Distributed-Parameter Port-Hamiltonian Systems

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Chapter 1
Introduction

In the first part of this chapter we introduce some motivation examples and show that these possesses a common structure. Finally, we indicate what we do in more detail in the chapters to follow.

1.1. Motivating examples

In this section we shall by using simple examples introduce our class and indicate what are natural (control) questions for these systems. We begin with the example of the transmission line. This model describes the charge density and magnetic flux in a cable, as is depictured below.

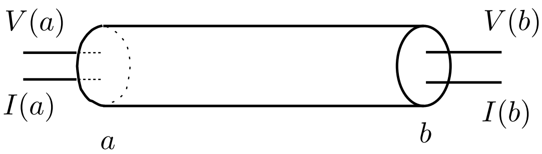

Figure 1.1.: Transmission line

Example 1.1.1 (Transmission line) Consider the *transmission line* on the spatial interval $[a, b]$

$$\begin{aligned} \frac{\partial Q}{\partial t}(\zeta, t) &= -\frac{\partial}{\partial \zeta}\frac{\phi(\zeta, t)}{L(\zeta)} \\ \frac{\partial \phi}{\partial t}(\zeta, t) &= -\frac{\partial}{\partial \zeta}\frac{Q(\zeta, t)}{C(\zeta)}. \end{aligned} \tag{1.1}$$

Here $Q(\zeta, t)$ is the charge at position $\zeta \in [a, b]$ and time $t > 0$, and $\phi(\zeta, t)$ is the flux at position ζ and time t. C is the (distributed) capacity and L is the (distributed) inductance.

The voltage and current are given by $V = Q/C$ and $I = \phi/L$, respectively. The energy of this system is given by

$$E(t) = \frac{1}{2}\int_a^b \frac{\phi(\zeta,t)^2}{L(\zeta)} + \frac{Q(\zeta,t)^2}{C(\zeta)} d\zeta. \tag{1.2}$$

If ϕ and Q satisfy the differential equation (1.1), then we find the following

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_a^b \frac{\phi(\zeta,t)}{L(\zeta)}\frac{\partial \phi}{\partial t}(\zeta,t) + \frac{Q(\zeta,t)}{C(\zeta)}\frac{\partial Q}{\partial t}(\zeta,t)d\zeta \\ &= \int_a^b -\frac{\phi(\zeta,t)}{L(\zeta)}\frac{\partial}{\partial\zeta}\frac{Q(\zeta,t)}{C(\zeta)} - \frac{Q(\zeta,t)}{C(\zeta)}\frac{\partial}{\partial\zeta}\frac{\phi(\zeta,t)}{L(\zeta)} d\zeta \\ &= -\int_a^b \frac{\partial}{\partial\zeta}\left[\frac{\phi(\zeta,t)}{L(\zeta)}\frac{Q(\zeta,t)}{C(\zeta)}\right] d\zeta \\ &= \frac{\phi(a,t)}{L(a)}\frac{Q(a,t)}{C(a)} - \frac{\phi(b,t)}{L(b)}\frac{Q(b,t)}{C(b)}.\end{aligned} \tag{1.3}$$

We see that the change of energy can only occur via the boundary. We can also write the expression of (1.3) using voltage and current, and we find that

$$\frac{d}{dt}E(t) = V(a,t)I(a,t) - V(b,t)I(b,t). \tag{1.4}$$

Since voltage times current equals power and the change of energy is also power, this equation represents a power balance. We interpret this equality by saying that the power of the systems equals the power flow at its boundary. Hence by choosing proper boundary conditions we may ensure that the energy stays within the system.

A natural control question would be how to stabilize this system. The power balance (1.4) is very useful for solving this question. Suppose that the voltage at $\zeta = a$ is set to zero, and at the other end we put a resistor, i.e., $V(b,t) = RI(b,t)$. We see from the power balance (1.4) that

$$\frac{d}{dt}E(t) = -RI(b,t)^2.$$

This implies that the energy decays. However, will the energy converge to zero, and if so how fast? These are stability/stabilizability question which we study in Chapter 6.

Since the power flows via the boundary, it is natural to control via the boundary. In fact we did this already in the previous paragraph when we put $V(b,t) = RI(b,t)$. Hence we come up with the question which and how many of the four variables, voltage and current at the boundary, we may choose as an input. It seems from (1.4) that we may take all four of them as (independent) inputs. As we shall see in Chapter 3, we may only choose at most two as inputs. A similar question can be asked for outputs. Since an output is dictated by the system, one may wonder if all four boundary conditions are dictated by the system, see Chapter 5 for the answer.

If the system dictates the output, there should be a unique solution for a given initial condition. In our partial differential equation (1.1), we have not given an initial condition, i.e., $\phi(\zeta,0)$ and $Q(\zeta,0)$. As one may expect, we choose these initial conditions in the energy space, meaning that the initial energy, $E(0)$, is finite. Giving only an initial condition is not sufficient for a partial differential equation (p.d.e.) to have a (unique) solution, one also has to impose boundary conditions. In Chapter 2 we answer the technical question for which boundary conditions the p.d.e. possesses a (unique) solution. □

The previous example is standard for the class of systems we are studying. There is an energy function (Hamiltonian), a power balance giving that the change of energy (power) goes via the boundary of the spatial domain. The example of the (undamped) vibrating string is very similar.

Example 1.1.2 (Wave equation) Consider a *vibrating string* of length $L = b - a$, held stationary at both ends and free to vibrate transversely subject to the restoring forces due to tension in the string. The vibrations on the system can be modeled by

$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = c\frac{\partial^2 w}{\partial \zeta^2}(\zeta, t), \qquad c = \frac{T}{\rho},\ t \geq 0, \tag{1.5}$$

where $\zeta \in [a, b]$ is the spatial variable, $w(\zeta, t)$ is the vertical position of the string, T is the Young's modulus of the string, and ρ is the mass density, which are assumed to be constant along the string. This model is a simplified version of other systems where vibrations occur, as in the case of large structures, and it is also used in acoustics. Although the wave equation is normally presented in the form (1.5), it is not the form we will be using. However, more importantly, it is not the right model when Young's modulus or the mass density are depending on the spatial coordinate. When the later happens, the correct model is given by

$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta, t)\right]. \tag{1.6}$$

It is easy to see that this equals (1.5) when the physical parameter are not spatially dependent. This system has the energy/Hamiltonian

$$E(t) = \frac{1}{2}\int_a^b \rho(\zeta)\left(\frac{\partial w}{\partial t}(\zeta, t)\right)^2 + T(\zeta)\left(\frac{\partial w}{\partial \zeta}(\zeta, t)\right)^2 d\zeta. \tag{1.7}$$

As we did in the previous example we can calculate the change of energy, i.e., power. This gives (see also Exercise 1.1)

$$\frac{d}{dt}E(t) = \frac{\partial w}{\partial t}(b, t)T(b)\frac{\partial w}{\partial \zeta}(b, t) - \frac{\partial w}{\partial t}(a, t)T(a)\frac{\partial w}{\partial \zeta}(a, t). \tag{1.8}$$

Again we see that the change of energy goes via the boundary of the spatial domain. One may notice that the position is not the (actual) variable used in the energy and the power expression. The variables are velocity ($\frac{\partial w}{\partial t}$) and strain ($\frac{\partial w}{\partial \zeta}$).

For this model one can pose similar questions as for the model from Example 1.1.1. In particular, a control problem could be to damp out the vibrations on the string. One approach to do this is to add damping along the spatial domain. This can also be done by interacting with the forces and velocities at the end of the string, i.e., at the boundary. □

Example 1.1.3 (Beam equations) In recent years the boundary control of flexible structures has attracted much attention with the increase of high technology applications such as space science and robotics. In these applications the control of vibrations is crucial. These vibrations can be modeled by beam equations. For instance, the *Euler-Bernoulli beam* equation models the transversal vibration of an elastic beam if the cross-section dimension of the beam is negligible in comparison with its length. If the cross-section dimension is not negligible, then it is necessary to consider the effect of the rotary inertia. In that case, the transversal vibration is better described by the *Rayleigh beam equation*. An improvement over these models is given by the *Timoshenko beam*, since it incorporates shear and rotational inertia effects, which makes it a more precise model. These equations are given, respectively, by

- Euler-Bernoulli beam:
$$\rho(\zeta)\frac{\partial^2 w}{\partial t^2}(\zeta,t) + \frac{\partial^2}{\partial \zeta^2}\left(EI(\zeta)\frac{\partial^2 w}{\partial \zeta^2}(\zeta,t)\right) = 0, \quad \zeta \in (a,b),\ t \geq 0,$$
where $w(\zeta,t)$ is the transverse displacement of the beam, $\rho(\zeta)$ is the mass per unit length, $E(\zeta)$ is the Young's modulus of the beam, and $I(\zeta)$ is the area moment of inertia of the beam's cross section.
- Rayleigh beam:
$$\rho(\zeta)\frac{\partial^2 w}{\partial t^2}(\zeta,t) - I_\rho(\zeta)\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial \zeta^2}(\zeta,t)\right) + \frac{\partial^2}{\partial \zeta^2}\left(EI(\zeta)\frac{\partial^2 w}{\partial z^2}(\zeta,t)\right) = 0,$$
where $\zeta \in (a,b)$, $t \geq 0$, $w(\zeta,t)$ is the transverse displacement of the beam, $\rho(\zeta)$ is the mass per unit length, I_ρ is the rotary moment of inertia of a cross section, $E(\zeta)$ is the Young's modulus of the beam, and $I(\zeta)$ is the area moment of inertia.
- Timoshenko beam:
$$\begin{aligned}\rho(\zeta)\frac{\partial^2 w}{\partial t^2}(\zeta,t) &= \frac{\partial}{\partial \zeta}\left[K(\zeta)\left(\frac{\partial w}{\partial \zeta}(\zeta,t) - \phi(\zeta,t)\right)\right], \quad \zeta \in (a,b),\ t \geq 0, \\ I_\rho(\zeta)\frac{\partial^2 \phi}{\partial t^2}(\zeta,t) &= \frac{\partial}{\partial \zeta}\left(EI(\zeta)\frac{\partial \phi}{\partial \zeta}(\zeta,t)\right) + K(\zeta)\left(\frac{\partial w}{\partial \zeta}(\zeta,t) - \phi(\zeta,t)\right),\end{aligned} \tag{1.9}$$
where $w(\zeta,t)$ is the transverse displacement of the beam and $\phi(\zeta,t)$ is the rotation angle of a filament of the beam. The coefficients $\rho(\zeta)$, $I_\rho(\zeta)$, $E(\zeta)$, $I(\zeta)$, and $K(\zeta)$ are the mass per unit length, the rotary moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus respectively.

For the last model we show that it has similar properties as found in the previous examples. The energy/Hamiltonian for this system is given by

$$\begin{aligned}E(t) = \frac{1}{2}\int_a^b \Bigg[&K(\zeta)\left(\frac{\partial w}{\partial \zeta}(\zeta,t) - \phi(\zeta,t)\right)^2 + \rho(\zeta)\left(\frac{\partial w}{\partial t}(\zeta,t)\right)^2 + \\ &E(\zeta)I(\zeta)\left(\frac{\partial \phi}{\partial \zeta}(\zeta,t)\right)^2 + I_\rho\left(\frac{\partial \phi}{\partial t}(\zeta,t)\right)^2\Bigg] d\zeta.\end{aligned} \tag{1.10}$$

Next we want to calculate the power. For this it is better to introduce some (physical) notation first.

$$\begin{aligned}
x_1(\zeta,t) &= \frac{\partial w}{\partial \zeta}(\zeta,t) - \phi(\zeta,t) && \text{shear displacement}\\
x_2(\zeta,t) &= \rho(\zeta)\frac{\partial w}{\partial t}(\zeta,t) && \text{momentum}\\
x_3(\zeta,t) &= \frac{\partial \phi}{\partial \zeta}(\zeta,t) && \text{angular displacement}\\
x_4(\zeta,t) &= I_\rho(\zeta)\frac{\partial \phi}{\partial t}(\zeta,t) && \text{angular momentum}
\end{aligned}$$

Using this notation and the model (1.9), we find that the power equals (Exercise 1.2)

$$\frac{dE}{dt}(t) = \left[K(\zeta)x_1(\zeta,t)\frac{x_2(\zeta,t)}{\rho(\zeta)} + E(\zeta)I(\zeta)x_3(\zeta,t)\frac{x_4(\zeta,t)}{I_\rho(\zeta)}\right]_a^b. \tag{1.11}$$

Again we see that the power goes via the boundary of the spatial domain. □

In the previous three examples we see that by imposing the right-boundary conditions no energy will be lost. In other words, these system cannot loose energy internally. However, there are many systems in which there is (internal) loss of energy. This may be caused by internal friction by internal friction, as is the case in the following example.

Example 1.1.4 (Damped wave equation) Consider the one-dimensional wave equation of Example 1.1.2. One cause of damping is known as structural damping. Structural damping arises from internal friction in a material converting vibrational energy into heat. In this case the vibrating string is modeled by

$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right] + \frac{k_s}{\rho(\zeta)}\frac{\partial^2}{\partial \zeta^2}\left[\frac{\partial w}{\partial t}(\zeta,t)\right], \quad \zeta \in [a,b],\ t \geq 0, \tag{1.12}$$

where k_s is a positive constant.

To see that the energy decays, we calculate the power, i.e., $\frac{dE}{dt}$, where the energy is given by (1.7).

$$\begin{aligned}
\frac{d}{dt}E(t) &= \frac{\partial w}{\partial t}(b,t)T(b)\frac{\partial w}{\partial \zeta}(b,t) - \frac{\partial w}{\partial t}(a,t)T(a)\frac{\partial w}{\partial \zeta}(a,t) + \\
&\quad \frac{\partial w}{\partial t}(b,t)k_s\frac{\partial^2 w}{\partial \zeta \partial t}(b,t) - \frac{\partial w}{\partial t}(a,t)k_s\frac{\partial^2 w}{\partial \zeta \partial t}(a,t) - k_s\int_a^b \left[\frac{\partial^2 w}{\partial \zeta \partial t}(\zeta,t)\right]^2 d\zeta.
\end{aligned} \tag{1.13}$$

From this equality, we see that if there is no energy flow through the boundary, the energy will still decay.

Although this system looses energy internally, questions like is the system decaying to zero, when no force is applied at the boundary are still valid for this model as well. □

The most standard example of a model with diffusion, is the model of heat distribution.

Example 1.1.5 (Heat conduction) The model of heat conduction consists of only one conservation law, that is the *conservation of energy.* It is given by the following conservation law:

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial \zeta} J_Q, \tag{1.14}$$

where $u(\zeta, t)$ is the energy density and $J_Q(\zeta, t)$ is the heat flux. This conservation law is completed by two closure equations. The first one expresses the calorimetric properties of the material:

$$\frac{\partial u}{\partial T} = c_V(T), \tag{1.15}$$

where $T(\zeta, t)$ is the temperature distribution and c_V is the heat capacity. The second closure equation defines heat conduction property of the material (Fourier's conduction law):

$$J_Q = -\lambda(T, \zeta)\frac{\partial T}{\partial \zeta}, \tag{1.16}$$

where $\lambda(T, \zeta)$ denotes the heat conduction coefficient. Assuming that the variations of the temperature are not too large, one may assume that the heat capacity and the heat conduction coefficient are independent of the temperature, one obtains the following partial differential equation:

$$\frac{\partial T}{\partial t} = \frac{1}{c_V}\frac{\partial}{\partial \zeta}\left(\lambda(\zeta)\frac{\partial T}{\partial \zeta}\right). \tag{1.17}$$

If we look at the (positive) quantity $E(t) = \frac{1}{2}\int_a^b c_V T(\zeta, t)^2 d\zeta$, then it is not hard to see that

$$\frac{dE}{dt}(t) = \left[T(\zeta, t)\lambda(\zeta)\frac{\partial T}{\partial \zeta}(\zeta, t)\right]_a^b - \int_a^b \lambda(\zeta)\left(\frac{\partial T}{\partial \zeta}(\zeta, t)\right)^2 d\zeta.$$

Hence even when there is no heat flow through the boundary of the spatial domain, the quantity $E(t)$ will decrease. It will decrease as long as the heat flux is non-zero. □

This later two examples are in nature completely different to the first examples. In the next section we show that the first three examples have a common format. We return to the example of the structural damped wave and the heat conduction only in Chapter 7 of these notes.

1.2. Class of PDE's

In this section we revisit the first examples of the previous section and show that they all lie in the same class of systems.

If we introduce the variable $x_1 = Q$ and $x_2 = \phi$ in the first example, see equation (1.1), then the p.d.e. can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\frac{\partial}{\partial \zeta}\left[\begin{pmatrix} \frac{1}{C(\zeta)} & 0 \\ 0 & \frac{1}{L(\zeta)} \end{pmatrix}\begin{pmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{pmatrix}\right]. \tag{1.18}$$

The Hamiltonian is written as, see (1.2)

$$\begin{aligned} E(t) &= \frac{1}{2}\int_a^b \frac{x_1(\zeta,t)^2}{C(\zeta)} + \frac{x_2(\zeta,t)^2}{L(\zeta)} d\zeta \\ &= \frac{1}{2}\int_a^b \begin{pmatrix} x_1(\zeta,t) & x_2(\zeta,t) \end{pmatrix} \begin{pmatrix} \frac{1}{C(\zeta)} & 0 \\ 0 & \frac{1}{L(\zeta)} \end{pmatrix} \begin{pmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{pmatrix} d\zeta. \end{aligned} \tag{1.19}$$

For the wave equation we can write down a similar form. We define $x_1 = \rho\frac{\partial w}{\partial t}$ (momentum) and $x_2 = \frac{\partial w}{\partial \zeta}$ (strain). The p.d.e. (1.6) can equivalently be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \zeta}\left[\begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{pmatrix}\begin{pmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{pmatrix}\right]. \tag{1.20}$$

The energy/Hamiltonian becomes in the new variables, see (1.7),

$$\begin{aligned} E(t) &= \frac{1}{2}\int_a^b \frac{x_1(\zeta,t)^2}{\rho(\zeta)} + T(\zeta)x_2(\zeta,t)^2 d\zeta \\ &= \frac{1}{2}\int_a^b \begin{pmatrix} x_1(\zeta,t) & x_2(\zeta,t) \end{pmatrix} \begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{pmatrix} \begin{pmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{pmatrix} d\zeta. \end{aligned} \tag{1.21}$$

For the model of Timoshenko beam, we have already introduced our variables in Example 1.1.3. We write the model and the energy using these new variables. Calculating the time derivative of the variables $x_1, \ldots, x_4$, we find by using (1.9)

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \\ x_3(\zeta,t) \\ x_4(\zeta,t) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial \zeta}\left(\frac{x_2(\zeta,t)}{\rho(\zeta)}\right) - \frac{x_4(\zeta,t)}{I_\rho(\zeta)} \\ \frac{\partial}{\partial z}\left(K(\zeta)x_1(\zeta,t)\right) \\ \frac{\partial}{\partial z}\left(\frac{x_4(\zeta,t)}{I_\rho(\zeta)}\right) \\ \frac{\partial}{\partial z}\left(E(\zeta)I(\zeta)x_3(\zeta,t)\right) + K(\zeta)x_1(\zeta,t) \end{pmatrix} \tag{1.22}$$

We can write this in a form similar to those presented in (1.18) and (1.20). However, as we shall see, we also need a "constant" term. Since this is a long expression, we will not write down the coordinates ζ and t.

$$\begin{aligned} \frac{\partial}{\partial t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = &\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \frac{\partial}{\partial \zeta}\left[\begin{pmatrix} K & 0 & 0 & 0 \\ 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & EI & 0 \\ 0 & 0 & 0 & \frac{1}{I_\rho} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right] + \\ &\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} K & 0 & 0 & 0 \\ 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & EI & 0 \\ 0 & 0 & 0 & \frac{1}{I_\rho} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \end{aligned} \tag{1.23}$$

Formulating the energy/Hamiltonian in the variables $x_1, \dots, x_4$ is easier, see (1.10)

$$\begin{aligned} E(t) =& \frac{1}{2}\int_a^b K(\zeta)x_1(\zeta,t)^2 + \frac{1}{\rho(\zeta)}x_2(\zeta,t)^2 + E(\zeta)I(\zeta)x_3(\zeta,t)^2 + \frac{1}{I_\rho(\zeta)}x_4(\zeta,t)^2 d\zeta \\ =& \frac{1}{2}\int_a^b \left[\begin{pmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \\ x_3(\zeta,t) \\ x_4(\zeta,t) \end{pmatrix}^T \begin{pmatrix} K(\zeta) & 0 & 0 & 0 \\ 0 & \frac{1}{\rho(\zeta)} & 0 & 0 \\ 0 & 0 & E(\zeta)I(\zeta) & 0 \\ 0 & 0 & 0 & \frac{1}{I_\rho(\zeta)} \end{pmatrix} \begin{pmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \\ x_3(\zeta,t) \\ x_4(\zeta,t) \end{pmatrix} \right] d\zeta. \end{aligned} \tag{1.24}$$

We see that in the new formulation these examples have a common structure. There is only one spatial and one time derivative, and the relation between these two derivatives is of the form

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}\left[\mathcal{H}(\zeta)x(\zeta,t)\right] + P_0\left[\mathcal{H}(\zeta)x(\zeta,t)\right]. \tag{1.25}$$

Furthermore, we have that P_1 is symmetric, i.e., $P_1^T = P_1$, P_0 is *anti-symmetric*, i.e., $P_0^T = -P_0$. Furthermore, they are both independent of ζ. Finally, $\mathcal{H}$ is a (strictly) positive symmetric multiplication operator, independent of t. The energy or Hamiltonian can be expressed by using x and $\mathcal{H}$. That is

$$E(t) = \frac{1}{2}\int_a^b x(\zeta,t)^T \mathcal{H}(\zeta) x(\zeta,t) d\zeta. \tag{1.26}$$

As we have seen in the examples, the change of energy (power) of these systems was only possible via the boundary of its spatial domain. In the following theorem we show that this is a general property for any system which is of the form (1.25) with Hamiltonian (1.26).

Theorem 1.2.1. *Consider the partial differential equation (1.25) in which P_0, P_1 are constant matrices satisfying $P_1^T = P_1$ and $P_0^T = -P_0$. Furthermore, $\mathcal{H}$ is independent on t and is symmetric, i.e. for all ζ's we have that $\mathcal{H}(\zeta)^T = \mathcal{H}(\zeta)$. For the Hamiltonian/energy given by (1.26) the following balance equation holds for all solutions of (1.25)*

$$\frac{dE}{dt}(t) = \frac{1}{2}\left[(\mathcal{H}x)^T(\zeta,t) P_1 (\mathcal{H}x)(\zeta,t) \right]_a^b. \tag{1.27}$$

PROOF: By using the partial differential equation, we find that

$$\begin{aligned} \frac{dE}{dt}(t) =& \frac{1}{2}\int_a^b \frac{\partial x}{\partial t}(\zeta,t)^T \mathcal{H}(\zeta)x(\zeta,t)d\zeta + \frac{1}{2}\int_a^b x(\zeta,t)^T\mathcal{H}(\zeta)\frac{\partial x}{\partial t}(\zeta,t)d\zeta \\ =& \frac{1}{2}\int_a^b \left[P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}x)(\zeta,t) + P_0(\mathcal{H}x)(\zeta,t)\right]^T \mathcal{H}(\zeta)x(\zeta,t)d\zeta + \\ & \frac{1}{2}\int_a^b x(\zeta,t)^T\mathcal{H}(\zeta,t)\left[P_1\frac{\partial}{\partial\zeta}(\mathcal{H}x)(\zeta,t) + P_0(\mathcal{L}x)(\zeta,t)\right] d\zeta. \end{aligned}$$

Using now the fact that $P_1, \mathcal{H}(\zeta)$ are symmetric, and P_0 is anti-symmetric, we write the last expression as

$$\begin{aligned}
&\frac{1}{2}\int_a^b \left[\frac{\partial}{\partial\zeta}\left(\mathcal{H}x\right)(\zeta,t)\right]^T P_1\mathcal{H}(\zeta)x(\zeta,t) + \left[\mathcal{H}(\zeta)x(\zeta,t)\right]^T \left[P_1\frac{\partial}{\partial\zeta}\left(\mathcal{H}x\right)(\zeta,t)\right] d\zeta + \\
&\frac{1}{2}\int_a^b -\left[\mathcal{H}(\zeta)x(\zeta,t)\right]^T P_0\mathcal{H}(\zeta)x(\zeta,t) + \left[\mathcal{H}(\zeta)x(\zeta,t)\right]^T \left[P_0\mathcal{H}(\zeta)x(\zeta,t)\right] d\zeta \\
=&\frac{1}{2}\int_a^b \frac{\partial}{\partial\zeta}\left[\left(\mathcal{H}x\right)^T(\zeta,t)P_1\left(\mathcal{H}x\right)(\zeta,t)\right] d\zeta \\
=&\frac{1}{2}\left[\left(\mathcal{H}x\right)^T(\zeta,t)P_1\left(\mathcal{H}x\right)(\zeta,t)\right]_a^b .
\end{aligned}$$

Hence we have proved the theorem. ■

The balance equation will turn out to be very important, and will guide us in many problems. An overview of the (control) problems which we study in the coming chapters is given in Section 1.4. First we concentrate a little bit more on the class of systems given by (1.25). We show that we have to see it as a combination of two structure. One given by P_1 and P_0, and the other given by $\mathcal{H}$. This is the subject of the following section, in which we also explain the name Port-Hamiltonian.

1.3. Dirac structures and port-Hamiltonian systems

In this section we show that we can identify a deeper underlying structure to the p.d.e. (1.25) and the balance equation (1.27). Therefore we look once more at the first displayed equation in the proof of Theorem 1.2.1. For $a = -\infty$ and $b = \infty$ this equation becomes

$$\frac{dE}{dt}(t) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial x}{\partial t}(\zeta,t)^T\mathcal{H}(\zeta)x(\zeta,t)d\zeta + \frac{1}{2}\int_{-\infty}^{\infty} x(\zeta,t)^T\mathcal{H}(\zeta)\frac{\partial x}{\partial t}(\zeta,t)d\zeta. \tag{1.28}$$

In the expression on the right-hand side we have $\frac{\partial x}{\partial t}$ and $\mathcal{H}x$. These are the same variables used to describe the p.d.e. (1.25). Let us rename these variables $f = \frac{\partial x}{\partial t}$ and $e = \mathcal{H}x$. Furthermore, we "forget" the time, i.e., we see e and f only as functions of the spatial variable. By doing so the p.d.e. becomes

$$f(\zeta) = P_1\frac{\partial e}{\partial\zeta}(\zeta) + P_0 e(\zeta) \tag{1.29}$$

and the right-hand side of (1.28) becomes

$$\frac{1}{2}\int_{-\infty}^{\infty} f(\zeta)^T e(\zeta)d\zeta + \frac{1}{2}\int_{-\infty}^{\infty} e(\zeta)^T f(\zeta)d\zeta. \tag{1.30}$$

Using the equation (1.29), we can rewrite the integrals in (1.30).

$$
\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty} f(\zeta)^T e(\zeta) d\zeta + \frac{1}{2}\int_{-\infty}^{\infty} e(\zeta)^T f(\zeta) d\zeta \\
&\qquad = \frac{1}{2}\int_{-\infty}^{\infty} \left[P_1 \frac{\partial e}{\partial \zeta}(\zeta) + P_0 e(\zeta)\right]^T e(\zeta) d\zeta + \\
&\qquad \quad \frac{1}{2}\int_{-\infty}^{\infty} e(\zeta)^T \left[P_1 \frac{\partial e}{\partial \zeta}(\zeta) + P_0 e(\zeta)\right] d\zeta \\
&\qquad = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial e}{\partial \zeta}(\zeta)^T P_1 e(\zeta) + e(\zeta)^T P_1 \frac{\partial e}{\partial \zeta}(\zeta) d\zeta \\
&\qquad = \frac{1}{2}\int_{-\infty}^{\infty} -e(\zeta)^T P_0 e(\zeta) + e(\zeta)^T P_0 e(\zeta) d\zeta \\
&\qquad = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial \zeta}\left[e(\zeta)^T P_1 e(\zeta)\right] d\zeta,
\end{aligned}
$$

where we have used that $P_1^T = P_1$ and $P_0^T = -P_0$. We remark that the above derivation is exactly the same as the one in the proof of Theorem 1.2.1. Now under the mild assumption that $e(\zeta)$ is zero in plus and minus infinity, we conclude that

$$\frac{1}{2}\int_{-\infty}^{\infty} f(\zeta)^T e(\zeta) d\zeta + \frac{1}{2}\int_{-\infty}^{\infty} e(\zeta)^T f(\zeta) d\zeta = 0. \tag{1.31}$$

for all e and f satisfying (1.29).

Based on (1.28) we call the expressions $\int f^T e d\zeta$, $\int e^T f d\zeta$, the power. Remember that the change of energy is by definition the power. Hence we see from (1.31) that the power is zero. This was already clear from the proof of Theorem 1.2.1, but we received it now for any pair of variables which satisfies (1.29).

To illustrate this observation we consider three systems which have the same P_1 and P_0, but are totally different in their time behavior.

Example 1.3.1 We consider the following simple p.d.e.

$$\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial x}{\partial \zeta}(\zeta, t), \qquad \zeta \in \mathbb{R}. \tag{1.32}$$

We regard this equation as the equation (1.29) with $P_1 = 1$ and $P_0 = 0$ and $f = \frac{\partial x}{\partial t}$, $e = x$. Hence if we define the energy $E(t) = \frac{1}{2}\int_{-\infty}^{\infty} x(\zeta, t)^2 d\zeta$ and we assume that $x(\pm\infty, t) = 0$, then we know that $\dot{E}(t)$ is zero along solutions. □

Example 1.3.2 Consider the following p.d.e.

$$\frac{\partial x}{\partial t}(\zeta, t) = x(\zeta, t)\frac{\partial x}{\partial \zeta}(\zeta, t), \qquad \zeta \in \mathbb{R}. \tag{1.33}$$

This equation is known as the *inviscid Burgers's equation*. If we define $f = \frac{\partial x}{\partial t}$ and $e = \frac{1}{2}x^2$, then we see that this equation equals (1.29) for the same P_1 and P_0 as in the previous example. Furthermore, we know that

$$0 = 2\int_{-\infty}^{\infty} f(\zeta,t)e(\zeta,t)d\zeta = \int_{-\infty}^{\infty} \frac{\partial x}{\partial t}(\zeta,t)x(\zeta,t)^2 d\zeta$$

provided $x(\infty,t) = x(-\infty,t) = 0$ for all t. The later integral can see as the time derivative of $H(t) = \frac{1}{3}\int_{-\infty}^{\infty} x(\zeta,t)^3 d\zeta$. Hence we find that this H is a conserved quantity, i.e.,

$$\frac{dH}{dt}(t) = -\int_{-\infty}^{\infty} \frac{\partial x}{\partial t}(\zeta,t)x(\zeta,t)^2 d\zeta = \int_{-\infty}^{\infty} f(\zeta,t)e(\zeta,t)d\zeta = 0. \tag{1.34}$$

The above holds for every x satisfying (1.33) with $x(\infty,t) = x(-\infty,t) = 0$. Since the p.d.e. (1.33) is non-linear, proving existence of solutions is much harder than in the previous example. However, as in the linear example we have found a conserved quantity. □

In the previous example we have chosen a different e, but the same f. We can also choose a different f.

Example 1.3.3 Consider the discrete-time implicit equation

$$x(\zeta,n+1) - x(\zeta,n) = \frac{\partial}{\partial\zeta}\left[x(\zeta,n+1) + x(\zeta,n)\right], \ \zeta \in \mathbb{R}, n \in \mathbb{Z}. \tag{1.35}$$

In this equation, we choose $f(\zeta,n) = x(\zeta,n+1)-x(\zeta,n)$ and $e(\zeta,n) = x(\zeta,n+1)+x(\zeta,n)$. For this choice, we see that (1.35) is the same as (1.29) with $P_1 = 1$ and $P_0 = 0$.

If we choose the energy to be same as in Example 1.3.1, i.e., $E(n) = \int_{-\infty}^{\infty} x(\zeta,n)^2 d\zeta$, then we find that

$$E(n+1) - E(n) = \int_{-\infty}^{\infty} x(\zeta,n+1)^2 - x(\zeta,n)^2 d\zeta = \int_{-\infty}^{\infty} f(\zeta,n)e(\zeta,n)d\zeta = 0, \tag{1.36}$$

provided $x(\pm\infty,n) = 0$ for all $n \in \mathbb{Z}$. So for the implicit difference equation (1.35) we have once more a conserved quantity without knowing the solutions, or even knowing existence. □

As is become clear in the previous examples we may distinguish between an underlying structure and the actual system. This underlying structure is named a Dirac structure and is defined next.

Definition 1.3.4. Let $\mathcal{E}$ and $\mathcal{F}$ be two Hilbert spaces with inner product $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{F}}$, respectively. Assume moreover that they are isometrically isomorphic, that is there exists a linear mapping $r_{\mathcal{F},\mathcal{E}} : \mathcal{F} \mapsto \mathcal{E}$ such that

$$\langle r_{\mathcal{F},\mathcal{E}} f_1, r_{\mathcal{F},\mathcal{E}} f_2\rangle_{\mathcal{E}} = \langle f_1, f_2\rangle_{\mathcal{F}} \tag{1.37}$$

for all $f_1, f_2 \in \mathcal{F}$. The *bond space* $\mathcal{B}$ is defined as $\mathcal{F} \times \mathcal{E}$. On $\mathcal{B}$ we define the following symmetric pairing

$$\left\langle \begin{pmatrix} f_1 \\ e_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ e_2 \end{pmatrix} \right\rangle_+ = \langle f_1, r_{\mathcal{E},\mathcal{F}} e_2\rangle_{\mathcal{F}} + \langle e_1, r_{\mathcal{F},\mathcal{E}} f_2\rangle_{\mathcal{E}}, \tag{1.38}$$

where $r_{\mathcal{E},\mathcal{F}} = r_{\mathcal{F},\mathcal{E}}^{-1}$.

Let $\mathcal{V}$ be a linear subspace of $\mathcal{B}$, then the orthogonal subspace with respect to the symmetric pairing (1.38) is defined as

$$\mathcal{V}^{\perp} = \{b \in \mathcal{B} \mid \langle b, v\rangle_+ = 0 \text{ for all } v \in \mathcal{V}\}. \tag{1.39}$$

A *Dirac structure* is a linear subspace of the bond space $\mathcal{D}$ satisfying

$$\mathcal{D}^{\perp} = \mathcal{D}. \tag{1.40}$$

♣

The variables e and f are called the *effort* and *flow*, respectively, and their spaces $\mathcal{E}$ and $\mathcal{F}$ are called the *effort* and *flow space*. The bilinear product $\langle f, r_{\mathcal{E},\mathcal{F}}e\rangle_{\mathcal{F}}$ is called the *power* or *power product*. Note that $\langle f, r_{\mathcal{E},\mathcal{F}}e\rangle_{\mathcal{F}} = \langle r_{\mathcal{F},\mathcal{E}}f, e\rangle_{\mathcal{E}}$.

Dirac structures are depictured in Figure 1.2. Finally, we mention that by (1.39), we

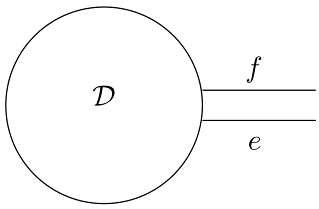

Figure 1.2.: Dirac structure

have that for any element of the Dirac structure

$$2\langle f, r_{\mathcal{E},\mathcal{F}}e\rangle_{\mathcal{F}} = \langle b, b\rangle_+ = 0. \tag{1.41}$$

This we interpret by saying that for any element of a Dirac structure the power is zero. A Dirac structure can been seen as the largest subspace which this holds i.e., if $\mathcal{V}$ is a subspace of $\mathcal{B}$ satisfying (1.41), then $\mathcal{V}$ is a Dirac structure if there does not exists a subspace $\mathcal{W}$ such that $\mathcal{V} \subset \mathcal{W}$, $\mathcal{V} \neq \mathcal{W}$, and the power of every element in $\mathcal{W}$ is zero.

Next we identify the Dirac structure associated to Examples 1.3.1–1.3.3.

Example 1.3.5 Choose the effort and flow space as $L^2(-\infty, \infty)$, and let $r_{\mathcal{F},\mathcal{E}} = I$. Define the following subspace of $\mathcal{B} = \mathcal{F} \times \mathcal{E}$

$$\mathcal{D} = \left\{ \begin{pmatrix} f \\ e \end{pmatrix} \in \mathcal{B} \mid e \text{ is absolutely continuous and } \frac{de}{d\zeta} \in L^2(-\infty, \infty), \right. \tag{1.42}$$
$$\left. e(-\infty) = e(\infty) = 0, \text{ and } f = \frac{de}{d\zeta} \right\}.$$

We claim that this subspace is a Dirac structure. Let $b = \left(\begin{smallmatrix} f \\ e \end{smallmatrix}\right) \in \mathcal{B}$ and let $\langle b, d\rangle_+ = 0$ for all $d = \left(\begin{smallmatrix} f_d \\ e_d \end{smallmatrix}\right) \in \mathcal{D}$. Using our power product this implies that

$$0 = \langle f, e_d\rangle + \langle f_d, e\rangle = \langle f, e_d\rangle + \langle \frac{de_d}{d\zeta}, e\rangle.$$

In other words

$$\langle \frac{de_d}{d\zeta}, e \rangle = -\langle e_d, f \rangle. \tag{1.43}$$

This is equivalent to saying that e lies in the domain of the dual of the differential operator, $\frac{d}{d\zeta}$. Similar to Example A.3.64 and A.3.66 we conclude from this equation that e is absolutely continuous and $e(-\infty) = e(\infty) = 0$. Furthermore, by integration by parts we see that

$$\langle \frac{de_d}{d\zeta}, e \rangle = \int_{-\infty}^{\infty} \frac{de_d}{d\zeta}(\zeta) e(\zeta) d\zeta = -\int_{-\infty}^{\infty} e_d(\zeta) \frac{de}{d\zeta}(\zeta) d\zeta = -\langle e_d, \frac{de}{d\zeta} \rangle.$$

Combining this with (1.43), we conclude that

$$-\langle e_d, \frac{de}{d\zeta} \rangle = -\langle e_d, f \rangle.$$

Since this holds for a dense set of e_d, we conclude that $f = \frac{de}{d\zeta}$. Concluding, we see that $\left(\begin{smallmatrix} f \\ e \end{smallmatrix}\right) \in \mathcal{D}$, and so $\mathcal{D}$ is a Dirac structure. $\square$

Now we have formally defined what is a Dirac structure, we can define our class of systems.

