colon (i \downarrow [n]) \longrightarrow (i \downarrow [n])$$

which sends a map $\sigma\colon [k] \to [n]$ to the map $F\sigma\colon [k+1] \to [n]$ given by

$$(F\sigma)(0) = 0, \qquad (F\sigma)(i) = \sigma(i-1) \text{ if } i \geq 1.$$

This becomes a functor in the evident way.

Let $e\colon [0] \to [n]$ denote the map whose image is 0, and let $E\colon (i \downarrow [n]) \to (i \downarrow [n])$ be the functor which sends every object to e and every map to the identity. We thus have three functors

$$F, id, e\colon (i \downarrow [n]) \longrightarrow (i \downarrow [n]).$$

The reader can check that there are natural transformations $id \to F$ and $e \to F$. This shows that upon taking classifying spaces the maps induced by F, id, and e are all homotopic. In particular, the identity map is null-homotopic; so $(i \downarrow [n])$ is contractible.

The argument from the above example actually shows the following. For each $\sigma\colon [k] \to [n]$ in Δ, let $sh(\sigma)$ denote the map $[k+1] \to [n+1]$ which sends $0 \mapsto 0$ and $i \mapsto \sigma(i-1)+1$ for $i \geq 1$. So $sh(\sigma)$ is a 'shift' of the map σ.

Proposition 21.3. *Let $J \hookrightarrow \Delta$ be a subcategory satisfying the following:*

(1) For each map σ in J, $sh(\sigma)$ is also in J;
(2) For each $n \geq 0$, the 'add 1 map' $[n] \to [n+1]$ given by $i \mapsto i+1$ belongs to J.
(3) For each $n \geq 0$, the map $[0] \to [n]$ whose image is $\{0\}$ belongs to J.

Then $J \hookrightarrow \Delta$ is homotopy initial.

Proof. Left to the reader. $\square$

The reader can check that Δ_f is the smallest subcategory of Δ satisfying the three conditions in the above proposition.

Exercise 21.4. Let Ω be the full subcategory of $\mathcal{S}et$ consisting of the objects $[0], [1], [2], \ldots$. Recall that $[n] = \{0, 1, \ldots, n\}$. Note that Δ is a subcategory of Ω, with the only maps in Δ being the monotone increasing functions.

Adapt the method used in Example 21.2 to prove that the inclusion $\Delta \hookrightarrow \Omega$ is homotopy initial.

Example 21.5. Consider the product category $\Delta\times\Delta$. Objects are pairs $([n_1], [n_2])$, and a map $([k_1], [k_2]) \to ([n_1], [n_2])$ simply consists of two maps $k_1 \to n_1$ and $k_2 \to n_2$.

Let $d\colon \Delta \to \Delta \times \Delta$ denote the diagonal functor. We claim that this is homotopy initial. As a consequence, d^{op} is homotopy terminal; so if $X_{*,*}$ is a bisimplicial space then its homotopy colimit is weakly equivalent to the homotopy colimit of the simplicial space $[n] \mapsto X_{n,n}$.

To justify the claim, we prove that $\big(d \downarrow ([p], [q])\big)$ is contractible for any p and q. The method is similar to that of the previous example. Recall that an object of $\big(d \downarrow ([p], [q])\big)$ consists of an object $[n]$ in Δ and a map $d([n]) \to ([p], [q])$. So we have an $[n]$ and two maps $[n] \to [p]$ and $[n] \to [q]$. Given an $[n']$ and two maps $[n'] \to [p]$ and $[n'] \to [q]$, a map from the first object to this one consists of a map $[n] \to [n']$ making the two evident triangles commute.

Let

$$F\colon \big(d \downarrow ([p], [q])\big) \longrightarrow \big(d \downarrow ([p], [q])\big)$$

be the functor which sends the object $([n], \sigma_1\colon [n] \to [p], \sigma_2\colon [n] \to [q])$ to the object $([n+1], [n+1] \to [p], [n+1] \to [q])$ where the first map sends $0 \mapsto 0$ and $i \mapsto \sigma_1(i-1)$ for $i \geq 1$, while the second map sends $0 \mapsto 0$ and $i \mapsto \sigma_2(i-1)$ for $i \geq 1$. The functor F has the evident behavior on maps.

Let $e\colon (d \downarrow ([p], [q])) \longrightarrow (d \downarrow ([p], [q]))$ denote the functor which sends all objects to $([0], e_0, e_0)$ where $e_0\colon [k] \to [n]$ always denotes the map whose image is $\{0\}$.

The reader can check that there are natural transformations $id \to F$ and $e \to F$. So after taking classifying spaces one finds that the identity is null-homotopic, and therefore $(d \downarrow ([p], [q]))$ is contractible.

21.6. **Truncated simplicial objects.** Let $\Delta_{\leq n}$ be the subcategory of Δ consisting of all objects $[k]$ where $k \leq n$. A functor $(\Delta_{\leq n})^{op} \to X$ is called an ***n*-truncated** simplicial space, or an ***n*-skeletal** simplicial space.

When taking homotopy colimits of an n-truncated simplicial space, one can no longer throw away the degeneracies and be guaranteed the same answer. That is, the subcategory of $(\Delta_{\leq n})^{op}$ consisting of the face maps is no longer homotopy final. One can see that the proof in Example 21.2 breaks down, as that proof used the infinite nature of the category Δ. Still, there is a nice reduction one can make.