Let $\mathcal{H}$ be a real-valued function of x, and let $\mathcal{D}$ be a Dirac structure. We furthermore assume that we have a mapping H from $\mathcal{F}$ to $\mathbb{R}$ which is Fréchet differentiable for any x, i.e., see Definition A.5.25.

$$H(x + \Delta x) - H(x) = (dH(x))\,(\Delta x) + R(x, \Delta x), \tag{1.44}$$

with $\frac{\|R(x,\Delta x)\|}{\|\Delta x\|}$ converges to zero when $\Delta x \to 0$. Since H takes values in $\mathbb{R}$, we can by Riesz representation theorem write the term $(dH(x))\,(\Delta x)$ as $\langle \Delta x, \tilde{f} \rangle_{\mathcal{F}}$ for some $\tilde{f}$. Note that $\tilde{f}$ still depends on x. Since $\mathcal{F}$ is isomorphically isomorf to $\mathcal{E}$, we can find an $\tilde{e}$ such that $\langle \Delta x, \tilde{f} \rangle_{\mathcal{F}} = \langle \Delta x, r_{\mathcal{E},\mathcal{F}} \tilde{e} \rangle_{\mathcal{F}}$. We denote this $\tilde{e}$ by $\frac{\partial \mathfrak{H}}{\partial x}(x)$. Combining this notation with equation (1.44), we find

$$H(x + \varepsilon \Delta x) - H(x) = \varepsilon \langle \Delta x, r_{\mathcal{E},\mathcal{F}} \frac{\partial \mathfrak{H}}{\partial x}(x) \rangle_{\mathcal{F}} + \varepsilon o(\varepsilon). \tag{1.45}$$

The system associated with H and $\mathcal{D}$ is defined as

$$\{x(\cdot, t) \mid \begin{pmatrix} \frac{\partial x}{\partial t}(\cdot, t) \\ \frac{\partial \mathfrak{H}}{\partial x}(\cdot, t) \end{pmatrix} \in \mathcal{D} \text{ for all } t\}. \tag{1.46}$$

Before we write our examples in this format, we show that the system defined above has H as a conserved quantity along its trajectory.

$$H(x(t + \varepsilon)) - H(x(t)) = H\left(x(t) + \varepsilon \dot{x}(t) + \varepsilon r(x(t), \varepsilon)\right) - H(x(t)),$$

where $\|r(x(t),\varepsilon)\| \to 0$ when $\varepsilon \to 0$. Since ε is small, we may ignore this term. Doing so, and using (1.45) we find

$$\frac{H(x(t+\varepsilon)) - H(x(t))}{\varepsilon} = \langle \dot{x}(t), r_{\mathcal{E},\mathcal{F}} \frac{\partial \mathfrak{H}}{\partial x}(x(t))\rangle_{\mathcal{F}} + o(\varepsilon).$$

By the definition of our system, we have that the power product is zero, see (1.41) and so, we find that

$$\frac{dH(x(t))}{dt} = 0.$$

Note that we normally write $H(t)$ instead of $H(x(t))$. Thus along trajectories, H is a conserved quantity. This conserved quantity is called the *Hamiltonian*. Equation (1.46) clearly indicates that the system is defined by two objects. Namely, the Dirac structure and the Hamiltonian.

We illustrate the above by looking once more to the Examples 1.3.1 and 1.3.2.

Example 1.3.6 From Example 1.3.5 we know which Dirac structure lies under the p.d.e.'s of Examples 1.3.1 and 1.3.2. Hence it remains to identify the Hamiltonians.

For Example 1.3.1 we easily see that $H = \frac{1}{2}\int_{-\infty}^{\infty} x^2 d\zeta$. If we define $\mathfrak{H} = \frac{1}{2}x^2$, then it by combining (1.46) with (1.42) gives the p.d.e. (1.32).

For Example 1.3.2 we find $H = \frac{1}{3}\int_{-\infty}^{\infty} x^3 d\zeta$ and $\mathfrak{H} = \frac{1}{3}x^3$. □

In contrast to our examples in Section 1.1, the examples 1.3.1–1.3.3 do not have a boundary. However, as seen in e.g. Example 1.1.1 the boundary is very useful for control purposes. So we have to re-think and adjust the theory as developed until now. To illustrate this, we consider Example 1.3.3 on a compact spatial interval.

Example 1.3.7 Consider the p.d.e.

$$\frac{\partial x}{\partial t}(\zeta,t) = \frac{\partial x}{\partial \zeta}(\zeta,t), \qquad \zeta \in (a,b). \tag{1.47}$$

As in Theorem 1.2.1,we take the energy equal to $E(t) = \frac{1}{2}\int_a^b x(\zeta,t)^2 d\zeta$. We find that

$$\frac{dE}{dt}(t) = \int_a^b \frac{\partial x}{\partial t}(\zeta,t)x(\zeta,t)d\zeta = \frac{1}{2}\left[x(b,t)^2 - x(a,t)^2\right]. \tag{1.48}$$

Or equivalently,

$$\frac{dE}{dt}(t) = \int_a^b \frac{\partial x}{\partial t}(\zeta,t)x(\zeta,t)d\zeta - \frac{1}{2}\left[x(b,t)^2 - x(a,t)^2\right] = 0. \tag{1.49}$$

As before, we want to see the middle expression as a power product. Hence if we introduce $f = \frac{\partial x}{\partial t}$ and $e = x$, then the integral induces a power product between f and e. However, we still have the boundary terms. We introduce the *boundary port variables* $f_\partial = \frac{1}{\sqrt{2}}[x(b) - x(a)]$, and $e_\partial = \frac{1}{\sqrt{2}}[x(b) + x(a)]$. With these variables, we see that we may write (1.49) as

$$\frac{dE}{dt} = \int_a^b f(\zeta)e(\zeta)d\zeta - f_\partial e_\partial = 0. \tag{1.50}$$

We can see this as a system like (1.46) by defining

$$\mathcal{E} = \mathcal{F} = L^2(a,b) \oplus \mathbb{R}$$

and

$$r_{\mathcal{F},\mathcal{E}} = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}.$$

The Dirac structure is given by

$$\mathcal{D} = \{ \begin{pmatrix} f \\ f_\partial \\ e \\ e_\partial \end{pmatrix} \in \mathcal{B} \mid e \text{ is absolutely continuous and } \frac{de}{d\zeta} \in L^2(a,b), \tag{1.51}$$

$$f = \frac{de}{d\zeta}, f_\partial = \frac{1}{\sqrt{2}}[e(b) - e(a)] \text{ and } e_\partial = \frac{1}{\sqrt{2}}[e(b) + e(a)]\}.$$

Again we have a Dirac structure, but now it contains boundary variables. We can formulate our system on this Dirac structure as

$$\{x(\cdot,t) \mid \begin{pmatrix} \frac{\partial x}{\partial t} \\ f_\partial \\ \frac{\partial \mathfrak{H}}{\partial x} \\ e_\partial \end{pmatrix} \in \mathcal{D}\}. \tag{1.52}$$

where $\mathcal{D}$ is given by (1.51) and $\mathfrak{H} = \frac{1}{2}x^2$. $\square$

From (1.52) we see that our system is defined via a Dirac structure, an Hamiltonian and port variables. This motivated the name "Port-Hamiltonian Systems".

One of the advantages of considering a system as a Dirac structure with a Hamiltonian is that coupling of systems is now very easy. Suppose we have two Dirac structures, $\mathcal{D}_1$ and $\mathcal{D}_2$ as depictured in Figure 1.3. We couple the structures by $f_2 = -\tilde{f}_2$ (the flow out

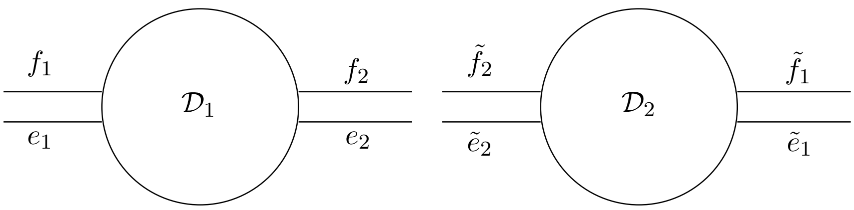

Figure 1.3.: Composition of Dirac structures

of the first system equals the incoming flow of the other system) and $e_2 = \tilde{e}_2$. Then the structure defined by

$$\mathcal{D} = \left\{ \begin{pmatrix} f_1 \\ \tilde{f}_1 \\ e_1 \\ \tilde{e}_1 \end{pmatrix} \mid \text{there exist } f_2, e_2 \text{ s.t. } \begin{pmatrix} f_1 \\ f_2 \\ e_1 \\ e_2 \end{pmatrix} \in \mathcal{D}_1 \text{ and } \begin{pmatrix} \tilde{f}_1 \\ -f_2 \\ \tilde{e}_1 \\ e_2 \end{pmatrix} \in \mathcal{D}_2 \right\}. \tag{1.53}$$

has zero power. To see this, we assume that for both systems $\langle f, e\rangle + \langle e, f\rangle$ denotes the power. We take as power for $\mathcal{D}$

$$\langle f_1, e_1\rangle + \langle e_1, f_1\rangle + \langle \tilde{f}_1, \tilde{e}_1\rangle + \langle \tilde{e}_1, \tilde{f}_1\rangle$$

We find for this power product

$$\begin{aligned}\langle f_1, e_1\rangle + \langle e_1, f_1\rangle + &\langle \tilde{f}_1, \tilde{e}_1\rangle + \langle \tilde{e}_1, \tilde{f}_1\rangle \\ &= \langle f_1, e_1\rangle + \langle e_1, f_1\rangle + \langle f_2, e_2\rangle + \langle e_2, f_2\rangle \\ &\quad - \langle f_2, e_2\rangle - \langle e_2, f_2\rangle + \langle \tilde{f}_1, \tilde{e}_1\rangle + \langle \tilde{e}_1, \tilde{f}_1\rangle \\ &= \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle + \\ &\quad \left\langle \begin{pmatrix} \tilde{f}_1 \\ -f_2 \end{pmatrix}, \begin{pmatrix} \tilde{e}_1 \\ e_2 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} \tilde{e}_1 \\ e_2 \end{pmatrix}, \begin{pmatrix} \tilde{f}_1 \\ -f_2 \end{pmatrix} \right\rangle.\end{aligned}$$

The last expressions are zero since $\begin{pmatrix} f_1 \\ f_2 \\ e_1 \\ e_2 \end{pmatrix} \in \mathcal{D}_1$ and $\begin{pmatrix} \tilde{f}_1 \\ -f_2 \\ \tilde{e}_1 \\ e_2 \end{pmatrix} \in \mathcal{D}_2$, respectively. Thus we see that the total power of the interconnected Dirac structure is zero. Although this is not sufficient to show that $\mathcal{D}$ defined by (1.53) is a Dirac structure, it indicates the promising direction. It only remains to show that $\mathcal{D}$ is maximal. For many coupled Dirac structures this holds. If the systems has the Hamiltonian H_1 and H_2 respectively, then the Hamiltonian of the coupled system is $H_1 + H_2$. Of course we can extend this to the coupling of more than two systems. We show a physical example next.

Example 1.3.8 (Suspension system) Consider a simplified version of a suspension system described by two strings connected in parallel through a distributed spring. This system can be modeled by

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial \zeta^2} + \alpha(v - u) \\ \frac{\partial^2 v}{\partial t^2} &= c^2 \frac{\partial^2 v}{\partial \zeta^2} + \alpha(u - v) \quad \zeta \in (-\infty, \infty),\ t \geq 0,\end{aligned} \tag{1.54}$$

where c and α are positive constants and $u(\zeta, t)$ and $v(\zeta, t)$ describe the displacement, respectively, of both strings. The use of this model has potential applications in isolation of objects from outside disturbances. As an example in engineering, rubber and rubber-like materials are used to absorb vibration or shield structures from vibration. As an approximation, these materials can be modeled as a distributed spring. We show that this system can be described as the interconnection of three subsystems, i.e., two vibrating strings and one distributed spring. Seeing the system as an interconnection of subsystems allows us to have some modularity in the modeling process, and because of this modularity, the modeling process can be performed in an iterative manner, gradually refining the model by adding other subsystems. □

1.4. Overview

Now we know the class of systems we will be working with we can give more details on the content of the coming chapters.

In Chapter 2 we study for which homogenous boundary conditions the p.d.e. (1.25) possesses a unique solution which has non-increasing energy. This we do by applying the general theory of infinite-dimensional systems. If these boundary conditions are non-zero, then in Chapter 3 we show that this p.d.e. still has well-defined solutions. Hence this enable us to apply a control (input) at the boundary to the system described by (1.25). Under this same conditions, we show that boundary output is also possible. The mapping between the input and the output can be described by the transfer function. This is the subject of Chapter 4. Till Chapter 5 we have only considered systems which are non-increasing in energy if no control is applied. Furthermore, the control has been restricted to smooth functions. In Chapter 5, we extend our class of systems in both directions. We show that many more boundary conditions are possible, and furthermore, we show that if the homogenous system is well-posed, then the same hold for the system when L^2-input are applied. Chapter 6 we can solve our first control problem. We can identify a large class of boundary feedback which stabilize the system. The stability is exponential, meaning that the energy decays exponentially fast to zero. In 7 we can treat a larger class of system. There we return to the examples 1.1.4 and 1.1.5. In this section, we really need the underlying Dirac structure. Hence till Chapter 7, the Dirac structure is underlying our system, and apart for using to define boundary port, we shall not use it very prominently. This changes in Chapter 7.

1.5. Exercises

1.1. In this exercise we check some integrals which appeared in our examples.

   a) Check equation (1.8).

   b) Check the equality (1.13).

1.2. Prove equation (1.11).

1.3. Show that equation (1.51) in Example 1.3.7 defines a Dirac structure.

Chapter 2
Homogeneous differential equation

2.1. Introduction

In this chapter we study the partial differential equation (1.25). In particular, we characterize boundary conditions such that this p.d.e. has a unique solution, and such that the energy decays along the solutions. In order to clarify the approach we take, let us consider the most simplified version of (1.25)

$$\frac{\partial x}{\partial t}(\zeta,t) = \alpha \frac{\partial x}{\partial \zeta}(\zeta,t), \qquad \zeta \in [0,1], \quad t \geq 0, \tag{2.1}$$

where α is a positive constant. As initial condition we take $x_0(\zeta)$.

If x_0 is continuously differentiable, then it is easy to see that a solution is given by, see Exercise 2.1.

$$x(\zeta,t) = \begin{cases} x_0(\zeta+\alpha t) & \zeta \in [0,1],\ \zeta+\alpha t < 1 \\ x_0(1) & \zeta \in [0,1],\ \zeta+\alpha t > 1 \end{cases} \tag{2.2}$$

However, this is not the only solution of (2.1). Another solution is given by

$$x(\zeta,t) = \begin{cases} x_0(\zeta+\alpha t) & \zeta \in [0,1],\ \zeta+\alpha t < 1 \\ g(\zeta+\alpha t) & \zeta \in [0,1],\ \zeta+\alpha t > 1 \end{cases} \tag{2.3}$$

where g is an arbitrary continuous differentiable function on $(1,\infty)$ satisfying $g(1) = x_0(1)$.

Hence we have that the p.d.e. does not possess a unique solution. The reason for this is that we did not impose a boundary condition. If we impose the boundary condition $x(1,t) = 0$ for all t, then the unique solution is given by, see also (2.2),

$$x(\zeta,t) = \begin{cases} x_0(\zeta+\alpha t) & \zeta \in [0,1],\ \zeta+\alpha t < 1 \\ 0 & \zeta \in [0,1],\ \zeta+\alpha t > 1 \end{cases} \tag{2.4}$$

Note that we assume that $x_0(1)$ satisfies the boundary condition as well. The rule of thumb is that for a first order p.d.e. one need one boundary condition. However, this is only a rule of thumb. If we impose to the p.d.e. (2.1) the boundary condition $x(0,t) = 0$, then this p.d.e. does not possess any solution if $x_0 \neq 0$, see Exercise 2.2.

We return to the p.d.e. (2.1) with boundary condition $x(1,t) = 0$. The solution is given by (2.4). We see that for any initial condition, even the ones which are only integrable, the function $x(\zeta,t)$ of (2.4) is well-defined. This function depends on x_0. In particular, we can define mappings

$$x_0 \mapsto x(\cdot,t), \qquad t \geq 0. \tag{2.5}$$

This mapping has the following properties

- It is linear, i.e., if x_0 is written as $c_1 f_0 + c_2 g_0$, then x can be written as $c_1 f + c_2 g$, where f_0, g_0 are mapped to f and g, respectively.
- If we take x_1 equal to $x(\cdot,\tau)$, then the (composed) mapping

  $$x_0 \mapsto x(\cdot,\tau) = x_1 \mapsto \tilde{x}(\cdot,t)$$

  equals the mapping

  $$x_0 \mapsto x(\cdot,t+\tau).$$

The reason for these properties lies in the linearity and time-invariance of the p.d.e.

Next we choose a (particular) class of initial conditions. It is easy to see that (2.1) is of the form (1.25), and so the energy associated to it equals $\frac{1}{2}\int_0^1 |x(\zeta)|^2 d\zeta$, or $\frac{1}{2}\int_0^1 \alpha |x(\zeta)|^2 d\zeta$. We take the class of functions with finite energy to be our class of initial conditions.

Thus we have defined for our simple example the space of initial conditions, and we have seen some nice properties of the solution. In the sequel we omit the spatial argument, when we write down an initial condition, or a solution, and we see (2.5) as a mapping in the energy space.

In the following section we first have to consider some abstract theory. The general result obtained there enables us to show that for certain boundary conditions the p.d.e. (1.25) possesses a unique solution.

2.2. Semigroup and infinitesimal generator

In this section, we recap some abstract differential theory. We denote by X an abstract Hilbert space, with inner product $\langle\cdot,\cdot\rangle_X$ and norm $\|\cdot\|_X = \sqrt{\langle\cdot,\cdot\rangle_X}$. For the simple p.d.e. considered in the previous section, we saw some nice properties of the mapping from the initial condition to the solution at time t. These properties are formalized in the following definition.

Definition 2.2.1. Let X be a Hilbert space. The operator valued function $t \mapsto T(t)$, $t \geq 0$ is a *strongly continuous semigroup* if the following holds

1. For all $t \geq 0$, $T(t)$ is a bounded linear operator on X, i.e., $T(t) \in \mathcal{L}(X)$;
2. $T(0) = I$;

3. $T(t+\tau) = T(t)T(\tau)$ for all $t, \tau \geq 0$.

4. For all $x_0 \in X$, we have that $\|T(t)x_0 - x_0\|_X$ converges to zero, when $t \downarrow 0$.

We sometimes abbreviate strongly continuous semigroup to C_0-*semigroup* and most times it will be denoted by $(T(t))_{t\geq 0}$. ♣

We call X the *state space*, and its elements *states*. To obtain a feeling for these defining properties assume that $T(t)$ denotes the mapping of initial condition to solution at time t of some linear, time-invariant, differential equation. We remark that this will always hold for any semigroup, see Lemma 2.2.6. Under this assumption, we can understand these defining properties of a strongly continuous semigroup much better.

1. That $T(t)$ is a bounded operator means that the solution at time t has not left the space of initial conditions, i.e., the state space. The linearity implies that the solution corresponding to the initial condition $x_0 + \tilde{x}_0$ equals $x(t) + \tilde{x}(t)$, where $x(t)$ and $\tilde{x}(t)$ are the solution corresponding to x_0 and $\tilde{x}_0$, respectively. This is logical, because we assumed that the underlying differential equation is linear.

2. This is trivial; the solution at time zero must be equal to the initial condition.

3. Let $x(\tau)$ be the state at time τ. If we take this as our new initial condition and proceed for t seconds, then by the time-invariance this must equal $x(t+\tau)$. Since we assume that $T(s)$ is the mapping from x_0 to $x(s)$, we see that the time-invariance of the underlying differential equation implies property 3. of Definition 2.2.1.

   Property 3. is known as a group property, and since it only holds for positive time, it motivates the name "semigroup".

4. This property tells you that if you go backward in time to zero, then $x(t)$ approaches the initial condition. This sounds very logical, but need not to hold for all operators satisfying 1.–3.

   Property 4. is known as strong continuity.

The easiest example of a strongly continuous semigroup is the exponential of a matrix. That is, let A be an $n \times n$ matrix, the matrix-valued function $T(t) = e^{At}$ satisfies the properties of Definition 2.2.1 on the Hilbert space $\mathbb{R}^n$, see Exercise 2.3. Clearly the exponential of a matrix is also defined for $t < 0$. If the semigroup can be extended to all $t \in \mathbb{R}$, then we say that $T(t)$ is a group. We present the formal definition next.

Definition 2.2.2. Let X be a Hilbert space. The operator valued function $t \mapsto T(t)$, $t \in \mathbb{R}$ is a *strongly continuous group*, or C_0-*group*, if the following holds

1. For all $t \in \mathbb{R}$, $T(t)$ is a bounded linear operator on X;

2. $T(0) = I$;

3. $T(t+\tau) = T(t)T(\tau)$ for all $t, \tau \in \mathbb{R}$.

4. For all $x_0 \in X$, we have that $\|T(t)x_0 - x_0\|_X$ converges to zero, when $t \to 0$. ♣

It is easy to see that the exponential of a matrix is a group. However, only a few semigroups are actually a group. In the study of p.d.e.'s you encounter semigroups more often than groups. The reason for that is that if you have a group, you may go from the initial condition backward in time. For our simple p.d.e. of Section 2.1 this is not possible, as it is shown at the end of the following example.

Example 2.2.3 In this example we show that the mapping defined by (2.5) defines a strongly continuous semigroup. As state space we choose $L^2(0,1)$. We see that its norm corresponds with the energy associated to this system.

Based on (2.4) and (2.5) we define the following (candidate) semigroup on $L^2(0,1)$

$$(T(t)x_0)(\zeta) = \begin{cases} x_0(\zeta+\alpha t) & \zeta \in [0,1],\ \zeta+\alpha t < 1 \\ 0 & \zeta \in [0,1],\ \zeta+\alpha t > 1 \end{cases} \tag{2.6}$$

This is clearly a linear mapping. It is also bounded since

$$\begin{aligned} \|T(t)x_0\|^2 &= \int_0^1 |(T(t)x_0)(\zeta)|^2 d\zeta \\ &= \int_0^{\max\{1-\alpha t,0\}} |x_0(\zeta+\alpha t)|^2 d\zeta \\ &= \int_{\alpha t}^{\max\{1,\alpha t\}} |x_0(\eta)|^2 d\eta \\ &\leq \int_0^1 |x_0(\eta)|^2 d\eta = \|x_0\|^2. \end{aligned} \tag{2.7}$$

From this we conclude that $T(t)$ is bounded with bound less or equal to one.

Using (2.6), we see that for all x_0 there holds $T(0)x_0 = x_0$. Thus Property 2. of Definition 2.2.1 holds.

We take an arbitrary function $x_0 \in L^2(0,1)$ and call $T(\tau)x_0 = x_1$, then for $\zeta \in [0,1]$, we have

$$\begin{aligned} (T(t)x_1)(\zeta) &= \begin{cases} x_1(\zeta+\alpha t) & \zeta+\alpha t < 1 \\ 0 & \zeta+\alpha t > 1 \end{cases} \\ &= \begin{cases} x_0(\zeta+\alpha t+\alpha\tau) & z+\alpha t+\alpha\tau < 1 \\ 0 & \zeta+\alpha t+\alpha\tau > 1 \\ 0 & \zeta+\alpha t > 1 \end{cases} \end{aligned}$$

Since the third case is already covered by the second one, we have that

$$(T(t)x_1)(\zeta) = \begin{cases} x_0(\zeta+\alpha t+\alpha\tau) & \zeta+\alpha t+\alpha\tau < 1 \\ 0 & \zeta+\alpha t+\alpha\tau > 1 \end{cases}$$

By (2.6) this equals $(T(t+\tau)x_0)(\zeta)$, and so we have proved the third property of Definition 2.2.1. It remains to show the fourth property, that is the strong continuity. Since we have to take the limit for $t \downarrow 0$, we may assume that $\alpha t < 1$. Then

$$\|T(t)x_0 - x_0\|^2 = \int_0^{1-\alpha t} |x_0(\zeta + \alpha t) - x_0(\zeta)|^2 d\zeta + \int_{1-\alpha t}^1 |x_0(\zeta)|^2 d\zeta.$$

Since $x_0 \in L^2(0,1)$, the last term converges to zero when $t \downarrow 0$. For the first term some more work is required.

First we assume that x_0 is a continuous function. Then for every $\zeta \in (0,1)$, the function $x_0(\zeta + \alpha t)$ converges to $x_0(\zeta)$ if $t \downarrow 0$. Furthermore, we have that $|x(\zeta + \alpha t)| \leq \max_{\zeta \in [0,1]} |x_0(\zeta)|$. Using Lebesgue dominated convergence theorem, we conclude that

$$\lim_{t \downarrow 0} \int_0^{1-\alpha t} |x_0(\zeta + \alpha t) - x_0(\zeta)|^2 d\zeta = 0.$$

Hence for continuous functions we have proved that property 4. of Definition 2.2.1 holds. This property remains to be shown for an arbitrary function in $L^2(0,1)$.

Let $x_0 \in L^2(0,1)$, and let $\varepsilon > 0$. We can find a continuous function $x_\varepsilon \in L^2(0,1)$ such that $\|x_0 - x_\varepsilon\| \leq \varepsilon$. Next we choose $t_\varepsilon > 0$ such that $\|T(t)x_\varepsilon - x_\varepsilon\| \leq \varepsilon$ for all $t \in [0, t_\varepsilon]$. By the previous paragraph this is possible. Combining this we find for that $t \in [0, t_\varepsilon]$

$$\begin{aligned} \|T(t)x_0 - x_0\| &= \|T(t)x_0 - T(t)x_\varepsilon + T(t)x_\varepsilon - x_\varepsilon + x_\varepsilon - x_0\| \\ &\leq \|T(t)(x_0 - x_\varepsilon)\| + \|T(t)x_\varepsilon - x_\varepsilon\| + \|x_\varepsilon - x_0\| \\ &\leq \|x_\varepsilon - x_0\| + \|T(t)x_\varepsilon - x_\varepsilon\| + \|x_\varepsilon - x_0\| \\ &\leq 3\varepsilon, \end{aligned}$$

where we used (2.7). Since this holds for all $\varepsilon > 0$, we have that Property 4. holds. Having checked all defining properties for a C_0-semigroup, we conclude that $(T(t))_{t \geq 0}$ given by (2.6) is a strongly continuous semigroup.

It is now easy to see that $(T(t))_{t \geq 0}$ cannot be extended to a group. If there would be a possibility to define $T(t)$ for negative t, then we must have that $T(-t)T(t) = T(-t+t) = T(0) = I$ for all $t > 0$. However, from (2.6) it is clear that $T(2/\alpha) = 0$, and so there exists no operator Q such that $QT(2/\alpha) = I$, and thus $(T(t))_{t \geq 0}$ cannot be extended to a group. $\square$

So we have seen some examples of strongly continuous semigroups. It is easily shown that the semigroup defined by (2.6) is strongly continuous for every t. This holds for any semigroup, as is shown in Theorem 2.5.1. In that theorem more properties of strongly continuous semigroups are listed.

Given the semigroup $\left(e^{At}\right)_{t \geq 0}$ with A being a square matrix, one may wonder how to obtain A. The easiest way to do this is by differentiating e^{At} and evaluating this at $t = 0$. We could try to do this with any semigroup. However, we only have that an arbitrary semigroup is continuous, see property 4, and so it may be hard (impossible) to differentiate at zero. The trivial solution is that we only differentiate $T(t)x_0$ when it is possible, as is shown next.

Definition 2.2.4. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Hilbert space X. If the following limit exists

$$\lim_{t\downarrow 0} \frac{T(t)x_0 - x_0}{t}, \tag{2.8}$$

then we say that x_0 is an element of the *domain* of A, shortly $x_0 \in D(A)$, and we define Ax_0 as

$$Ax_0 = \lim_{t\downarrow 0} \frac{T(t)x_0 - x_0}{t}. \tag{2.9}$$

We call A *infinitesimal generator* of the strongly continuous semigroup $(T(t))_{t\geq 0}$. ♣

We may wonder what is the operator A for the semigroup of Example 2.2.3. We derive this next.

Example 2.2.5 Consider the C_0-semigroup defined by (2.6). For this semigroup, we would like to obtain A, and see how it is related to the original p.d.e. which we started with (2.1).

For the semigroup of Example 2.2.3, we want to calculate (2.8). We consider first the limit for a fixed $\zeta \in [0,1)$. Since $\zeta < 1$, there exists a small time interval $[0, t_0)$ such that $\zeta + \alpha t < 1$ for all t in this interval. Evaluating (2.8) at ζ and assuming that t is in the prescribed interval, we have

$$\lim_{t\downarrow 0} \frac{(T(t)x_0)(\zeta) - x_0(\zeta)}{t} = \lim_{t\downarrow 0} \frac{x_0(\zeta + \alpha t) - x_0(\zeta)}{t}.$$

The later limit exists, when x_0 is differentiable, and for these functions the limit equals $\alpha \frac{dx_0}{d\zeta}(\zeta)$. So we find an answer for $\zeta < 1$. For $\zeta = 1$, the limit (2.8) becomes

$$\lim_{t\downarrow 0} \frac{0 - x_0(1)}{t}.$$

We see that this limit will never exist, except if $x_0(1) = 0$.

Hence we find that the domain of A will consists of all functions which are differentiable and which are zero at $\zeta = 1$. Furthermore,

$$Ax_0 = \alpha \frac{dx_0}{d\zeta}. \tag{2.10}$$

Since A has to map into $L^2(0,1)$, we see that the domain consists of functions in $L^2(0,1)$ which are differentiable and whose derivative lies in $L^2(0,1)$. This space is known as the *Sobolev space* $H^1(0,1)$. With this notation, we can write down the domain of A

$$D(A) = \{x_0 \in L^2(0,1) \mid x_0 \in H^1(0,1) \text{ and } x_0(1) = 0\}. \tag{2.11}$$

□

As in our example of Section 2.1, it turns out that every semigroup is related to a differential equation. This we state next. The proof of this lemma can be found in Theorem 2.5.2.

Lemma 2.2.6. *Let A be the infinitesimal generator of the strongly continuous semigroup $(T(t))_{t\geq 0}$. Then for every $x_0 \in D(A)$, we have that $T(t)x_0 \in D(A)$ and*

$$\frac{d}{dt}T(t)x_0 = AT(t)x_0. \tag{2.12}$$

So combining this with property 2. of Definition 2.2.1, we see that for $x_0 \in D(A)$, $x(t) := T(t)x_0$ is a solution of the (abstract) differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0. \tag{2.13}$$

Although $T(t)x_0$ only satisfies (2.13) if $x_0 \in D(A)$, we call $T(t)x_0$ the *solution* for any x_0.

It remains to formulate the abstract differential equation for our semigroup of Examples 2.2.3 and 2.2.5.

Example 2.2.7 We have calculated A in (2.10), and so we can now easily write down the abstract differential equation (2.13). Since we have time and spatial dependence, we use the partial derivatives. Doing so (2.13) becomes

$$\frac{\partial x}{\partial t} = \alpha \frac{\partial x}{\partial \zeta},$$

which is our p.d.e. of Section 2.1. Note that the boundary condition at $\zeta = 1$ is not explicitly visible. It is hidden in the domain of A. □

We know now how to get an infinitesimal generator for a semigroup, but normally, we want to go into the other direction. Given a differential equation, we want to find the solution, i.e., the semigroup. There exists a general theorem for showing this, but since we shall not use it in the sequel, we don't include it here. We concentrate on a special class of semigroups and groups, and hence generators.

Definition 2.2.8. A strongly continuous semigroup $(T(t))_{t\geq 0}$ is called a *contraction semigroup* if $\|T(t)x_0\|_X \leq \|x_0\|_X$ for all $x_0 \in X$ and all $t \geq 0$.

A strongly continuous group is called a *unitary group* if $\|T(t)x_0\|_X = \|x_0\|_X$ for all $x_0 \in X$ and all $t \in \mathbb{R}$. ♣

If $(T(t))_{t\geq 0}$ is a contraction semigroup, then the function $f(t) := \|T(t)x_0\|_X^2$ must have a non-positive derivative at $t = 0$, provided this derivative exists. Using the fact that $f(t) = \langle T(t)x_0, T(t)x_0\rangle_X$, and that (2.13) holds for $x_0 \in D(A)$, it is easy to show that the derivative of f equals for $x_0 \in D(A)$

$$\dot{f}(t) = \langle AT(t)x_0, T(t)x_0\rangle_X + \langle T(t)x_0, AT(t)x_0\rangle_X.$$

Hence if A is the infinitesimal generator of a contraction semigroup, then

$$\langle Ax_0, x_0\rangle_X + \langle x_0, Ax_0\rangle_X = \dot{f}(0) \leq 0.$$

It is not hard to show that if $(T(t))_{t\geq 0}$ is a contraction semigroup, then the same holds for $(T(t)^*)_{t\geq 0}$. Hence a similar inequality as derived above holds for A^* as well. Both conditions are sufficient as well.

Theorem 2.2.9. *An operator A defined on the Hilbert space X is the infinitesimal generator of a contraction semigroup on X if and only if the following conditions hold*

1. *for all $x_0 \in D(A)$ we have that $\langle Ax_0, x_0\rangle + \langle x_0, Ax_0\rangle \le 0$;*
2. *for all $x_0 \in D(A^*)$ we have that $\langle A^*x_0, x_0\rangle + \langle x_0, A^*x_0\rangle \le 0$.*

For unitary groups there is a similar theorem.

Theorem 2.2.10. *An operator A defined on the Hilbert space X is the infinitesimal generator of a unitary group on X if and only if $A = -A^*$.*

2.3. Homogeneous solutions to the port-Hamiltonian system

In this section, we apply the general result presented in the previous section to our p.d.e., i.e., we consider

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}\left[\mathcal{H}(\zeta)x(\zeta,t)\right] + P_0\left[\mathcal{H}(\zeta)x(\zeta,t)\right]. \tag{2.14}$$

with the boundary condition

$$\tilde{W}_B \begin{pmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{pmatrix} = 0. \tag{2.15}$$

In order to apply the theory of the previous section, we do not regard $x(\cdot,\cdot)$ as a function of place and time, but as a function of time, which takes values in a function space, i.e., we see $x(\zeta,t)$ as the function $x(t,\cdot)$ evaluated at ζ. With a little bit of misuse of notation, we write $x(t,\cdot) = (x(t))(\cdot)$. Hence we "forget" the spatial dependence, and we write the p.d.e. as the (abstract) ordinary differential equation

$$\frac{dx}{dt}(t) = P_1 \frac{\partial}{\partial \zeta}\left[\mathcal{H}x(t)\right] + P_0\left[\mathcal{H}x(t)\right]. \tag{2.16}$$

Hence we consider the operator

$$Ax := P_1 \frac{d}{d\zeta}\left[\mathcal{H}x\right] + P_0\left[\mathcal{H}x\right] \tag{2.17}$$

on a domain which includes the boundary conditions. The domain should be a part of the *state space* X, which we identify next. For our class of p.d.e.'s we have a natural energy function, see (1.26). Hence it is quite natural to consider only states which have a finite energy. That is we take as our state space all functions for which $\int_a^b x(\zeta)^T\mathcal{H}(\zeta)x(\zeta)d\zeta$ is finite. We assume that for every $\zeta \in [a,b]$, $\mathcal{H}(\zeta)$ a symmetric matrix and there exist m, M, independent of ζ, with $0 < m \le M < \infty$ and $mI \le \mathcal{H}(\zeta) \le MI$. Under these assumptions it is easy to see that the integral $\int_a^b x(\zeta)^T\mathcal{H}(\zeta)x(\zeta)d\zeta$ is finite if only if x is square integrable over $[a,b]$. We take as our state space

$$X = L^2((a,b);\mathbb{R}^n) \tag{2.18}$$

with inner product

$$\langle f, g\rangle_X = \frac{1}{2}\int_a^b f(\zeta)^T \mathcal{H}(\zeta) g(\zeta) d\zeta. \tag{2.19}$$

This implies that the squared norm of a state x equals the energy of this state.

So we have found our operator A and our state space. As mentioned above, the domain of A will include the boundary conditions.

It turns out that formulating the boundary conditions directly in x at $\zeta = a$ and $\zeta = b$ is not the best choice. It is better to formulate them in the *boundary effort* and *boundary flow*, which are defined as

$$e_\partial = \frac{1}{\sqrt{2}}\left[\mathcal{H}(b)x(b) + \mathcal{H}(a)x(a)\right] \quad \text{and} \quad f_\partial = \frac{1}{\sqrt{2}}\left[P_1\mathcal{H}(b)x(b) - P_1\mathcal{H}(a)x(a)\right], \tag{2.20}$$

respectively.

We show some properties of this transformation.

Lemma 2.3.1. *Let P_1 be symmetric and invertible, then the matrix R_0 defined as*

$$R_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix} \tag{2.21}$$

is invertible, and satisfies

$$\begin{pmatrix} P_1 & 0 \\ 0 & -P_1 \end{pmatrix} = R_0^T \Sigma R_0, \tag{2.22}$$

where

$$\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{2.23}$$

All possible matrices R which satisfies (2.22) are given by the formula

$$R = UR_0,$$

with U satisfying $U^T \Sigma U = \Sigma$.

Proof: We have that

$$\frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & I \\ -P_1 & I \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}\frac{1}{\sqrt{2}} = \begin{pmatrix} P_1 & 0 \\ 0 & -P_1 \end{pmatrix}.$$

Thus using the fact that P_1 is symmetric, we have that $R_0 := \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} P_1 & -P_1 \\ I & I \end{smallmatrix}\right)$ satisfies (2.22). Since P_1 is invertible, the invertibility of R_0 follows from equation (2.22).

Let R be another solution of (2.22). Hence

$$R^T \Sigma R = \begin{pmatrix} P_1 & 0 \\ 0 & -P_1 \end{pmatrix} = R_0^T \Sigma R_0.$$

This can be written in the equivalent form

$$R_0^{-T} R^T \Sigma R R_0^{-1} = \Sigma.$$

Calling $RR_0^{-1} = U$, we have that $U^T \Sigma U = \Sigma$ and $R = UR_0$, which proves the assertion.■

Combining (2.20) and (2.21) we see that

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = R_0 \begin{pmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{pmatrix}. \tag{2.24}$$

Since the matrix R_0 is invertible, we can write any condition which is formulated in $(\mathcal{H}x)(b)$ and $(\mathcal{H}x)(a)$ into an equivalent condition which is formulated in f_∂ and e_∂. Furthermore, we see from (2.20) that the following holds

$$(\mathcal{H}x)(b)^T P_1 (\mathcal{H}x)(b) - (\mathcal{H}x)(a)^T P_1 (\mathcal{H}x)(a) = f_\partial^T e_\partial + e_\partial^T f_\partial. \tag{2.25}$$

Using (2.24), we write the boundary condition (2.15) (equivalently) as

$$W_B \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = 0, \tag{2.26}$$

where $W_B = \tilde{W}_B R_0^{-1}$.

Thus we have formulated our state space X, see (2.18) and (2.19), and our operator, A, see (2.17). The domain of this operator is given by, see (2.15) and (2.26),

$$D(A) = \{x \in L^2((a,b);\mathbb{R}^n) \mid \mathcal{H}x \in H^1((a,b);\mathbb{R}^n),\ W_B \left(\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right) = 0\}. \tag{2.27}$$

Here $H^1((a,b);\mathbb{R}^n)$ are all functions from (a,b) to $\mathbb{R}^n$ which are square integrable and have a derivative which is again square integrable.