Let Sub_n be the poset of subsets of $\{0, 1, \ldots, n\}$, ordered by inclusion, regarded as a category in the usual way. A picture of this category would look like an n-cube, hence the name. Note that Sub_n can also be thought of as the category of sub-simplices of Δ^n—so the sub-simplices of a simplex form a cube! Let iSub_n be the full subcategory consisting of all objects except $\{0, 1, \ldots, n\}$ (the 'i' is for 'initial').

Notice that there is a functor $\Gamma\colon \mathrm{iSub}_n \to \Delta_{\leq n}$, defined as follows. For any subset $S = \{i_0, \dots, i_k\}$ of $[n]$, there is a unique order-preserving bijection between S and $[k]$. Using this, an inclusion of subsets gives rise to an inclusion in $\Delta_{\leq n}$. The map Γ just sends the subset S to $[k]$, and has the evident behavior on maps. For instance, the inclusion $\{1\} \hookrightarrow \{0,1\}$ is sent to the map $[0] \mapsto [1]$ whose image is 1; the inclusion $\{1,3\} \hookrightarrow \{1,2,3\}$ is send to the map $[1] \to [2]$ whose image is $\{0,2\}$.

Proposition 21.7. *The functor* $\Gamma\colon \mathrm{iSub}_n \to \Delta_{\leq n}$ *is homotopy initial. So* $\Gamma^{op}\colon (\mathrm{iSub}_n)^{op} \to (\Delta_{\leq n})^{op}$ *is homotopy terminal.*

Remark 21.8. Let X be an n-truncated simplicial object. The above proposition shows that when computing $\mathrm{hocolim}\, X$ the degeneracies don't really matter—in the sense that one can write down a cubical diagram, using only face maps, whose homotopy colimit is $\mathrm{hocolim}\, X$. However, this does *not* say that if you look at the subdiagram of X consisting only of face maps that the homotopy colimit of *that* diagram is also the same as $\mathrm{hocolim}\, X$. The subcategory of $(\Delta_{\leq n})^{op}$ consisting of the face maps is not homotopy final!

The proof of the above proposition is more involved than what we have done so far. The classifying spaces of the overcategories are somewhat complicated, and their contractibility has to be proven by a combinatorial argument. A nice reference in the literature is [Si, Section 6].

Let $I_{n,k}$ denote the overcategory $(\mathrm{iSub}_n \downarrow [k])$, where $k \leq n$. Note that an object of $I_{n,k}$ is a pair (σ, ϕ) where $\sigma \subseteq [n]$ and $\phi\colon \Gamma(\sigma) \to [k]$ is an order-preserving map. It is useful to drop the 'Γ', and regard ϕ just as an order-preserving map $\sigma \to [k]$. To have a map $(\sigma, \phi) \to (\sigma', \phi')$ means that $\sigma \subseteq \sigma'$ and ϕ is the restriction of ϕ'. From this it is easy to see that $I_{n,k}$ is a poset.

We wish to ultimately show that each $I_{n,k}$ is contractible, but we'll start by describing a certain stratification of $I_{n,k}$. For each order-preserving map $\alpha\colon [n] \to [k]$, let J_α denote the full subcategory of $I_{n,k}$ consisting of pairs (σ, ϕ) such that ϕ is the restriction of α. It's easy to check that J_α is isomorphic to the category iSub_n (in effect, the data in ϕ is redundant), and so the nerve of J_α is $\mathrm{sd}\,\Delta^n$.

If α and β are maps $[n] \to [k]$ in Δ, then $J_\alpha \cap J_\beta$ consists of pairs (σ, ϕ) such that ϕ is the restriction of both α and β. If we let S denote the maximal subset of $[n]$ on which α and β agree, then $J_\alpha \cap J_\beta$ is isomorphic to the category of subsets of S; hence $J_\alpha \cap J_\beta$ is $\mathrm{sd}\,\Delta^i$ for some i (or empty). This same reasoning applies to any iterated intersection $J_{\alpha_1} \cap J_{\alpha_2} \cap \ldots \cap J_{\alpha_l}$.

Order-preserving maps $[n] \to [k]$ are in bijective correspondence with monotone increasing sequences of length $n+1$, with values in $\{0, 1, \dots, k\}$. There are $\binom{n+k+1}{k}$ such sequences (they are in bijective correspondence with monomials of degree $n+1$ in the variables $X_0, X_1, \dots, X_k$, where the exponent of X_i is the number of times i appears in the sequence). So we have seen how to decompose the nerve of $I_{n,k}$ into $\binom{n+k+1}{k}$ copies of $\mathrm{sd}\,\Delta^n$, and the intersection of any number of these copies is a copy of $\mathrm{sd}\,\Delta^i$ for some i (or else empty).

Exercise 21.9. Using the above description, work out explicit pictures of $I_{1,1}$, $I_{2,1}$, and $I_{2,2}$. The first, for instance, is the union of 3 copies of $\mathrm{sd}\,\Delta^1$, glued together in a certain way.