The following theorem shows that this operator generates a contraction semigroup precisely when the power (2.25) is negative, see (1.27).

Theorem 2.3.2. *Consider the operator A defined in (2.17) and (2.27), where we assume the following*

- *P_1 is an invertible, symmetric real $n \times n$ matrix;*
- *P_0 is an anti-symmetric real $n \times n$ matrix;*
- *For all $\zeta \in [a,b]$ the $n \times n$ matrix $\mathcal{H}(\zeta)$ is real, symmetric, and $mI \le \mathcal{H}(\zeta) \le MI$, for some $M, m > 0$ independent of ζ;*
- *W_B is a full rank real matrix of size $n \times 2n$.*

Then A is the infinitesimal generator of a contraction semigroup on X if and only if $W_B \Sigma W_B^T \ge 0$.

Furthermore, A is the infinitesimal generator of a unitary group on X if and only if W_B satisfies $W_B \Sigma W_B^T = 0$.

Proof: The proof is divided in several steps. In the first step we simplify the expression $\langle Ax, x\rangle + \langle x, Ax\rangle_X$. We shall give the proof for the contraction semigroup in full detail. The proof for the unitary group follows easily from it, see also Exercise 2.4. We write $W_B = S(I+V, I-V)$, see Lemma 2.4.1.

Step 1. For the differential operator (2.17) we have

$$\begin{aligned}\langle Ax, x\rangle_X + \langle x, Ax\rangle_X =& \frac{1}{2}\int_a^b \left[P_1\frac{\partial}{\partial\zeta}\left(\mathcal{H}x\right)(\zeta) + P_0\left(\mathcal{L}x\right)(\zeta)\right]^T \mathcal{H}(\zeta)x(\zeta)d\zeta + \\ & \frac{1}{2}\int_a^b x(\zeta)^T\mathcal{H}(\zeta)\left[P_1\frac{\partial}{\partial\zeta}\left(\mathcal{H}x\right)(\zeta) + P_0\left(\mathcal{H}x\right)(\zeta)\right]d\zeta.\end{aligned}$$

Using now the fact that $P_1, \mathcal{H}(\zeta)$ are symmetric, and P_0 is anti-symmetric, we write the last expression as

$$\begin{aligned}&\frac{1}{2}\int_a^b \left[\frac{d}{d\zeta}\left(\mathcal{H}x\right)(\zeta)\right]^T P_1\mathcal{H}(\zeta)x(\zeta) + \left[\mathcal{H}(\zeta)x(\zeta)\right]^T\left[P_1\frac{d}{d\zeta}\left(\mathcal{H}x\right)(\zeta)\right]d\zeta + \\ &\frac{1}{2}\int_a^b -\left[\mathcal{H}(\zeta)x(\zeta)\right]^T P_0\mathcal{H}(\zeta)x(\zeta) + \left[\mathcal{H}(\zeta)x(\zeta,t)\right]^T\left[P_0\mathcal{H}(\zeta)x(\zeta)\right]d\zeta \\ =&\frac{1}{2}\int_a^b \frac{d}{d\zeta}\left[\left(\mathcal{H}x\right)^T(\zeta)P_1\left(\mathcal{H}x\right)(\zeta)\right]d\zeta \\ =&\frac{1}{2}\left[\left(\mathcal{H}x\right)^T(b)P_1\left(\mathcal{H}x\right)(b) - \left(\mathcal{H}x\right)^T(a)P_1\left(\mathcal{H}x\right)(a)\right].\end{aligned}$$

Combining this with (2.25), we see that

$$\langle Ax, x\rangle_X + \langle x, Ax\rangle_X = \frac{1}{2}\left[f_\partial^T e_\partial + e_\partial^T f_\partial\right]. \tag{2.28}$$

By assumption, the vector $\left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right]$ lies in the kernel of W_B. Hence by using Lemma 2.4.2 we know that $\left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right]$ equals $\left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right]\ell$ for some $\ell \in \mathbb{R}^n$. Substituting this in the above, we find

$$\begin{aligned}\langle Ax, x\rangle_X + \langle x, Ax\rangle_X =& \frac{1}{2}\left[f_\partial^T e_\partial + e_\partial^T f_\partial\right] \\ =&\frac{1}{2}\left[\ell^T(I - V^T)(-I - V)\ell + \ell^T(-I - V^T)(I - V)\ell\right] \\ =&\ell^T(-I + V^T V)\ell. \end{aligned}\tag{2.29}$$

By the assumption and Lemma 2.4.1, we have that this is less or equal to zero. Hence we have proved the first condition in Theorem 2.2.9.

Step 2. Let $g \in L^2((a,b),\mathbb{R}^n)$ be given. If for every $f \in H^1((a,b);\mathbb{R}^n)$ which is zero at $\zeta = a$ and $\zeta = b$, the following equality hold for some $\tilde{g} \in L^2((a,b);\mathbb{R}^n)$

$$\int_a^b g(\zeta)^T\frac{df}{d\zeta}(\zeta)d\zeta = \int_a^b \tilde{g}(\zeta)^T f(\zeta)d\zeta,$$

then $g \in H^1((a,b);\mathbb{R}^n)$ and $\frac{dg}{d\zeta} = -\tilde{g}$.

Step 3. In this step we determine the adjoint operator of A. By definition, $y \in X$ lies in the domain of A^* if there exists a $\tilde{y} \in X$, such that

$$\langle y, Ax\rangle_X = \langle \tilde{y}, x\rangle_X \tag{2.30}$$

for all $x \in D(A)$.

Let $x \in D(A)$, then (2.30) becomes

$$\frac{1}{2}\int_a^b y(\zeta)^T \mathcal{H}(\zeta)\left[P_1 \frac{d}{d\zeta}[\mathcal{H}x](\zeta) + P_0 \mathcal{H}x(\zeta)\right] d\zeta = \frac{1}{2}\int_a^b \tilde{y}(\zeta)^T \mathcal{H}(\zeta) x(\zeta) d\zeta. \tag{2.31}$$

Since the half is on both sides, we neglect it and write the the left-hand side of this equation as

$$\begin{aligned}
\int_a^b & y(\zeta)^T \mathcal{H}(\zeta)\left[P_1 \frac{d}{d\zeta}[\mathcal{H}x](\zeta) + P_0 \mathcal{H}x(\zeta)\right] d\zeta \\
&= \int_a^b y(\zeta)^T \mathcal{H}(\zeta) P_1 \frac{d}{d\zeta}[\mathcal{H}x](\zeta) d\zeta + \int_a^b y(\zeta)^T \mathcal{H}(\zeta) P_0 \mathcal{H}x(\zeta) d\zeta \\
&= \int_a^b \left[P_1 \mathcal{H}(\zeta) y(\zeta)\right]^T \frac{d}{d\zeta}[\mathcal{H}x](\zeta) d\zeta - \int_a^b \left[P_0 \mathcal{H}(\zeta) y(\zeta)\right]^T \mathcal{H}x(\zeta) d\zeta,
\end{aligned} \tag{2.32}$$

where we have used that $\mathcal{H}$ and P_1 are symmetric, and that P_0 is anti-symmetric. The last integral is already of the form $\langle y_2, x\rangle_X$. Hence it remains to write the second last integral in a similar form. Since we are assuming that $y \in D(A^*)$ this is possible. In particular, there exists a $y_1 \in X$ such that

$$\int_a^b \left[P_1 \mathcal{H}(\zeta) y(\zeta)\right]^T \frac{d}{d\zeta}[\mathcal{H}x](\zeta) d\zeta = \int_a^b y_1(\zeta)^T \mathcal{H}(\zeta) x(\zeta) d\zeta \tag{2.33}$$

for all $x \in D(A)$. The set of functions $x \in H^1((a,b);\mathbb{R}^n)$ which are zero at $\zeta = a$ and $\zeta = b$ forms a subset of $D(A)$, and so by step 2, and (2.33) we conclude that

$$P_1(\mathcal{H}y)(\cdot) \in H^1((a,b);\mathbb{R})^n \text{ and } y_1 = -\frac{d}{d\zeta}\left[P_1 \mathcal{H}y\right].$$

Since P_1 is constant and invertible, we have that $\mathcal{H}y \in H^1((a,b);\mathbb{R})^n$. Integrating by part we find that for $x \in D(A)$,

$$\begin{aligned}
\int_a^b \left[P_1(\mathcal{H}y)(\zeta)\right]^T \frac{d}{d\zeta}[\mathcal{H}x](\zeta) d\zeta = &- \int_a^b \frac{d}{d\zeta}\left[P_1 \mathcal{H}(\zeta) y(\zeta)\right]^T [\mathcal{H}x](\zeta) d\zeta \\
&+ \left[\left[P_1(\mathcal{H}y)(\zeta)\right]^T (\mathcal{H}x)(\zeta)\right]_a^b.
\end{aligned} \tag{2.34}$$

The boundary term can be written as

$$\begin{aligned}
\left[\left[P_1(\mathcal{H}y)(\zeta)\right]^T (\mathcal{H}x)(\zeta)\right]_a^b &= \begin{pmatrix} (\mathcal{H}y)(b) \\ (\mathcal{H}y)(a) \end{pmatrix}^T \begin{pmatrix} P_1 & 0 \\ 0 & -P_1 \end{pmatrix} \begin{pmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{pmatrix} \\
&= \begin{pmatrix} (\mathcal{H}y)(b) \\ (\mathcal{H}y)(a) \end{pmatrix}^T R_0^T \Sigma R_0 \begin{pmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{pmatrix} \\
&= \begin{pmatrix} (\mathcal{H}y)(b) \\ (\mathcal{H}y)(a) \end{pmatrix}^T R_0^T \Sigma \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix},
\end{aligned} \tag{2.35}$$

where we used (2.22) and (2.24).

Combining (2.32), (2.34), and (2.35) we find that

$$\begin{aligned}\int_a^b y(\zeta)^T \mathcal{H}(\zeta)\left[P_1 \frac{d}{d\zeta}[\mathcal{H}x](\zeta) + P_0 \mathcal{H}x(\zeta)\right] d\zeta = &- \int_a^b \frac{d}{d\zeta}\left[P_1 \mathcal{H}(\zeta) y(\zeta)\right]^T (\mathcal{H}x)(\zeta) d\zeta \\ &- \int_a^b \left[P_0 \mathcal{H}(\zeta) y(\zeta)\right]^T (\mathcal{H}x)(\zeta) d\zeta \\ &+ \begin{pmatrix} (\mathcal{H}y)(b) \\ (\mathcal{H}y)(a) \end{pmatrix}^T R_0^T \Sigma \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}. \end{aligned} \tag{2.36}$$

The right-hand side of this equation must equal the inner product of x with some function $\tilde{y}$. The first two terms on the right-hand side are already in this form, and hence it remains to write the last term in the requested form. However, since this last term only depends on the boundary variables, this is not possible. Hence this term has to disappear, i.e., has to be zero for all $x \in D(A)$. Combining (2.27) and Lemma 2.4.2 we see that $\left(\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right)$ can be written as $\left(\begin{smallmatrix} I-V \\ I+V \end{smallmatrix}\right) l$, $l \in \mathbb{R}^n$. We define

$$\begin{pmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{pmatrix} = R_0 \begin{pmatrix} (\mathcal{H}y)(b) \\ (\mathcal{H}y)(a) \end{pmatrix}. \tag{2.37}$$

Using this notation, the last term of (2.36) equals

$$\begin{pmatrix} (\mathcal{H}y)(b) \\ (\mathcal{H}y)(a) \end{pmatrix}^T R_0^T \Sigma \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{pmatrix}^T \Sigma \begin{pmatrix} I - V \\ -I - V \end{pmatrix} l.$$

The first expression is zero for all $x \in D(A)$, i.e., all $\left(\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right)$ in the kernel of W_B if and only if the second expression is zero for all $l \in \mathbb{R}^n$. By taking the transpose of this last expression, we see that this holds if and only if $\left(\begin{smallmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{smallmatrix}\right) \in \ker(-I - V^T,\ I - V^T)$.

Combining this with equation (2.36) we see that

$$A^* y = -\frac{d}{d\zeta}\left[P_1 \mathcal{H} y\right] - P_0 \mathcal{H} y = -P_1 \frac{d}{d\zeta}\left[\mathcal{H} y\right] - P_0 \mathcal{H} y \tag{2.38}$$

and its domain equals

$$\begin{aligned} D(A^*) = \{y \in L^2((a,b);\mathbb{R}^n) \mid \mathcal{H}y \in H^1((a,b);\mathbb{R}^n), \\ (-I - V^T,\ I - V^T) \left(\begin{smallmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{smallmatrix}\right) = 0\}, \end{aligned} \tag{2.39}$$

where $\left(\begin{smallmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{smallmatrix}\right)$ is given by (2.37).

Step 4. In this step we show that $\langle A^* y, y\rangle_X + \langle y, A^* y\rangle \leq 0$ for all $y \in D(A^*)$. Since the expression of A^*, see Step 3, is minus the expression for A, we can proceed as in Step 1. By doing so, we find

$$\langle A^* y, y\rangle_X + \langle y, A^* y\rangle = -\frac{1}{2}\left[(\mathcal{H}y)^T(b) P_1 (\mathcal{H}y)(b) - (\mathcal{H}y)^T(a) P_1 (\mathcal{H}y)(a)\right]. \tag{2.40}$$

Using (2.37), we can rewrite this as

$$\langle A^* y, y\rangle_X + \langle y, A^* y\rangle = -\frac{1}{2}\left[\tilde{f}_\partial^T \tilde{e}_\partial - \tilde{e}_\partial^T \tilde{f}_\partial\right]. \tag{2.41}$$

Since $\left(\begin{smallmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{smallmatrix}\right) \in \ker(-I - V^T,\ I - V^T)$, or equivalently, $\left(\begin{smallmatrix} -\tilde{f}_\partial \\ \tilde{e}_\partial \end{smallmatrix}\right) \in \ker(I + V^T,\ I - V^T)$, we conclude by Lemma 2.4.2 that

$$\begin{pmatrix} -\tilde{f}_\partial \\ \tilde{e}_\partial \end{pmatrix} = \begin{pmatrix} I - V^T \\ -I - V^T \end{pmatrix} \ell$$

for some $\ell \in \mathbb{R}^n$. Substituting this in (2.41) gives

$$\begin{aligned}\langle A^* y, y\rangle_X + \langle y, A^* y\rangle =& \frac{1}{2}\left[\ell^T (I - V)(-I - V^T)\ell + \ell^T(-I - V)(I - V^T)\ell\right] \\ =& \ell^T[-I + VV^T]\ell.\end{aligned}$$

From our condition on V, see the beginning of this proof, we see that the above expression is negative. Hence using Theorem 2.2.9 we conclude that A generates a contraction semigroup.

Step 5. It remains to show that A generates a unitary group when $W_B \Sigma W_B = 0$. By Lemma 2.4.1 we see that this condition on W_B is equivalent to V being unitary. Now we show that the domain of A^* equals the domain of A. Comparing (2.27) and (2.39) we have that the domain are equal if and only if the kernel of W_B equals the kernel of $(-I - V^T,\ I - V^T)$. Since V is unitary, we have

$$\ker \begin{pmatrix} -I - V^T & I - V^T \end{pmatrix} = \ker(-V^T \begin{pmatrix} I + V & I - V \end{pmatrix}) = \ker(-V^T S^{-1} W_B) = \ker W_B,$$

where in the last equality we used that S is invertible. Comparing (2.17), (2.27) with (2.38), (2.39), and using the above equality, we conclude that $A = -A^*$. Applying Theorem 2.2.10 we see that A generates a unitary group. ■

We apply this Theorem to our standard example from the introduction, see also Example 2.2.5 and 2.2.7.

Example 2.3.3 Consider the homogeneous p.d.e. on the spatial interval $[0, 1]$.

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \frac{\partial x}{\partial \zeta}(\zeta, t), && \zeta \in [0, 1],\ t \geq 0 \\ x(\zeta, 0) &= x_0(\zeta), && \zeta \in [0, 1] \\ 0 &= x(1, t), && t \geq 0. \end{aligned}$$

We see that the first equation can be written in the from (2.14) by choosing $P_1 = 1$, $\mathcal{H} = 1$ and $P_0 = 0$. Using this and equation (2.20) the boundary variables are given by

$$f_\partial = \frac{1}{\sqrt{2}}[x(1) - x(0)], \qquad e_\partial = \frac{1}{\sqrt{2}}[x(1) + x(0)]. \tag{2.42}$$

The boundary condition becomes in these variables

$$0 = x(1,t) = \frac{1}{\sqrt{2}}[f_\partial(t) + e_\partial(t)] = W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \tag{2.43}$$

with $W_B = \left(\frac{1}{\sqrt{2}} \; \frac{1}{\sqrt{2}} \right)$.

Since $W_B \Sigma W_B^T = 1$, we conclude from Theorem 2.3.2 that the operator associated to the p.d.e. generates a contraction semigroup on $L^2(0,1)$. The expression for this contraction semigroup is given in Example 2.2.3. □

2.4. Technical lemma's

This section contains two technical lemma's on representation of matrices. They are important for the proof, but not for the understanding of the examples.

Lemma 2.4.1. *Let W be a $n \times 2n$ matrix and let $\Sigma = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. Then W has rank n and $W\Sigma W^T \geq 0$ if and only if there exist a matrix $V \in \mathbb{R}^{n\times n}$ and an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that*

$$W = S \begin{pmatrix} I+V & I-V \end{pmatrix} \tag{2.44}$$

with $VV^T \leq I$, or equivalently $V^TV \leq I$.

Furthermore, $W\Sigma W^T > 0$ if and only if $VV^T < I$ and $W\Sigma W^T = 0$ if and only if $VV^T = I$, i.e., V is unitary.

Proof: If W is of the form (2.44), then we find

$$W\Sigma W^T = S \begin{pmatrix} I+V & I-V \end{pmatrix} \Sigma \begin{pmatrix} I+V^T \\ I-V^T \end{pmatrix} S^T = S[2I - 2VV^T]S^T,$$

which is non-negative, since $VV^T \leq I$.

Now we prove that if W is of full rank and is such that $W\Sigma W^T \geq 0$, then relation (2.44) holds. Writing W as $W = \begin{pmatrix} W_1 & W_2 \end{pmatrix}$, we see that $W\Sigma W^T \geq 0$ is equivalent to $W_1W_2^T + W_2W_1^T \geq 0$. Hence

$$(W_1+W_2)(W_1+W_2)^T \geq (W_1-W_2)(W_1-W_2)^T \geq 0. \tag{2.45}$$

If $x \in \ker((W_1+W_2)^T)$, then the above inequality implies that $x \in \ker((W_1-W_2)^T)$. Thus $x \in \ker(W_1^T) \cap \ker(W_2^T)$. Since W has full rank, this implies that $x = 0$. Hence $W_1 + W_2$ is invertible.

Using (2.45) once more, we see that

$$(W_1+W_2)^{-1}(W_1-W_2)(W_1-W_2)^T(W_1+W_2)^{-T} \leq I$$

and thus $V := (W_1+W_2)^{-1}(W_1-W_2)$ satisfies $VV^T \leq I$. Summarizing, we have

$$\begin{aligned} \begin{pmatrix} W_1 & W_2 \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} W_1+W_2+W_1-W_2 & W_1+W_2-W_1+W_2 \end{pmatrix} \\ &= \frac{1}{2}(W_1+W_2)\begin{pmatrix} I+V & I-V \end{pmatrix}. \end{aligned}$$

Defining $S := \frac{1}{2}(W_1 + W_2)$, we have shown the representation (2.44).

If instead of inequality, we have equality for W, then it is easy to show that we have equality in the equation for V as well. Thus V is unitary. ■

Lemma 2.4.2. *Suppose that the $n \times 2n$ matrix W can be written in the format of equation (2.44), i.e., $W = S(I+V,\ I-V)$ with S and V square matrices, and S invertible. Then the kernel of W equals the range of $\left(\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right)$.*

If V is unitary, then the kernel of W equals the range of ΣW^T.

PROOF: Let $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ be in the range of $\left(\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right)$. By the equality (2.44), we have that

$$\begin{aligned} W\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= S\begin{pmatrix} I+V & I-V \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= S\begin{pmatrix} I+V & I-V \end{pmatrix}\begin{pmatrix} I-V \\ -I-V \end{pmatrix} l = 0. \end{aligned}$$

Hence we see that the range of $\left(\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right)$ lies in the kernel of W. It is easy to show that W has rank n, and so the kernel of W has dimension n. Thus if we can show that the $2n \times n$ matrix $\left(\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right)$ has full rank, then we have proved the first assertion. If this matrix would not have full rank, then there should be a non-trivial element in its kernel. It is easy to see that the kernel consists of zero only, and so we have proved the the first part of the lemma.

Suppose now that V is unitary, then

$$\begin{pmatrix} I-V \\ -I-V \end{pmatrix} = \begin{pmatrix} -I+V^T \\ -I-V^T \end{pmatrix} V = -\Sigma W^T S^{-T} V.$$

Since the range of ΣW^T equals the range of $-\Sigma W^T S^{-T} V$, we have proved the second assertion. ■

2.5. Properties of semigroups and their generators

This section contains some nice properties of semigroup and generators. Since the proofs are rather long, we decided to put them separately. However, form time to time we shall refer to one of these properties.

Theorem 2.5.1. *A strongly continuous semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X has the following properties:*

a. $\|T(t)\|$ is bounded on every finite subinterval of $[0,\infty)$;

b. $T(t)$ is strongly continuous for all $t \in [0,\infty)$;

c. For all $x \in X$ we have that $\frac{1}{t}\int_0^t T(s)x ds \to x$ as $t \to 0^+$;

d. If $\omega_0 = \inf_{t>0}(\frac{1}{t}\log\|T(t)\|)$, then $\omega_0 = \lim_{t\to\infty}(\frac{1}{t}\log\|T(t)\|) < \infty$;

e. $\forall \omega > \omega_0$, *there exists a constant* M_ω *such that* $\forall t \geq 0$, $\|T(t)\| \leq M_\omega e^{\omega t}$.

This constant ω_0 *is called the* growth bound *of the semigroup.*

PROOF: *a.* First we show that $\|T(t)\|$ is bounded on some neighborhood of the origin, that is, there exist $\delta > 0$ and $M > 1$ depending on δ such that

$$\|T(t)\| \leq M \qquad \text{for } t \in [0, \delta].$$

If this does not hold, then there exists a sequence $\{t_n\}$, $t_n \to 0^+$ such that $\|T(t_n)\| \geq n$. Hence, by the Uniform Boundedness Theorem A.3.19, there exists one x such that $\{\|T(t_n)x\|\}$ is unbounded; but this contradicts the strong continuity at the origin. If we set $t = m\delta + \tau$ with $0 \leq \tau \leq \delta$, then

$$\|T(t)\| \leq \|T(\delta)\|^m \|T(\tau)\| \leq M^{1+m} \leq M M^{t/\delta} = M e^{\omega t},$$

where $\omega = \delta^{-1} \log M$.

b. For fixed $t > 0$, $s \geq 0$ we have

$$\|T(t+s)x - T(t)x\| \leq \|T(t)\| \|T(s)x - x\| \leq M e^{\omega t} \|T(s)x - x\|.$$

Hence we may conclude that

$$\lim_{s \to 0^+} \|T(t+s)x - T(t)x\| = 0.$$

Moreover, for $t > 0$ and $s \geq 0$ sufficiently small, we have

$$\|T(t-s)x - T(t)x\| \leq \|T(t-s)\| \|x - T(s)x\|.$$

Thus $\lim\limits_{s \to 0^-} \|T(t+s)z - T(t)x\| = 0$, and $T(t)x$ is continuous.

c. Let $x \in X$ and $\varepsilon > 0$. By the strong continuity of $(T(t))_{t \geq 0}$ we can choose a $\tau > 0$ such that $\|T(s)x - x\| \leq \varepsilon$ for all $s \in [0, \tau]$. For $t \in [0, \tau]$ we have that

$$\begin{aligned} \|\frac{1}{t} \int_0^t T(s)x ds - x\| &= \|\frac{1}{t} \int_0^t [T(s)x - x] ds\| \\ &\leq \frac{1}{t} \int_0^t \|T(s)x - x\| ds \leq \frac{1}{t} \int_0^t \varepsilon ds = \varepsilon. \end{aligned}$$

d. Let $t_0 > 0$ be a fixed number and $M = \sup\limits_{t \in [0, t_0]} \|T(t)\|$; then for every $t \geq t_0$ there exists $n \in \mathbb{N}$ such that $n t_0 \leq t < (n+1) t_0$. Consequently,

$$\begin{aligned} \frac{\log \|T(t)\|}{t} &= \frac{\log \|T^n(t_0) T(t - n t_0)\|}{t} \\ &\leq \frac{n \log \|T(t_0)\|}{t} + \frac{\log M}{t} \\ &= \frac{\log \|T(t_0)\|}{t_0} \cdot \frac{n t_0}{t} + \frac{\log M}{t}. \end{aligned}$$

The latter is smaller than or equal to $\frac{\log \|T(t_0)\|}{t_0} + \frac{\log M}{t}$ if $\log \|T(t_0)\|$ is positive, and it is smaller than or equal to $\frac{\log \|T(t_0)\|}{t_0} \frac{t-t_0}{t} + \frac{\log M}{t}$ if $\log \|T(t_0)\|$ is negative. Thus

$$\limsup_{t\to\infty} \frac{\log \|T(t)\|}{t} \leq \frac{\log \|T(t_0)\|}{t_0} < \infty,$$

and since t_0 is arbitrary, we have that

$$\limsup_{t\to\infty} \frac{\log \|T(t)\|}{t} \leq \inf_{t>0} \frac{\log \|T(t)\|}{t} \leq \liminf_{t\to\infty} \frac{\log \|T(t)\|}{t}.$$

Thus

$$\omega_0 = \inf_{t>0} \frac{\log \|T(t)\|}{t} = \lim_{t\to\infty} \frac{\log \|T(t)\|}{t} < \infty.$$

e. If $\omega > \omega_0$, there exists a t_0 such that

$$\frac{\log \|T(t)\|}{t} < \omega \qquad \text{for } t \geq t_0;$$

that is,

$$\|T(t)\| \leq e^{\omega t} \qquad \text{for } t \geq t_0.$$

But

$$\|T(t)\| \leq M_0 \qquad \text{for } 0 \leq t \leq t_0,$$

and so with

$$M_\omega = M_0, \qquad \text{for the case that } \omega > 0,$$

and

$$M_\omega = e^{-\omega t_0} M_0 \quad \text{for the case that } \omega < 0,$$

we obtain the stated result. ■

Theorem 2.5.2. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Hilbert space X with infinitesimal generator A. Then the following results hold:*

a. *For $x_0 \in D(A)$, $T(t)x_0 \in D(A)$ $\forall t \geq 0$;*

b. $\frac{d}{dt}(T(t)x_0) = AT(t)x_0 = T(t)Ax_0$ *for $x_0 \in D(A)$, $t > 0$;*

c. $\frac{d^n}{dt^n}(T(t)x_0) = A^n T(t)x_0 = T(t)A^n x_0$ *for $x_0 \in D(A^n)$, $t > 0$;*

d. $T(t)x_0 - x_0 = \int_0^t T(s)Ax_0 ds$ *for $x_0 \in D(A)$;*

e. $\int_0^t T(s)x ds \in D(A)$ *and* $A\int_0^t T(s)x ds = T(t)x - x$ *for all $x \in X$, and $D(A)$ is dense in X;*

f. *A is a closed linear operator;*

g. $\bigcap_{n=1}^{\infty} D(A^n)$ *is dense in* Z.

PROOF: *a,b,c.* First we prove *a* and *b*. Let $s > 0$ and consider

$$\frac{T(t+s)x_0 - T(t)x_0}{s} = T(t)\frac{(T(s)-I)x_0}{s} = \frac{T(s)-I}{s}T(t)x_0.$$

If $x_0 \in D(A)$, the middle limit exists as $s \to 0^+$, and hence the other limits also exist. In particular, $T(t)x_0 \in D(A)$ and the strong right derivative of $T(t)x_0$ equals $AT(t)x_0 = T(t)Ax_0$.

For $t > 0$ and s sufficiently small, we have

$$\frac{T(t-s)x_0 - T(t)x_0}{-s} = T(t-s)\frac{(T(s)-I)x_0}{s}.$$

Hence the strong left derivative exists and equals $T(t)Ax_0$. Part *c* follows by induction on this result.

d. Take any $x^* \in X$ and $x_0 \in D(A)$. Then

$$\langle x^*, T(t)x_0 - x_0\rangle = \int_0^t \frac{d}{du}\langle x^*, T(u)x_0\rangle du,$$

and hence

$$\begin{aligned}\langle z^*, T(t)x_0 - x_0\rangle &= \int_0^t \langle z^*, T(u)Ax_0\rangle du \qquad \textit{for } x_0 \in D(A)\\ &= \langle x^*, \int_0^t T(u)Ax_0 du\rangle.\end{aligned}$$

x^* was arbitrary and so this proves *d*.

e. We first show that $D(A)$ is dense in Z. Consider the following for any $x \in X$

$$\frac{T(s)-I}{s}\int_0^t T(u)xdu = \frac{1}{s}\int_0^t T(s+u)xdu - \frac{1}{s}\int_0^t T(u)xdu.$$

These integrals are well defined, since $T(t)$ is strongly continuous (Lemma A.5.5 and Example A.5.15). Letting $\rho = s + u$ in the second integral, we have

$$\begin{aligned}\frac{T(s)-I}{s}\int_0^t T(u)xdu &= \frac{1}{s}\int_s^{t+s} T(\rho)xd\rho - \frac{1}{s}\int_0^t T(u)xdu\\ &= \frac{1}{s}\Big[\int_t^{t+s} T(\rho)xd\rho + \int_s^t T(\rho)xd\rho -\\ &\qquad \int_s^t T(u)xdu - \int_0^s T(u)xdu\Big]\\ &= \frac{1}{s}\Big[\int_0^s (T(t+u) - T(u))xdu\Big]\\ &= \frac{1}{s}\int_0^s T(u)(T(t)-I)xdu.\end{aligned}$$

Now, as $s \to 0^+$, the right-hand side tends to $(T(t)-I)x$ (see Theorem 2.5.1.c). Hence

$$\int_0^t T(u)xdu \in D(A) \quad \text{and} \quad A\int_0^t T(u)xdu = (T(t)-I)x.$$

Furthermore, $\frac{1}{t}\int_0^t T(u)xdu \to x$ as $t \to 0^+$, and hence for any $x \in X$, there exists a sequence in $D(A)$ that tends to x. This shows that $\overline{D(A)} = X$.

f. To prove that A is closed, we let $\{n_n\}_{n\in\mathbb{N}}$ be a sequence in $D(A)$ converging to x such that Ax_n converges to y. Then $\|T(s)Ax_n - T(s)y\| \leq Me^{\omega s}\|Ax_n - y\|$ and so $T(s)Ax_n \to T(s)y$ uniformly on $[0,t]$. Now, since $x_n \in D(A)$, we have that

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n ds.$$

Using the Lebesgue dominated convergence Theorem A.5.21, we see that

$$T(t)x - x = \int_0^t T(s)yds,$$

and so

$$\lim_{t\downarrow 0} \frac{T(t)x - x}{t} = \lim_{t\downarrow 0} \frac{1}{t}\int_0^t T(s)yds = y.$$

Hence $x \in D(A)$ and $Ax = y$, which proves that A is closed.

g. Let $C_0^\infty([0,\infty))$ be the class of all real-valued functions on $[0,\infty)$ having continuous derivatives of all orders and having compact support contained in the open right half-line $(0,\infty)$. If $\psi \in C_0^\infty([0,\infty))$, then so does $\psi^{(r)}$, the rth derivative of ψ, and $\psi(u)T(u)x$ is a continuous vector-valued function from $[0,\infty)$ to X. Let X_0 be the set of all elements of the form

$$g = \int_0^\infty \psi(u)T(u)xdu \qquad x \in X,\ \psi \in C_0^\infty([0,\infty)).$$

These are well defined by Lemma A.5.5. We shall show that $X_0 \subset D(A^r)$ for $r \geq 1$ and that X_0 is dense in X. For sufficiently small s, we have

$$\begin{aligned}
\frac{T(s)-I}{s}g &= \frac{1}{s}\int_0^\infty \psi(u)[T(u+s)z - T(u)x]du \\
&= \frac{1}{s}\int_s^\infty [\psi(u-s) - \psi(u)]T(u)xdu - \frac{1}{s}\int_0^s \psi(u)T(u)xdu.
\end{aligned}$$

But $\frac{\psi(u-s)-\psi(u)}{s} \to -\dot{\psi}(u)$ as $s \to 0^+$, uniformly with respect to u, and the last expression is zero for sufficiently small s, since the support ψ is contained in $(0,\infty)$. Thus $g \in D(A)$ and $Ag = -\int_0^\infty \dot{\psi}(u)T(u)xdu$. Repeating this argument, we see that $g \in D(A^r)$ for all $r > 0$, and

$$A^r g = (-1)^r \int_0^\infty \psi^{(r)}(u)T(u)xdu$$

which shows that $X_0 \subset \bigcap_{r=1}^{\infty} D(A^r)$. Suppose now that the closure of X_0 is not X. Then there must exist a $x_0 \in X$ such that

$$\langle x_0, g \rangle = 0 \quad \forall g \in X_0 \quad \text{and} \quad \|x_0\| = 1.$$

Thus

$$\langle x_0, \int_0^\infty \psi(u) T(u) x du \rangle = \int_0^\infty \psi(u) \langle x_0, T(u) x \rangle du = 0$$

$\forall \psi \in C_0^\infty([0,\infty))$ and $x \in X$. But $\langle x_0, T(u)x_0 \rangle$ is continuous with $\|x_0\| = 1$. Hence there exists a $\psi \in C_0^\infty([0,\infty))$ such that $\int_0^\infty \psi(u)\langle x_0, T(u)x_0\rangle du \neq 0$. This is a contradiction, and so $\overline{X_0} = X$. ∎

2.6. Exercises

2.1. Check equation (2.2).

2.2. In this exercise you will prove that the partial differential equation

$$\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial x}{\partial \zeta}(\zeta, t), \qquad \zeta \in [0,1], \quad t \geq 0,$$

with non-zero initial condition $x(\zeta, 0) = x_0(\zeta)$ and boundary condition

$$x(0,t) = 0$$

does not possess a solution.

a) Assume that the p.d.e. possesses a solution, show that for any continuously differentiable f satisfying $f(\eta) = 0$ for $\eta \geq 1$ the function $q(t) = \int_0^1 f(\zeta + t)x(\zeta,t)d\zeta$ has derivative zero for $t \geq 0$.

b) For the function defined in the previous item, show that $q(1) = 0$, independently of the value of f in the interval $[0,1)$.

c) Conclude from the previous two items that $\int_0^1 f(\zeta)x_0(\zeta)d\zeta$ is zero for all continuously differentiable functions f.

d) Prove that for any non-zero initial condition the p.d.e. with the chosen boundary condition doe not possess a solution in positive time.

2.3. Let A be a real $n \times n$ matrix, and define $T(t)$ as e^{At}.

a) Show that $(T(t))_{t\geq 0}$ is a strongly continuous semigroup on $\mathbb{R}^n$.

b) Show that $(T(t))_{t\in\mathbb{R}}$ is a strongly continuous group on $\mathbb{R}^n$.

2.4. In this exercise we show that A generates a unitary group if and only if A and $-A$ generate a contraction semigroup.

a) Prove that if $(T(t))_{t\in\mathbb{R}}$ is a strongly continuous group satisfying $\|T(t)\| \leq 1$ for all $t \in \mathbb{R}$, then $(T(t))_{t\in\mathbb{R}}$ is a unitary group.

b) Prove Theorem 2.2.10.

2.5. Consider differential operator associated to the p.d.e.

$$\frac{\partial x}{\partial t}(\zeta, t) = \alpha \frac{\partial x}{\partial \zeta}(\zeta, t), \qquad \zeta \in [0,1], \quad t \geq 0, \tag{2.46}$$

with $\alpha > 0$ generates a strongly continuous semigroup on $L^2(0,1)$ for any of the following boundary conditions.

a) The state at the right-hand side is set to zero, i.e., $x(1,t) = 0$.

b) The states at both ends are equal, i.e., $x(1,t) = x(0,t)$.

c) Determine for $\alpha = 1$, the boundary conditions for the semigroup associated to the p.d.e. (2.46) is a unitary group.

2.6. Consider differential operator associated to the p.d.e.

$$\frac{\partial x}{\partial t}(\zeta, t) = -\frac{\partial x}{\partial \zeta}(\zeta, t), \qquad \zeta \in [0,1], \quad t \geq 0, \tag{2.47}$$

generates a strongly continuous semigroup on $L^2(0,1)$ for any of the following boundary conditions.

a) The state at the left-hand side is set to zero, i.e., $x(0,t) = 0$.

b) The states at both ends are equal, i.e., $x(1,t) = x(0,t)$.

c) Determine the boundary conditions for the semigroup associated to the p.d.e. (2.47) is a unitary group.

2.7. Consider the transmission line of Example 1.1.1. To this transmission line we put a resistor at the right-end, i.e., $V(b,t) = RI(b,t)$, and at the left-end we put the voltage equal to zero. Show that the operator associated to this p.d.e. generates a contraction semigroup on the energy space.

2.8. Consider the vibrating string of Example 1.1.2.

a) Formulate this in our general form (2.14). Determine P_1, P_0 and $\mathcal{H}$.

b) Reformulate the condition on $\mathcal{H}$, see Theorem 2.3.2 in conditions on T and ρ. Are these conditions restrictive from a physical point of view?

c) Show that by imposing the no-force boundary conditions, i.e., $\frac{\partial w}{\partial \eta}(b,t) = \frac{\partial w}{\partial \eta}(a,t) = 0$ the system generates a unitary group on the energy space.

d) If the endpoints of string are hold at a constant position, i.e., $w(a,t) = w(b,t) = p$, $p \in \mathbb{R}$ independent of time, does the operator associated to the p.d.e. generate a strongly continuous semigroup on the energy space?

2.9. Consider the Timoshenko beam from Example 1.1.3. Show that the operator associated to this p.d.e. generates a C_0-semigroup on the energy space, when

a) $\frac{\partial w}{\partial t}(a,t) = \frac{\partial w}{\partial t}(b,t) = 0$, and $\frac{\partial \phi}{\partial t}(a,t) = \frac{\partial \phi}{\partial t}(b,t) = 0$.

b) $\frac{\partial w}{\partial t}(a,t) = 0$, $\frac{\partial \phi}{\partial t}(a,t) = 0$, $\frac{\partial w}{\partial t}(b,t) = -\frac{\partial \phi}{\partial \zeta}(b,t) + \phi(b,t)$, and $\frac{\partial \phi}{\partial \zeta}(b,t) = -Q\frac{\partial \phi}{\partial t}(b,t)$, $Q \geq 0$.