The above description tells us that the nerve of $I_{n,k}$ is the barycentric subdivision of a certain complex we'll call $L_{n,k}$. We can describe this complex as follows:

(1) The n-simplices correspond to monotone increasing sequences $a_0 \dots a_n$ whose values are in $\{0, \dots, k\}$ (i.e., to maps $[n] \to [k]$).
(2) The $(n-i)$-simplices correspond to sequences as in (1) except where i of the a_j's have been replaced by the symbol '?'.
(3) The face-map d_i corresponds to replacing the ith entry of the sequence with a '?'.

For instance, in $L_{1,1}$ there are three 1-simplices, indexed by the sequences 00, 01, and 11. We have that $d_1(00) = 0?$ and $d_1(01) = 0?$, etc. So $L_{1,1}$ consists of three 1-simplices which are glued together sequentially, with one pair head-to-head and the other pair tail-to-tail: $\cdot \to \cdot \leftarrow \cdot \to \cdot$

We need to show that $L_{n,k}$ is contractible.

Lemma 21.10. *Let X be a simplicial complex which is purely of dimension d (meaning that every simplex is contained in a d-simplex). Suppose the d-simplices can be ordered as $F_1, F_2, \dots, F_M$ in such a way that for each $i \geq 1$*

(a) the subcomplex $F_{i+1} \cup F_{i+2} \cup \cdots \cup F_M$ intersects F_i purely in dimension $d-1$, and

(b) F_i has at least one face which is not in $F_{i+1} \cup \cdots \cup F_M$.

Then X is contractible.

Proof. The main point is that if σ is an n-simplex and S is any union of codimension one faces forming a *proper* subset of $\partial\sigma$, then there is a deformation retraction of σ onto S. Since the n-simplex F_1 has a face which is not in $F_2 \cup \cdots \cup F_M$, we can therefore deformation-retract X down to $F_2 \cup \cdots \cup F_M$. Now proceed by induction, at each step choosing a deformation retration of $F_k \cup \cdots \cup F_M$ down to $F_{k+1} \cup \cdots \cup F_M$. $\square$

Proposition 21.11. *If $k \leq n$, the nerve of $L_{n,k}$ is contractible.*

Proof. We have already seen that $L_{n,k}$ is purely n-dimensional. The n-simplices of $L_{n,k}$ are indexed by monotone increasing sequences of length $n+1$ with values in $\{0, 1, \dots, k\}$, and we can order these lexicographically. We claim that this ordering satisfies the conditions of the lemma.

Let F be the n-simplex corresponding to a sequence $a_0 a_1 \dots a_n$. If $a_i = a_{i+1}$ for some i, then the face of F corresponding to $a_0 a_1 \dots a_{i-1} ? a_{i+1} \dots a_n$ only belongs to n-simplices which come before F in the ordering (because such an n-simplex would have the "?" replaced with a number $j \leq a_{i+1}$, and we would then have $j \leq a_i$ as well). On the other hand, if the sequence $a_0 a_1 \dots a_n$ has no repeats then it means that $n = k$ and we are looking at the sequence $0, 1, \dots, n$. In this case, the face of F corresponding to $0, 1, \dots, n-1, ?$ only belongs to n-simplices which come before F in the ordering. This proves property (b).

To prove property (a), suppose that F meets an n-simplex G corresponding to the sequence $b_0 b_1 \dots b_n$, where $\{b\}$ is lexicographically greater than $\{a\}$. We need to find a sequence $\{c\}$ which is also lexicographically greater than $\{a\}$, such that the intersection of F with the $\{c\}$-simplex contains an $(n-1)$-simplex which itself contains $F \cap G$.

Let j be the smallest index for which $a_j \neq b_j$. Then $F \cap G$ is contained entirely in the $(n-1)$-simplex $a_0 \dots a_{j-1} ? a_{j+1} \dots a_n$. Note that we cannot have $a_j = k$, since $a_j < b_j$. If $a_0 a_1 \dots a_{j-1}(a_j + 1) a_{j+1} \dots a_n$ is a monotone increasing sequence then we can take it as our $\{c\}$—the corresponding n-simplex intersects F in the codimension 1 face $a_0 \dots a_{j-1} ? a_{j+1} \dots a_n$, and this face contains $F \cap G$.

If the sequence $a_0 a_1 \dots a_{j-1}(a_j+1)a_{j+1} \dots a_n$ is *not* monotone increasing then this means $a_j = a_{j+1} = \cdots = a_{j+p}$ for some $p \geq 1$ (where we choose p as large as possible). Since $a_j < b_j$ we must have $a_i \neq b_i$ for $i \in [j, j+p]$—in particular, $a_{j+p} \neq b_{j+p}$. In this case take $\{c\}$ to be the sequence $a_0 a_1 \dots a_{j+p-1}(a_{j+p}+1)a_{j+p+1} \dots a_n$. Then the intersection of F and the simplex corresponding to $\{c\}$ contains the codimension one face $a_0 a_1 \dots a_{j+p-1} ? a_{j+p+1} \dots a_n$, which in turn contains $F \cap G$. □

21.12. **Homotopical symmetric products.** This will be the final example of this section. Let (n) denote the finite set $\{1, 2, \dots, n\}$, and let I denote the category whose objects are all such sets (with $n \geq 1$) and where the maps are monomorphisms. The category I is similar to Δ_f, except that we have now expanded the morphisms to include permutations.

Let X be a pointed space. For every map $\sigma\colon (n) \to (k)$ in I, there is an induced map $\sigma_*\colon X^n \to X^k$ sending $(x_1, \dots, x_n)$ to the tuple with x_i in spot $\sigma(i)$ and the basepoint in all other spots. This gives a diagram $X^*\colon I \to \mathcal{T}op$.

Our first goal will be to show that the colimit of X^* is isomorphic to something more familiar, namely the infinite symmetric product of X. The latter is the space $\mathrm{SP}^\infty(X) = X^\infty/\Sigma_\infty$, where X^∞ is the colimit of the sequence

$$X \hookrightarrow X^2 \hookrightarrow X^3 \hookrightarrow \cdots$$

in which each map sends $(x_1, \dots, x_n)$ to $(x_1, \dots, x_n, *)$. To see that $\operatorname{colim}_I X^*$ and $\mathrm{SP}^\infty(X)$ are isomorphic, follow the steps in the exercise below.

Exercise 21.13. Let $\omega = \{1, 2, 3, \dots\}$, and let I_∞ be the subcategory of $\mathcal{S}et$ consisting of the objects (n) (for all $n \geq 1$) and ω, where the maps as follows:

- maps from (n) to (k) are the monomorphisms;
- maps from (n) to ω are the monomorphisms;
- maps from ω to ω are the elements of Σ_∞.

Let $j\colon I \hookrightarrow I_\infty$ be the inclusion. Finally, let I_{std} be the subcategory of I consisting of all objects (n) but where the morphisms are the *standard* inclusions $(n) \hookrightarrow (k)$.

(a) If $D\colon I \to \mathcal{T}op$, let $L_j D = \operatorname{colim}_{I \to I_\infty} D$ be the relative colimit (or left Kan extension) of D along j. Recall that $[L_j D](\omega) \cong \operatorname{colim}_{n \in (j \downarrow \omega)} D_n$. Note that there is an evident functor $I_{std} \to (j \downarrow \omega)$, and prove that this is terminal. Deduce that $[L_j D](\omega) \cong \operatorname{colim}_{I_{std}} D$.

(b) Let $B\Sigma_\infty$ denote the category with one object and endomorphism set Σ_∞. Note that there is an evident functor $B\Sigma_\infty \to I_\omega$ sending the unique object to ω. Prove that this functor is terminal.

(c) If $D\colon I \to \mathcal{T}op$ is any diagram, argue that $\operatorname{colim} D$ is isomorphic to $\operatorname{colim}[L_j D]$. Use (b) to deduce that the latter is is isomorphic to $[L_j D](\omega)/\Sigma_\infty$, and use (a) to replace $[L_j D](\omega)$ with $\operatorname{colim}_{I_{std}} D$. When $D = X^*$, deduce that $\operatorname{colim}_I X^* \cong X^\infty/\Sigma_\infty$.

We now wish to consider the homotopy colimit $\operatorname{hocolim}_I X^*$; it is natural to call this the *homotopical infinite symmetric product* of X. We'll use the notation $\mathrm{SP}^h(X) = \operatorname{hocolim}_I X^*$. This construction was probably first considered by Jeff Smith, who used it in the context of symmetric spectra—the first reference I know in print is [Sh, Section 1]. The spaces $\mathrm{SP}^h(X)$ were later intensively studied in

[Schl], where it was shown that if X is path-connected then $\mathrm{SP}^h(X) \simeq \Omega^\infty\Sigma^\infty X$; this is related to the Barratt-Priddy-Quillen theorem.

We proved in the previous exercise that $\mathrm{colim}_I X^* \cong (X^\infty)/\Sigma_\infty$, and from this it would be a natural guess that $\mathrm{hocolim}_I X^* \simeq (X^\infty)_{h\Sigma^\infty}$. However, this guess is incorrect (for reasons which we will see below). To correct the guess, recall that $\omega = \{1, 2, 3, \ldots\}$. Let M denote the injective self-maps of ω, which form a monoid under composition. Clearly we have $\Sigma_\infty \subseteq M$, and if X is a pointed space there is a natural action of M on X^∞ which extends the action of Σ_∞. We have the following nice result, which is [Schl, Proposition 3.7].

Proposition 21.14. *If X is a well-pointed CW-complex, then $\mathrm{SP}^h(X) \simeq (X^\infty)_{hM}$.*

We'll outline the proof of this following [Schl], leaving most steps as exercises for the reader. First, let I_ω denote the subcategory of $\mathcal{S}et$ whose objects are the sets (n) together with ω, and where the maps are the monomorphisms. Let $j\colon I \hookrightarrow I_\omega$ denote the evident inclusion. Finally, let BM denote the category with one object and endomorphism set M, and let $i\colon BM \to I_\omega$ denote the inclusion sending the unique object of BM to ω.

The following exercise contains a key result due to J. Smith, which first appeared in [Sh, Lemma 2.2.9]. I owe my understanding of this proof to Stefan Schwede, and the proof we outline below is entirely from [Sch].

Exercise 21.15. Prove that $j\colon BM \hookrightarrow I_\omega$ is homotopy terminal by following the steps below.