2.10. In the theory developed in this chapter, we considered the p.d.e.'s of the spatial interval $[0,1]$. However, the theory is independent of this spatial interval. In this exercise, we show that if we have proved a theorem for the spatial interval $[0,1]$, then one can easily formulate the result for the general interval $[a,b]$.

a) Assume that the spatial coordinate ζ lies in the interval $[a,b]$. Introduce the new spatial coordinate η as

$$\eta = \frac{\zeta - a}{b - a}.$$

Reformulate the p.d.e. (2.14) in the new spatial coordinate.

b) What are the new $\mathcal{H}$, P_0, when P_1 remains the same?

c) Determine the boundary effort and flow in the new coordinate.

d) How do the boundary conditions (2.15) and (2.26) change when using the new spatial variable?

2.11. Consider coupled vibrating strings as given in the figure below. We assume that

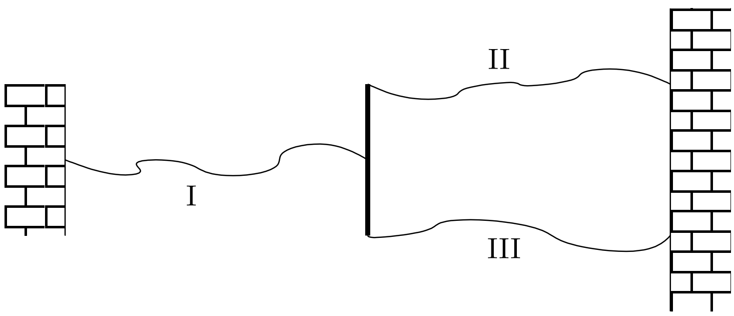

Figure 2.1.: Coupled vibrating strings

all the length of the strings are equal. The model for every vibrating string is given by (1.6) with physical parameters, $\rho_{\rm I}, T_{\rm I}, \rho_{\rm II}$, etc. Furthermore, we assume that the three strings are connected via a (massless) bar, as shown in Figure 2.1. This bar can only move in the vertical direction. This implies that the velocity of string I at its right-hand side equals those of the other two strings at their left-hand side. Furthermore, the force of string I at its right-end side equals the sum of the forces of the other two at their left-hand side, i.e.,

$$T_{\rm I}(b)\frac{\partial w_{\rm I}}{\partial \zeta}(b) = T_{\rm II}(a)\frac{\partial w_{\rm II}}{\partial \zeta}(a) + T_{\rm III}(a)\frac{\partial w_{\rm III}}{\partial \zeta}(a).$$

As depictured, the strings are attached to a wall.

a) Identify the boundary conditions for the system given in Figure 2.1.
b) Formulate the coupled strings as depictured in Figure 2.1 as a Port-Hamiltonian system (2.14) and (2.15). Furthermore, determine the energy space X.
c) Show that the differential operator associated to the above system generates a contraction semigroup on the energy space X.

2.12. Consider transmission lines in a network as depictured in Figure 2.2. In the cou-

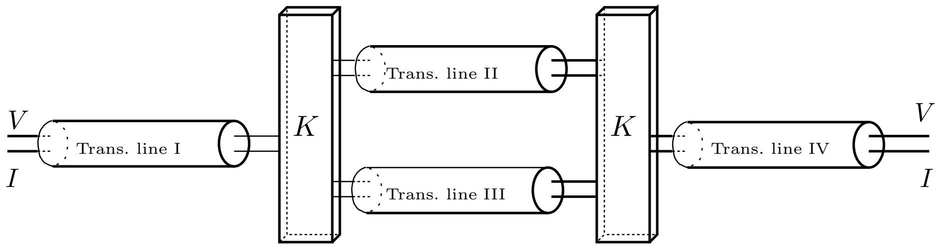

Figure 2.2.: Coupled transmission lines

pling parts K, we have that Kirchhoff laws holds. Hence charge flowing out of the transmission line I, enters II and III, etc. Thus for the coupling between I and II and III, there holds

$$V_{\mathrm{I}}(b) = V_{\mathrm{II}}(a) = V_{\mathrm{III}}(a)$$

and

$$I_{\mathrm{I}}(b) = I_{\mathrm{II}}(a) + I_{\mathrm{III}}(a).$$

a) Identify the boundary conditions for the system induced by the second coupling in Figure 2.2.
b) Formulate the coupled transmission lines as depictured in Figure 2.2 as a Port-Hamiltonian system (2.14). Furthermore, identify the boundary effort and flow, and the energy space X.
c) If the voltage at both ends is set to zero, i.e., $V_{\mathrm{I}}(a) = 0, V_{\mathrm{IV}}(b) = 0$, then show that the differential operator associated to the above system generates a contraction semigroup on the energy space X.
d) If left end side is coupled to the right-hand side, i.e., the voltage and current at both ends are equal, then show that the differential operator associated to the above system generates a unitary group on the energy space X.

2.7. Notes and references

More on semigroups can be found in [5]. The result of Section 2.3 are taken from [9].

Chapter 3
Boundary Control Systems

In this chapter we add a control function to the partial differential equation (1.25), see also (2.14). In particular, we are interested in boundary controls and we will show that these systems have well-defined solutions. We explain our ideas by means of an example, the controlled transport equation, which is given by

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial x}{\partial \zeta}(\zeta,t), && \zeta \in [0,1],\ t \geq 0 \\ x(\zeta,0) &= x_0(\zeta), && \zeta \in [0,1] \\ x(1,t) &= u(t), && t \geq 0 \end{aligned} \tag{3.1}$$

for a control function u. In Chapter 2 we have solved the partial differential equation for the specific choice $u = 0$. In this chapter we show that the solution of (3.1) is given by

$$x(\zeta,t) = \begin{cases} x_0(\zeta+t) & \zeta+t \leq 1 \\ u(\zeta+t-1) & \zeta+t > 1 \end{cases}$$

and that in a similar manner the partial differential equation (1.25) with a boundary control can be treated.

In Section 3.1 we first have to study some abstract theory, which enables us to show that for certain boundary controls the partial differential equation (1.25) possesses a unique solution.

3.1. Inhomogeneous differential equations

In Chapter 2 we have studied homogeneous (abstract) ordinary differential equations of the form

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0. \tag{3.2}$$

Under the assumption that A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ we showed that the solution of (3.2) is given by

$$x(t) = T(t)x_0.$$

In order to be able to add control operators to equation (3.2) we have to study inhomogeneous (abstract) ordinary differential equations first. Such an inhomogeneous equation is given by

$$\dot{x}(t) = Ax(t) + f(t), \quad x(0) = x_0. \tag{3.3}$$

We assume that f is a continuous function with values in the Hilbert space X. Later, when we deal with input and control functions $f(t)$ has usually the form $Bu(t)$.

First we have to define what we mean by a solution of (3.3), and we begin with the notion of a classical solution. The function $x(t)$ is called a *classical solution* of (3.3) *on an interval* $[0, \tau]$ if $x(t)$ is a continuous function on $[0, \tau]$ whose derivative is again continuous on $[0, \tau]$, $x(t) \in D(A)$ for all $t \in [0, \tau]$ and $x(t)$ satisfies (3.3) for all $t \in [0, \tau]$.

Assume that $f \in C([0, \tau]; X)$ and that x is a classical solution of (3.3) on $[0, \tau]$. Then formally we have

$$\begin{aligned} \frac{d}{ds}[T(t-s)x(s)] &= T(t-s)\dot{x}(s) - AT(t-s)x(s) \\ &= T(t-s)[Ax(s) + f(s)] - AT(t-s)x(s) \\ &= T(t-s)f(s), \end{aligned}$$

which implies

$$\begin{aligned} \int_0^t T(t-s)f(s)\,ds &= \int_0^t \frac{d}{ds}[T(t-s)x(s)]\,ds = T(t-t)x(t) - T(t-0)x(0) \\ &= x(t) - T(t)x_0. \end{aligned}$$

Equivalently,

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds.$$

This equation is known as the *variation of constant formula*. The next lemma shows that the formal argument which we used to derive this formula can be made precise. For the proof we refer to Section 3.5.

Lemma 3.1.1. *Assume that $f \in C([0, \tau]; X)$ and that x is a classical solution of (3.3) on $[0, \tau]$. Then $Ax(\cdot)$ is an element of $C([0, \tau]; X)$, and*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds. \tag{3.4}$$

This lemma tells us how a classical solution will look like, but does not tells us anything about existence and uniqueness of solutions. This is the subject of the following theorem, whose proof is given in Section 3.5

Theorem 3.1.2. *If A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t \geq 0}$ on a Hilbert space X, $f \in C^1([0, \tau]; X)$ and $x_0 \in D(A)$, then (3.4) is continuously differentiable on $[0, \tau]$ and it is the unique classical solution of (3.3).*

By taking $f = 0$ and $x_0 \notin D(A)$, it is clear that (3.4) will not be the classical solution of (3.3) in general. Although (3.4) is only a classical solution for special initial states x_0 and smooth functions f, we see that this equation is well-defined for every $t \geq 0$ and every $x_0 \in X$ and $f \in L^2([0,\tau]; X)$. Furthermore, for $x_1 \in D(A^*)$, one can show that $\langle x_1, x(t)\rangle$ is continuous differentiable, and satisfies (3.4) weakly. Therefore, for any $x_0 \in X$ and every $f \in L^2([0,\tau]; X)$ we call (3.4) the *solution* of (3.3). Since we sometimes want to distinguish between this solution and the classical solution, we use the name *mild solution* for the variation of constant formula (3.4).

Most times the inhomogeneous differential equation (3.3) is of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{3.5}$$

where B is a linear bounded operator from the input space U to the state space X. The spaces U and X are Hilbert spaces. u is considered as the (control) input and B is the control operator. This is a special case of (3.3) via $f(t) = Bu(t)$. Hence the (mild) solution of (3.5) is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)\, ds \tag{3.6}$$

for every $x_0 \in X$ and every $u \in L^2([0,\tau]; X)$. We apply this to our simple example of the transport equation.

Example 3.1.3 We study the following controlled partial differential equation

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial x}{\partial \zeta}(\zeta,t) + u(t) \\ x(1,t) &= 0. \end{aligned}$$

We can write this partial differential equation as

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with, see also Examples 2.2.7 and 2.3.3,

$$\begin{aligned} Ax &= \frac{dx}{d\zeta}, \\ D(A) &= \left\{x \in L^2(0,1) \mid x \in H^1(0,1) \text{ and } x(1) = 0\right\}, \\ Bu &= 1 \cdot u. \end{aligned}$$

In Chapter 2 we have shown that A generates a C_0-semigroup on $L^2(0,1)$. Using the formula for the semigroup generated by A, see Examples 2.2.3 and 2.2.5 we get

$$x(\zeta,t) = x_0(\zeta+t)\mathbb{1}_{[0,1]}(\zeta+t) + \int_0^t \mathbb{1}_{[0,1]}(\zeta+t-\tau)u(\tau)\, d\tau,$$

where $\mathbb{1}_{[0,1]}(\zeta) = 1$ if $\zeta \in [0,1]$ and $\mathbb{1}_{[0,1]}(\zeta) = 0$ otherwise. $\square$

In this example we applied a control within the spatial domain. More logical for this example is to apply a control at the boundary. However, when doing so, we cannot rewrite this system in our standard form (3.5), see Example 3.2.3. This is general the case when controlling a p.d.e. via its boundary. Thus systems with control at the boundary form a new class of systems, and are introduced next.

3.2. Boundary control systems

We first explain the idea by means of the controlled transport equation (3.1). Consider the following system

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial x}{\partial \zeta}(\zeta,t), && \zeta \in [0,1],\ t \geq 0 \\ x(\zeta,0) &= x_0(\zeta), && \zeta \in [0,1] \\ x(1,t) &= u(t), && t \geq 0. \end{aligned} \tag{3.7}$$

for an input $u \in L^2(0,\tau)$.

Boundary control problems like the one above occur frequently in our applications, but unfortunately they do not fit into our standard formulation (3.5). However, for sufficiently smooth inputs it is possible to reformulate such problems so that they do lead to an associated system in the standard form (3.5). In order to find the associated system for the controlled transport equation we use the following trick. Assume that x is a classical solution of the p.d.e. (3.7) and that u is continuously differentiable. Defining

$$v(\zeta,t) = x(\zeta,t) - u(t),$$

we obtain the following partial differential equation for v

$$\begin{aligned} \frac{\partial v}{\partial t}(\zeta,t) &= \frac{\partial v}{\partial \zeta}(\zeta,t) - \dot{u}(t), && \zeta \in [0,1],\ t \geq 0 \\ v(1,t) &= 0, && t \geq 0. \end{aligned}$$

This partial differential equation for v can be written in the standard form as

$$\dot{v}(t) = Av(t) + B\tilde{u}(t)$$

for $\tilde{u} = \dot{u}$. Hence by applying a simple trick, we can reformulate a p.d.e. with boundary control into a p.d.e. with internal control. The price we have to pay is that u has to be smooth.

The trick applied to (3.7) can be extended to abstract boundary control systems:

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), && x(0) = x_0, \\ \mathfrak{B}x(t) &= u(t), \end{aligned} \tag{3.8}$$

where $\mathfrak{A} : D(\mathfrak{A}) \subset X \mapsto X$, $u(t) \in U$, U is a Hilbert space, and the *boundary operator* $\mathfrak{B} : D(\mathfrak{B}) \subset X \mapsto U$ satisfies $D(\mathfrak{A}) \subset D(\mathfrak{B})$.

In order to reformulate equation (3.8) into an abstract form (3.5), we need to impose extra conditions on the system.

Definition 3.2.1. The control system (3.8) is a *boundary control system* if the following hold:

a. The operator $A : D(A) \mapsto X$ with $D(A) = D(\mathfrak{A}) \cap \ker(\mathfrak{B})$ and
$$Ax = \mathfrak{A}x \qquad \text{for } x \in D(A) \tag{3.9}$$
is the infinitesimal generator of a C_0-semigroup on X;

b. There exists a $B \in \mathcal{L}(U, X)$ such that for all $u \in U$, $Bu \in D(\mathfrak{A})$, the operator $\mathfrak{A}B$ is an element of $\mathcal{L}(U, X)$ and
$$\mathfrak{B}Bu = u, \qquad u \in U. \tag{3.10}$$

♣

Part b. of the definition is equivalent to the fact that the range of the operator $\mathfrak{B}$ equals U. Note that Part a. of the definition guarantees that the system possesses a unique solution for the choice $u(t) = 0$, i.e., the homogeneous equation is well-posed. Part b. allows us to choose every value in U for $u(t)$. In other words, the values of inputs are not restricted, which is a logical condition for inputs.

We say that the function $x(t)$ is a *classical solution* of the boundary control system of Definition 3.2.1 if $x(t)$ is a continuously differentiable function, $x(t) \in D(\mathfrak{A})$ for all t, and $x(t)$ satisfies (3.8) for all t.

For a boundary control system, we can apply a similar trick as the one applied in the beginning of this section. This is the subject of the following theorem. It turns out that $v(t) = x(t) - Bu(t)$ is a solution of the abstract differential equation
$$\begin{aligned} \dot{v}(t) &= Av(t) - B\dot{u}(t) + \mathfrak{A}Bu(t), \\ v(0) &= v_0. \end{aligned} \tag{3.11}$$

Since A is the infinitesimal generator of a C_0-semigroup and B and $\mathfrak{A}B$ are bounded linear operators, we have from Theorem 3.1.2 that equation (3.11) has a unique classical solution for $v_0 \in D(A)$. Furthermore, we can prove the following relation between the (classical) solutions of (3.8) and (3.11).

Theorem 3.2.2. *Consider the boundary control system (3.8) and the abstract Cauchy equation (3.11). Assume that $u \in C^2([0, \tau]; U)$ for all $\tau > 0$. Then, if $v_0 = x_0 - Bu(0) \in D(A)$, the classical solutions of (3.8) and (3.11) are related by*
$$v(t) = x(t) - Bu(t). \tag{3.12}$$

Furthermore, the classical solution of (3.8) is unique.

Proof: Suppose that $v(t)$ is a classical solution of (3.11). Then $v(t) \in D(A) \subset D(\mathfrak{A}) \subset D(\mathfrak{B})$, $Bu(t) \in D(\mathfrak{B})$, and so
$$\mathfrak{B}x(t) = \mathfrak{B}[v(t) + Bu(t)] = \mathfrak{B}v(t) + \mathfrak{B}Bu(t) = u(t),$$

where we have used that $v(t) \in D(A) \subset \ker \mathfrak{B}$ and equation (3.10). Furthermore, from (3.12) we have

$$\begin{aligned}
\dot{x}(t) &= \dot{v}(t) + B\dot{u}(t) \\
&= Av(t) - B\dot{u}(t) + \mathfrak{A}Bu(t) + B\dot{u}(t) && \text{by (3.11)} \\
&= Av(t) + \mathfrak{A}Bu(t) \\
&= \mathfrak{A}(v(t) + Bu(t)) && \text{by (3.9)} \\
&= \mathfrak{A}x(t) && \text{by (3.12).}
\end{aligned}$$

Thus, if $v(t)$ is a classical solution of (3.11), then $x(t)$ defined by (3.12) is a classical solution of (3.8).

The other implication is proved similarly. The uniqueness of the classical solutions of (3.8) follows from the uniqueness of the classical solutions of (3.11). ∎

The (mild) solution of (3.11) is given by

$$v(t) = T(t)v(0) + \int_0^t T(t-\tau)[\mathfrak{A}Bu(\tau) - B\dot{u}(\tau)]\, d\tau \tag{3.13}$$

for every $v(0) \in X$ and every $u \in H^1(0,\infty); U)$. Therefore, the function

$$x(t) = T(t)(x_0 - Bu(0)) + \int_0^t T(t-\tau)[\mathfrak{A}Bu(\tau) - B\dot{u}(\tau)]\, d\tau + Bu(t) \tag{3.14}$$

is called the *mild solution* of the abstract boundary control system (3.8) for every $x_0 \in X$ and every $u \in H^1((0,\infty); U)$.

As an example we study again the controlled transport equation from the beginning of this section.

Example 3.2.3 We consider the following system

$$\begin{aligned}
\frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial x}{\partial \zeta}(\zeta,t), && \zeta \in [0,1],\ t \geq 0 \\
x(\zeta,0) &= x_0(\zeta), && \zeta \in [0,1] \\
x(1,t) &= u(t), && t \geq 0.
\end{aligned}$$

for an input $u \in H^1(0,\tau)$. In order to write this example in the form (3.8) we choose $X = L^2(0,1)$ and

$$\begin{aligned}
\mathfrak{A}x &= \frac{dx}{d\zeta}, && D(\mathfrak{A}) = \left\{x \in L^2(0,1) \mid x \in H^1(0,1)\right\}, \\
\mathfrak{B}x &= x(1), && D(\mathfrak{B}) = D(\mathfrak{A}).
\end{aligned}$$

These two operators satisfy the assumption of a boundary control system. More precisely: the operators $\mathfrak{A}$ and $\mathfrak{B}$ are linear, $\mathfrak{A}$ restricted to the domain $D(\mathfrak{A}) \cap \ker \mathfrak{B}$ generates a C_0-semigroup, see Example 2.2.7. Furthermore, the range of $\mathfrak{B}$ is $\mathbb{C} = U$

and the choice $B = 1$ implies $\mathfrak{B}Bu = u$. Using $\mathfrak{A}B = 0$, we conclude from equation (3.14) that the solution is given by

$$\begin{aligned} x(t) &= v(t) + Bu(t) \\ &= T(t)v(0) + \int_0^t T(t-\tau)[\mathfrak{A}Bu(\tau) - B\dot{u}(\tau)]\,d\tau + Bu(t) \\ &= T(t)v(0) - \int_0^t T(t-\tau)\dot{u}(\tau)\,d\tau + u(t). \end{aligned}$$

Using the precise form of the shift-semigroup, see Example 2.2.3, we can write the solution of the boundary controlled partial differential equation as

$$x(\zeta,t) = v_0(\zeta+t)\mathbb{1}_{[0,1]}(\zeta+t) - \int_0^t \mathbb{1}_{[0,1]}(\zeta+t-\tau)\dot{u}(\tau)\,d\tau + u(t).$$

If $\zeta + t > 1$, we have

$$x(\zeta,t) = -[u(\tau)]|_{\zeta+t-1}^t + u(t) = u(\zeta+t-1),$$

and if $\zeta + t \leq 1$, then

$$x(\zeta,t) = v_0(\zeta+t) - [u(\tau)]|_0^t + u(t) = v_0(\zeta+t) + u(0) = x_0(\zeta+t).$$

Or equivalently,

$$x(\zeta,t) = \begin{cases} x_0(\zeta+t) & \zeta+t \leq 1 \\ u(\zeta+t-1) & \zeta+t > 1 \end{cases} \tag{3.15}$$

which proves our claim made on the first page of this chapter.

Now it is not hard to show that the controlled p.d.e. form this example cannot be written as the abstract control system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{3.16}$$

with B a bounded operator. Since for $u = 0$, we have that the p.d.e. becomes the homogeneous equation of Examples 2.2.5 and 2.2.7, we have that A can only be $\mathfrak{A}$ restricted to $D(\mathfrak{A}) \cap \ker \mathfrak{B}$. Hence the semigroup is the shift semigroup.

If our controlled p.d.e. would be of the form (3.16), then by Theorem 3.1.2, we would have that $x(t) \in D(A)$, whenever $x_0 \in D(A)$ and $f \in C^1((0,\tau;X)$. Choosing $x_0 \in D(A)$ and $u = 1$, we see by (3.15), that $x(1,t) = u(t)$. Since this is unequal to zero, we have that $x(t) \notin D(A)$. Concluding we find that the boundary controlled p.d e. of this example cannot be written in the form (3.16). □

The controlled transport equation is a simple example of our general class of port-Hamiltonian systems. This example could be written as a boundary control system. In the following section, we show that this holds in general for a port-Hamiltonian system.

3.3. Port-Hamiltonian systems as boundary control systems

In this section we add a boundary control to our Hamiltonian system and we show that the assumptions of a boundary control system are satisfied. The port-Hamiltonian system with control is given by

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= P_1 \frac{\partial}{\partial \zeta}[\mathcal{H}x(t)] + P_0[\mathcal{H}x(t)] && (3.17)\\ u(t) &= W_{B,1}\begin{pmatrix} f_\partial(t) \\ e_\partial(t)\end{pmatrix} && (3.18)\\ 0 &= W_{B,2}\begin{pmatrix} f_\partial(t) \\ e_\partial(t)\end{pmatrix}. && (3.19)\end{aligned}$$

We first recall the defining operators of the port-Hamiltonian system. As in Section 2.3 we assume that

- P_1 is an invertible, symmetric real $n \times n$-matrix;
- P_0 is an anti-symmetric real $n \times n$-matrix;
- $\mathcal{H}(\zeta)$ is a symmetric, invertible $n \times n$-matrix for every $\zeta \in [a,b]$ and $mI \leq \mathcal{H}(\zeta) \leq MI$ for some $m, M > 0$ independent of ζ;
- $W_B := \left(\begin{smallmatrix} W_{B,1} \\ W_{B,2}\end{smallmatrix}\right)$ is a full rank real matrix of size $n \times 2n$.

We recall that the boundary effort and flow are given by, see (2.24)

$$\begin{pmatrix} f_\partial(t) \\ e_\partial(t)\end{pmatrix} = R_0 \begin{pmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t)\end{pmatrix}$$

where R_0 is the invertible $n \times n$-matrix defined in (2.21).

We can write the Hamiltonian system as a boundary control system

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0,\\ \mathfrak{B}x(t) &= u(t),\end{aligned}$$

by defining

$$\begin{aligned} \mathfrak{A}x &= P_1 \frac{\partial}{\partial \zeta}[\mathcal{H}x] + P_0[\mathcal{H}x], && (3.20)\\ D(\mathfrak{A}) &= \left\{ x \in L^2((a,b);\mathbb{R}^n) \mid \mathcal{H}x \in H^1((a,b);\mathbb{R}^n), W_{B,2}\begin{pmatrix} f_\partial \\ e_\partial\end{pmatrix} = 0\right\}, && (3.21)\\ \mathfrak{B}x &= W_{B,1}\begin{pmatrix} f_\partial \\ e_\partial\end{pmatrix}, && (3.22)\\ D(\mathfrak{B}) &= D(\mathfrak{A}). && (3.23)\end{aligned}$$

As in Section 2.3 we have chosen the Hilbert space $X = L^2((a,b);\mathbb{R}^n)$ equipped with the inner product

$$\langle f,g\rangle_X := \frac{1}{2}\int_a^b f(\zeta)^T\mathcal{H}(\zeta)g(\zeta)\,d\zeta$$

as our state space. The input space U equals $\mathbb{R}^k$, where k is the number of rows of $W_{B,1}$. We are now in a position to show that the controlled port-Hamiltonian system is indeed a boundary control system.

Theorem 3.3.1. *If*

$$\begin{pmatrix} W_{B,1} \\ W_{B,2}\end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0\end{pmatrix}\begin{pmatrix} W_{B,1}^T & W_{B,2}^T\end{pmatrix} \geq 0, \tag{3.24}$$

then the system (3.17)–(3.19) is a boundary control system on X. Furthermore, the operator

$$Ax = P_1\frac{\partial}{\partial\zeta}[\mathcal{H}x] + P_0[\mathcal{H}x] \tag{3.25}$$

with the domain

$$D(A) = \left\{\mathcal{H}x \in H^1((a,b);\mathbb{R}^n \mid \begin{pmatrix} f_\partial \\ e_\partial\end{pmatrix} \in \ker\begin{pmatrix} W_{B,1} \\ W_{B,2}\end{pmatrix}\right\} \tag{3.26}$$

generates a contraction semigroup on X.

Proof: We begin with the simple observation that

$$\begin{pmatrix} f_\partial \\ e_\partial\end{pmatrix} \in \ker\begin{pmatrix} W_{B,1} \\ W_{B,2}\end{pmatrix}$$

is equivalent to

$$W_B\begin{pmatrix} f_\partial \\ e_\partial\end{pmatrix} = \begin{pmatrix} W_{B,1} \\ W_{B,2}\end{pmatrix}\begin{pmatrix} f_\partial \\ e_\partial\end{pmatrix} = 0.$$

From Theorem 2.3.2 follows that the operator A defined in (3.25) and (3.26) is the infinitesimal generator of a contraction semigroup on X. Moreover, by (3.21) and (3.22) we have that $D(A) = D(\mathfrak{A}) \cap \ker\mathfrak{B}$. Hence part a. of Definition 3.2.1 is satisfied.

The $n\times 2n$-matrix W_B is of full rank n and R_0 is an invertible matrix. Thus there exists a $2n\times n$-matrix S such that

$$W_BR_0S = \begin{pmatrix} W_{B,1} \\ W_{B,2}\end{pmatrix}R_0S = \begin{pmatrix} I_k & 0 \\ 0 & 0\end{pmatrix}, \tag{3.27}$$

where I_k is the identity matrix on $\mathbb{R}^k$. We write $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22}\end{pmatrix}$ and we define the operator $B \in \mathcal{L}(U,X)$ by

$$(Bu)(\zeta) := \mathcal{H}(\zeta)^{-1}\left(S_{11}\frac{\zeta-a}{b-a} + S_{21}\frac{b-\zeta}{b-a}\right)u.$$

The definition of B implies that Bu is a square integrable function and that $\mathcal{H}x \in H^1((a,b);\mathbb{R}^n)$. Furthermore, from (3.27) it follows that $W_{B,2}R_0\left(\begin{smallmatrix} S_{11} \\ S_{21}\end{smallmatrix}\right) = 0$. Combining this with the definition of the boundary effort and flow, we obtain that $Bu \in D(\mathfrak{A})$. Furthermore, B and $\mathfrak{A}B$ are linear bounded operators from U to X and using (3.27) once more we see that

$$\mathfrak{B}Bu = W_{B,1}R_0 \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} u = u.$$

Thus the Hamiltonian system is indeed a boundary control system. ■

As an example we once more study the controlled transport equation.

Example 3.3.2 We consider the system

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial x}{\partial \zeta}(\zeta,t), \qquad && \zeta \in [0,1],\ t \geq 0 \\ x(\zeta,0) &= x_0(\zeta), && \zeta \in [0,1]. \end{aligned} \tag{3.28}$$

This system can be written in form (3.17) by choosing $n = 1$, $P_0 = 0$, $P_1 = 1$ and $\mathcal{H} = 1$. Therefore, we have

$$R_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} x(1,t) - x(0,t) \\ x(1,t) + x(0,t) \end{pmatrix}.$$

Since $n = 1$, we can either apply one control or no control at all. The control free case has been treated in Chapter 2, and so we choose one control. By using the boundary variables, the control is written as, see (3.18)

$$u(t) = \begin{pmatrix} a & b \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} x(1,t) - x(0,t) \\ x(1,t) + x(0,t) \end{pmatrix} = \frac{1}{\sqrt{2}}\left[(a+b)x(1,t) + (b-a)x(0,t)\right]. \tag{3.29}$$

Note that $W_B = (a,b)$ has full rank if and only if $a^2 + b^2 \neq 0$.

By Theorem 3.3.1 gives that the p.d.e. (3.28) together with (3.29) is a boundary control system if $a^2 + b^2 \neq 0$ and $2ab \geq 0$, see (3.24). Thus possible boundary controls are for example

$$\begin{aligned} u(t) &= x(1,t), \qquad && (a = b = \frac{\sqrt{2}}{2}), \\ u(t) &= 3x(1,t) - x(0,t), && (a = \sqrt{2}, b = 2\sqrt{2}). \end{aligned}$$

For the control $u(t) = -x(1,t) + 3x(0,t)$ we don't know the answer. □

3.4. Outputs

In the previous sections we have added a control function to our systems. In this section additionally an output is added. We follow the line laid out in the previous sections. We start by assuming that the output equation can be represented via a bounded operator. Since this is normally not the case for observations at the boundary, we have to consider boundary observation separately. For this we directly formulate the result for our port-Hamiltonian systems.

We start with the control system 3.5 to which we add a output equation.

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{3.30}$$

$$y(t) = Cx(t) + Du(t), \tag{3.31}$$

where B is a linear bounded operator from the input space U to the state space X, C is a linear bounded operator from X to the output space Y, and D is a linear operator from U to Y. All Spaces U, X and Y are Hilbert spaces.

In Section 3.1 we showed that the solution of (3.30) is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)\,ds.$$

This function is well-defined for every $x_0 \in X$ and every $u \in L^2([0,\tau];U)$. The output equation (3.31) only contains bounded operators. Hence there is no difficulty in "solving" this equation. We summarize the answer in the following theorem

Theorem 3.4.1. *Consider the abstract equation (3.30)–(3.31), with A the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$, and $B, C,$ and D bounded.*

The solution of (3.30)–(3.31) is given by the variation of constant formula (3.6)

$$\begin{aligned} x(t) &= T(t)x_0 + \int_0^t T(t-s)Bu(s)\,ds, \\ y(t) &= CT(t)x_0 + C\int_0^t T(t-s)Bu(s)\,ds + Du(t) \end{aligned}$$

for every $x_0 \in X$ and every $u \in L^2([0,\tau];U)$.

As said in the beginning of this section, our main focus lies on boundary observation for port-Hamiltonian systems. We develop conditions on the boundary observation guaranteeing that a certain balance equation is satisfied, which are important for Chapter 4 and 5. The standard Hamiltonian system with boundary control and boundary observation is given by

$$\dot{x}(t) = P_1\frac{\partial}{\partial\zeta}[\mathcal{H}x(t)] + P_0[\mathcal{H}x(t)] \tag{3.32}$$

$$u(t) = W_B\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \tag{3.33}$$

$$y(t) = W_C\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}. \tag{3.34}$$

It is assumed that (3.32)–(3.33) satisfy the assumption of a port-Hamiltonian system, see Section 3.3. Further we assume that $W_B\Sigma W_B^T \geq 0$, where $\Sigma = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. This guarantees that (3.32)–(3.33) is a boundary control system, see Theorem 3.3.1. Note that we have taken $W_B = W_{B,1}$ or equivalently $W_{B,2} = 0$. In other words, we are using the maximal number of control.

The output equation is formulated very similar to the input equation. As for the input we assume that we use full measurements, i.e., W_C be a full rank matrix of size $n \times 2n$. Since we do not want to measure quantities that we already have chosen as inputs, see (3.33), we assume that $\begin{pmatrix} W_B \\ W_C \end{pmatrix}$ is of full rank, or equivalently this matrix is invertible.

Combining this assumption with the fact that Σ is invertible, we see that the product $\begin{pmatrix} W_B \\ W_C \end{pmatrix} \Sigma \begin{pmatrix} W_B^T & W_C^T \end{pmatrix}$ is invertible as well. Its inverse is defined as

$$P_{W_B,W_C} = \left(\begin{pmatrix} W_B \\ W_C \end{pmatrix} \Sigma \begin{pmatrix} W_B^T & W_C^T \end{pmatrix} \right)^{-1} = \begin{pmatrix} W_B \Sigma W_B^T & W_B \Sigma W_C^T \\ W_C \Sigma W_B^T & W_C \Sigma W_C^T \end{pmatrix}^{-1}. \tag{3.35}$$

Theorem 3.4.2. *Consider the system (3.32)–(3.34) with W_B a full rank $n \times 2n$ matrix satisfying $W_B \Sigma W_B^T \geq 0$, and W_C a full rank $n \times 2n$ matrix such that $\begin{pmatrix} W_B \\ W_C \end{pmatrix}$ is invertible.*

For every $u \in C^2(0,\infty;\mathbb{R}^n)$, $\mathcal{H}x(0) \in H^1((a,b);\mathbb{R}^n)$, and $u(0) = W_B \begin{pmatrix} f_\partial(0) \\ e_\partial(0) \end{pmatrix}$, the system (3.32)–(3.34) has a unique (classical) solution, with $\mathcal{H}x(t) \in H^1((a,b);\mathbb{R}^n)$. The output $y(\cdot)$ is continuous, and the following balance equation is satisfied:

$$\frac{d}{dt}\|x(t)\|_X^2 = \frac{1}{2} \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_{W_B,W_C} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}. \tag{3.36}$$

PROOF: See Exercise 3.1. ∎

As an example we return to the controlled transport equation of Example 3.3.2.

Example 3.4.3 We consider the system

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial x}{\partial \zeta}(\zeta,t), && \zeta \in [0,1],\ t \geq 0 \\ x(\zeta,0) &= x_0(\zeta), && \zeta \in [0,1] \\ u(t) &= x(1,t), && t \geq 0. \end{aligned} \tag{3.37}$$

In Example 3.3.2 we have already seen that this system can be written in form (3.32)–(3.33) by choosing $n = 1$, $P_0 = 0$, $P_1 = 1$, $\mathcal{H} = 1$ and $W_B = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$.

Now we add the output equation

$$y(t) = \begin{pmatrix} c & d \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} x(1,t) - x(0,t) \\ x(1,t) + x(0,t) \end{pmatrix} = \frac{1}{\sqrt{2}}(c+d)x(1,t) + (c-d)x(0,t)). \tag{3.38}$$

Since $W_C = \begin{pmatrix} c & d \end{pmatrix}$ must have full rank, we find that $c^2 + d^2 \neq 0$. Furthermore, since $\begin{pmatrix} W_B \\ W_C \end{pmatrix}$ must be invertible, we find that $c \neq d$.

Hence all conditions of Theorem 3.4.2 are satisfying for this system whenever $c \neq d$.

For the particular choice $c = -d = \frac{\sqrt{2}}{2}$, that is $y(t) = x(0,t)$, we find $P_{W_B,W_C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, or equivalently

$$\frac{d}{dt}\|x(t)\|_X^2 = \frac{1}{2}\left(|u(t)|^2 - |y(t)|^2\right).$$ □

3.5. Some proofs

This section contains the proofs of Lemma 3.1.1 and Theorem 3.1.2.

PROOF: (of Lemma 3.1.1) From (3.3), we have that $Ax(t) = \dot{x}(t) - f(t)$ and $\dot{x} \in C([0,\tau];X)$ shows that $Ax(\cdot) \in C([0,\tau];X)$.

We now prove (3.4). Let t be an arbitrary, but fixed, element of $(0,\tau]$ and consider the function $T(t-s)x(s)$ for $s \in [0,t)$. We shall show that this function is differentiable in s. Let h be sufficiently small and consider

$$\begin{aligned} & \frac{T(t-s-h)x(s+h) - T(t-s)x(s)}{h} \\ = & \frac{T(t-s-h)x(s+h) - T(t-s-h)x(s)}{h} + \\ & \frac{T(t-s-h)x(s) - T(t-s)x(s)}{h}. \end{aligned}$$

If h converges to zero, then the last term converges to $-AT(t-s)x(s)$, since $x(s) \in D(A)$. Thus it remains to show that the first term converges. We have the following equality

$$\begin{aligned} & \frac{T(t-s-h)x(s+h) - T(t-s-h)x(s)}{h} - T(t-s)\dot{x}(s) \\ = \; & T(t-s-h)\frac{x(s+h) - x(s)}{h} - T(t-s-h)\dot{x}(s) + \\ & T(t-s-h)\dot{x}(s) - T(t-s)\dot{x}(s). \end{aligned}$$

The uniform boundedness of $T(t)$ on any compact interval and the strong continuity allow us to conclude from the last equality that

$$\lim_{h\to 0} \|T(t-s-h)\frac{x(s+h)-x(s)}{h} - T(t-s)\dot{x}(s)\| = 0.$$

So we have proved that

$$\begin{aligned} \frac{d}{ds}[T(t-s)x(s)] &= T(t-s)\dot{x}(s) - AT(t-s)x(s) \\ &= T(t-s)[Ax(s) + f(s)] - AT(t-s)x(s) \\ &= T(t-s)f(s). \end{aligned}$$

This implies

$$\begin{aligned} \int_0^t T(t-s)f(s)\,ds &= \int_0^t \frac{d}{ds}[T(t-s)x(s)]\,ds = T(t-t)x(t) - T(t-0)x(0) \\ &= x(t) - T(t)x_0. \end{aligned}$$

Thus a classical solution to (3.3) necessarily has the form (3.4). ■

PROOF: (of Theorem 3.1.2) *Uniqueness*: If z_1 and z_2 are two different solutions, then their difference $\Delta(t) = z_1(t) - z_2(t)$ satisfies the differential equation

$$\frac{d\Delta}{dt} = A\Delta, \qquad \Delta(0) = 0$$

and so we need to show that its only solution is $\Delta(t) \equiv 0$. To do this, define $y(s) = T(t-s)\Delta(s)$ for a fixed t and $0 \leq s \leq t$. Clearly, $\frac{dy}{ds} = 0$ and so $y(s) =$ constant $= T(t)\Delta(0) = 0$. However, $y(t) = \Delta(t)$ shows that $\Delta(t) = 0$.
Existence: Clearly, all we need to show now is that $v(t) = \int_0^t T(t-s)f(s)ds$ is an element of $C^1([0,\tau];X) \cap D(A)$ and satisfies differential equation (3.3). Now

$$\begin{aligned} v(t) &= \int_0^t T(t-s)[f(0) + \int_0^s \dot{f}(\alpha)d\alpha]ds \\ &= \int_0^t T(t-s)f(0)ds + \int_0^t \int_\alpha^t T(t-s)\dot{f}(\alpha)dsd\alpha, \end{aligned}$$

where we have used Fubini's Theorem A.5.22. From Theorem 2.5.2.e, it follows that

$$T(t-\alpha)z - z = A\int_\alpha^t T(t-s)zds \qquad \text{for all } z \in Z.$$

Hence $v(t) \in D(A)$, and $\int_0^t \|A\int_\alpha^t T(t-s)\dot{f}(\alpha)ds\|d\alpha = \int_0^t \|T(t-\alpha)\dot{f}(\alpha) - \dot{f}(\alpha)\|d\alpha < \infty$. Thus, since A is closed, by Theorem A.5.23 we have that

$$\begin{aligned} Av(t) &= [T(t) - I]f(0) + \int_0^t [T(t-\alpha) - I]\dot{f}(\alpha)d\alpha \\ &= T(t)f(0) + \int_0^t T(t-\alpha)\dot{f}(\alpha)d\alpha - f(t). \end{aligned}$$

Now, since the convolution product is commutative, i.e., $\int_0^t g(t-s)h(s)ds = \int_0^t g(s)h(t-s)ds$, we have that

$$v(t) = \int_0^t T(s)f(t-s)ds$$

and so

$$\begin{aligned} \frac{dv}{dt}(t) &= T(t)f(0) + \int_0^t T(s)\dot{f}(t-s)ds \\ &= T(t)f(0) + \int_0^t T(t-s)\dot{f}(s)ds, \end{aligned}$$

once again using commutativity of the convolution product. It follows that $\frac{dv}{dt}$ is continuous and

$$\frac{dv}{dt}(t) = Av(t) + f(t).$$

■

3.6. Exercises

3.1. Prove Theorem 3.4.2. Hint: Use the calculations of the proof of Theorem 2.3.2.

3.2. Consider the transmission line of Example 1.1.1 for any of the following boundary condition:

a) At the left-end the voltage equal $u_1(t)$ and at the right-end the voltage equals the input $u_2(t)$.

b) At the left-end we put the voltage equal to zero and at the right-end the voltage equals the input $u(t)$.

c) At the left-end we put the voltage equal to $u(t)$ and at the right-end the voltage equals R times the current, for some $R > 0$.