(a) Define a functor $c\colon M \to M$ by the following formula: if $f \in M$, then
$$(cf)(i) = \begin{cases} i & \text{if } i \text{ is odd,} \\ 2\cdot f(i/2) & \text{if } i \text{ is even.}\end{cases}$$
Verify that c is a homomorphism of monoids, and therefore induces a functor $Bc\colon BM_{cat} \to BM_{cat}$.

(b) Construct a natural transformation $id \to Bc$, as well as a natural transformation $Bc \to *$ where $*$ is the functor which sends all morphisms to the identity. Conclude that on classifying spaces one has $id \simeq Bc \simeq *$ as maps $BM \to BM$, and therefore BM is contractible.

(c) Fix $n \geq 1$. For any $\alpha \in M$, let $\alpha + n$ be the element of M which is the identity on the numbers $1, 2, \ldots, n$ and sends $n+i$ to $n+\alpha(i)$ for $i \geq 1$. Define a functor $BM_{cat} \to ((n) \downarrow j)$ which sends the unique object to the standard inclusion $(n) \hookrightarrow \omega$ and which sends the morphism $\alpha \in M$ to $\alpha + n$. Verify that that this is indeed a functor, that it is fully faithful, and that it is surjective on isomorphism classes—so conclude that it as an equivalence of categories.

(d) Deduce that $((n) \downarrow j)$ is contractible, for all $n \geq 1$. Prove that $(\omega \downarrow j)$ has an initial object and is therefore also contractible. Conclude that j is homotopy terminal.

Exercise 21.16. Now let $D\colon I \to \mathcal{T}op$ be any diagram. Let L_jD denote the homotopy left Kan extension of D along the inclusion $j\colon I \hookrightarrow I_\omega$.

(a) Prove that there are weak equivalences
$$\operatorname*{hocolim}_I D \simeq \operatorname*{hocolim}_{I_\omega}(L_jD) \simeq \operatorname*{hocolim}_{BM}(L_jD)(\omega) = [(L_jD)(\omega)]_{hM}.$$

(b) Prove that there is a weak equivalence $(L_jD)(\omega) \simeq \mathrm{hocolim}_{I_{std}} D$.

(c) Prove that if the maps in $D\colon I_{std} \to \mathcal{T}op$ are all cofibrations, then $\operatorname{hocolim}_{I_{std}} D \simeq \operatorname{colim}_{I_{std}} D$.
(d) Conclude that if X is a well-pointed CW-complex then $\operatorname{hocolim}_I X^* \simeq (X^\infty)_{hM}$.

Remark 21.17. The main difference between the work in Exercises 21.13 and 21.15 is that in the latter we must use I_ω instead of I_∞. The reason is that although $B\Sigma_\infty \to I_\infty$ is terminal, it is not homotopy terminal; this is why the monoid M, rather than Σ_∞, appears in Proposition 21.14.

Exercise 21.18. Prove that $B\Sigma_\infty \to I_\infty$ is not homotopy terminal.

22. Homotopical decompositions of spaces

By a "homotopical decomposition" of a space X we mean a diagram $D\colon I \to \mathcal{T}op$ together with a map $\operatorname{colim}_I D \to X$, such that the composite $\operatorname{hocolim}_I D \to \operatorname{colim}_I D \to X$ is a weak equivalence. Note that by Proposition 18.14 a homotopical decomposition yields, in particular, a spectral sequence for computing the cohomology groups $\mathcal{E}^*(X)$ from the groups $\mathcal{E}^*(D_i)$.

We have already seen one example of a homotopical decomposition, back in Section 14.15. If $\{A_1, \dots, A_n\}$ is a closed cover of X then one can form the cubical diagram $A\colon P_n \to \mathcal{T}op$ sending a subset $\{i_1, \dots, i_k\}$ to $A_{i_1} \cap \cdots \cap A_{i_k}$. Under the condition of certain inclusions being cofibrations, this is a homotopical decomposition.

Note that giving a diagram $D\colon I \to \mathcal{T}op$ together with a map $\operatorname{colim}_I D \to X$ is the same as giving a diagram $I \to (\mathcal{T}op \downarrow X)$. If we let $\Gamma\colon (\mathcal{T}op \downarrow X) \to \mathcal{T}op$ denote the forgetful functor sending the pair $[Y, Y \to X]$ to Y, then D is just the composite

$$I \longrightarrow (\mathcal{T}op \downarrow X) \xrightarrow{\Gamma} \mathcal{T}op.$$

In many applications I is actually a subcategory of $(\mathcal{T}op \downarrow X)$.

Homotopical decompositions seem to be useful in a variety of situations. In this section we will give a few examples of these decompositions.

Here is one example worth recording:

Proposition 22.1. *Let $\{U_\alpha\}$ be an open cover of X. Let I be the subcategory of $(\mathcal{T}op \downarrow X)$ consisting of the U_α's and all their finite intersections. Then*

$$\operatorname*{hocolim}_I \Gamma \to X$$

is a weak equivalence.

Proof. See [DI]. □

Before proceeding to another important example, we need a new tool. To set this in context, all of the theorems we stated in Parts 1–3 about homotopy colimits are actually generic results which work basically the same in any model category (not just in $\mathcal{T}op$). The following result is very particular to $\mathcal{T}op$, however.

Let $D\colon I \to \mathcal{T}op$ and suppose one has a map $p\colon \operatorname{colim}_I D \to X$, where X is some space. For each n and each map $\sigma\colon \Delta^n \to X$, consider the category $F(D)_\sigma$ whose objects are tuples

$$[i, \alpha\colon \Delta^n \to D_i]$$

such that $p \circ \alpha = \sigma$. A map from this object to $[j, \beta\colon \Delta^n \to D_j]$ is a map $i \to j$ making the evident triangle commute. We call $F(D)_\sigma$ the **fiber category** of D over σ.

Theorem 22.2. *In the above setting, suppose that for each $n \geq 0$ and each $\sigma\colon \Delta^n \to X$, the category $F(D)_\sigma$ is contractible. Then the composite* $\operatorname{hocolim}_I D \to \operatorname{colim}_I D \to X$ *is a weak equivalence.*

Now assume in addition that there is a diagram $\tilde{D}\colon I \to s\mathcal{S}et$ and a natural isomorphism $\phi_i\colon |\tilde{D}_i| \to D_i$. For each $\sigma\colon \Delta^n \to X$ we can define a new category $\tilde{F}(D)_\sigma$ as follows. Objects of this category are pairs $[i, \Delta^n_s \to \tilde{D}_i]$ such that the composite $|\Delta^n_s| \to |\tilde{D}_i| \to D_i \to X$ is equal to α, and maps are the expected things. Here Δ^n_s denote the canonical n-simplex $\Delta^n_s \in s\mathcal{S}et$. Note that there is a map of categories $\tilde{F}(D)_\sigma \to F(D)_\sigma$, but there's no reason to suspect that this is a weak equivalence.

We have the following refinement of the previous theorem:

Theorem 22.3. *In the above setting, suppose that for each $n \geq 0$ and each $\sigma\colon \Delta^n \to X$, the category $\tilde{F}(D)_\sigma$ is contractible. Then the composite* $\operatorname{hocolim}_I D \to \operatorname{colim}_I D \to X$ *is a weak equivalence.*

We will give the proofs of Theorems 22.2 and 22.3 in Section 22.8 below. But first we record some useful applications. These are all inspired by the discussion in [J2, Section 2].

Proposition 22.4. *Let $\Delta \downarrow X$ denote the overcategory $(j \downarrow X)$, where $j\colon \Delta \to \mathcal{T}op$ is the usual functor. The functor j gives us a map $(\Delta \downarrow X) \to (\mathcal{T}op \downarrow X)$, and the natural map*

$$\operatorname*{hocolim}_{(\Delta\downarrow X)} j^*\Gamma \to \operatorname*{colim}_{(\Delta\downarrow X)} j^*\Gamma \to X$$

is a weak equivalence.

Proof. Note that the diagram $j^*\Gamma$ lifts to a diagram $\tilde{\Gamma}\colon (\Delta \downarrow X) \to s\mathcal{S}et$. So we can attempt to use Theorem 22.3.

Let $I = (\Delta \downarrow X)$, and let $\sigma\colon \Delta^n \to X$. An object of $\tilde{F}(\Gamma)_\sigma$ consists of an object $[k]$, a map $f\colon \Delta^k \to X$, and a simplicial map $\Delta^n \to \Delta^k$ whose composite with f is σ. But note that this category has an initial object, given by $[n]$, the map $\sigma\colon \Delta^n \to X$, and the identity map $\Delta^n \to \Delta^n$. So $\tilde{F}(\Gamma)_\sigma$ is contractible, and we are done. □

Proposition 22.5. *Let $\Delta_c(X)$ be the full subcategory of $\mathcal{T}op \downarrow X$ consisting of all maps whose domain is a simplex. Then* $\operatorname{hocolim}_{\Delta_c(X)} \Gamma \to X$ *is a weak equivalence.*

Proof. This is a consequence of Theorem 22.2. The same kind of argument as in the previous proof shows that the fiber categories $F(\Gamma)_\sigma$ are all contractible. □

Now let $p\colon E \to B$ be a map, and let $\alpha\colon I \to \mathcal{T}op \downarrow B$ be a functor. Let Γ_p denote the diagram $I \to \mathcal{T}op$ sending i to $\alpha(i)^*E$, the pullback of $E \to B$ along the map $\alpha(i)$. Clearly there is a map $\operatorname{colim}_I \Gamma_p \to E$, and so we may consider the composite

$$\operatorname*{hocolim}_I \Gamma_p \to \operatorname*{colim}_I \Gamma_p \to E.$$

Proposition 22.6. *In the above setting, let $I = (\Delta \downarrow B)$. Then the map* $\operatorname{hocolim}_I \Gamma_p \to E$ *is a weak equivalence, for any map p which is a fibration.*

Proof. Consider the diagram $D\colon I \to \mathcal{T}op$ which sends a pair $([k], \Delta^k \to B)$ to the geometric realization of the simplicial set obtained as the pullback $\Delta^k_s \to SB \leftarrow SE$, where $S(-)$ is the singular functor.

There is an evident map of diagrams $|D| \to \Gamma_p$, and the fact that p is a fibration implies that this map is an objectwise weak equivalence. One uses here that $SE \to SB$ is a fibration of simplicial sets, and that in $s\mathcal{S}et$ a pullback of a weak equivalence along a fibration is another weak equivalence.