Show that these systems can be written as a boundary control system.

3.3. Consider the vibrating string of Example 1.1.2 with the boundary conditions

$$\frac{\partial x}{\partial \zeta}(0,t) = 0 \quad \text{and} \quad \frac{\partial x}{\partial \zeta}(1,t) = u(t) \quad t \geq 0.$$

Reformulate this system as a boundary control system.

3.4. Consider the Timonshenko beam from Example 1.1.3 for any of the following boundary condition:

a) $\frac{\partial w}{\partial t}(a,t) = u_1(t)$, $\frac{\partial w}{\partial t}(b,t) = u_2(t)$, and $\frac{\partial \phi}{\partial t}(a,t) = \frac{\partial \phi}{\partial t}(b,t) = 0$.

b) $\frac{\partial w}{\partial t}(a,t) = u(t)$, $\frac{\partial \phi}{\partial t}(a,t) = 0$, $\frac{\partial w}{\partial t}(b,t) = -\frac{\partial \phi}{\partial \zeta}(b,t) + \phi(b,t)$, and $\frac{\partial \phi}{\partial \zeta}(b,t) = -Q\frac{\partial \phi}{\partial t}(b,t)$, $Q \geq 0$.

Reformulate these systems as boundary control systems.

3.5. We consider again the transmission line of Exercise 3.2 with boundary conditions a). Define an output to the system such that

$$\frac{d}{dt}\|x(t)\|_X^2 = u(t)^T y(t).$$

3.6. In the formulation of port-Hamiltonian systems as boundary control systems, we have the possibility that some boundary conditions are set to zero, see (3.19). However, when we add an output, this possibility was excluded, see (3.32)–(3.34). Furthermore, we assumed that we had n outputs. In this exercise we show that this did not pose a restriction to the theory.

a) Show that if $W_{B,2} = 0$, i.e., $W_B = W_{B,1}$, and $W_B \Sigma W_B^T \geq 0$, then (3.17) with control (3.18) is a well-defined boundary control system. Given the expression for the domain of the infinitesimal generator A.

b) Let W_B be a $n \times 2n$ matrix of full rank satisfying $W_B \Sigma W_B^T \geq 0$. We decompose $u = W_B \left(\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right)$ as

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} W_{B,1} \\ W_{B,2} \end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}.$$

By using the previous item, show that putting $u_2 = 0$ is allowed. Furthermore, show that it leads to the same boundary control system as in (3.17)–(3.19).

Consider the following system

$$\begin{aligned}
\frac{\partial x}{\partial t}(\zeta, t) &= P_1 \frac{\partial}{\partial \zeta}[\mathcal{H}x(t)] + P_0[\mathcal{H}x(t)] \\
u(t) &= W_{B,1} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \\
0 &= W_{B,2} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \\
y(t) &= W_C \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}
\end{aligned}$$

where we assume that $W_B = \left(\begin{smallmatrix} W_{B,1} \\ W_{B,2} \end{smallmatrix}\right)$ is a full rank matrix of size $n \times 2n$ satisfying $W_B \Sigma W_B^T \geq 0$. Furthermore, W_C is a full-rank matrix of size $k \times 2n$ satisfying

$$\operatorname{rank} \begin{pmatrix} W_B \\ W_C \end{pmatrix} = n + k.$$

c) Explain why the above rank conditions are logical.

d) Show that the above system is well-defined. In particular show that for smooth inputs and initial conditions, we have a unique solution, with the state and output trajectory continuous.

3.7. Consider the vibrating string of Exercise 3.3 with the boundary conditions

$$\frac{\partial x}{\partial \zeta}(0, t) = 0 \quad \text{and} \quad \frac{\partial x}{\partial \zeta}(1, t) = u(t) \quad t \geq 0.$$

and we measure the velocity at $\zeta = 0$.

a) Prove that this is a well-defined input-output system.
Hint: Use the previous exercise.

b) Does a balance equation like (3.36) hold?

c) Repeat the above two questions for the measurement $y(t) = \frac{\partial w}{\partial t}(1, t)$.

3.8. Consider the coupled strings of Exercise 2.11. Now we apply a force $u(t)$, to the bar in the middle, see Figure 3.1. This implies that the force balance in the middle becomes

$$T_{\mathrm{I}}(b) \frac{\partial w_{\mathrm{I}}}{\partial \zeta}(b) = T_{\mathrm{II}}(a) \frac{\partial w_{\mathrm{II}}}{\partial \zeta}(a) + T_{\mathrm{III}}(a) \frac{\partial w_{\mathrm{III}}}{\partial \zeta}(a) + u(t).$$

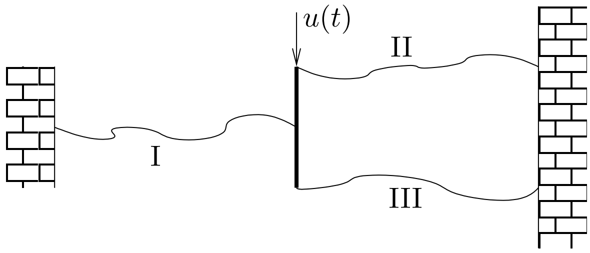

Figure 3.1.: Coupled vibrating strings with external force

a) Formulate the coupled vibrating strings with external force as a boundary control system.

b) Additionally, we measure the velocity of the bar in the middle. Reformate the system with this output as (3.32)–(3.34).

c) For the input and output defined above, determine the power balance in terms of the input and output, see (3.36).

Chapter 4
Transfer Functions

In this chapter we discuss the concept of transfer functions. Let us first recapitulate the concept for finite-dimensional systems. Consider the ordinary differential equation

$$\ddot{y}(t) + 3\dot{y}(t) - 7y(t) = -\dot{u}(t) + 2u(t), \tag{4.1}$$

where the dot denotes the derivative with respect to time. In many textbooks one derives the transfer function by taking the Laplace transform of this differential equation under the assumption that the initial conditions are zero. Since the following rules hold for the Laplace transform

$$\begin{aligned} \dot{f}(t) &\to F(s) - f(0) \\ \ddot{f}(t) &\to s^2F(s) - sf(0) - \dot{f}(0), \end{aligned}$$

we have that after Laplace transformation the differential equation becomes the algebraic equation:

$$s^2Y(s) + 3sY(s) - 7Y(s) = -sU(s) + 2U(s). \tag{4.2}$$

This implies that

$$Y(s) = \frac{-s+2}{s^2+3s-7}U(s). \tag{4.3}$$

The rational function in front of $U(s)$ is called the transfer function associated with the differential equation (4.1).

This is a standard technique, but there are some difficulties with it if we want to extend it to partial differential equations. One of the difficulties is that one has to assume that u and y are Laplace transformable. Since u is chosen, this is not a strong assumption, but once u is chosen, y is dictated by the differential equation, and it is not known a priory whether it is Laplace transformable. Furthermore, the Laplace transform only exists in some right half-plane of the complex plane. This implies that we have the equality (4.3) for those s in the right-half plane for which the Laplace transform of u and y both exist. The right-half plane in which the Laplace transform exists is named the region of convergence. Even for the simple differential equation (4.1) equality (4.3) does not hold everywhere. Taking into account the region of convergence of both u and y, we find that (4.3) only holds for those s which lies right of the poles, i.e., the zeros of $s^2 + 3s - 7$.

To overcome all these difficulties we define the transfer function in a different way. We shall look for solutions of the differential equation which are exponentials. Let us illustrate this for the simple differential equation of (4.1). Given $s \in \mathbb{C}$, we look for a solution pair of the form $(u(t), y(t)) = (e^{st}, y_s e^{st})$. If for an s such a solution exists, and it is unique, then we call y_s the transfer function of (4.1) in the point s. Substituting this pair into our differential equation, we have

$$s^2 y_s e^{st} + 3s y_s e^{st} - y_s e^{st} = -s e^{st} + 2e^{st}. \tag{4.4}$$

We recognize the common term e^{st} which is never zero, and so we may divide by it. After this division, we obtain

$$s^2 y_s + 3s y_s - y_s = -s + 2. \tag{4.5}$$

This is uniquely solvable for y_s if and only if $s^2 + 3s - 7 \neq 0$.

We see that we have obtained, without running into mathematical difficulties, the transfer function. Since for p.d.e.'s or for abstract differential equations the concept of a solution is well-defined, we may define transfer function via exponential functions.

4.1. Basic definition and properties

In this section we start with a very general definition of a transfer function, which even applies to systems not described by a p.d.e, but via e.g. a difference differential equation or an integral equation. To formulate this definition, we first have to introduce a general system. In a general system, we have a time axis, $\mathbb{T}$, which is assumed to be a subset of $\mathbb{R}$. Furthermore, we distinguish three spaces, U, Y, and R. U and Y are the input- and output space, respectively, whereas R contains the rest of the variables. In our examples, R will become the state space. A *system* $\mathfrak{S}$ is a subset of $(U \times R \times Y)^{\mathbb{T}}$, i.e., a subset of all functions from $\mathbb{T}$ to $U \times R \times Y$.

Definition 4.1.1. Let $\mathfrak{S}$ be a system, let s be an element of $\mathbb{C}$, and let $u_0 \in U$. We say that $(u_0 e^{st}, r(t), y(t))$ is an *exponential solution* in $\mathfrak{S}$ if there exist $r_0 \in R$, $y_0 \in Y$, such that $(u_0 e^{st}, r_0 e^{st}, y_0 e^{st}) = (u_0 e^{st}, r(t), y(t))$, $t \in \mathbb{T}$.

If for every $u_0 \in U$ the output trajectory, $y_0 e^{st}$, corresponding to an exponential solution is unique, then we call the mapping $u_0 \mapsto y_0$ the *transfer function at s*. We denote this mapping by $G(s)$. Let $\Omega \subset \mathbb{C}$ be the set consisting of all s for which the transfer function at s exists. The mapping $s \in \Omega \mapsto G(s)$ is defined as the *transfer function* of the system $\mathfrak{S}$. ♣

Recall that a system is said to be *linear* if U, R, and Y are linear spaces, and if $(\alpha u_1 + \beta u_2, \alpha r_1 + \beta r_2, \alpha y_1 + \beta y_2) \in \mathfrak{S}$ whenever (u_1, r_1, y_1) and (u_2, r_2, y_2) are in $\mathfrak{S}$. The system is *time-invariant*, when $\mathbb{T}$ is an interval of the form (t_0, ∞), $t_0 \geq -\infty$, and $(u(\cdot + \tau), r(\cdot + \tau), y(\cdot + \tau))$ is in $\mathfrak{S}$ for all $\tau > 0$, whenever $(u, r, y) \in \mathfrak{S}$.

Lemma 4.1.2. *Consider the abstract system $\mathfrak{S}$ which is linear and time-invariant on the time axis $[0,\infty)$. If $\mathfrak{S}$ is such that $(0,0,y(t)) \in \mathfrak{S}$ implies that $y(t)$ is the zero function, then exponential u and r implies exponential y, i.e., if $(u_0e^{st}, r_0e^{st}, y(t)) \in \mathfrak{S}$, then $y(t) = y_0e^{st}$ for some $y_0 \in Y$.*

Additionally, assume that for a given $s \in \mathbb{C}$ there exists an exponential solution for all $u_0 \in U$, and assume that $(0, r_0e^{st}, y(t)) \in \mathfrak{S}$ implies that $r_0 = 0$. Then the transfer function at s exists, and is a linear mapping.

PROOF: Let $(u_0e^{st}r_0e^{st}, y(t))$ be an element of $\mathfrak{S}$, and let $\tau \geq 0$. Combining the linearity and time-invariance, we see that

$$(0,0,y(t+\tau) - e^{s\tau}y(t)) = (u_0e^{s(t+\tau)}, r_0e^{s(t+\tau)}, y(t+\tau)) - (u_0e^{s\tau}e^{st}, r_0e^{s\tau}e^{st}, e^{s\tau}y(t))$$

is an element of $\mathfrak{S}$. By assumption, this implies that $y(t+\tau) = e^{s\tau}y(t)$ for all τ. Choosing $t=0$, gives the desired result.

Now assume that for a given $s \in \mathbb{C}$ there exists an exponential solution for all $u_0 \in U$. First we show that the exponential solution is unique. When $(u_0e^{st}, r_0e^{st}, y_0e^{st})$ and $(u_0e^{st}, \tilde{r}_0e^{st}, \tilde{y}_0e^{st})$ are both in $\mathfrak{S}$, then by the linearity $(0, (r_0 - \tilde{r}_0)e^{st}, (y_0 - \tilde{y}_0)e^{st}) \in \mathfrak{S}$. By our assumption this implies that $r_0 = \tilde{r}_0$ and $y_0 = \tilde{y}_0$.

From this we see that we can define a mapping $u_0 \mapsto y_0$. It remains to show that this mapping is linear. Let $(u_{10}e^{st}, r_{10}e^{st}, y_{10}e^{st})$ and $(u_{20}e^{st}, r_{20}e^{st}, y_{20}e^{st})$ be two exponential solutions. By the linearity, it is easy to see that

$$((\alpha u_{10} + \beta u_{20})e^{st}, (\alpha r_{10} + \beta r_{20})e^{st}, (\alpha y_{10} + \beta y_{20})e^{st}) \in \mathfrak{S}$$

Hence this implies that $\alpha u_{10} + \beta u_{20}$ is mapped to $\alpha y_{10} + \beta y_{20}$. In other words, the mapping is linear. ■

In our applications r will be the state x. It turns out that for the class of systems we are considering, the conditions in the above lemma are very weak, see Exercise 4.1. Thus the transfer function exists, and is a linear operator.

We begin by showing that the transfer function for the system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4.6}$$

$$y(t) = Cx(t) + Du(t), \tag{4.7}$$

where B, C and D are bounded operators and A generates a strongly continuous semigroup, is given by the familiar formula $G(s) = C(sI - A)^{-1}B + D$, for s in the resolvent set, $\rho(A)$, of A.

Theorem 4.1.3. *Consider the state linear system (4.6)–(4.7), with B, C, and D bounded operators. As solutions of this system we take the mild solutions, see Theorem 3.4.1.*

If $(u(t), x(t), y(t))$ is an exponential solution, then it is a classical solution as well. Furthermore, for $s \in \rho(A)$, the transfer function exists and is given by

$$G(s) = C(sI - A)^{-1}B + D. \tag{4.8}$$

PROOF: The mild solution of (4.6) is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)Bu(\tau)d\tau \tag{4.9}$$

For an exponential solution this becomes

$$x_0 e^{st} = T(t)x_0 + \int_0^t T(t-\tau)Bu_0 e^{s\tau}d\tau. \tag{4.10}$$

By taking Laplace transforms, this equality becomes

$$\frac{x_0}{\lambda - s} = (\lambda I - A)^{-1}x_0 + (\lambda I - A)^{-1}Bu_0\frac{1}{\lambda - s} \tag{4.11}$$

for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \max\{\mathrm{Re}(s), \omega_0\}$, where ω_0 is the growth bound of $(T(t))_{t\geq 0}$.

Since for fixed λ the right-hand side of (4.11) is an element of $D(A)$, the same holds for the left-hand side. In other words, $x_0 \in D(A)$. By Theorem 3.1.2, we conclude that $x(t)$ is a classical solution.

Equation (4.11) can be rewritten as

$$x_0 - (\lambda - s)(\lambda I - A)^{-1}x_0 = (\lambda I - A)^{-1}Bu_0. \tag{4.12}$$

Multiplying this by $(\lambda I - A)$, we obtain $(sI - A)x_0 = Bu_0$. Thus for $s \in \rho(A)$,

$$x_0 = (sI - A)^{-1}Bu_0. \tag{4.13}$$

Using the output equation of our system, we have that

$$y_0 e^{st} = y(t) = Cx(t) + Du(t) = Cx_0 e^{st} + Du_0 e^{st} = C(sI - A)^{-1}Bu_0 e^{st} + Du_0 e^{st}.$$

Hence $y_0 = C(sI - A)^{-1}Bu_0 + Du_0$ which proves (4.8). ■

The above theorem shows that for state-space systems of the form (4.6)–(4.7) the transfer function exists, and is given by the formula well-known from finite-dimensional system theory. Our main class of systems, the port-Hamiltonian systems have their control and observation at the boundary, and are not of the form (4.6)–(4.7). As proved in Theorem 3.3.1, they form a subclass of the boundary control systems. In the following theorem, we formulate transfer functions for boundary control systems. As in the previous theorem, exponential solutions are always classical solutions.

Theorem 4.1.4. *Consider the system*

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0 \\ u(t) &= \mathfrak{B}x(t) \\ y(t) &= \mathfrak{C}x(t) \end{aligned} \tag{4.14}$$

where $(\mathfrak{A}, \mathfrak{B})$ satisfies the conditions for boundary control system, see Definition 3.2.1 and $\mathfrak{C}$ is a bounded linear operator from $D(\mathfrak{A})$ to Y, with Y a Hilbert space.

As the class of solutions we take the mild solutions, see equation (3.14).

If $(u(t), x(t), y(t))$ is an exponential solution, then it is a classical solution as well. Furthermore, for $s \in \rho(A)$, the transfer function exists and is given by

$$G(s) = \mathfrak{C}(sI - A)^{-1} [\mathfrak{A}B - sB] + \mathfrak{C}B \tag{4.15}$$

for $s \in \rho(A)$.

For $s \in \rho(A)$ and $u_0 \in U$, $G(s)$ can also be found as the (unique) solution of

$$\begin{aligned} sx_0 &= \mathfrak{A}x_0 \\ u_0 &= \mathfrak{B}x_0 \\ G(s)u_0 &= \mathfrak{C}x_0, \end{aligned} \tag{4.16}$$

with $x_0 \in D(\mathfrak{A})$.

PROOF: The proof is very similar to that of Theorem 4.1.3. By (3.14) we know that the mild solution is given by

$$x(t) = T(t)(x_0 - Bu(0)) + \int_0^t T(t-\tau) [\mathfrak{A}Bu(\tau) - B\dot{u}(\tau)] \, d\tau + Bu(t).$$

Assuming that $(u(t), x(t), y(t))$ is an exponential solution, the above equation becomes

$$e^{st}x_0 = T(t)(x_0 - Bu_0) + \int_0^t T(t-\tau) [\mathfrak{A}Be^{s\tau}u_0 - Bse^{s\tau}u_0] \, d\tau + Be^{st}u_0.$$

Taking Laplace transforms, we find for $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > \max\{\operatorname{Re}(s), \omega_0\}$, where ω_0 is the growth bound of the semigroup $(T(t))_{t\geq 0}$,

$$\frac{x_0}{\lambda - s} = (\lambda I - A)^{-1}(x_0 - Bu_0) + (\lambda I - A)^{-1} \left[\mathfrak{A}B\frac{u_0}{\lambda - s} - B\frac{su_0}{\lambda - s}\right] + B\frac{u_0}{\lambda - s}.$$

Or equivalently,

$$\frac{x_0 - Bu_0}{\lambda - s} = (\lambda I - A)^{-1}(x_0 - Bu_0) + (\lambda I - A)^{-1} \left[\mathfrak{A}B\frac{u_0}{\lambda - s} - B\frac{su_0}{\lambda - s}\right].$$

This implies that $x_0 - Bu_0 \in D(A)$, and

$$(\lambda I - A)[x_0 - Bu_0] = [x_0 - Bu_0](\lambda - s) + \mathfrak{A}Bu_0 - Bsu_0.$$

Subtracting the term $\lambda[x_0 - Bu_0]$ from both sides, this becomes

$$(sI - A)[x_0 - Bu_0] = \mathfrak{A}Bu_0 - Bsu_0. \tag{4.17}$$

For $s \in \rho(A)$, this gives

$$x_0 = (sI - A)^{-1} [\mathfrak{A}B - sB] \, u_0 + Bu_0. \tag{4.18}$$

Since $x_0 - Bu_0 \in D(A)$ and since $u \in C^2([0,\infty); U)$ we have by Theorem 3.2.2 that $x(t)$ is a classical solution of (4.14). In particular, for all $t \geq 0$, $x(t) \in D(\mathfrak{A})$. By assumption the domain of $\mathfrak{C}$ contains the domain of $\mathfrak{A}$. Hence $y_0 e^{st} = y(t) = \mathfrak{C}x(t)$, holds point-wise in t. Taking $t = 0$, gives $y_0 = \mathfrak{C}x_0$. Combining this equality with (4.18), we obtain (4.15).

Since $x(t) = x_0 e^{st}$ is a classical solution for $u(t) = u_0 e^{st}$, we can substitute this in the differential equation (4.14). By doing so we find

$$\begin{aligned} s x_0 e^{st} &= \mathfrak{A} x_0 e^{st} \\ u_0 e^{st} &= \mathfrak{B} x_0 e^{st} \\ y_0 e^{st} &= \mathfrak{C} x_0 e^{st} \end{aligned}$$

Removing the exponential term, we find equation (4.16). The uniqueness of x_0 follows by (4.18). ■

We close this section by calculating the transfer function for the simple Example 3.4.3.

Example 4.1.5 Consider the system

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial x}{\partial \zeta}(\zeta,t), && \zeta \in [0,1],\ t \geq 0 \\ x(\zeta,0) &= x_0(\zeta), && \zeta \in [0,1] \\ u(t) &= x(1,t), && t \geq 0. \\ y(t) &= x(0,t), && t \geq 0. \end{aligned} \tag{4.19}$$

If we define $\mathfrak{C}x = x(0)$, then it is easy to see that all assumptions in Theorem 4.1.4 are satisfied, see Theorem 3.3.1. Hence we can calculate the transfer function, we do this via the equation (4.16). For the system (4.19) this becomes

$$\begin{aligned} s x_0(\zeta) &= \frac{\partial x_0}{\partial \zeta}(\zeta) = \frac{d x_0}{d\zeta}(\zeta) \\ u_0 &= x_0(1) \\ G(s)u_0 &= x_0(0). \end{aligned}$$

The above differential equation has the solution $x_0(\zeta) = \alpha e^{s\zeta}$. Using the other two equations, we see that $G(s) = e^{-s}$. □

4.2. Transfer functions for port-Hamiltonian systems

In this section we apply the results found in the previous section to our class of port-Hamiltonian systems. Since we have obtained Theorem 4.1.4 describing transfer functions for general boundary control system, the application to port-Hamiltonian system is straightforward. We obtain the transfer function for the system defined in (3.32)–(3.34). That is, the system is given by

$$\begin{aligned} \dot{x}(t) &= P_1 \frac{\partial}{\partial \zeta}[\mathcal{H}x(t)] + P_0[\mathcal{H}x(t)] && (4.20)\\ u(t) &= W_B \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} && (4.21)\\ y(t) &= W_C \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}. && (4.22) \end{aligned}$$

It is assumed that (4.20)–(4.21) satisfy the assumption of a port-Hamiltonian system, see Section 3.3. Further we assume that $W_B \Sigma W_B^T \geq 0$, where $\Sigma = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. This guarantees that (4.20)–(4.21) is a boundary control system, see Theorem 3.3.1.

As for the input we assume that we use full measurements, i.e., W_C be a full rank matrix of size $n \times 2n$ and $\left(\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right)$ is of full rank, or equivalently this matrix is invertible.

Theorem 4.2.1. *Consider the system (4.20)–(4.22). This system has the transfer function $G(s)$, which is determined by*

$$\begin{aligned} s x_0 &= P_1 \frac{d}{d\zeta}[\mathcal{H}x_0] + P_0[\mathcal{H}x_0] && (4.23)\\ u_0 &= W_B \begin{pmatrix} f_{\partial,0} \\ e_{\partial,0} \end{pmatrix} && (4.24)\\ G(s)u_0 &= W_C \begin{pmatrix} f_{\partial,0} \\ e_{\partial,0} \end{pmatrix}, && (4.25) \end{aligned}$$

where

$$\begin{pmatrix} f_{\partial,0} \\ e_{\partial,0} \end{pmatrix} = R_0 \begin{pmatrix} (\mathcal{H}x_0)(b) \\ (\mathcal{H}x_0)(a) \end{pmatrix}. \tag{4.26}$$

Furthermore, the transfer function satisfies the following equality

$$Re\,(s)\|x_0\|^2 = \frac{1}{2} \begin{pmatrix} u_0^T & u_0^T G(s)^* \end{pmatrix} P_{W_B,W_C} \begin{pmatrix} u_0 \\ G(s)u_0 \end{pmatrix}. \tag{4.27}$$

with P_{W_B,W_C} the inverse of $\left(\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right) \Sigma \left(\begin{smallmatrix} W_B^T & W_C^T \end{smallmatrix}\right)$.

Proof: The proof is a direct combination of Theorems 3.4.2 and 4.1.4. By the first theorem, we know that the system (4.20)–(4.22) is a well-defined boundary control system and that the output equation is well-defined in the domain of the system operator $\mathfrak{A}$. Hence all conditions of Theorem 4.1.4 are satisfied, and the defining relation for the transfer function, equation (4.15), becomes (4.23)–(4.25).

The transfer function is by definition related to the exponential solution $(u_0 e^{st}, x_0 e^{st}, G(s)u_0 e^{st})$. Substituting this solution in (3.36) gives (4.27). ■

Looking at (4.23)–(4.25) we see that the calculation of the transfer function is equivalent to solving an ordinary differential equation. If $\mathcal{H}$ is constant, i.e., independent of ζ, this is easy. However, in general it can be very hard to solve this ordinary differential equation by hand, see Exercise 4.4.

In the above theorem we assumed that we had n controls and n measurements. However, this needs not to hold. If the system has some of its boundary conditions set to zero, and/or less than n measurements, then one can take two approaches for obtaining the transfer function. As is shown in Exercise 3.6 this system satisfies all the conditions of Theorem 4.1.4, and hence one can use that theorem to obtain the differential equation determining the transfer function. Another approach is to regard the zero boundary conditions as inputs set to zero, and to add extra measurements such that we have n controls and n measurements. The transfer function one is looking for is now a sub-block of the $n \times n$ transfer function. We explain this in more detail by means of an example. However, before we do this we remark that (4.27) equals the balance equation (1.27) which lies at the hart of our class of systems.

Example 4.2.2 Consider the transmission line of Example 1.1.1 for which we assume that we control the voltages at both ends, and measure the currents at the same points. Furthermore, we assume that the spatial interval equals $[0, 1]$. Hence the model becomes

$$\begin{aligned} \frac{\partial Q}{\partial t}(\zeta, t) &= -\frac{\partial}{\partial \zeta} \frac{\phi(\zeta, t)}{L(\zeta)} && (4.28)\\ \frac{\partial \phi}{\partial t}(\zeta, t) &= -\frac{\partial}{\partial \zeta} \frac{Q(\zeta, t)}{C(\zeta)} \\ u(t) &= \begin{pmatrix} \frac{Q(1,t)}{C(1)} \\ \frac{Q(0,t)}{C(0)} \end{pmatrix} && (4.29)\\ y(t) &= \begin{pmatrix} \frac{\phi(1,t)}{L(1)} \\ \frac{\phi(0,t)}{L(0)} \end{pmatrix}. && (4.30) \end{aligned}$$

For the transfer function, this p.d.e. is replaced by the ordinary differential equation

$$\begin{aligned} sQ_0(\zeta) &= -\frac{d}{d\zeta} \frac{\phi_0(\zeta)}{L(\zeta)} && (4.31)\\ s\phi_0(\zeta) &= -\frac{d}{d\zeta} \frac{Q_0(\zeta)}{C(\zeta)} \\ u_0 &= \begin{pmatrix} u_{10} \\ u_{20} \end{pmatrix} = \begin{pmatrix} \frac{Q_0(1)}{C(1)} \\ \frac{Q_0(0)}{C(0)} \end{pmatrix} && (4.32)\\ y_0 &= \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} = \begin{pmatrix} \frac{\phi_0(1)}{L(1)} \\ \frac{\phi_0(0)}{L(0)} \end{pmatrix}. && (4.33) \end{aligned}$$

Since we want to illustrate transfer functions, and their properties, we make the simplifying assumption that $C(\zeta) = L(\zeta) = 1$ for all $\zeta \in [0, 1]$. With this assumption, it is easy to see that the solution of (4.31) is given by

$$Q_0(\zeta) = \alpha e^{s\zeta} + \beta e^{-s\zeta}, \qquad \phi_0(\zeta) = -\alpha e^{s\zeta} + \beta e^{-s\zeta}, \tag{4.34}$$

where α, β are complex constants. Using (4.32) we can related these constant to u_0,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{e^s - e^{-s}} \begin{pmatrix} 1 & -e^{-s} \\ -1 & e^s \end{pmatrix} u_0. \tag{4.35}$$

Combining this with (4.33) gives

$$y_0 = \frac{1}{e^s - e^{-s}} \begin{pmatrix} -e^s - e^{-s} & 2 \\ -2 & e^s + e^{-s} \end{pmatrix} u_0.$$

Thus the transfer function is given by

$$G(s) = \begin{pmatrix} -\tanh(s) & -\frac{1}{\sinh(s)} \\ \frac{1}{\sinh(s)} & \tanh(s) \end{pmatrix}. \tag{4.36}$$

Using now the balance equation (1.4), we find

$$\begin{aligned} \mathrm{Re}\,(s)\|x_0\|^2 &= \mathrm{Re}\,(V_0(0)I_0(0)) - \mathrm{Re}\,(V_0(1)I_0(1)) \\ &= \mathrm{Re}\,(u_{20}G_{21}(s)u_{10} + u_{20}G_{22}(s)u_{20}) - \\ &\quad\ \mathrm{Re}\,(u_{10}G_{11}(s)u_{10} + u_{10}G_{12}(s)u_{20}). \end{aligned} \tag{4.37}$$

By taking $u_{10} = 0$, we conclude that the real part of G_{22} is positive for $\mathrm{Re}(s) > 0$. Combined with the fact that G_{22} is analytic for $s \in \mathbb{C}_0^+ := \{s \in \mathbb{C} \mid \mathrm{Re}\,(s) > 0\}$, we have that G_{22} is *positive real*. This can also be checked by direct calculation on $G_{22}(s) = \tanh(s)$.

Consider next the system defined by the p.d.e. (4.28) with input $u(t) = Q(1,t)$, output $I(1,t)$ and boundary condition $Q(0,t) = 0$. We can proceed like we did above, but we see that we have already obtained the transfer function by putting $u_{20} = 0$ in (4.32) and only look at y_{10} in (4.33). Hence the transfer function of this single input single output system is $-\tanh(s)$.

The transfer functions (4.36) and $-\tanh(s)$ have their poles on the imaginary axis, and so one cannot draw a *Bode* or *Nyquist plot*. In order to show these concept known from classical control theory can also be used for system described by p.d.e.'s we add a damping such that we obtain a system with no poles on the imaginary axis.

We consider the p.d.e. (4.28) with the following conditions

$$V(1,t) = RI(1,t) \tag{4.38}$$
$$u(t) = V(0,t) \tag{4.39}$$
$$y(t) = I(0,t). \tag{4.40}$$

Again we take the simplifying assumption that $C(\zeta) = L(\zeta) = 1$, $\zeta \in [0,1]$. Calculation the transfer function leads to the o.d.e. (4.31) with the boundary conditions

$$\begin{aligned} V_0(1) &= RI_0(1) && (4.41)\\ u_0 &= V_0(0) && (4.42)\\ y_0 &= I_0(0). && (4.43) \end{aligned}$$

The o.d.e. has as solution (4.34). The equations (4.41)–(4.43) imply

$$\begin{aligned} \alpha e^{s} + \beta e^{-s} &= R\left(-\alpha e^{s} + \beta e^{-s}\right)\\ u_0 &= \alpha + \beta\\ y_0 &= -\alpha + \beta. \end{aligned}$$

Solving this equation gives the following transfer function,

$$G(s) = \frac{\cosh(s) + R\sinh(s)}{\sinh(s) + R\cosh(s)}. \tag{4.44}$$

The Nyquist plot of this is a perfect circle, see Figure 4.1 Again using the balance

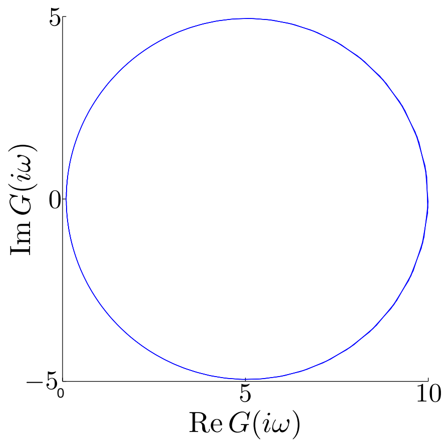

Figure 4.1.: Nyquist plot of (4.44) for $R = 10$

equation (1.4), we find that for this system

$$\begin{aligned} \mathrm{Re}\,(s)\|x_0\|^2 &= \mathrm{Re}\,(V_0(0)I_0(0)) - \mathrm{Re}\,(V_0(1)I_0(1))\\ &= \mathrm{Re}\,(u_0G(s)u_0) - \mathrm{Re}\,(I_0(1)RI_0(1)). \end{aligned}$$

Hence for $\mathrm{Re}(s) > 0$, we have

$$\mathrm{Re}\,(G(s))u_0^2 = \mathrm{Re}\,(s)\|x_0\|^2 + RI_0(1)^2, \tag{4.45}$$

and so G is positive real. $\square$

4.3. Exercises

4.1. Show that the assumption of Lemma 4.1.2 holds for the system (4.6)–(4.7), whenever $s \in \rho(A)$.

4.2. Show that the assumption of Lemma 4.1.2 holds for the system (4.14), whenever $s \in \rho(A)$.

4.3. Determine the transfer function of the system

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial \lambda x}{\partial \zeta}(\zeta,t), && \zeta \in [a,b],\ t \geq 0 \\ x(\zeta,0) &= x_0(\zeta), && \zeta \in [0,1] \\ u(t) &= \lambda(b)x(b,t), && t \geq 0. \\ y(t) &= \lambda(a)x(a,t), && t \geq 0, \end{aligned} \tag{4.46}$$

where λ is a (strictly) positive continuous function not depending on t.

4.4. Consider the system of the transmission line given by (4.28)–(4.30).

a) Show that even when the physical parameter C and L are spatial dependent, the equality (4.37) still holds.

b) Choose $L(\zeta) = e^{\zeta}$, and $C(\zeta) = 1$, and determine the transfer function. Hint: You may use a computer package like Maple or Mathematica.

4.5. Our standard port-Hamiltonian system is defined on the spatial interval $[a,b]$. In Exercise 2.10 we have shown that it can easily be transformed to a port-Hamiltonian system on the spatial interval $[0,1]$. How does the transfer function change?

4.6. Show that the boundary control system (4.14) defines a linear and time-invariant system, see below Definition 4.1.1.

4.7. Show that the Nyquist plot of transfer function (4.44) is a circle. Furthermore, show that G restricted to the imaginary axis is periodic, and determine the period.

4.8. Consider the vibrating string of Example 1.1.2. We assume that the mass density and Young's modulus are constant. We control this system by controlling the velocity at $\zeta = b$ and the strain at $\zeta = a$, i.e., $u(t) = \begin{pmatrix} \frac{\partial w}{\partial t}(t,b) \\ \frac{\partial w}{\partial \zeta}(t,a) \end{pmatrix}$. We observe the same quantities, but at the opposite ends, i.e., $y(t) = \begin{pmatrix} \frac{\partial w}{\partial t}(t,a) \\ \frac{\partial w}{\partial \zeta}(t,b) \end{pmatrix}$.

Determine the transfer function of this system.

4.9. Since we have defined transfer functions via a different way, it may be good to check some well-known property of it. Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be two linear and time-invariant systems, with input-output pair u_1, y_1 and u_2, y_2, respectively. Assume that for a given $s \in \mathbb{C}$ both systems have a transfer function.

a) Show that the series connection, i.e., $u_2 = y_2$ has the transfer function $G(s) = G_2(s)G_1(s)$.

b) Show that the parallel connection, i.e., $u_1 = u_2 = u$, and $y = y_1 + y_2$ has the transfer function $G_1(s) + G_2(s)$.

c) Show that the feedback connection, i.e., $u_1 = u - y_2$, $y = y_1$ has the transfer function $G_1(s)\left[I + G_2(s)\right]^{-1}$ provided $I + G_2(s)$ is invertible.

4.10. Consider the coupled strings of Exercise 3.8. As input we apply a force to the bar in the middle, and as output we measure the velocity of this bar. Assuming that all physical parameters are not depending on ζ, determine the transfer function.

4.4. Notes and references

The ideas for defining the transfer function in the way we did is old, but has hardly been investigated for distributed parameter system. [32] was the first paper where this approach has been used for infinite-dimensional systems. In that paper the concept we named transfer function was called a characteristic function.

One may find the exponential solution in Polderman and Willems [20], where all solutions of this type are called the exponential behavior.

Chapter 5
Well-posedness

5.1. Introduction

Consider the abstract linear differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \tag{5.1}$$

$$y(t) = Cx(t) + Du(t). \tag{5.2}$$

where x is assumed to take values in the Hilbert space X, u in the Hilbert space U, and y is the Hilbert space Y. The operator A is assumed to be the infinitesimal generator of a C_0-semigroup, and B, C, and D are bounded linear operators. Under these assumptions we know that the abstract differential equation (5.1)–(5.2) possesses for every $u \in L^2((0,t_f);U)$ a unique (mild) solution, see Theorem 3.4.1. Existence of solutions for an arbitrary initial condition $x_0 \in X$ and input $u \in L^2((0,t_f);U)$, such that x is continuous and $y \in L^2((0,t_f);Y)$ will be called *well-posedness*. Hence if B, C, and D are bounded linear operators, then the system (5.1)–(5.2) is well-posed if and only if A is the infinitesimal generator of a C_0-semigroup.