So we are reduced to showing that $\operatorname{hocolim}_I |D| \to X$ is a weak equivalence. This is an easy application of Theorem 22.3, very similar to the proof of Proposition 22.4. □

The following corollary is now immediate from Proposition 18.14.

Corollary 22.7. *If $p\colon E \to B$ is a fibration, then for any cohomology theory $\mathcal{E}$ there is a spectral sequence*

$$E_2^{p,q} = H^p(\Delta \downarrow B; \mathcal{E}^q(\Gamma_p)) \Rightarrow \mathcal{E}^{p+q}(E).$$

Note that for each simplex $\sigma\colon \Delta^n \to B$, the space $\Gamma_p(\sigma) = \sigma^* E$ is weakly equivalent to the fiber F of p. So the diagram $\mathcal{E}^q\Gamma_p$ is a diagram of abelian groups where all the abelian groups are isomorphic. One can also check that every map in the diagram is an isomorphism. So this is something which should be called a "local coefficient system". The above spectral sequence is a version of the generalized Atiyah-Hirzebruch/Leray-Serre spectral sequence.

22.8. **Proofs of the two main theorems.** The two proofs are both based on an analogous theorem about simplicial sets. Let $D\colon I \to s\mathcal{S}et$ be a diagram of simplicial sets, let $X \in s\mathcal{S}et$, and suppose there is a map $\operatorname{colim}_I D \to X$. For each simplex $\sigma \in X_n$, let $F(D)_\sigma$ denote the category whose objects are pairs $[i, \alpha \in (D_i)_n]$ such that the map $D_i \to X_i$ sends α to σ. A map in $F(D)_\sigma$ from $[i, \alpha \in (D_i)_n]$ to $[j, \beta \in (D_j)_n]$ is a map $i \to j$ such that $D_i \to D_j$ sends α to β. We call $F(D)_\sigma$ the "fiber category" of D over σ.

The following result is a slight generalization of [J2, Lemma 2.7]. The proof, however, is exactly the same.

Proposition 22.9. *Suppose that $D\colon I \to s\mathcal{S}et$ and X are as above, and assume that for every $n \geq 0$ and every $\sigma \in X_n$, the fiber category $F(D)_\sigma$ is contractible. Then the map* $\operatorname{hocolim}_I D \to X$ *is a weak equivalence of simplicial sets.*

Proof. Consider the simplicial replacement $\operatorname{srep}(D)$, and observe that this is a bisimplicial set. Let us write $\operatorname{srep}(D)_{p,q}$ for the q-simplices in the pth level of $\operatorname{srep}(D)$; that is so say,

$$\operatorname{srep}(D)_{p,q} = \coprod_{i_0 \leftarrow \cdots \leftarrow i_p} D(i_p)_q.$$

When drawing the bisimplicial set we draw the q-direction vertically and the p-direction horizontally.

If $B_{*,*}$ is a bisimplicial set, then there are two geometric realizations of B, depending on whether we realize vertically or horizontally. Define

$$|B|_h = \operatorname{coeq}\left[\coprod_{[n]\to[k]} B_{k,*} \times \Delta^n \rightrightarrows \coprod_n B_{n,*} \times \Delta^n\right]$$

and

$$|B|_v = \operatorname{coeq}\left[\coprod_{[n]\to[k]} B_{*,k} \times \Delta^n \rightrightarrows \coprod_n B_{*,n} \times \Delta^n.\right]$$

Note that $\operatorname{hocolim}_I D = |\operatorname{srep}(D)|_h$ in this notation.

Let $d(B)$ denote the diagonal simplicial set of B. Then we know there are natural maps $|B|_h \to d(B) \leftarrow |B|_v$ and that these are both isomorphisms.

Let $c_h X$ denote the bisimplicial set with $(c_h X)_{p,q} = X_q$, where all the horizontal faces and degeneracies are the identity map. This bisimplicial set is 'horizontally constant'.

There is a natural map of bisimplicial sets $\operatorname{srep}(D) \to c_h X$. This gives a commutative diagram

$$\begin{tikzcd}
|\operatorname{srep}(D)|_h \arrow[r, "\cong"] \arrow[d] & d(\operatorname{srep}(D)) \arrow[d] & |\operatorname{srep}(D)|_v \arrow[l, "\cong"'] \arrow[d] \\
|c_h X|_h \arrow[r, "\cong"'] & d(c_h X) & |c_h X|_v. \arrow[l, "\cong"]
\end{tikzcd}$$

Our goal is to show that the left vertical map is a weak equivalence, and so it will suffice to show that the right vertical map is a weak equivalence.

We will argue that each map of simplicial sets $\operatorname{srep}(D)_{*,q} \to (c_h X)_{*,q}$ is a weak equivalence. This will imply that we get a weak equivalence after applying the vertical geometric realization.

Note that $(c_h X)_{*,q}$ is just the discrete simplicial set corresponding to the set X_q. So it will suffice to prove that the fiber of the map $\pi_q\colon \operatorname{srep}(D)_{*,q} \to X_q$ over any point is contractible. But if $\sigma \in X_q$, then one readily checks that the fiber of π_q over σ is the nerve of the category $F(D)_\sigma$, and hence is contractible by assumption. $\square$

We can now give the proofs of our two theorems:

Proof of Theorem 22.2. Let $\operatorname{Sing}\colon \mathcal{T}op \to s\mathcal{S}et$ denote the usual singular functor. Applying this to D gives a diagram $\operatorname{Sing} D\colon I \to s\mathcal{S}et$, together with an induced map $\operatorname{colim}(\operatorname{Sing} D) \to \operatorname{Sing} X$. An n-simplex of $\operatorname{Sing} X$ is just a map $\sigma\colon \Delta^n \to X$, and the fiber category $F(\operatorname{Sing} D)_\sigma$ from Proposition 22.9 is precisely the fiber category $F(D)_\sigma$ from the statement of the theorem. Sincee these fiber categories are assumed to be contractible, Proposition 22.9 says that $\operatorname{hocolim}_I(\operatorname{Sing} D) \to \operatorname{Sing} X$ is a weak equivalence of simplicial sets.

The final step is to apply geometric realization to the above map, and then to use the following commutative diagram:

$$\begin{tikzcd}
|\operatorname{hocolim}(\operatorname{Sing} D)| \arrow[r] & |\operatorname{colim}(\operatorname{Sing} D)| \arrow[r] & |\operatorname{Sing} X| \\
\operatorname{hocolim}|\operatorname{Sing} D| \arrow[r] \arrow[u, "\cong"'] \arrow[d, "\sim"] & \operatorname{colim}|\operatorname{Sing} D| \arrow[r] \arrow[u, "\cong"'] \arrow[d] & |\operatorname{Sing} X| \arrow[u, equal] \arrow[d, "\sim"] \\
\operatorname{hocolim} D \arrow[r] & \operatorname{colim} D \arrow[r] & X.
\end{tikzcd}$$

We know from the previous paragraph that the composite across the top row is a weak equivalence. The two-out-of-three property then shows that the composite across the bottom row is also a weak equivalence. $\square$

Proof of Theorem 22.3. This proof is similar to the preceeding one. The natural maps $|\tilde{D}_i| \to D_i$ and $D_i \to X$ allow us to consider the composites

$$\tilde{D}_i \to \operatorname{Sing}|\tilde{D}_i| \to \operatorname{Sing} D_i \to \operatorname{Sing} X.$$

These are compatible as i varies, so we have a map $\operatorname{colim}_I \tilde{D} \to \operatorname{Sing} X$. The assumptions of the theorem say precisely that the fiber categories $F(\tilde{D})_\sigma$ are contractible, for every simplex σ of $\operatorname{Sing} X$. By Proposition 22.9 we therefore have that $\operatorname{hocolim}_I \tilde{D} \to \operatorname{Sing} X$ is a weak equivalence.

To complete the proof one considers the following diagram:

$$\begin{tikzcd}
{|\operatorname{hocolim} \tilde{D}|} \arrow[r, "\sim"] \arrow[d, "\cong"] & {|\operatorname{hocolim}(\operatorname{Sing} D)|} \arrow[r] \arrow[d, "\cong"] & {|\operatorname{Sing} X|} \arrow[d, equal] \\
\operatorname{hocolim} |\tilde{D}| \arrow[r, "\sim"] & \operatorname{hocolim} |\operatorname{Sing} D| \arrow[r] \arrow[d] & {|\operatorname{Sing} X|} \arrow[d, "\sim"] \\
 & \operatorname{hocolim} D \arrow[r] & X.
\end{tikzcd}$$

We have proven that $\operatorname{hocolim}_I \tilde{D} \to \operatorname{Sing} X$ is a weak equivalence. Our assumption that the maps $|\tilde{D}_i| \to D_i$ are weak equivalences implies that $\operatorname{hocolim} |\tilde{D}| \to \operatorname{hocolim} D$ is a weak equivalence. The two-out-of-three property, applied several times, now gives that $\operatorname{hocolim} D_i \to X$ is a weak equivalence. □

23. A survey of other applications

23.1. Telescopes and the localization of spaces. ????

23.2. Homotopy decompositions of classifying spaces. ?????

23.3. Homotopical sheaf theory. ?????

23.4. Further directions. In this final section we mention aspects of the theory of homotopy limits and colimits which we have not addressed here. We also suggest some other references.

(1) A very general approach to homotopy limits and colimits, and particularly their role as derived functors, can be found in [DHKS].

(2) Let I be a topological category—that is, a category where the morphism sets have the structure of topological spaces, and where composition is continuous. An **enriched diagram** $X: I \to \mathcal{T}op$ consists of a topological space $X(i)$ for every $i \in I$, together with continuous maps of spaces $I(i,j) \to \mathrm{Map}(X(i), X(j))$ which are compatible with composition and identities.

One important example of this is when G is a topological group, and I is the topological category with one object whose endomorphisms are G. An enriched diagram $X: I \to \mathcal{T}op$ consists of a space $X(*)$ and a continuous group action $G \times X(*) \to X(*)$.

One can ask for a theory of enriched homotopy colimits and limits. This has been developed recently in [S].

(3) Section 5 of Thomason's paper [T] contains a very compact and appealing treatment of homotopy limits and colimits, their associated spectral sequences, as well as a "Scholium of Great Enlightenment". We highly recommend it.

Appendix A. The simplicial cone construction

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Department of Mathematics, University of Oregon, Eugene, OR 97403

E-mail address: `ddugger@math.uoregon.edu`