As we have seen in Chapter 3 our class of port-Hamiltonian systems cannot be written in the format (5.1)–(5.2) with B, C and D bounded. However, we know that for every initial condition and every smooth input function we have a mild solution of the state differential equation, see (3.14). Furthermore, for smooth initial conditions and smooth inputs, the output equation is well-defined, see Theorem 3.4.2. Note that this was only obtained under the condition $W_B \Sigma W_B^T \geq 0$. This inequality is equivalent to the fact that we have a contraction semigroup, see Theorem 2.3.2. That a larger class of inputs might be possible, can be seen in the following example.

Consider the controlled transport equation on the interval $[0,1]$ with scalar control and observation on the boundary

$$\frac{\partial x}{\partial t}(\zeta,t) = \frac{\partial x}{\partial \zeta}(\zeta,t), \quad x(\zeta,0) = x_0(\zeta), \qquad \zeta \in [0,1] \tag{5.3}$$

$$u(t) = x(1,t), \tag{5.4}$$

$$y(t) = x(0,t). \tag{5.5}$$

From Example 3.2.3 we know that the mild solution of (5.3)–(5.4) is given by

$$x(\zeta,t) = \begin{cases} x_0(\zeta+t) & \zeta+t \le 1 \\ u(\zeta+t-1) & \zeta+t > 1. \end{cases} \tag{5.6}$$

We see that for every $t \ge 0$ the function $x(\cdot,t) \in X = L^2(0,1)$, whenever $u \in L^2(0,t_f)$. Furthermore, $x(\cdot,t)$ is a continuous function in t, i.e., $\|x(\cdot,t) - x((\cdot,t+h)$ converges to zero when h converges to zero, see Exercise 5.1. Hence the mild solution (5.6) can be extended from controls in $H^1(0,t_f)$ to $L^2(0,t_f)$. If x_0 and u are smooth, then we clearly see that $y(t)$ is well-defined for every $t \ge 0$ and it is given by

$$y(t) = \begin{cases} x_0(t) & 0 \le t \le 1 \\ u(t-1) & t > 1. \end{cases} \tag{5.7}$$

However, when $x_0 \in L^2(0,1)$ and $u \in L^2(0,t_f)$, the expression (5.7) still gives that y is well-defined as an L^2-function.

Summarizing, we see that we can define a (mild) solution for (5.3)–(5.5) for all $x_0 \in X$ and all $u \in L^2(0,t_f)$. This solution gives a state trajectory in the state space which is continuous, and an output trajectory which is square integrable on every compact time interval. Hence this system is well-posed.

Suppose next that we are applying a feedback of the form $u(t) = 2y(t)$, this leads to the new p.d.e.

$$\frac{\partial x}{\partial t}(\zeta,t) = \frac{\partial x}{\partial \zeta}(\zeta,t), \quad x(\zeta,0) = x_0(\zeta), \qquad \zeta \in [0,1] \tag{5.8}$$

$$x(1,t) = 2x(0,t). \tag{5.9}$$

We would like to know whether the system has a unique (mild) solution, and so we try to apply Theorem 2.3.2. From Example 2.3.3 we have that the boundary effort and flow are given by

$$f_\partial = \frac{1}{\sqrt{2}}[x(1)+x(0)], \qquad e_\partial = \frac{1}{\sqrt{2}}[x(1)-x(0)].$$

Hence the boundary condition (5.9) can be written as $f_\partial - 3e_\partial = 0$, i.e., $W_B = (1,-3)$. Calculation $W_B \Sigma W_B^T$ gives -6. Since this is negative, we know by Theorem 2.3.2) that the operator associated to the p.d.e. (5.8)–(5.9) does not generate a contraction semigroup. However, it does generate a C_0-semigroup. It is not hard to see that the solution of (5.8)–(5.9) is given by

$$x(\zeta,t) = 2^{n+1} x_0(\tau), \tag{5.10}$$

where $\zeta + t = n + \tau$, $\tau \in [0,1]$, $n \in \mathbb{N}$, see Exercise 5.2.

Concluding, we see from this very simple port-Hamiltonian system that the class of controls can be extended, and that it is possible to have a semigroup, even when the condition $W_B \Sigma W_B^T \ge 0$ is not satisfied. The aim of this chapter is to show that these results hold for any port-Hamiltonian system in our class.

5.2. Well-posedness for port-Hamiltonian systems

In this section we define formally what we mean by well-posedness. Although well-posedness can be defined very generally, see the Notes and reference section, we do it for our class of systems introduced in Sections 3.3 and 3.4.

$$\frac{\partial x}{\partial t}(\zeta,t) \quad = \quad P_1\frac{\partial}{\partial \zeta}[\mathcal{H}x(t)] + P_0[\mathcal{H}x(t)] \tag{5.11}$$

$$u(t) \quad = \quad W_{B,1}\begin{pmatrix} f_\partial(t) \\ e_\partial(t)\end{pmatrix} \tag{5.12}$$

$$0 \quad = \quad W_{B,2}\begin{pmatrix} f_\partial(t) \\ e_\partial(t)\end{pmatrix} \tag{5.13}$$

$$y(t) \quad = \quad W_{C}\begin{pmatrix} f_\partial(t) \\ e_\partial(t)\end{pmatrix}. \tag{5.14}$$

The precise assumptions on this system will be given in Theorem 5.2.6. However, as before, we assume that $\mathcal{H}$ is a symmetric $n\times n$ matrix satisfying $0 < mI \le \mathcal{H}(\zeta) \le MI$. Furthermore, $W_B := \left(\begin{smallmatrix} W_{B,1} \\ W_{B,2}\end{smallmatrix}\right)$ is a full rank matrix of size $n \times 2n$. We assume that $W_{B,1}$ is a $k \times 2n$ matrix. Our state space is the Hilbert space $X = L^2((a,b);\mathbb{R}^n)$ with the inner product

$$\langle f,g\rangle_X = \frac{1}{2}\int_a^b f(\zeta)^T\mathcal{H}(\zeta)g(\zeta)d\zeta. \tag{5.15}$$

The following lemma follows easily from Theorems 3.3.1 and 3.4.2. The proof is left as an exercise, see Exercise 5.3.

Lemma 5.2.1. *Let t_f be a positive real number. Assume that the operator A defined as $P_1\frac{\partial}{\partial\zeta}\mathcal{H} + P_0\mathcal{H}$ with domain,*

$$D(A) = \{x_0 \in X \mid \mathcal{H}x_0 \in H^1((a,b);\mathbb{R}^n), W_B\left(\begin{smallmatrix} f_{\partial,0} \\ e_{\partial,0}\end{smallmatrix}\right) = 0\}$$

is the infinitesimal generator of a C_0-semigroup on X. Then the system (5.11)–(5.13) is a boundary control system.

In particular, for every $\mathcal{H}x_0 \in H^1((a,b);\mathbb{R}^n)$ and every $u(\cdot) \in C^2([0,t_f);\mathbb{R}^k)$ with $u(0) = W_{B,1}\left(\begin{smallmatrix} f_{\partial,0} \\ e_{\partial,0}\end{smallmatrix}\right)$, and $0 = W_{B,2}\left(\begin{smallmatrix} f_{\partial,0} \\ e_{\partial,0}\end{smallmatrix}\right)$, there exists a unique classical solution of (5.11)–(5.13) on $[0,t_f]$. Furthermore, the output (5.14) is well-defined and $y(\cdot)$ is continuous on $[0,t_f]$.

This lemma tells us that under the existence assumption of a semigroup, we have a solution for smooth inputs and initial conditions. The concept of well-posedness implies that we have solutions for every initial condition and every square integrable input.

Definition 5.2.2. Consider the system (5.11)–(5.14) and let k be the dimension of u. This system is *well-posed* if there exists a $t_f > 0$ and $m_f \ge 0$ such that the following holds:

1. The operator A defined as $P_1 \frac{\partial}{\partial \zeta}\mathcal{H} + P_0\mathcal{H}$ with domain,

$$D(A) = \{x_0 \in X \mid \mathcal{H}x_0 \in H^1((a,b);\mathbb{R}^n), W_B \left(\begin{smallmatrix} f_{\partial,0} \\ e_{\partial,0} \end{smallmatrix}\right) = 0\} \tag{5.16}$$

is the infinitesimal generator of a C_0-semigroup on X.

2. The following inequality holds for all $\mathcal{H}x_0 \in H^1((a,b);\mathbb{R}^n)$ and $u \in C^2([0,t_f);\mathbb{R}^k)$ with $u(0) = W_{B,1}\left(\begin{smallmatrix} f_{\partial,0} \\ e_{\partial,0} \end{smallmatrix}\right)$, and $0 = W_{B,2}\left(\begin{smallmatrix} f_{\partial,0} \\ e_{\partial,0} \end{smallmatrix}\right)$

$$\|x(t_f)\|_X^2 + \int_0^{t_f} \|y(t)\|^2 dt \leq m_f \left[\|x_0\|_X^2 + \int_0^{t_f} \|u(t)\|^2 dt\right]. \tag{5.17}$$

♣

If the system is well-posed, then we can define a (mild) solution of (5.11)–(5.14) for all $x_0 \in X$ and $u \in L^2((0,t_f),\mathbb{R}^k)$ such that $x(t)$ is a continuous function in X and y is square integrable.

Theorem 5.2.3. *If the system (5.11)–(5.14) is well-posed, then the mild solution (3.14) can be extended to hold for all $u \in L^2((0,t_f);\mathbb{R}^k)$. Furthermore, the state trajectory remains continuous, and the output is square integrable.*

For the (extended) solution, the inequality (5.17) holds.

Furthermore, if the system is well-posed for some $t_f > 0$, then it is well-posed for all $t_f > 0$.

Proof: See Section 5.6. ∎

As we have seen in Chapter 4, the system (5.11)–(5.14) has a transfer function. From Theorem 4.1.4 we conclude that this function exists in the resolvent set of A. Since A generates a C_0-semigroup, this resolvent set contains a right half-plane. So the transfer function exists on some right-half plane. Furthermore, the transfer function is bounded for $\mathrm{Re}(s) \to \infty$.

Lemma 5.2.4. *Let G be the transfer function of a well-posed system, then*

$$\limsup_{\mathrm{Re}(s)\to\infty} \|G(s)\| \leq \sqrt{m_f}, \tag{5.18}$$

where m_f is the constant from equation (5.17).

Proof: If (5.18) does not hold, then we can find an s such that $|e^{2st_f}| > m_f$ and $\|G(s)\| > \sqrt{m_f}$. Next choose $u_0 \in U$ such that $\|u_0\| = 1$ and $\|G(s)u_0\| = \|G(s)\|$. The exponential solution $(u_0e^{st}, x_0e^{st}, G(s)u_0e^{st})$ satisfies

$$\begin{aligned}
\|x(t_f)\|_X^2 + \int_0^{t_f} \|y(t)\|^2 dt &= \|x(t_f)\|_X^2 + \int_0^{t_f} \|G(s)u_0e^{st}\|^2 dt \\
&= |e^{2st_f}|\|x_0\|_X^2 + \|G(s)\|^2 \int_0^{t_f} \|e^{st}u_0\|^2 dt \\
&> m_f \left[\|x_0\|_X^2 + \int_0^{t_f} \|u(t)\|^2 dt\right].
\end{aligned}$$

This contradicts (5.17), and so (5.18) must hold. ∎

Although the transfer function is bounded on some right half-plane, this does not imply that it will converge along the real axis. If this happens, we call the transfer function regular.

Definition 5.2.5. Let $G(s)$ be the transfer function of (5.11)–(5.14). The system (5.11)–(5.14) is *regular* when $\lim_{s\in\mathbb{R}, s\to\infty} G(s)$ exists. If the system (5.11)–(5.14) is regular, then the *feed-through* term D is defined as $D = \lim_{s\in\mathbb{R}, s\to\infty} G(s)$. ♣

Now we have all the ingredients to formulate the main result of this chapter.

Theorem 5.2.6. *Consider the partial differential equation (5.11)–(5.14) on the spatial interval $[a, b]$, with $x(\zeta, t)$ taking values in $\mathbb{R}^n$. Let X be the Hilbert space $L^2((a, b); \mathbb{R}^n)$ with inner product (5.15). Furthermore, assume that*

- *P_1 is real-valued, invertible, and symmetric, i.e., $P_1^T = P_1$,*
- *$\mathcal{H}(\zeta)$ is a (real) symmetric matrix satisfying $0 < mI \le \mathcal{H}(\zeta) \le MI$, $\zeta \in [a, b]$.*
- *The multiplication operator $P_1\mathcal{H}$ can be written as*

$$P_1\mathcal{H}(\zeta) = S^{-1}(\zeta)\Delta(\zeta)S(\zeta), \tag{5.19}$$

  with $\Delta(\cdot)$ a diagonal multiplication operator, and both $\Delta(\cdot)$ and $S(\cdot)$ are continuously differentiable,

- *$W_B := \begin{pmatrix} W_{B,1} \\ W_{B,2} \end{pmatrix}$ is a $n \times 2n$ matrix with rank n,*
- $\operatorname{rank} \begin{pmatrix} W_{B,1} \\ W_{B,2} \\ W_C \end{pmatrix} = n + \operatorname{rank}(W_C)$.

If the homogeneous p.d.e., i.e., $u \equiv 0$, generates a C_0-semigroup on X, then the system (5.11)–(5.14) is well-posed, and the corresponding transfer function G is regular. Furthermore, we have that $\lim_{\mathrm{Re}(s)\to\infty} G(s) = \lim_{s\to\infty, s\in\mathbb{R}} G(s)$.

This theorem tells us that if A defined in Lemma 5.2.1 generates a C_0-semigroup, then the system is well-posed. In particular, we have a mild solution for all square integrable inputs. From Chapter 2 we know that if $W_B\Sigma W_B^T \ge 0$, then A generates a (contraction) semigroup, and so in this situation the system is well-posed.

In the coming sections we prove this result. Here we comment on the conditions.

- The first two conditions are very standard, and are assumed to be satisfied for all our port-Hamiltonian systems until now.
- Note that we do not have a condition on P_0. In fact the term $P_0\mathcal{H}$ may be replaced by any bounded operator on X, see Lemma 5.4.1.
- The third condition is not very strong, and will almost always be satisfied if $\mathcal{H}(\cdot)$ is continuously differentiable. Note that Δ contains the eigenvalues of $P_1\mathcal{H}$, whereas S^{-1} contains the eigenvectors.

- The fourth condition tells us that we have n boundary conditions, when we put the input to zero. This very logical, since we have an n'th order p.d.e.
- The last condition tells that we are not measuring quantities that are set to zero, or set to be an input. This condition is not important for the proof, and will normally follow from correct modeling.

As mentioned, the third condition is not very strong. We prove some properties of Δ next.

Lemma 5.2.7. *Let P_1 and $\mathcal{H}$ satisfy the conditions of Theorem 5.2.6. Then Δ can be written as*

$$\Delta(\zeta) = \begin{pmatrix} \Lambda(\zeta) & 0 \\ 0 & \Theta(\zeta) \end{pmatrix}, \tag{5.20}$$

where $\Lambda(\cdot)$ is a diagonal (real) matrix, with (strictly) positive functions on the diagonal, and $\Theta(\cdot)$ is a diagonal (real) matrix, with (strictly) negative functions on the diagonal.

Proof: Let $\zeta \in [a,b]$ be fixed. Since $\mathcal{H}(\zeta) > mI$, we can take the square root of it. By the law of inertia, we know that the inertia of $\mathcal{H}(\zeta)^{\frac{1}{2}} P_1 \mathcal{H}(\zeta)^{\frac{1}{2}}$ equals the inertia of P_1. This implies that the inertia of $\mathcal{H}(z)^{\frac{1}{2}} P_1 \mathcal{H}(\zeta)^{\frac{1}{2}}$ is independent of ζ. Furthermore, since P_1 is invertible, we conclude that the number of negative eigenvalues of $\mathcal{H}(z)^{\frac{1}{2}} P_1 \mathcal{H}(\zeta)^{\frac{1}{2}}$ equals the number of negative eigenvalues of P_1. A similar statement holds for the positive eigenvalues.

A simple calculation gives that the eigenvalues of $\mathcal{H}(\zeta)^{\frac{1}{2}} P_1 \mathcal{H}(\zeta)^{\frac{1}{2}}$ are equal to the eigenvalues of $P_1\mathcal{H}(\zeta)$. Concluding, we see that for all $\zeta \in [a,b]$ zero is not an eigenvalue of $P_1\mathcal{H}(\zeta)$, and that the number of negative and positive eigenvalues of $P_1\mathcal{H}(\zeta)$ is independent of ζ. We can regroup the eigenvalues such that first are positive. By doing so, we obtain (5.20). ∎

We illustrate the conditions in the theorem by proving that they are easily satisfied for the example of the wave equation. From Example 1.1.2 together with (1.20) we know that the model of the wave equation written in the port-Hamiltonian form is given by

$$\frac{\partial}{\partial t} \begin{pmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \zeta} \left[\begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{pmatrix} \begin{pmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{pmatrix} \right], \tag{5.21}$$

where $x_1 = \rho \frac{\partial w}{\partial t}$ is the momentum and $x_2 = \frac{\partial w}{\partial \zeta}$ is the strain.

Hence we have that

$$P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{H}(\zeta) = \begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{pmatrix}.$$

Being physical constants, the Young's modulus T and the mass density ρ are positive. Hence it is clear that P_1 and $\mathcal{H}$ satisfy the first two conditions of Theorem 5.2.6. We shall show that (5.19) holds.

$$P_1\mathcal{H}(\zeta) = \begin{pmatrix} 0 & T(\zeta) \\ \frac{1}{\rho(\zeta)} & 0 \end{pmatrix}. \tag{5.22}$$

satisfies the third condition, i.e., relation (5.19). The eigenvalues of this matrix are $\pm\gamma$ with $\gamma(\zeta) = \frac{T(\zeta)}{\rho(\zeta)}$. The corresponding eigenvectors are

$$\begin{pmatrix} \gamma(\zeta) \\ \frac{1}{\rho(\zeta)} \end{pmatrix} \text{ and } \begin{pmatrix} -\gamma(\zeta) \\ \frac{1}{\rho(\zeta)} \end{pmatrix}. \tag{5.23}$$

Hence

$$P_1\mathcal{H} = S^{-1}\Delta S = \begin{pmatrix} \gamma & -\gamma \\ \frac{1}{\rho} & \frac{1}{\rho} \end{pmatrix} \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix} \begin{pmatrix} \frac{1}{2\gamma} & \frac{\rho}{2} \\ \frac{-1}{2\gamma} & \frac{\rho}{2} \end{pmatrix}, \tag{5.24}$$

where we have omitted the dependence on ζ. The idea of the proof is the following. Since S is invertible, well-posedness will not change if we perform a basis transformation, $\tilde{x} = Sx$. After this basis transformation, the p.d.e. becomes

$$\begin{aligned} \frac{\partial \tilde{x}}{\partial t}(\zeta,t) =& \frac{\partial}{\partial \zeta}\left(\Delta\tilde{x}\right)(\zeta,t) + S(\zeta)\frac{dS^{-1}(\zeta)}{d\zeta}\Delta(\zeta)\tilde{x}(\zeta,t) \\ =& \frac{\partial}{\partial\zeta}\begin{pmatrix} \gamma(\zeta)\tilde{x}_1(\zeta,t) \\ -\gamma(\zeta)\tilde{x}_2(\zeta,t) \end{pmatrix} + S(\zeta)\frac{dS^{-1}(\zeta)}{d\zeta}\Delta(\zeta)\tilde{x}(\zeta,t). \end{aligned} \tag{5.25}$$

We see that we have a very nice set of simple p.d.e.'s, just two simple delay line, but they are corrupted by the term $S(\zeta)\frac{dS^{-1}(\zeta)}{d\zeta}\Delta(\zeta)\tilde{x}(\zeta,t)$. We first assume that this term is not present, and so we study the well-posedness of the collection of delay lines

$$\frac{\partial}{\partial t}\begin{pmatrix} \tilde{x}_1(\zeta,t) \\ \tilde{x}_2(\zeta,t) \end{pmatrix} = \frac{\partial}{\partial\zeta}\begin{pmatrix} \gamma(\zeta)\tilde{x}_1(\zeta,t) \\ -\gamma(\zeta)\tilde{x}_2(\zeta,t) \end{pmatrix} \tag{5.26}$$

Although it seems that these p.d.e.'s are uncoupled, they are coupled via the boundary conditions. In Section 5.3 we investigate when a p.d.e. like the one given in (5.26) with control and observation at the boundary is well-posed. In Section 5.4 we return to the original p.d.e., and show that ignoring bonded terms, like we did in (5.25) and (5.26) does not influence the well-posedness of the system. Since a basis transformation does not effect it either, we have proved Theorem 5.2.6.

5.3. The operator $P_1\mathcal{H}$ is diagonal.

In this section, we prove Theorem 5.2.6 if $P_1\mathcal{H}$ is diagonal, i.e., when $S = I$. For this we need the following two lemma's.

Lemma 5.3.1. *Let $\lambda(\zeta)$ be a positive continuous differentiable function on the interval $[a,b]$. With this function we define the scalar system*

$$\frac{\partial w}{\partial t}(\zeta,t) = \frac{\partial}{\partial\zeta}\left(\lambda(\zeta)w(\zeta,t)\right), \qquad w(\zeta,0) = w_0(\zeta) \qquad \zeta\in[a,b] \tag{5.27}$$

The value at b we choose as input

$$u(t) = \lambda(b)w(b,t) \tag{5.28}$$

and as output we choose the value on the other end

$$y(t) = \lambda(a)w(a,t). \tag{5.29}$$

The system (5.27)–(5.29) is a well-posed system on the state space $L^2(a,b)$. Its transfer function is given by

$$G(s) = e^{-p(b)s}, \tag{5.30}$$

where p is defined as

$$p(\zeta) = \int_a^\zeta \lambda(\zeta)^{-1} d\zeta \qquad \zeta \in [a,b]. \tag{5.31}$$

This transfer function satisfies

$$\lim_{\mathrm{Re}(s)\to\infty} G(s) = 0. \tag{5.32}$$

Proof: It is easy to see that the system (5.27)–(5.29) is a very simple version of the general Port-Hamiltonian system (3.32)–(3.34) with $P_1 = 1$, $\mathcal{H} = \lambda$, $W_B = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $W_C = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$, see Exercise 5.4. Since $W_B \Sigma W_B^T = 1 > 0$, we conclude by Theorem 3.4.2 that (5.27)–(5.29) has a well-defined solution provided the initial condition and the input are smooth. For this class, the balance equation (3.36) holds. However, this is the same as (1.27). Using that form of the power balance, we obtain

$$\begin{aligned} \frac{d}{dt}\int_a^b w(\zeta,t)\lambda(\zeta)w(\zeta,t)d\zeta &= \left[\left[\lambda(\zeta)w(\zeta,t)\right]\lambda(\zeta)w(\zeta,t)\right]_a^b \\ &= |u(t)|^2 - |y(t)|^2, \end{aligned}$$

where we used (5.28) and (5.29). Thus for all $t_f > 0$ we have that

$$\begin{aligned} \int_a^b w(\zeta,t_f)\lambda(\zeta)w(\zeta,t_f)d\zeta - &\int_a^b w(\zeta,0)\lambda(\zeta)w(\zeta,0)d\zeta \\ &= \int_0^{t_f} |u(\tau)|^2 d\tau - \int_0^{t_f} |y(\tau)|^2 d\tau. \end{aligned} \tag{5.33}$$

Since λ is strictly positive, we have that the energy norm $\int_a^b w(\zeta,t)\lambda(\zeta)w(\zeta,t)d\zeta$ is equivalent to the $L^2(a,b)$-norm, and so on a dense set an inequality like (5.17) is satisfied. Thus the system is well-posed.

As we have seen in Chapter 4, the transfer function $G(s)$ is constructed by finding for $s \in \mathbb{C}$ and for all u_0 a triple $(u_s(t), w_s(\zeta,t), y(t)) = (u_0 e^{st}, w_0(\zeta)e^{st}, y_0 e^{st})$ satisfying (5.27)–(5.29). Substituting a triple of this form in the p.d.e., gives

$$s w_0(\zeta) = \frac{\partial}{\partial\zeta}\left(\lambda(\zeta)w_0(\zeta)\right), \quad u_0 = \lambda(b)w_0(b), \quad y_0 = \lambda(a)w_0(a).$$

Thus $w_0(\zeta) = u_0\lambda(\zeta)^{-1}\exp(s(p(\zeta)-p(b)))$, and $y_0 = u_0\exp(-sp(b))$. This proves (5.30). The property (5.32) follows directly from (5.30) and the fact that $p(b) > 0$. ∎

For a p.d.e. with negative coefficient, we obtain a similar result.

Lemma 5.3.2. *Let $\theta(\zeta)$ be a negative continuous function on the interval $[a,b]$. With this function we define the scalar system*

$$\frac{\partial w}{\partial t}(\zeta,t) = \frac{\partial}{\partial z}\left(\theta(\zeta)w(\zeta,t)\right), \qquad w(\zeta,0) = w_0(\zeta) \qquad \zeta \in [a,b] \tag{5.34}$$

The value at a we choose as input

$$u(t) = \theta(a)w(a,t) \tag{5.35}$$

and as output we choose the value on the other end

$$y(t) = \theta(b)w(b,t). \tag{5.36}$$

The system (5.34)–(5.36) is a well-posed system on the state space $L^2(a,b)$. Its transfer function is given by

$$G(s) = e^{n(b)s}, \tag{5.37}$$

where

$$n(\zeta) = \int_a^\zeta \theta(\zeta)^{-1} d\zeta, \qquad \zeta \in [a,b]. \tag{5.38}$$

This transfer function satisfies

$$\lim_{\mathrm{Re}(s)\to\infty} G(s) = 0. \tag{5.39}$$

We use these two lemmas to prove Theorem 5.2.6 when $P_1\mathcal{H}$ is diagonal and the input space has dimension n.

Consider the following diagonal hyperbolic system on the spatial interval $\zeta \in [a,b]$

$$\frac{\partial}{\partial t}\begin{pmatrix} x_+(\zeta,t) \\ x_-(\zeta,t) \end{pmatrix} = \frac{\partial}{\partial \zeta}\left[\begin{pmatrix} \Lambda(\zeta) & 0 \\ 0 & \Theta(\zeta) \end{pmatrix}\begin{pmatrix} x_+(\zeta,t) \\ x_-(\zeta,t) \end{pmatrix}\right] \tag{5.40}$$

where $\Lambda(\zeta)$ is a diagonal (real) matrix, with positive functions on the diagonal, and $\Theta(\zeta)$ is a diagonal (real) matrix, with negative functions on the diagonal. Furthermore, we assume that Λ and Θ are continuously differentiable.

With this p.d.e. we associate the following boundary control and observation

$$u_s(t) := \begin{pmatrix} \Lambda(b)x_+(b,t) \\ \Theta(a)x_-(a,t) \end{pmatrix}, \tag{5.41}$$

$$y_s(t) := \begin{pmatrix} \Lambda(a)x_+(a,t) \\ \Theta(b)x_-(b,t) \end{pmatrix}. \tag{5.42}$$

Theorem 5.3.3. *Consider the p.d.e. (5.40) with u_s and y_s as defined in (5.41) and (5.42), respectively.*

- *The system defined by (5.40)–(5.42) is well-posed and regular. Furthermore, its transfer function converges to zero for* $\mathrm{Re}(s) \to \infty$.
- *To the p.d.e. (5.40) we define a new set of inputs and outputs. The new input* $u(t)$ *is written as*
$$u(t) = Ku_s(t) + Qy_s(t), \tag{5.43}$$
where K *and* Q *are two square matrices, with* $[K, Q]$ *of rank* n. *The new output is written as*
$$y(t) = O_1 u_s(t) + O_2 y_s(t). \tag{5.44}$$
where O_1 *and* O_2 *are some matrices. For the system (5.40) with input* $u(t)$ *and output* $y(t)$, *we have the following possibilities:*
  1. *If* K *is invertible, then the system (5.40), (5.43), and (5.44) is well-posed and regular. Furthermore, its transfer function converges to* $O_1 K^{-1}$ *for* $\mathrm{Re}(s) \to \infty$
  2. *If* K *is not invertible, then the operator* A_K *defined as*
$$A_K \begin{pmatrix} g_+(\zeta) \\ g_-(\zeta) \end{pmatrix} = \frac{\partial}{\partial \zeta} \left[\begin{pmatrix} \Lambda(\zeta) & 0 \\ 0 & \Theta(\zeta) \end{pmatrix} \begin{pmatrix} g_+(\zeta) \\ g_-(\zeta) \end{pmatrix} \right] \tag{5.45}$$
with domain
$$D(A_K) = \left\{ \begin{pmatrix} g_+(\zeta) \\ g_-(\zeta) \end{pmatrix} \in H^1((a,b), \mathbb{R}^n) \mid \right. \\ \left. K \begin{pmatrix} \Lambda(b) g_+(b) \\ \Theta(a) g_-(a) \end{pmatrix} + Q \begin{pmatrix} \Lambda(a) g_+(a) \\ \Theta(b) g_-(b) \end{pmatrix} = 0 \right\} \tag{5.46}$$
does not generate a C_0*-semigroup on* $L^2((a,b); \mathbb{R}^n)$.

Note that the last item implies that the homogeneous p.d.e. does not have a well-defined solution, when K is not invertible.

PROOF: The first item is a direct consequence of Lemma 5.3.1 and 5.3.2 by noticing that the system (5.40)–(5.42) is built out of copies of the system (5.27)–(5.29) and the system (5.34)–(5.36). Furthermore, these sub-systems do not interact with each other.

For the proof of the first part of the second assertion, with K invertible, we rewrite the new input, as $u_s(t) = K^{-1}u(t) - K^{-1}Qy_s(t)$. This can be seen as a feedback interconnection on the system (5.40)–(5.42), as is depicted in Figure 5.1. The system contains one feedback loop with gain matrix $K^{-1}Q$. By Theorem 5.6.1, we have that if $I + G_s(s)K^{-1}Q$ is invertible on some right half-plane and if this inverse is bounded on a right-half plane, then the closed loop system is well-posed. Since $\lim_{\mathrm{Re}(s)\to\infty} G_s(s) = 0$, we see that this holds for every K^{-1} and Q. So under the assumption that K is invertible, we find that (5.40) with input and output given by (5.43) and (5.44) is well-posed. The regularity follows easily. By regarding the loops in Figure 5.1, we see that the feed-though term is $O_1 K^{-1}$.

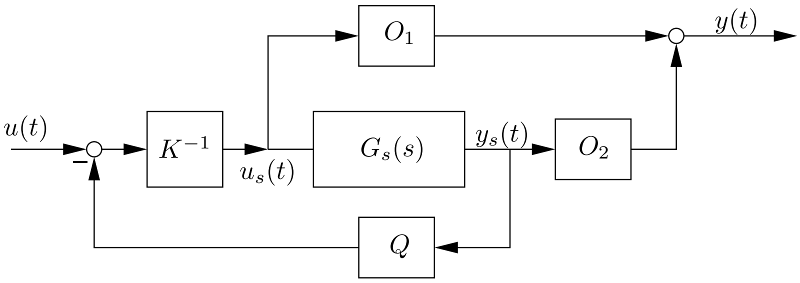

Figure 5.1.: The system (5.40) with input (5.43) and output (5.44)

So it remains to show that there is no C_0-semigroup when K is non-invertible. Since K is singular, there exists a non-zero $v \in \mathbb{R}^n$ such that $v^T K = 0$. Since $[K, Q]$ has full rank, we know that $q^T := v^T Q \neq 0$. So at least one of the components of q is unequal to zero. For the sake of the argument, we assume that this holds for the first one.

If A_K would be the infinitesimal generator of a C_0-semigroup, then for all $x_0 \in D(A_K)$ the abstract differential equation

$$\dot{x}(t) = A_K x(t), \qquad x(0) = x_0 \tag{5.47}$$

would have classical solution, i.e., for all $t > 0$, $x(t)$ is differentiable, it is an element of $D(A_K)$, and it satisfies (5.47). Hence by (5.46), we have that $x(t)$ is an element of H^1. Since we are working in a one dimensional spatial domain, we have that functions in H^1 are continuous. So we have that for every t, $x(t)$ is a continuous function of ζ satisfying the boundary conditions in (5.46).

So if A_K would generate a C_0-semigroup, then for every $x_0 \in D(A_K)$ there would be a function $x(\zeta, t) := \begin{pmatrix} x_+(\zeta,t) \\ x_-(\zeta,t) \end{pmatrix}$ which is a (mild) solution to the p.d.e. (5.40), and satisfies for all $t > 0$ the boundary condition

$$K \begin{pmatrix} \Lambda(b)x_+(b,t) \\ \Theta(a)x_-(a,t) \end{pmatrix} + Q \begin{pmatrix} \Lambda(a)x_+(a,t) \\ \Theta(b)x_-(b,t) \end{pmatrix} = 0.$$

Using the vectors v and q, we see that this $x(\zeta, t)$ must satisfy

$$0 = q^T \begin{pmatrix} \Lambda(a)x_+(a,t) \\ \Theta(b)x_-(b,t) \end{pmatrix}, \quad t > 0. \tag{5.48}$$

Now we construct an initial condition in $D(A_K)$, for which this equality does not hold. Note that we have chosen the first component of q unequal to zero.

The initial condition x_0 is chosen to have all components zero except for the first one. For this first component we choose an arbitrary function in $H^1(a,b)$ which is zero at a and b, but nonzero everywhere else on the open set (a,b). It is clear that this initial condition is in the domain of A_K. Now we solve (5.40).

Standard p.d.e. theory gives that the solution of (5.40) can be written as

$$x_{+,m}(\zeta,t) = f_{+,m}(p_m(\zeta) + t)\lambda_m(\zeta)^{-1}, \tag{5.49}$$

$$x_{-,\ell}(\zeta,t) = f_{-,\ell}(n_\ell(\zeta) + t)\theta_\ell(\zeta)^{-1}, \tag{5.50}$$

where λ_m and θ_ℓ are the m-th and the ℓ-th diagonal element of Λ and Θ, respectively. Furthermore, $p_m(\zeta) = \int_a^\zeta \lambda_m(\zeta)^{-1} d\zeta$, $n_l(\zeta) = \int_a^\zeta \theta_l(\zeta)^{-1} d\zeta$, see also Exercises 5.4 and 5.5. The functions f_+, f_- need to be determined from the boundary and initial conditions.

Using the initial condition we have that $f_{+,m}(p_m(\zeta)) = \lambda_m(\zeta)x_{0,+,m}(\zeta)$ and $f_{-,\ell}(n_\ell(\zeta)) = \theta_\ell(\zeta)x_{0,-,\ell}(\zeta)$. Since $p_m > 0$, and $n_\ell < 0$, we see that the initial condition determines f_+ on a (small) positive interval, and f_- on a small negative interval. By our choice of the initial condition, we find that

$$\begin{aligned} f_{+,1}(\xi) &= \lambda_1(\xi)x_{0,+,1}(\xi) && \xi \in [0, p_1(b)), \\ f_{+,m}(\xi) &= 0 && \xi \in [0, p_m(b)), \quad m \geq 2, \\ f_{-,\ell}(\xi) &= 0 && \xi \in [n_\ell(b), 0), \quad \ell \geq 1. \end{aligned} \tag{5.51}$$

The solution $x(\zeta, t)$ must also satisfy (5.48), thus for all $t > 0$ we have that

$$0 = q^T \begin{pmatrix} f_+(t) \\ f_-(n(b) + t) \end{pmatrix} \tag{5.52}$$

Combining this with (5.51), we find

$$0 = q_1 f_{+,1}(p_1(\zeta)) = q_1 x_{0,+,1}(\zeta)\lambda_1^{-1}(\zeta)$$

on some interval $[a, \beta]$. Since q_1 and λ_1 are unequal to zero, we find that x_0 must be zero on some interval. This is in contradiction with our choice of the initial condition. Thus A_K cannot be the infinitesimal generator of a C_0-semigroup. ∎

5.4. Proof of Theorem 5.2.6.

In this section we use the results of the previous section to prove Theorem 5.2.6. We begin with a useful lemma, which proof can be found in Section 5.6.

Lemma 5.4.1. *The system (5.11)–(5.14) is well-posed if and only if the system*

$$\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta} (\mathcal{H}x)(\zeta, t), \tag{5.53}$$

with inputs, outputs given by (5.12)–(5.14) is well-posed.

Let $G(s)$ denote the transfer function of (5.11)–(5.14) and $G_0(s)$ the transfer function of (5.53) with (5.12)–(5.14). Then

$$\lim_{s \to \infty} G(s) = \lim_{s \to \infty} G_0(s), \tag{5.54}$$

and

$$\lim_{\mathrm{Re}(s) \to \infty} G(s) = \lim_{\mathrm{Re}(s) \to \infty} G_0(s), \tag{5.55}$$

This lemma tells us that we may ignore any bounded linear term involving x.

Now we have all the results needed to prove Theorem 5.2.6.

By the third assumption of Theorem 5.2.6, the matrices P_1 and $\mathcal{H}$ satisfy the equation (5.19):

$$P_1\mathcal{H}(\zeta) = S^{-1}(\zeta)\Delta(\zeta)S(\zeta).$$

With this we introduce the new state vector

$$\tilde{x}(\zeta,t) = S(\zeta)x(\zeta,t), \qquad \zeta \in [a,b] \tag{5.56}$$

Under this basis transformation, the p.d.e. (5.11) becomes

$$\begin{aligned}\frac{\partial \tilde{x}}{\partial t}(\zeta,t) =& \frac{\partial}{\partial \zeta}\left(\Delta\tilde{x}\right)(\zeta,t) + S(\zeta)\frac{dS^{-1}(\zeta)}{d\zeta}\Delta(\zeta)\tilde{x}(\zeta,t)+\\ & S(\zeta)P_0(\zeta)S(\zeta)^{-1}\tilde{x}(\zeta,t), \qquad x(\zeta,0) = S(\zeta)x_0(\zeta) = \tilde{x}_0(\zeta).\end{aligned} \tag{5.57}$$

The relations (5.12)–(5.14) become

$$\begin{aligned}0 =& M_{11}P_1^{-1}S^{-1}(b)\Delta(b)\tilde{x}(b,t) + M_{12}P_1^{-1}S^{-1}(a)\Delta(a)\tilde{x}(a,t)\\ =& \tilde{M}_{11}\Delta(b)\tilde{x}(b,t) + \tilde{M}_{12}\Delta(a)\tilde{x}(a,t) \end{aligned}\tag{5.58}$$

$$\begin{aligned}u(t) =& M_{21}P_1^{-1}S^{-1}(b)\Delta(b)\tilde{x}(b,t) + M_{22}P_1^{-1}S^{-1}(a)\Delta(a)\tilde{x}(a,t)\\ =& \tilde{M}_{21}\Delta(b)\tilde{x}(b,t) + \tilde{M}_{22}\Delta(a)\tilde{x}(a,t) \end{aligned}\tag{5.59}$$

$$\begin{aligned}y(t) =& C_1P_1^{-1}S^{-1}(b)\Delta(b)\tilde{x}(b,t) + C_2P_1^{-1}S^{-1}(a)\Delta(a)\tilde{x}(a,t)\\ =& \tilde{C}_1\Delta(b)\tilde{x}(b,t) + \tilde{C}_2\Delta(a)\tilde{x}(a,t). \end{aligned}\tag{5.60}$$

We introduce $\tilde{M} = \left(\begin{smallmatrix}\tilde{M}_{11} & \tilde{M}_{12}\\ \tilde{M}_{21} & \tilde{M}_{22}\end{smallmatrix}\right)$ with

$$\begin{pmatrix}\tilde{M}_{j1} & \tilde{M}_{j2}\end{pmatrix} = \begin{pmatrix}M_{j1} & M_{j2}\end{pmatrix}\begin{pmatrix}P_1^{-1}S(b)^{-1} & 0\\ 0 & P_1^{-1}S(a)^{-1}\end{pmatrix}, \qquad j = 1,2.$$

and

$$\tilde{C} = \begin{pmatrix}\tilde{C}_1 & \tilde{C}_2\end{pmatrix} = \begin{pmatrix}C_1 & C_2\end{pmatrix}\begin{pmatrix}P_1^{-1}S(b)^{-1} & 0\\ 0 & P_1^{-1}S(a)^{-1}\end{pmatrix}.$$

Since the matrix $\left(\begin{smallmatrix}P_1^{-1}S(b)^{-1} & 0\\ 0 & P_1^{-1}S(a)^{-1}\end{smallmatrix}\right)$ has full rank, we see that the rank conditions in Theorem 5.2.6 imply similar rank conditions for $\tilde{M}$ and $\tilde{C}$.

Using Lemma 5.4.1 we see that we only have to prove the result for the p.d.e.

$$\frac{\partial \tilde{x}}{\partial t}(\zeta,t) = \frac{\partial}{\partial \zeta}\left(\Delta\tilde{x}\right)(\zeta,t). \tag{5.61}$$

with boundary conditions, inputs, and outputs as described in (5.58)–(5.60).

It is clear that if condition (5.58) is not present, then Theorem 5.3.3 gives that the above system is well-posed and regular if and only if the homogeneous p.d.e. generates a C_0-semigroup on $L^2((a,b);\mathbb{R}^n)$. Since the state transformation (5.56) defines a bounded mapping on $L^2((a,b);\mathbb{R}^n)$, we have proved Theorem 5.2.6 provided there is no condition (5.12).

Thus it remains to prove Theorem 5.2.6 if we have put part of the boundary conditions to zero. Or equivalently, to prove that the system (5.58)–(5.61) is well-posed and regular if and only if the homogeneous p.d.e. generates a C_0-semigroup.

We replace (5.58) by

$$v(t) = \tilde{M}_{11}\Delta(b)\tilde{x}(b,t) + \tilde{M}_{12}\Delta(a)\tilde{x}(a,t), \tag{5.62}$$

where we regard v as a new input. Hence we have the system (5.61) with the new extended input

$$\begin{pmatrix} v(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} \tilde{M}_{11} \\ \tilde{M}_{21} \end{pmatrix} \Delta(b)\tilde{x}(t,b) + \begin{pmatrix} \tilde{M}_{12} \\ \tilde{M}_{22} \end{pmatrix} \Delta(a)\tilde{x}(t,a). \tag{5.63}$$

and the output (5.60). By doing so, we have obtained a system without a condition (5.58). For this system we know that it is well-posed and regular if and only if the homogeneous equation generates a C_0-semigroup.

Assume that the system (5.61), (5.63) and (5.60) is well-posed, then we may choose any (locally) square input. In particular, we may choose $v \equiv 0$. Thus the system (5.57)–(5.61) is well-posed and regular as well.

Assume next that the p.d.e. with the extended input in (5.63) set to zero, does not generate a C_0-semigroup. Since this gives the same homogeneous p.d.e. as (5.61) with (5.58) and u in (5.59) set to zero, we know that this p.d.e. does not generate a C_0-semigroup. This finally proves Theorem 5.2.6.

5.5. Well-posedness of the vibrating string.

In this section we illustrate the usefulness of Theorem 5.2.6 by applying it to the vibrating string of Example 1.1.2.

By equation (1.20) we know that for the vibrating string there holds

$$P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{H}(\zeta) = \begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{pmatrix}.$$

Since we want to illustrate the theory and proofs derived in the previous sections, we do not directly check if for a (to-be-given) set of boundary conditions the semigroup condition is satisfied. Instead of that, we rewrite the system in its diagonal form, and check the conditions using Theorem 5.3.3. As we have seen in Section 5.4, the proof of Theorem 5.2.6 follows after a basis transformation directly from Theorem 5.3.3. Hence we start by diagonalizing $P_1\mathcal{H}$. Although all the results hardly change, we assume for simplicity of notation that Young's modulus T and the mass density ρ are constant. Being physical constants, they are naturally positive.

From equation (5.24 we know that the operator $P_1\mathcal{H}$ is diagonalizable:

$$P_1\mathcal{H} = S^{-1}\Delta S = \begin{pmatrix} \gamma & -\gamma \\ \frac{1}{\rho} & \frac{1}{\rho} \end{pmatrix} \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix} \begin{pmatrix} \frac{1}{2\gamma} & \frac{\rho}{2} \\ \frac{-1}{2\gamma} & \frac{\rho}{2} \end{pmatrix}, \tag{5.64}$$

where γ is positive and satisfies $\gamma^2 = c = \frac{T}{\rho}$.

Hence the state transformation under which the p.d.e. becomes diagonal is

$$\tilde{x} = \frac{1}{2}\begin{pmatrix} \frac{1}{\gamma} & \rho \\ \frac{-1}{\gamma} & \rho \end{pmatrix} x.$$

Since we assumed that $\gamma > 0$, we see that $\tilde{x}_1$, $\tilde{x}_2$ correspond to x_+, and x_- in equation (5.40), respectively and Λ, Θ to γ and $-\gamma$, respectively. Hence we have that the input and output u_s and y_s defined for the diagonal system (5.40) by the equations (5.41)–(5.42) are expressed in the original coordinates by

$$u_s(t) = \frac{1}{2}\begin{pmatrix} x_1(b,t) + \gamma\rho x_2(b,t) \\ x_1(a,t) - \gamma\rho x_2(a,t) \end{pmatrix}, \tag{5.65}$$

$$y_s(t) = \frac{1}{2}\begin{pmatrix} x_1(a,t) + \gamma\rho x_2(a,t) \\ x_1(b,t) - \gamma\rho x_2(b,t) \end{pmatrix}. \tag{5.66}$$

This pair of boundary input and output variables consists in complementary linear combinations of the momentum x_1 and the strain x_2 at the boundaries: however they lack an obvious physical interpretation. One could consider another choice of boundary input and outputs, for instance the velocity and the strain at the boundary points and choose as input

$$u_1(t) = \begin{pmatrix} \frac{x_1}{\rho}(b,t) \\ x_2(a,t) \end{pmatrix} \tag{5.67}$$

and as output

$$y_1(t) = \begin{pmatrix} \frac{x_1}{\rho}(a,t) \\ x_2(b,t) \end{pmatrix}. \tag{5.68}$$

We may apply Theorem 5.3.3 to check whether this system is well-posed, and to find the feed-through. Expressing the input-output pair (u_1, y_1) in (u_s, y_s) gives

$$u_1(t) = \begin{pmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{-1}{\sqrt{T\rho}} \end{pmatrix} u_s(t) + \begin{pmatrix} 0 & \frac{1}{\rho} \\ \frac{1}{\sqrt{T\rho}} & 0 \end{pmatrix} y_s(t), \tag{5.69}$$

$$y_1(t) = \begin{pmatrix} 0 & \frac{1}{\rho} \\ \frac{1}{\sqrt{T\rho}} & 0 \end{pmatrix} u_s(t) + \begin{pmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{-1}{\sqrt{T\rho}} \end{pmatrix} y_s(t). \tag{5.70}$$

Hence

$$K = \begin{pmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{-1}{\sqrt{T\rho}} \end{pmatrix} \quad Q = \begin{pmatrix} 0 & \frac{1}{\rho} \\ \frac{1}{\sqrt{T\rho}} & 0 \end{pmatrix}, \tag{5.71}$$

$$O_1 = \begin{pmatrix} 0 & \frac{1}{\rho} \\ \frac{1}{\sqrt{T\rho}} & 0 \end{pmatrix}, \qquad O_2 = \begin{pmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{-1}{\sqrt{T\rho}} \end{pmatrix}. \tag{5.72}$$

Since K is invertible, the system with the input-output pair (u_1, y_1) is well-posed and regular, and the feed-through term is given by $O_1 K^{-1} = \begin{pmatrix} 0 & -\gamma \\ \frac{1}{\gamma} & 0 \end{pmatrix}$.

Since the states are defined as $x_1 = \rho \frac{\partial w}{\partial t}$ and $x_2 = \frac{\partial w}{\partial \zeta}$, the control and observation are easily formulated using w. Namely, $u_1(t) = \begin{pmatrix} \frac{\partial w}{\partial t}(t,b) \\ \frac{\partial w}{\partial \zeta}(t,a) \end{pmatrix}$ and $y_1(t) = \begin{pmatrix} \frac{\partial w}{\partial t}(t,a) \\ \frac{\partial w}{\partial z}(t,b) \end{pmatrix}$, respectively. Hence we observe the velocity and strain at opposite ends.

Next we show that if we would control the velocity and strain at the same end, this does not give a well-posed system. The control and observation are given by

$$u_2(t) = \begin{pmatrix} \frac{\partial w}{\partial t}(b,t) \\ \frac{\partial w}{\partial \zeta}(b,t) \end{pmatrix} = \begin{pmatrix} \frac{x_1}{\rho}(b,t) \\ x_2(b,t) \end{pmatrix} \tag{5.73}$$

and as output

$$y_2(t) = \begin{pmatrix} \frac{\partial w}{\partial t}(a,t) \\ \frac{\partial w}{\partial \zeta}(a,t) \end{pmatrix} = \begin{pmatrix} \frac{x_1}{\rho}(a,t) \\ x_2(a,t) \end{pmatrix}. \tag{5.74}$$

It is easy to see that

$$u_2(t) = \begin{pmatrix} \frac{1}{\rho} & 0 \\ \frac{1}{\gamma\rho} & 0 \end{pmatrix} u_s(t) + \begin{pmatrix} 0 & \frac{1}{\rho} \\ 0 & -\frac{1}{\gamma\rho} \end{pmatrix} y_s(t). \tag{5.75}$$

Clearly the matrix in front of u_s is not invertible, and hence we conclude by Theorem 5.3.3 that the wave equation with the homogeneous boundary conditions $u_2 = 0$ does not generate a C_0-semigroup. Hence this system is not well-posed.

Until now we have been controlling velocity and strain at the end points. However, for the wave equation, it seems very naturally to control the position, i.e., $w(\cdot, t)$. So we consider the wave equation (1.5) with the following control and observation.

$$u_3(t) = \begin{pmatrix} w(b,t) \\ \frac{\partial w}{\partial \zeta}(a,t) \end{pmatrix} \tag{5.76}$$

$$y_3(t) = \begin{pmatrix} w(a,t) \\ \frac{\partial w}{\partial z}(b,t) \end{pmatrix}. \tag{5.77}$$

Since the first control and first observation cannot be written as linear combination of our boundary effort and flow, we find that this system is not of the form (5.11)–(5.14). However, we still can investigate the well-posedness of the system. For this we realize that the first element in u_3 is the time derivative of the first element of u_1. So we can see the wave equation with the input (5.76) and output (5.77) as the following series connection.

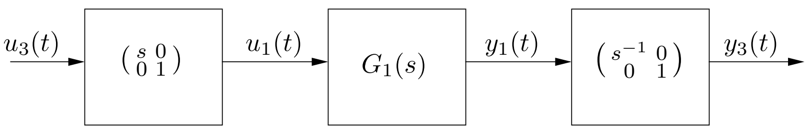

Figure 5.2.: The wave equation with input and output (5.76) and (5.77)

From this it is clear that the transfer function, $G_3(s)$ of the system with input u_3 and output y_3 is given by

$$G_3(s) = \begin{pmatrix} s^{-1} & 0 \\ 0 & 1 \end{pmatrix} G_1(s) \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}.$$

Since for large real s the transfer function $G_1(s)$ is approximately equal to $\begin{pmatrix} 0 & -\gamma \\ \gamma^{-1} & 0 \end{pmatrix}$, we see that $G_3(s)$ grows like s for large s. By Lemma 5.2.4 we know that any well-posed system has a transfer function which is bounded in some right half-plane. Thus the wave equation with input (5.76) and output (5.77) is not well-posed.

The reason for the well-posedness of this system is different than for the choice u_2 and y_2. Since if we put u_3 to zero, then this implies that u_1 is zero as well, and so we know that this homogeneous equation is well-defined and has a unique solution. So if the controls and/or observations are not formulated in the boundary effort and flow, then we may loose well-posedness even if there is a semigroup.

5.6. Technical lemma's

In this section we present the proofs for the technical results which we needed. We begin by considering the boundary control system

The following result is essential in our proof, and it has been proved by G. Weiss in [30].

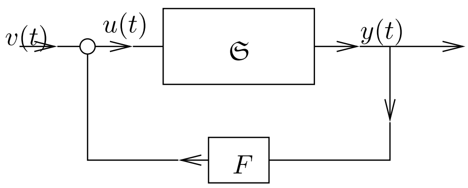

Figure 5.3.: The closed loop system

Theorem 5.6.1. *Let $\mathfrak{S}$ be a well-posed system with input space U and output space Y, both being Hilbert spaces. Denote the transfer function by $G(s)$. Let F be a bounded linear operator from Y to U such that $I - G(s)F$ is invertible in some right half-plane. If the inverse is bounded in a right half-plane, then the closed loop system as depicted in Figure 5.3 is again well-posed. If G is regular, then so is the closed loop transfer function.*

5.7. Exercises

5.1. Prove that the function defined by (5.6) is a continuous function in t with values in $L^2(0,1)$ for any $x_0 \in L^2(0,1)$ and any $u \in L^2(0,\infty)$. That is prove that

$$\int_0^1 |x(\zeta,t) - x(\zeta,t+h)|^2 d\zeta \to 0$$

for $h \to 0$.

5.2. In this exercise we study the p.d.e. (5.8) with boundary condition (5.9).

a) Show that for an initial condition x_0 which is continuously differentiable and satisfies $x_0(1) = 2x_0(0)$ the (classical) solution of (5.8)–(5.9) is given by (5.10).

b) Show that the mapping $x_0 \mapsto x(\cdot, t)$ with $x(\zeta, t)$ given by (5.10) defines a C_0-semigroup on $L^2(0, 1)$.

c) Conclude that (5.10) is the mild solution of (5.8)–(5.9) for any initial condition $x_0 \in L^2(0, 1)$.

5.3. Prove Lemma 5.2.1.

5.4. In this exercise, we show some more results for the system defined by (5.27)–(5.29).

a) Show that the system (5.27)–(5.29) is a port-Hamiltonian system of the form (3.32)–(3.34).

b) Show that the solution of (5.27)–(5.28) is given by

$$w(\zeta, t) = f(p(\zeta) + t)\lambda(\zeta)^{-1}, \tag{5.78}$$

where

$$\begin{aligned} p(\zeta) &= \int_a^\zeta \lambda(\zeta)^{-1} d\zeta \qquad && \zeta \in [a, b] && (5.79)\\ f(p(\zeta)) &= \lambda(\zeta) w_0(\zeta), && \zeta \in [a, b] && (5.80)\\ f(p(b) + t) &= u(t), && t > 0. && (5.81) \end{aligned}$$

5.5. Lemma 5.3.2: its solution is given as

$$w(\zeta, t) = f(n(\zeta) + t)\theta(\zeta)^{-1}, \tag{5.82}$$

where

$$\begin{aligned} n(\zeta) &= \int_a^z \theta(\zeta)^{-1} d\zeta && && (5.83)\\ f(n(\zeta)) &= \theta(\zeta) w_0(\zeta), \qquad && z \in [a, b] && (5.84)\\ f(t) &= u(t), \qquad t > 0. && && (5.85) \end{aligned}$$

5.6. Show that the transfer function is regular with feed-though term zero if the constant m_f in equation (5.17) can be chosen such that $m_f \to 0$ if $t_f \downarrow 0$ for $x_0 = 0$.

5.8. Notes and references

The third condition tells us that $P_1\mathcal{L}$ is diagonalizable via a continuously differentiable basis transformation. In Kato [14, chapter II], one can find conditions on $P_1\mathcal{L}(\zeta)$ such that this is possible. For simplicity, we have assumed that $\Delta(\zeta)$ is continuously differentiable.

From the proof of Theorem 5.2.6, we see that we obtain an equivalent matrix condition for condition 1., i.e., item 1. of Theorem 5.2.6 holds if and only if K is invertible, see Theorem 5.3.3. Since the matrix K is obtained after a basis transformation, and depends on the negative and positive eigenvalues of $P_1\mathcal{H}$, it is not easy to rewrite this condition in a condition for M_{ij}.

A semigroup can be extended to a group, if the homogeneous p.d.e. has for every initial condition a solution for negative time. Using once more the proof of Theorem 5.3.3, we see that A in item 1. of Theorem 5.2.6 generates a group if and only if K and Q are invertible matrices.

That the system remains well-posed after feedback was proved by Weiss [30].

This chapter is completely based on the paper by Zwart et al, [33].

Chapter 6
Stability and Stabilizability

6.1. Introduction

In this chapter we study the stability of our systems. We study the stability of the state, i.e., we only look at the solutions of the homogeneous differential equation. As for non-linear systems there are two different notions of stability. Namely strong stability and exponential stability, which are defined next. Let A be the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X. We know that $x(t) := T(t)x_0$ is the (unique) mild solution of the differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0. \tag{6.1}$$

For the solutions we define two concepts of stability.

Definition 6.1.1. The system (6.1) is *strongly stable* if for every $x_0 \in X$ the state trajectory $x(t)$ converges to zero for t going to infinity. ♣

Definition 6.1.2. The system (6.1) is *exponentially stable*, if there exists a $M > 0$, $\omega < 0$ such that

$$\|T(t)\| \leq Me^{\omega t}, \qquad t \geq 0. \tag{6.2}$$

♣

Since $x(t) = T(t)x_0$, it is easy to see that exponential stability implies strong stability. The converse does not hold as is shown in the following example.

Example 6.1.3 Let X be the Hilbert space $L^2(0,\infty)$ and let the C_0-semigroup be given as

$$\left(T(t)f\right)(\zeta) = f(t+\zeta). \tag{6.3}$$

As in Example 2.2.3 it is not hard to show that this is a strongly continuous semigroup. Furthermore, for all $t \geq 0$ we have $\|T(t)\| = 1$, see Exercise 6.1. The later implies that this semigroup is not exponentially stable. It remains to show that it is strongly stable.

Let $\varepsilon > 0$ and let $f \in X = L^2(0,\infty)$. There exists a function f_ε such that $\|f - f_\varepsilon\| \leq \varepsilon$ and $f_\varepsilon(\zeta)$ is zero for ζ sufficiently large. Let denote ζ_f the point from which $f_\varepsilon(\zeta)$ equals zero. For $t > \zeta_f$, we have that $T(t)f_\varepsilon = 0$ and so

$$\begin{aligned}\|T(t)f\| \leq & \|T(t)f - T(t)f_\varepsilon\| + \|T(t)f_\varepsilon\| \\ = & \sqrt{\int_0^\infty |f(t+\zeta) - f_\varepsilon(t+\zeta)|^2 d\zeta} \\ \leq & \sqrt{\int_0^\infty |f(z) - f_\varepsilon(z)|^2 dz} \leq \varepsilon.\end{aligned}$$

Since this holds for any ε, we conclude that $(T(t))_{t\geq 0}$ is strongly stable. □

In this book we study systems in strong connection with their energy. This energy serves as our norm. In the following section we show that if the energy is decaying we have (under mild conditions) exponential stability. We urge to say that this holds for our nice class of port-Hamiltonian system, and does not hold generally as the following example shows.

Example 6.1.4 In this example we construct a contraction semigroup whose norm is strictly decreasing, but the semigroup is not strongly stable.

We take the Hilbert space $L^2(0,\infty)$. However, not with its standard inner product, but we choose as inner product

$$\langle f, g\rangle = \int_0^\infty f(\zeta)\overline{g(\zeta)} \left[e^{-\zeta} + 1\right] d\zeta. \tag{6.4}$$

As semigroup, we choose the right shift semigroup:

$$\begin{aligned}(T(t)f)(\zeta) &= \begin{cases} f(\zeta - t) & \zeta > t \\ 0 & \zeta \in [0,t)\end{cases} \\ &= f(\zeta - t)\mathbb{1}_{[0,\infty)}(\zeta - t).\end{aligned}$$

Using the formula for the norm and the formula of the semigroup, we see that

$$\begin{aligned}\|T(t)f\|^2 &= \int_0^\infty |f(\zeta - t)\mathbb{1}_{[0,\infty])}(\zeta - t)|^2 \left[e^{-\zeta} + 1\right] d\zeta \\ &= \int_0^\infty |f(\xi)|^2 \left[e^{-(\xi + t)} + 1\right] d\xi \\ &\geq \int_0^\infty |f(\xi)|^2 d\xi \\ &\geq \frac{1}{2}\int_0^\infty |f(\xi)|^2 \left[e^{-\xi} + 1\right] d\xi \\ &= \frac{1}{2}\|f\|^2.\end{aligned} \tag{6.5}$$

Hence $(T(t))_{t\geq 0}$ cannot be strongly stable. Even more importantly, with the exception of $f=0$, there is no initial condition for which $T(t)f \to 0$ as $t \to \infty$.

Next we show that the norm of the trajectory is always decreasing. Let $t_2 > t_1$ and let $f \neq 0$. From (6.5) we know that

$$\begin{aligned}\|T(t_2)f\|^2 &= \int_0^\infty |f(\xi)|^2 \left[e^{-(\xi+t_2)}+1\right] d\xi \\ &< \int_0^\infty |f(\xi)|^2 \left[e^{-(\xi+t_1)}+1\right] d\xi \\ &= \|T(t_1)f\|^2,\end{aligned}$$

where we have used that the negative exponential is strictly decreasing. Hence the norm of any trajectory is decaying, but the system is not strongly stable. □

Note that the above example shows that the second method of Lyapunov[1] is not directly applicable for p.d.e.'s[2].

We end with a small technical lemma, which will be useful later on.

Lemma 6.1.5. *Let $(T(t))_{\geq 0}$ be a strongly continuous semigroup on the Hilbert space X. If for some $t_1 > 0$ we have that $\|T(t_1)\| < 1$, then the C_0-semigroup is exponentially stable.*

PROOF: If there exists a $t_1 > 0$ such that $\|T(t_1)\| < 1$, then we have that $\frac{1}{t_1}\log\|T(t_1)\| < 0$. Hence $\omega_0 = \inf_{t>0} \frac{1}{t}\log\|T(t)\| < 0$. By Theorem 2.5.1.e, we can find a negative ω such that $\|T(t)\| \leq M_\omega e^{\omega t}$. Hence we have exponential stability. ■

In the following section we consider our class of port-Hamiltonian system, and we show that a simple condition is guaranteeing exponential stability.

6.2. Exponential stability of port-Hamiltonian systems

We return to our homogeneous port-Hamiltonian system of Section 2.3. That is we consider the p.d.e.

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}\left[\mathcal{H}(\zeta)x(\zeta,t)\right] + P_0\left[\mathcal{H}(\zeta)x(\zeta,t)\right]. \tag{6.6}$$

with the boundary condition

$$W_B \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} = 0, \tag{6.7}$$

where

$$\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}\begin{pmatrix} (\mathcal{H}x)(b,t) \\ (\mathcal{H}x)(a,t) \end{pmatrix}. \tag{6.8}$$

As in Theorem 2.3.2 we assume that the following holds

[1] Given a Lyapunov function V such that $\dot{V} < 0$, implies (asymptotic) stability

[2] One additionally needs pre-compactness of the trajectories

Assumption 6.2.1:

- P_1 is an invertible, symmetric real $n \times n$ matrix;
- P_0 is an anti-symmetric real $n \times n$ matrix;
- For all $\zeta \in [a,b]$ the $n \times n$ matrix $\mathcal{H}(\zeta)$ is real, symmetric, and $mI \leq \mathcal{H}(\zeta) \leq MI$, for some $M, m > 0$ independent of ζ;
- $\mathcal{H}$ is continuously differentiable on the interval $[a,b]$;
- W_B be a full rank real matrix of size $n \times 2n$;
- $W_B \Sigma W_B^T \geq 0$, where $\Sigma = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. ♡

The above, with the exception of the differentiability of $\mathcal{H}$, have been our standard assumptions in many previous chapters. However, we would like to remark that our main theorem 6.2.3 also holds if P_0 satisfies $P_0 + P_0^T \leq 0$. Under the conditions as listed in Assumption 6.2.1 we know that the operator A given by

$$Ax := P_1 \frac{d}{d\zeta} [\mathcal{H}x] + P_0 [\mathcal{H}x] \tag{6.9}$$

with domain

$$D(A) = \{x \in L^2((a,b);\mathbb{R}^n) \mid \mathcal{H}x \in H^1((a,b);\mathbb{R}^n),\ W_B \left(\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right) = 0\} \tag{6.10}$$

generates a contraction semigroup on the state space

$$X = L^2((a,b);\mathbb{R}^n) \tag{6.11}$$

with inner product

$$\langle f, g \rangle_X = \frac{1}{2} \int_a^b f(\zeta)^T \mathcal{H}(\zeta) g(\zeta) d\zeta. \tag{6.12}$$

In the following lemma, we show that the norm/energy of a state trajectory can be bounded by the energy at one of the boundaries. The proof is based on an idea of Cox and Zuazua in [4].

Lemma 6.2.2. *Consider the generator A of the contraction semigroup $(T(t))_{t \geq 0}$ given by (6.9) and (6.10). Let $x_0 \in X$ be any initial condition, then for sufficiently large $\tau > 0$ the state trajectory $x(t) := T(t)x_0$ satisfies*

$$\|x(\tau)\|_X^2 \leq c \int_0^\tau \|(\mathcal{H}x)(b,t)\|^2 \, dt \qquad \text{and} \tag{6.13}$$

$$\|x(\tau)\|_X^2 \leq c \int_0^\tau \|(\mathcal{H}x)(a,t)\|^2 \, dt, \tag{6.14}$$

where $c > 0$ is a constant that only depends on τ and not on x_0.

Proof: Let x_0 be an element in the domain of A and let $x(t) := T(t)x_0$. Since $x_0 \in D(A)$, we know that $x(t) \in D(A)$ for all t. Furthermore, $x(t)$ is the classical solution of (6.6).

For this trajectory, we define the (positive) function for $\zeta \in [a,b]$

$$F(\zeta) = \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^T(\zeta,t)\mathcal{H}(\zeta)x(\zeta,t)\,dt, \tag{6.15}$$

where we assume that $\gamma > 0$ and $\tau > 2\gamma(b-a)$. This last condition implies that we are not integrating over a negative time interval.

Differentiating this function with respect to ζ gives

$$\begin{aligned}\frac{dF}{d\zeta}(\zeta) = &\int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^T(\zeta,t)\frac{\partial}{\partial\zeta}(\mathcal{H}(\zeta)x(\zeta,t))\,dt + \\ &\int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} \left(\frac{\partial}{\partial\zeta}x(\zeta,t)\right)^T \mathcal{H}(\zeta)x(\zeta,t)\,dt + \\ &\gamma x^T(\zeta,\gamma(b-\zeta))\mathcal{H}(\zeta)x(\zeta,\gamma(b-\zeta)) + \\ &\gamma x^T(\zeta,\tau-\gamma(b-\zeta))\mathcal{H}(\zeta)x(\zeta,\tau-\gamma(b-\zeta)).\end{aligned}$$

Since P_1 is non-singular and since x satisfies (6.6), we obtain (for simplicity we omit the dependence on ζ and t)

$$\begin{aligned}\frac{dF}{d\zeta}(\zeta) = &\int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^T P_1^{-1}\left(\frac{\partial x}{\partial t} - P_0\mathcal{H}x\right)dt \\ &+ \gamma x^T(\zeta,\tau-\gamma(b-\zeta))\mathcal{H}(\zeta)x(\zeta,\tau-\gamma(b-\zeta)) \\ &+ \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} \left(P_1^{-1}\frac{\partial x}{\partial t} - \frac{d\mathcal{H}}{d\zeta}x - P_1^{-1}P_0\mathcal{H}x\right)^T x\,dt \\ &+ \gamma x^T(\zeta,\gamma(b-\zeta))\mathcal{H}(\zeta)x(\zeta,\gamma(b-\zeta)) \\ = &\int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^T P_1^{-1}\frac{\partial x}{\partial t} + \frac{\partial x}{\partial t}^T P_1^{-1}x\,dt \\ &- \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^T\frac{d\mathcal{H}}{d\zeta}x\,dt \\ &- \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^T\left(\mathcal{H}P_0^TP_1^{-1} + P_1^{-1}P_0\mathcal{H}\right)x\,dt \\ &+ \gamma x^T(\zeta,\tau-\gamma(b-\zeta))\mathcal{H}(\zeta)x(\zeta,\tau-\gamma(b-\zeta)) \\ &+ \gamma x^T(\zeta,\gamma(b-\zeta))\mathcal{H}(\zeta)x(\zeta,\gamma(b-\zeta))\end{aligned}$$

where we have used that $P_1^T = P_1$, $\mathcal{H}^T = \mathcal{H}$. The first integral can be solved, and so we find

$$\begin{aligned}\frac{dF}{d\zeta}(\zeta) = &\ x^T(\zeta,t)P_1^{-1}x(\zeta,t)\Big|_{t=\gamma(b-\zeta)}^{t=\tau-\gamma(b-\zeta)} \\ &- \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^T \frac{d\mathcal{H}}{d\zeta}\, x\, dt \\ &- \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^T \left(\mathcal{H}P_0^T P_1^{-1} + P_1^{-1}P_0\mathcal{H}\right) x\, dt \\ &+ \gamma x^T(\zeta,\tau-\gamma(b-\zeta))\mathcal{H}(\zeta)x(\zeta,\tau-\gamma(b-\zeta)) \\ &+ \gamma x^T(\zeta,\gamma(b-\zeta))\mathcal{H}(\zeta)x(\zeta,\gamma(b-\zeta)).\end{aligned}$$

By simplifying the equation above one obtains

$$\begin{aligned}\frac{dF}{d\zeta}(\zeta) = &- \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^T \left(\mathcal{H}P_0^T P_1^{-1} + P_1^{-1}P_0\mathcal{H} + \frac{d\mathcal{H}}{d\zeta}\right) x\, dt \\ &+ x^T(\zeta,\tau-\gamma(b-\zeta))\left[P_1^{-1}+\gamma\mathcal{H}(\zeta)\right]x(\zeta,\tau-\gamma(b-\zeta)) \\ &+ x^T(\zeta,\gamma(b-\zeta))\left[-P_1^{-1}+\gamma\mathcal{H}(\zeta)\right]x(\zeta,\gamma(b-\zeta)).\end{aligned}$$

By choosing γ large enough, i.e., by choosing τ large, we get that $P_1^{-1}+\gamma\mathcal{H}$ and $-P_1^{-1}+\gamma\mathcal{H}$ are coercive (positive definite). This in turn implies that (for τ large enough)

$$\frac{dF}{d\zeta}(\zeta) \geq -\int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^T \big(\mathcal{H}P_0^T P_1^{-1} + P_1^{-1}P_0\mathcal{H} + \frac{d\mathcal{H}}{d\zeta}\big)\, x\, dt.$$

Since P_1 and P_0 are constant matrices and, by assumption, $\frac{d\mathcal{H}}{d\zeta}(\zeta)$ is bounded, we can find a $\kappa > 0$ such that for all $\zeta \in [a,b]$ there holds

$$\mathcal{H}(\zeta)P_0^T P_1^{-1} + P_1^{-1}P_0\mathcal{H}(\zeta) + \frac{d\mathcal{H}}{d\zeta} \leq \kappa\mathcal{H}(\zeta).$$

Thus we find that

$$\frac{dF}{d\zeta}(\zeta) \geq -\kappa \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^T(\zeta,t)\mathcal{H}(\zeta)x(\zeta,t)\, dt = -\kappa\, F(\zeta), \tag{6.16}$$

where we used (6.15).

This inequality implies an inequality for F at different points. To prove this, we denote (for simplicity) the derivative of F by F'. From (6.16) we have that for all $\zeta_1, \zeta_2 \in [a,b]$ with $\zeta_1 \leq \zeta_2$

$$\int_{\zeta_1}^{\zeta_2} \frac{F'(\zeta)}{F(\zeta)}\, d\zeta \geq -\kappa \int_{\zeta_1}^{\zeta_2} d\zeta \tag{6.17}$$

Or, equivalently,

$$\ln\big(F(\zeta_2)\big) - \ln\big(F(\zeta_1)\big) \geq -\kappa\,(\zeta_2-\zeta_1). \tag{6.18}$$

Taking the exponential of this expression, and choosing $\zeta_2 = b$, gives:

$$F(b) \geq F(\zeta_1)\, e^{-\kappa\,(b-\zeta_1)} \geq F(\zeta_1)\, e^{-\kappa\,(b-a)} \quad \text{for} \quad \zeta_1 \in [a,b]. \tag{6.19}$$

On the other hand, since $\|x(t_2)\|_X \leq \|x(t_1)\|_X$ for any $t_2 \geq t_1$ (by the contraction property of the semigroup), we deduce that

$$\begin{aligned}\int_{\gamma(b-a)}^{\tau-\gamma(b-a)} \|x(t)\|_X^2\, dt &\geq \|x(\tau-\gamma(b-a))\|_X^2 \int_{\gamma(b-a)}^{\tau-\gamma(b-a)} dt \\ &= (\tau - 2\gamma(b-a))\|x(\tau-\gamma(b-a))\|_X^2.\end{aligned}$$

Using the definition of $F(\zeta)$ and $\|x(t)\|_X^2$, see (6.15) and (6.12), together with the equation above, the estimate (6.19), and the coercivity of $\mathcal{H}$ we obtain

$$\begin{aligned}2(\tau-2\gamma(b-a))\|x(\tau)\|_X^2 &\leq 2(\tau-2\gamma(b-a))\|x(\tau-\gamma(b-a))\|_X^2 \\ &\leq \int_a^b \int_{\gamma(b-a)}^{\tau-\gamma(b-a)} x^T(\zeta,t)\mathcal{H}(\zeta)x(\zeta,t)\, dt\, d\zeta \\ &\leq \int_a^b \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^T(\zeta,t)\mathcal{H}(\zeta)x(\zeta,t)\, dt\, d\zeta \\ &= \int_a^b F(\zeta)\, d\zeta \leq (b-a)F(b)\, e^{\kappa\,(b-a)} \\ &= (b-a)\, e^{\kappa\,(b-a)} \int_0^\tau x^T(b,t)\mathcal{H}(b)x(b,t)\, dt \\ &\leq Mm^{-1}(b-a)\, e^{\kappa\,(b-a)} \int_0^\tau \|(\mathcal{H}x)(b,t)\|^2\, dt.\end{aligned}$$

Hence for our choice of τ we have that

$$\|x(\tau)\|_X^2 \leq c \int_0^\tau \|(\mathcal{H}x)(b,t)\|^2\, dt, \tag{6.20}$$

where $c = \frac{M(b-a)\, e^{\kappa\,(b-a)}}{2(\tau-2\gamma(b-a))m}$. This proves estimate (6.13) for $x_0 \in D(A)$. Although, in Theorem 6.2.3 we only need inequality (6.13) for $x_0 \in D(A)$, with the help of the previous chapter we can obtain it for all $x_0 \in X$.

We replace (6.7) by the relation $W_B \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} = u(t)$. Furthermore, we define the output as $y(t) = (\mathcal{H}x)(b,t)$. Since the homogeneous equation, i.e., u=0, generates a (contraction) semigroup, we have by Theorem 5.2.6 that this system is well-posed. In particular, this means that if $u = 0$ small changes in the initial condition, gives small changes in the state at time τ and the L^2-norm of the output. More specifically, see (5.17),

$$\|x(\tau)\|_X^2 + \int_0^\tau \|y(t)\|^2 dt \leq m_f \|x_0\|_X^2.$$

Since the domain of A is dense in X, and since c was not depending on the initial condition, we conclude that (6.13) holds for all initial conditions.

The second estimate follows by replacing $F(\zeta)$ in the argument above by

$$\tilde{F}(\zeta) = \int_{\gamma(\zeta-a)}^{\tau-\gamma(\zeta-a)} x^T(\zeta,t)\mathcal{H}(\zeta)x(\zeta,t)\,dt.$$

■

With this technical lemma, the proof of exponential stability is easy.

Theorem 6.2.3. *Consider the operator A defined in (6.9) and (6.10). Furthermore, we assume that the conditions in Assumption 6.2.1 are satisfied. If some some positive constant k one of the following conditions is satisfied for all $x_0 \in D(A)$*

$$\langle Ax_0, x_0\rangle_X + \langle x_0, Ax_0\rangle_X \leq -k\|(\mathcal{H}x_0)(b)\|^2 \tag{6.21}$$

$$\langle Ax_0, x_0\rangle_X + \langle x_0, Ax_0\rangle_X \leq -k\|(\mathcal{H}x_0)(a)\|^2, \tag{6.22}$$

then the system is exponentially stable.

Proof: Without loss of generality we assume that the first inequality (6.21) holds. Let $x_0 \in D(A)$, and let τ be the same as in Lemma 6.2.2. We denote $T(t)x_0$ by $x(t)$, and since $x_0 \in D(A)$ we know that $x(t) \in D(A)$ and $\dot{x}(t) = Ax(t)$. Using this differential equation, it is easy to see that

$$\frac{d\|x(t)\|_X^2}{dt} = \frac{d\langle x(t), x(t)\rangle}{dt} = \langle Ax(t), x(t)\rangle + \langle x(t)Ax(t)\rangle. \tag{6.23}$$

Using this and equation (6.21), we have that

$$\begin{aligned}\|x(\tau)\|_X^2 - \|x(0)\|_X^2 &= \int_0^\tau \frac{d\|x(t)\|_X^2}{dt}(t)\,dt \\ &\leq -k\int_0^\tau \|(\mathcal{H}x)(b,t)\|^2 dt.\end{aligned}$$

Combining this with (6.13), we find that

$$\|x(\tau)\|_X^2 - \|x(0)\|_X^2 \leq \frac{-k}{c}\|x(\tau)\|_X^2.$$

Thus $\|x(\tau)\|_X^2 \leq \frac{c}{c+k}\|x(0)\|_X^2$. From this we see that the semigroup $(T(t))_{t\geq 0}$ generated by A satisfies $\|T(\tau)\| < 1$, from which we obtain exponential stability, see Lemma 6.1.5. ■

Estimate (6.21) provides a simple way to prove the exponential stability property. We note that Theorem 1.2.1 implies

$$\langle Ax, x\rangle_X + \langle x, Ax\rangle_X = (\mathcal{H}x)^T(b)P_1(\mathcal{H}x)(b) - (\mathcal{H}x)^T(a)P_1(\mathcal{H}x)(a). \tag{6.24}$$

This equality can be used on a case by case to show exponential stability. However, when $W_B\Sigma W_B^T > 0$, then the system is exponentially stable.

Lemma 6.2.4. *Consider the system (6.6)–(6.7). If $W_B \Sigma W_B^T > 0$ and the conditions of Assumption 6.2.1 hold, then the system is exponentially stable.*

PROOF: By Lemma 2.4.1 we know that W_B can be written as $W_B = S\,(I+V, I-V)$. Since $W_B \Sigma W_B^T > 0$ we have that $VV^T < I$.

We define $W_C = \left(I + V^T, -I + V^T\right)$. It is easy to see that with this choice, W_C is a $n \times 2n$ matrix with rank n. Furthermore, since $VV^T < I$, the matrix $\left(\begin{smallmatrix} I+V & I-V \\ I+V^T & -I+V^T \end{smallmatrix}\right)$ is invertible. This implies that $\left(\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right)$ is invertible.

With the matrices W_B and W_C we define a system. Namely, we take the p.d.e. (6.6) with input and output

$$u(t) = W_B \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}, \qquad y(t) = W_C \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}.$$

Applying Theorem 3.4.2 to this system, we see that

$$\frac{d}{dt}\|x(t)\|_X^2 = \frac{1}{2} \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_{W_B,W_C} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}.$$

Choosing $u = 0$, $x \in D(A)$, and $t = 0$ is the above equation, we find that

$$\langle Ax, x\rangle_X + \langle x, Ax\rangle_X = \frac{1}{2} \begin{pmatrix} 0 & y^T \end{pmatrix} P_{W_B,W_C} \begin{pmatrix} 0 \\ y \end{pmatrix}. \tag{6.25}$$

The matrix P_{W_B,W_C} is the inverse of the matrix $\left(\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right) \Sigma \left(W_B^T \; W_C^T \right)$. By the choice of W_C we find that

$$\begin{pmatrix} W_B \\ W_C \end{pmatrix} \Sigma \begin{pmatrix} W_B^T & W_C^T \end{pmatrix} = \begin{pmatrix} S[2I - 2VV^T]S^T & 0 \\ 0 & -2I + 2V^T V \end{pmatrix}.$$

Combining this with (6.25) we find that

$$\langle Ax, x\rangle_X + \langle x, Ax\rangle_X = \frac{1}{4} y^T[-I + V^T V]^{-1} y \le -m_1 \|y\|^2 \tag{6.26}$$

for some $m_1 > 0$. Here we have used that $VV^T < I$, and hence $V^T V - I < 0$.

The relation between u, y and x is given by

$$\begin{pmatrix} 0 \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} W_B \\ W_C \end{pmatrix} \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix} \begin{pmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{pmatrix} := W \begin{pmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{pmatrix},$$

where we have used that $u = 0$. Since P_1 and $\left(\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right)$ are non-singular, it follows that the matrix W is invertible and, in particular, $\|W\,w\|^2 \ge m_2 \|w\|^2$ for some real $m_2 > 0$. Taking norms on both sides yields

$$\|y\|^2 = \|W \begin{pmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{pmatrix}\|^2 \ge m_2 \left\| \begin{pmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{pmatrix} \right\|^2 \ge m_2 \|(\mathcal{H}x)(b)\|^2. \tag{6.27}$$

Combining the estimates (6.26) with (6.27) gives

$$\langle Ax, x\rangle_X + \langle x, Ax\rangle_X \le -m_1\|y\|^2 \le -m_1 m_2 \|(\mathcal{H}x)(b)\|^2.$$

Similarly, we find

$$\langle Ax, x\rangle_X + \langle x, Ax\rangle_X \le -m_1 m_2 \|(\mathcal{H}x)(a)\|^2.$$

Concluding we see that (6.21) holds, and so by Theorem 6.2.3) we conclude exponential stability. ■

The situation as described in this lemma will not happen often. It implies that you have as many dampers as boundary controls. In practice less dampers are necessary as is shown in the example of the following section.

6.3. Examples

In this section we show how to apply the results of the previous section. We show that once the input (boundary conditions) and the output are selected, a simple matrix condition allows to conclude on the exponential stability.

Example 6.3.1 Consider the *transmission line* on the spatial interval $[a, b]$

$$\begin{aligned} \frac{\partial Q}{\partial t}(\zeta, t) &= -\frac{\partial}{\partial \zeta}\frac{\phi(\zeta, t)}{L(\zeta)} \\ \frac{\partial \phi}{\partial t}(\zeta, t) &= -\frac{\partial}{\partial \zeta}\frac{Q(\zeta, t)}{C(\zeta)}. \end{aligned} \tag{6.28}$$

Here $Q(\zeta, t)$ is the charge at position $\zeta \in [a, b]$ and time $t > 0$, and $\phi(\zeta, t)$ is the flux at position ζ and time t. C is the (distributed) capacity and L is the (distributed) inductance. This example we studied in Example 1.1.1 and 4.2.2. To the p.d.e. we add the following input and output, see also Example E:4.2.2

$$u(t) = \begin{pmatrix} \frac{Q(b,t)}{C(b)} \\ \frac{Q(a,t)}{C(a)} \end{pmatrix} = \begin{pmatrix} V(b, t) \\ V(a, t) \end{pmatrix} \tag{6.29}$$

$$y(t) = \begin{pmatrix} \frac{\phi(b,t)}{L(b)} \\ \frac{\phi(a,t)}{L(a)} \end{pmatrix} = \begin{pmatrix} I(b, t) \\ I(a, t) \end{pmatrix}. \tag{6.30}$$

First we want to know whether the homogeneous system is (exponentially) stable. There for the determine the W_B associated to (6.29). The boundary effort and flow are given by

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -I(b) + I(a) \\ -V(b) + V(a) \\ V(b) + V(a) \\ I(b) + I(a) \end{pmatrix}$$

Hence W_B is given as
$$W_B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$
A simple calculation gives that $W_B \Sigma W_B^T = 0$. Hence the homogeneous p.d.e. generates a unitary group, and cannot be (strongly) stable, see Exercise ??

Now we apply an output feedback. If we apply a full output feedback, then it is not hard to show that we have obtained an exponentially stable system, see Exercise ??

We want to consider a more interesting example, in which we only apply a feedback on one of the boundaries. This is, we set the first input to zero, and put a resistor at the other end. This implies that we have the p.d.e. (6.28) with boundary conditions
$$V(a,t) = 0, \qquad V(b,t) = RI(b,t), \tag{6.31}$$
with $R > 0$. This leads to the following (new) W_B;
$$W_B = \frac{1}{\sqrt{2}} \begin{pmatrix} R & -1 & 1 & -R \\ 0 & 1 & 1 & 0 \end{pmatrix}. \tag{6.32}$$
Hence we have that $W_B \Sigma W_B^T = \left(\begin{smallmatrix} 2R & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Since this is not positive definite, we may not apply Lemma 6.2.4. Hence we must check whether one of the condition of Theorem 6.2.3 is satisfied. For this we return to the original balance equation, equation (6.24).

Using (6.24), we get
$$\langle Ax, x\rangle_X + \langle x, Ax\rangle_X = V(a)I(a) - V(b)I(b) = -RI(b)^2. \tag{6.33}$$
Furthermore, we have that $(\mathcal{H}x)(b) = \left(\begin{smallmatrix} V(b) \\ I(b) \end{smallmatrix}\right)$. Thus
$$\|(\mathcal{H}x)(b)\|^2 = V(b)^2 + I(b)^2 = (R^2+1)I(b)^2. \tag{6.34}$$
Combining the two previous equations, we find that
$$\langle Ax, x\rangle_X + \langle x, Ax\rangle_X \le -\frac{R}{1+R^2}\|(\mathcal{H}x)(b)\|^2. \tag{6.35}$$
Hence by Theorem 6.2.3 we conclude that putting a resistor at one end of the transmission line stabilizes the system exponentially. $\square$

6.4. Exercises

6.1. Prove that the expression given in (6.3) defines a strongly continuous semigroup on $L^2(0,\infty)$.

6.2. Show that a unitary group cannot be strongly stable.

6.5. Notes and references

The results in this chapter can be found in [29].

Chapter 7
Systems with Dissipation

7.1. Introduction

As the title indicates, in this chapter we study systems with dissipation. These systems appear naturally in many physical situations. For instance, by internal damping in a vibrating string, or by diffusion of heat in a metal bar. The behavior of these models will be different in nature than the model we have seen until now. For instance, since energy/heat dissipates, and hence cannot be recovered, we will not have a group, i.e., cannot go backward in time. Although the behavior is different, the results as obtained in the previous chapters can be used to prove existence and uniqueness for our class of dissipative systems. We begin by recapitulating two examples from Chapter 1.

Example 7.1.1 (Damped wave equation) Consider the one-dimensional wave equation of Example 1.1.2. One cause of damping is known as structural damping. Structural damping arises from internal friction in a material converting vibrational energy into heat. In this case the vibrating string is modeled by

$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right] + \frac{k_s}{\rho(\zeta)}\frac{\partial^2}{\partial \zeta^2}\left[\frac{\partial w}{\partial t}(\zeta,t)\right], \quad \zeta \in [a,b],\ t \geq 0, \tag{7.1}$$

where k_s is a positive constant and the other variables have the same meaning as in Example 1.1.2. □

In the previous example, we can still recognize a system which we have studied before, namely the (undamped) wave equation. In the model of the heat conduction the relation with our class of port-Hamiltonian systems seems completely lost.

Example 7.1.2 (Heat conduction) The model of heat conduction is given by

$$\frac{\partial T}{\partial t}(\zeta,t) = \frac{1}{c_V}\frac{\partial}{\partial \zeta}\left(\lambda(\zeta)\frac{\partial T(\zeta,t)}{\partial \zeta}\right). \tag{7.2}$$

where $T(\zeta,t)$ denotes the temperature at position $\zeta \in [a,b]$ and time t, c_V is the heat capacity, and $\lambda(\zeta)$ denotes the heat conduction coefficient. □

As we did for the undamped system, we see these models as examples of a general class of models. For this new model class, we investigate existence of solutions, boundary control, etc. We begin by showing that this new model class is the class of port-Hamiltonian system with an extra closure relation. Based on the underlying port-Hamiltonian system, proving existence of solutions, etc. will be easy.

7.2. General class of system with dissipation.

The general equation describing our class of port-Hamiltonian system was given by

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1 \frac{\partial \mathcal{H}x}{\partial \zeta}(\zeta,t) + P_0\left(\mathcal{H}x\right)(\zeta,t) \tag{7.3}$$

Now we add a dissipation term, and we obtain the following p.d.e.

$$\frac{\partial x}{\partial t}(\zeta,t) = \left(\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*\right)\left(\mathcal{H}x\right)(\zeta,t), \quad x(\zeta,0) = x_0(\zeta),\ \zeta \in [a,b], \tag{7.4}$$

where

$$\mathcal{J}x = P_1 \frac{\partial x}{\partial \zeta} + P_0\, x, \quad \mathcal{G}_R f = G_1 \frac{\partial f}{\partial \zeta} + G_0\, f, \quad \mathcal{G}_R^* x = -G_1^T \frac{\partial x}{\partial \zeta} + G_0^T\, x, \tag{7.5}$$

G_R^* is known as the *formal adjoint* of G_R. As before, we assume

Assumption 7.2.1:

- P_1 is a symmetric real $n \times n$ matrix;
- P_0 is an anti-symmetric real $n \times n$ matrix;
- For all $\zeta \in [a,b]$ the $n \times n$ matrix $\mathcal{H}(\zeta)$ is real, symmetric, and $mI \leq \mathcal{H}(\zeta) \leq MI$, for some $M, m > 0$ independent of ζ.

On the new term we assume

- G_1 and G_0 are real $n \times r$ matrices;
- For all $\zeta \in [a,b]$ the $r \times r$ matrix $S(\zeta)$ is real, symmetric, and $m_1 I \leq S(\zeta) \leq M_1 I$, for some $M_1, m_1 > 0$ independent of ζ. ♡

Note that we have removed the invertibility assumption on P_1. This assumption will be replaced by another assumption, see Assumption 7.2.7.

First we check whether the two examples from the introduction are in the class defined by (7.4) and (7.5).

Example 7.2.2 For the damped wave equation of Example 7.1.1 we have that the state is given by $x = \begin{pmatrix} \rho \frac{\partial w}{\partial t} \\ \frac{\partial w}{\partial \zeta} \end{pmatrix}$. The matrices $\mathcal{H}$, P_1 and P_0 are the ones found in (1.20), i.e.,

$$\mathcal{H}(\zeta) = \begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad P_0 = 0.$$

Using this and the formula for the state we find that (here omit ζ and t)

$$\begin{aligned}
\frac{\partial}{\partial t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \zeta}\left[\begin{pmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] + \begin{pmatrix} k_s \frac{\partial^2 \rho^{-1} x_1}{\partial \zeta^2} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \zeta}\left[\begin{pmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] + k_s \begin{pmatrix} \frac{\partial^2}{\partial \zeta^2} & 0 \\ 0 & 0 \end{pmatrix}\left[\begin{pmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \zeta}\left[\begin{pmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] + \\
&\quad - \begin{pmatrix} \frac{\partial}{\partial \zeta} \\ 0 \end{pmatrix} k_s \begin{pmatrix} -\frac{\partial}{\partial \zeta} & 0 \end{pmatrix}\left[\begin{pmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right]
\end{aligned}$$

Hence

$$G_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad G_0 = 0 \quad \text{and} \quad S = k_s.$$

Concluding, we see that the damped wave equation can be written in the form (7.4)–(7.5). Furthermore, it is easy to see that the conditions in Assumption 7.2.1 are satisfied. □

Example 7.2.3 The equation for heat condition can also be written in the format (7.4)–(7.5). We choose as state $x(\zeta, t)$ the temperature $T(\zeta, t)$ and furthermore, we choose

$$P_1 = P_0 = G_0 = 0, \quad G_1 = 1, \quad \mathcal{H}(\zeta) = \frac{1}{c_V} \text{ and } S(\zeta) = \lambda(\zeta).$$

Then we easily see that (7.4) becomes (7.2). The conditions of Assumption 7.2.1 are trivially satisfied. □

Similar to Theorem 1.2.1, there holds a balance equation for the system (7.4)–(7.5) for the "energy"

$$E(t) = \frac{1}{2}\int_a^b x(\zeta, t)^T \mathcal{H}(\zeta) x(\zeta, t) d\zeta. \tag{7.6}$$

Lemma 7.2.4. *Under the assumptions 7.2.1 the following balance equation holds for classical solution to the p.d.e. (7.4) and (7.5)*

$$\begin{aligned}
\frac{dE}{dt}(t) =& \frac{1}{2}\left[(\mathcal{H}x)^T(\zeta, t) P_1 (\mathcal{H}x)(\zeta, t)\right]_a^b - \qquad (7.7)\\
& \frac{1}{2}\left[(\mathcal{H}x)^T(\zeta, t) G_1 (S\mathcal{G}_R^* \mathcal{H}x)(\zeta, t) + (S\mathcal{G}_R^*\mathcal{H}x)^T(\zeta, t) G_1^T (\mathcal{H}x)(\zeta, t)\right]_a^b - \\
& \int_a^b (\mathcal{G}_R^*\mathcal{H}x)^T(\zeta, t) S(\zeta) (\mathcal{G}_R^*\mathcal{H}x)(\zeta, t) d\zeta,
\end{aligned}$$

where $E(t)$ is given by (7.6).

PROOF: The proof is very similar to that of Theorem 1.2.1. From that proof we easily see that

$$2\frac{dE}{dt}(t) = \left[(\mathcal{H}x)^T(\zeta,t)P_1(\mathcal{H}x)(\zeta,t)\right]_a^b - \int_a^b \left[(\mathcal{G}_R S\mathcal{G}_R^*)(\mathcal{H}x)(\zeta,t)\right]^T(\mathcal{H}x)(\zeta,t) + x(\zeta,t)^T\mathcal{H}(\zeta)\left[(\mathcal{G}_R S\mathcal{G}_R^*)(\mathcal{H}x)(\zeta,t)\right]d\zeta. \tag{7.8}$$

We concentrate on the second term, and we introduce some notation to simplify the formula. We write $z = \mathcal{H}x$, $q = \mathcal{G}_R^* z$. Furthermore, we omit the t. Using this combined with the fact that $\mathcal{H}(\zeta)$ is symmetric, we find that the second term of (7.8) becomes

$$\int_a^b \left[\mathcal{G}_R Sq(\zeta)\right]^T z(\zeta) + z(\zeta)^T\left[\mathcal{G}_R Sq(\zeta)\right]d\zeta. \tag{7.9}$$

Using integration by parts, we have that

$$\begin{aligned}
\int_a^b z(\zeta)^T\left[\mathcal{G}_R Sq(\zeta)\right]d\zeta &= \int_a^b z(\zeta)^T\left[G_1\frac{\partial}{\partial\zeta}(Sq)(\zeta) + G_0 q(\zeta)\right]d\zeta \\
&= \int_a^b -\frac{\partial z}{\partial\zeta}(\zeta)^T\left[G_1(\mathcal{S}q)(\zeta) + G_0(Sq)(\zeta)\right]d\zeta + \left[z(\zeta)^T G_1(Sq)(\zeta)\right]_a^b \\
&= \int_a^b \left[-G_1^T\frac{\partial z}{\partial\zeta}(\zeta) + G_0^T z(\zeta)\right]^T(Sq)(\zeta)d\zeta + \left[z(\zeta)^T G_1(Sq)(\zeta)\right]_a^b \\
&= \int_a^b (\mathcal{G}_R^* z)^T(\zeta)(Sq)(\zeta)d\zeta + \left[z(\zeta)^T G_1(Sq)(\zeta)\right]_a^b \\
&= \int_a^b (\mathcal{G}_R^* z)^T(\zeta)S(\zeta)(\mathcal{G}_R^* z)(\zeta)d\zeta + \left[z(\zeta)^T G_1(S\mathcal{G}_R^* z)(\zeta)\right]_a^b.
\end{aligned}$$

Using this and its transpose in (7.9) we find that (7.8) becomes

$$\begin{aligned}
2\frac{dE}{dt}(t) = {}&\left[(\mathcal{H}x)^T(\zeta,t)P_1(\mathcal{H}x)(\zeta,t)\right]_a^b - \\
&2\int_a^b (\mathcal{G}_R^*\mathcal{H}x)^T(\zeta)S(\zeta)(\mathcal{G}_R^*\mathcal{H}x)(\zeta)d\zeta - \\
&\left[(\mathcal{H}x)^T(\zeta)G_1(S\mathcal{G}_R^*\mathcal{H}x)(\zeta) + (S\mathcal{G}_R^*\mathcal{H}x)^T(\zeta)G_1^T(\mathcal{H}x)(\zeta).\right]_a^b
\end{aligned} \tag{7.10}$$ ■

From (7.7) we clearly see that if the boundary conditions are such that the term

$$\Big[(\mathcal{H}x)^T(\zeta,t)P_1(\mathcal{H}x)(\zeta,t) - (\mathcal{H}x)^T(\zeta,t)G_1(S\mathcal{G}_R^*\mathcal{H}x)(\zeta,t) + (S\mathcal{G}_R^*\mathcal{H}x)^T(\zeta,t)G_1^T(\mathcal{H}x)(\zeta,t)\Big]_a^b \leq 0,$$

then the energy is decaying. However, this is under the assumption of the existence of a solution.

As said in the introduction, we want to use the results form the previous chapters for proving existence and uniqueness of solutions. However, the nature of the system with dissipation seems completely different to that of the system we studied till now. The following example indicates how we may use an extended port-Hamiltonian system together with a closure relation for obtaining a system with dissipation.

Example 7.2.5 (General idea) Consider the following model. This could be the model of the transmission line, equation (1.1), in which we have taken all physical parameters equal to one,

$$\frac{\partial x_1}{\partial t}(\zeta,t) = -\frac{\partial x_2}{\partial \zeta}(\zeta,t) \tag{7.11}$$

$$\frac{\partial x_2}{\partial t}(\zeta,t) = -\frac{\partial x_1}{\partial \zeta}(\zeta,t). \tag{7.12}$$

Instead of looking at it as a differential equation, we regard this as an relation between the variables e and f, given by

$$\begin{pmatrix} f_1 \\ f_p \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial}{\partial\zeta} \\ -\frac{\partial}{\partial\zeta} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_p \end{pmatrix} := \mathcal{J}_e e \tag{7.13}$$

in which we substituted $f = \dot{x}$ and $e = x$. The power balance for the p.d.e. came directly from the following relation, see Section 1.3

$$\int f^T(\zeta)e(\zeta) + e^T(\zeta)f(\zeta)d\zeta = 0. \tag{7.14}$$

Or equivalently,

$$\int f_1(\zeta)e_1(\zeta) + f_p(\zeta)e_p(\zeta)d\zeta = 0. \tag{7.15}$$

If we add the closure relation

$$e_p = Sf_p \tag{7.16}$$

with $S = S(\zeta)$ a bounded and strictly positive function, to the equation (7.13), then we see that for a pair (f, e) satisfying both equations

$$\begin{aligned}\int f_1(\zeta)e_1(\zeta)d\zeta &= \int f_1(\zeta)e_1(\zeta) + f_p(\zeta)e_p(\zeta)d\zeta - \int f_p(\zeta)e_p(\zeta)d\zeta \\ &= 0 - \int f_p(\zeta)S(\zeta)f_p(\zeta)d\zeta \leq 0, \end{aligned} \tag{7.17}$$

where we used (7.15).

Next we defined a new system using (7.13) and (7.16). We take $f_1 = \frac{\partial x}{\partial t}$ and $e_1 = x$. Using equations (7.13) and (7.16), we find that

$$\frac{\partial x}{\partial t} = f_1 = \frac{\partial e_p}{\partial \zeta} = \frac{\partial S f_p}{\partial \zeta} = \frac{\partial}{\partial \zeta}\left(S\frac{\partial e_1}{\partial \zeta}\right) = \frac{\partial}{\partial \zeta}\left(S\frac{\partial x}{\partial \zeta}\right). \tag{7.18}$$

Hence we have obtained the diffusion equation. Furthermore, from (7.17) we find that

$$\frac{d}{dt}\int x(\zeta,t)^2 d\zeta = 2\int \frac{\partial x}{\partial t}(\zeta,t)x(\zeta,t)d\zeta = 2\int f_1 e_1 d\zeta \le 0.$$

Hence the "energy" $\int x(\zeta,t)^2 d\zeta$ is dissipating. $\square$

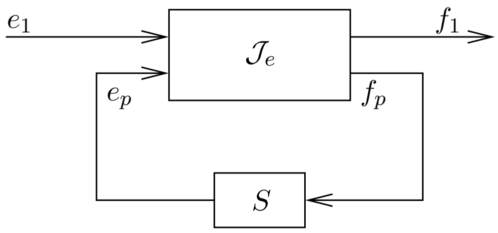

Figure 7.1.: Interconnection structure.

In the example we see that we can obtain a system with dissipation by adding a closure relation to a larger port-Hamiltonian system. Our system (7.4)–(7.5) can be seen in the same way.

Lemma 7.2.6. *The operator $\mathcal{J} - \mathcal{G}_R\mathcal{S}\mathcal{G}_R^*$ defined (7.5) can be seen as the mapping $\mathcal{J}_e$ together with the closure relation $e_p = \mathcal{S}f_p$, where $\mathcal{J}_e$ is defined by*

$$\mathcal{J}_e\begin{pmatrix} e_1 \\ e_p \end{pmatrix} = \begin{pmatrix} P_1 & G_1 \\ G_1^T & 0 \end{pmatrix}\frac{\partial}{\partial\zeta}\begin{pmatrix} e_1 \\ e_p \end{pmatrix} + \begin{pmatrix} P_0 & G_0 \\ -G_0^T & 0 \end{pmatrix}\begin{pmatrix} e \\ e_p \end{pmatrix} \tag{7.19}$$

Proof: Define the image of $\mathcal{J}_e\left(\begin{smallmatrix} e_1 \\ e_p \end{smallmatrix}\right)$ as $\left(\begin{smallmatrix} f_1 \\ f_p \end{smallmatrix}\right)$, see also Figure 7.1. Then

$$\begin{aligned} f_1 &= P_1\frac{\partial e}{\partial\zeta} + G_1\frac{\partial e_p}{\partial\zeta} + P_0 e_1 + G_0 e_p = \mathcal{J}e_1 + \left(G_1\frac{\partial}{\partial\zeta} + G_0\right)e_p \\ &= \mathcal{J}e_1 + \mathcal{G}_R e_p = \mathcal{J}e_1 + \mathcal{G}_R\mathcal{S}f_p \\ &= \mathcal{J}e_1 + \mathcal{G}_R\mathcal{S}\left(G_1^T\frac{\partial e_1}{\partial\zeta} - G_0^T e_1\right) \\ &= \mathcal{J}e_1 - \mathcal{G}_R\mathcal{S}\mathcal{G}_R^* e_1. \end{aligned}$$

This proves the assertion. $\blacksquare$

From equation (7.19) we see that it is natural to introduce a new P_1 and P_0 as

$$P_{1,\text{ext}} = \begin{pmatrix} P_1 & G_1 \\ G_1^T & 0 \end{pmatrix}, \qquad P_{0,\text{ext}} = \begin{pmatrix} P_0 & G_0 \\ -G_0^T & 0 \end{pmatrix}. \tag{7.20}$$

By the conditions on P_1 and P_0, see Assumption 7.2.1, we have that $P_{1,\text{ext}}$ is symmetric, and $P_{0,\text{ext}}$ is anti-symmetric. To the conditions listed in Assumption 7.2.1 we add the following.

Assumption 7.2.7: The matrix $P_{1,\text{ext}}$ as defined in (7.20) is invertible. ♡

Under this assumption, we see that our operator (7.19) fits perfectly in the theory as developed in Section 2.3. The following theorem is a direct consequence of Theorem 2.3.2. By the Hilbert space X_{ext} we denote the space $L^2((a,b);\mathbb{R}^{n+r})$ with inner product

$$\left\langle \begin{pmatrix} x \\ x_p \end{pmatrix}, \begin{pmatrix} z \\ z_p \end{pmatrix} \right\rangle_{X_{\text{ext}}} = \int_a^b x(\zeta)^T \mathcal{H}(\zeta) z(\zeta) d\zeta + \int_a^b x_p(\zeta)^T z_p(\zeta) d\zeta. \tag{7.21}$$

Hence $X_{\text{ext}} = X \oplus L^2((a,b);\mathbb{R}^r)$. Furthermore, we define

$$\begin{pmatrix} f_{\partial,\mathcal{H}x,x_p} \\ e_{\partial,\mathcal{H}x,x_p} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} P_{1,\text{ext}} & -P_{1,\text{ext}} \\ I_{n+r} & I_{n+r} \end{pmatrix} \left(\begin{array}{c} \begin{pmatrix} (\mathcal{H}x)(b) \\ x_p(b) \end{pmatrix} \\ \begin{pmatrix} (\mathcal{H}x)(a) \\ x_p(a) \end{pmatrix} \end{array} \right). \tag{7.22}$$

The matrix W_B is a full rank, real matrix of size $(n+r) \times 2(n+r)$. With this notation, we can formulate the following theorem, which is an adaptation of Theorem 2.3.2 to our extended setting.

Theorem 7.2.8. *Under the conditions of Assumptions 7.2.1 and 7.2.7 we have that the operator A_{ext} defined as*

$$A_{\text{ext}} \begin{pmatrix} x \\ x_p \end{pmatrix} = P_{1,\text{ext}} \frac{\partial}{\partial \zeta} \begin{pmatrix} (\mathcal{H}x) \\ x_p \end{pmatrix} + P_{0,\text{ext}} \begin{pmatrix} (\mathcal{H}x) \\ x_p \end{pmatrix} \tag{7.23}$$

with domain given by

$$D(A_{\text{ext}}) = \left\{ \left(\begin{smallmatrix} x \\ x_p \end{smallmatrix} \right) \in X_{\text{ext}} \mid \left(\begin{smallmatrix} \mathcal{H}x \\ x_p \end{smallmatrix} \right) \in H^1((a,b);\mathbb{R}^{n+r}) \text{ with } W_B \left(\begin{smallmatrix} f_{\partial,\mathcal{H}x,x_p} \\ e_{\partial,\mathcal{H}x,x_p} \end{smallmatrix} \right) = 0 \right\} \tag{7.24}$$

generates a contraction semigroup on X_{ext} if and only if $W_B \Sigma W_B^T \geq 0$.

Based on this theorem and the fact S dissipates energy, we can prove that the operator associated to p.d.e. (7.4) generates a contraction semigroup provided that one uses the correct boundary conditions. The proof of this result is an application of the general result Theorem 7.3.3 combined with the previous theorem.

Theorem 7.2.9. *Denote by $\mathcal{J}$, $\mathcal{G}_R$ and $\mathcal{G}_R^*$ the operators as defined in (7.7). Furthermore, let the Assumptions 7.2.1 and 7.2.7 be satisfied and let W_B be a $(n+r) \times 2(n+r)$ matrix of full rank such that $W_B \Sigma W_B^T \geq 0$. Then the operator A_S defined as*

$$A_S x = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*) (\mathcal{H}x) \tag{7.25}$$

with domain

$$D(A_S) = \{x \in X \mid \mathcal{H}x \in H^1((a,b);\mathbb{R}^n) \text{ with } W_B \left(\begin{smallmatrix} f_{\partial,\mathcal{H}x,S\mathcal{G}_{R^*}(\mathcal{H}x)} \\ e_{\partial,\mathcal{H}x,-S\mathcal{G}_{R^*}(\mathcal{H}x)} \end{smallmatrix} \right) = 0\} \tag{7.26}$$

generates a contraction semigroup on X.

PROOF: If we can show that the operator A_S with its domain can be written as (7.39)–(7.40), and if the other conditions of Theorem 7.3.3 holds, then the proof becomes a straightforward application of this general theorem.

We take A_{ext} to be the operator defined by (7.23) with domain given by (7.24). Using the notation $\mathcal{J}$, $\mathcal{G}_R$ and $\mathcal{G}_R^*$ we see that this operator can be written as

$$A_{\mathrm{ext}} = \begin{pmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{pmatrix} \begin{pmatrix} \mathcal{H} & 0 \\ 0 & I \end{pmatrix}. \tag{7.27}$$

This is easy formulated into the set-up of Theorem 7.3.3 be defining

$$A_2 = -\mathcal{G}_R^* \mathcal{H}, \qquad D(A_2) = \{x \in L^2((a,b);\mathbb{R}^n) \mid \mathcal{H}x \in H^1((a,b);\mathbb{R}^n)\} \tag{7.28}$$

and

$$A_1 = \begin{pmatrix} \mathcal{J}\mathcal{H} & \mathcal{G}_R \end{pmatrix}, \qquad D(A_1) = D(A_{\mathrm{ext}}). \tag{7.29}$$

Hence A_1 has the domain given by (7.24). Note the domain of A_2 imposes no extra restriction the domain of A_1.

Next we define S to be the multiplication operator

$$(Sf)(\zeta) = S(\zeta) f(\zeta). \tag{7.30}$$

Using the assumption on S it is easy to see that $S \in \mathcal{L}(L^2((a,b);\mathbb{R}^r))$ and it satisfies (7.37).

By Theorem 7.3.3 we have that $A_1 \begin{bmatrix} h \\ S(A_2 h) \end{bmatrix}$ with domain $D(A_S) = \{h \in H_1 \mid \begin{bmatrix} h \\ S(A_2 h) \end{bmatrix} \in D(A_{\mathrm{ext}})\}$ generates a contraction semigroup on X. From (7.26) and (7.27) we see that this A_S is also given by (7.25) with domain (7.26). This concludes the proof.■

We begin by applying the theorem to the example of the heat conduction, see Example 7.1.2.

Example 7.2.10 In this example we want to investigate for which boundary conditions the p.d.e. (7.2) describing the heat conductivity generates a contraction semigroup. We begin by identifying the extended state. Using Example 7.5 and equation (7.20) we see that

$$P_{1,\mathrm{ext}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad P_{0,\mathrm{ext}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies that

$$\begin{pmatrix} f_{\partial,g,x_p} \\ e_{\partial,g,x_p} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} g(b) \\ x_p(b) \\ g(a) \\ x_p(a) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x_p(b) - x_p(a) \\ g(b) - g(a) \\ g(b) + g(a) \\ x_p(b) + x_p(a) \end{pmatrix}. \tag{7.31}$$

For the matrix W_B describing the boundary condition, we observe the following. It should be full rank, and $W_B \Sigma W_B^T \geq 0$, and we describe the boundary conditions as being in its kernel. By Lemma 2.4.1 we see that without loss of generality, we may assume that

$$W_B = (I+V \quad I-V)$$

with $VV^T \leq I$. Using (7.31) we see that $W_B \begin{pmatrix} f_{\partial,g,x_p} \\ e_{\partial,g,x_p} \end{pmatrix} = 0$ is equivalent to

$$(I+V) \begin{pmatrix} x_p(b) - x_p(a) \\ g(b) - g(a) \end{pmatrix} = (V-I) \begin{pmatrix} g(b) + g(a) \\ x_p(b) + x_p(a) \end{pmatrix}. \tag{7.32}$$

From the theory of Chapter 2 we know that with the above boundary conditions the extended operator generates a contraction semigroup, provided $VV^T \leq I$. Using Theorem 7.2.9, we see that under the same conditions on V the operator

$$A_S := \frac{1}{c_V} \frac{d}{d\zeta} \left(\lambda(\zeta) \frac{d}{d\zeta} \right) \tag{7.33}$$

with domain

$$D(A_S) = \{ x \in L^2((a,b);\mathbb{R}) \mid x \in H^1((a,b);\mathbb{R}), \lambda \frac{dx}{d\zeta} \in H^1((a,b);\mathbb{R}) \text{ and} \tag{7.34}$$
$$(I+V) \begin{pmatrix} \frac{\lambda(b)}{c_v} \frac{dx}{d\zeta}(b) - \frac{\lambda(a)}{c_v} \frac{dx}{d\zeta}(a) \\ \frac{1}{c_v} x(b) - \frac{1}{c_v} x(a) \end{pmatrix} = (V-I) \begin{pmatrix} \frac{\lambda(b)}{c_v} \frac{dx}{d\zeta}(b) + \frac{\lambda(a)}{c_v} \frac{dx}{d\zeta}(a) \\ \frac{1}{c_v} x(b) + \frac{1}{c_v} x(a) \end{pmatrix} \Bigg\}$$

generates a contraction semigroup on $L^2((a,b);\mathbb{R})$. This implies that the homogeneous p.d.e. (7.2) with boundary conditions

$$(I+V) \begin{pmatrix} \lambda(b)\frac{dx}{d\zeta}(b) - \lambda(a)\frac{dx}{d\zeta}(a) \\ x(b) - x(a) \end{pmatrix} = (V-I) \begin{pmatrix} \lambda(b)\frac{dx}{d\zeta}(b) + \lambda(a)\frac{dx}{d\zeta}(a) \\ x(b) + x(a) \end{pmatrix} \tag{7.35}$$

has a mild solution for every initial condition in $L^2((a,b);\mathbb{R})$. Choosing $V=0$, we find as boundary conditions

$$\lambda(b) \frac{dx}{d\zeta}(b) = 0 \qquad x(b) = 0,$$

whereas $V = I$ gives

$$\lambda(b) \frac{dx}{d\zeta}(b) = \lambda(a) \frac{dx}{d\zeta}(a) \qquad x(b) = x(a).$$

□

We end by saying that Theorem 7.2.9 cannot be used for the damped wave equation of Example 7.1.1. The reason this lies in the fact that the extended P_1 matrix is non-invertible. Namely, using Example 7.2.2 we find that

$$P_{1,\mathrm{ext}} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

which is clearly no invertible. We remark that Theorem 7.2.8 and hence Theorem 2.3.2 can be extended such that non-invertible P_1's are possible, see [28].

7.3. General result

In this section we prove a general result, which is very useful in proving existence of the (homogeneous) differential equation. For the proof of this theorem we need the following two lemma's

Lemma 7.3.1. *Let A be a linear operator from $D(A) \subset X$ to X, where X is a Hilbert space, then A generates a contraction semigroup if and only if the following two conditions are satisfied:*

1. *For all $x_0 \in D(A)$*

$$2\mathrm{Re}\left(\langle Ax_0, x_0\rangle_X\right) = \langle Ax_0, x_0\rangle_X + \langle x_0, Ax_0\rangle_X \leq 0, \tag{7.36}$$

2. *The range of $\lambda I - A$ equals X for some $\lambda > 0$.*

Lemma 7.3.2. *Let A be the generator of a contraction semigroup on the Hilbert space X, and let $P \in \mathcal{L}(X)$ satisfy Re$\left(\langle Px, x\rangle\right) \leq 0$ for all $x \in X$, then $A + P$ generates a contraction semigroup.*

Theorem 7.3.3. *Let H_1 and H_2 be two Hilbert spaces. Furthermore, Let A_1 be a linear operator from $D(A_1) \subset H_1 \times H_2$ to H_1 and A_2 is a linear operator from $D(A_2) \subset H_1$ to H_2, and let $S \in \mathcal{L}(H_2)$ such that it is invertible and satisfies*

$$\mathrm{Re}\left(\langle Sx_2, x_2\rangle\right) \geq m_2\|x_2\|^2, \qquad x_2 \in H_2 \tag{7.37}$$

for some $m_2 > 0$ independent of x_2.

If the operator

$$A_{\mathrm{ext}} := \begin{bmatrix} & A_1 & \\ A_2 & & 0 \end{bmatrix} \tag{7.38}$$

with the domain $D(A_{\mathrm{ext}}) = \{(h_1, h_2) \in H_1 \times H_2 \mid h_1 \in D(A_2)$ and $(h_1, h_2) \in D(A_1)\}$ generates a contraction semigroup on $H_1 \times H_2$, then

$$A_S h = A_1 \begin{bmatrix} h \\ S\left(A_2 h\right) \end{bmatrix} \tag{7.39}$$

with domain

$$D(A_S) = \{h \in H_1 \mid \begin{bmatrix} h \\ S\left(A_2 h\right) \end{bmatrix} \in D(A_{\mathrm{ext}})\} \tag{7.40}$$

generates a contraction semigroup on H_1.

Proof: By Lemma 7.3.1 we have to check two conditions for A_S; that is

$$\mathrm{Re}\left(\langle A_S x_1, x_1\rangle\right) \leq 0 \qquad \text{for all } x_1 \in D(A_S) \tag{7.41}$$

and

$$\mathrm{ran}(\lambda I - A_S) = H_1 \qquad \text{for some } \lambda > 0. \tag{7.42}$$

We start by showing (7.41). Let $x_1 \in D(A_S)$

$$\begin{aligned}
\langle A_S x_1, x_1\rangle &= \langle A_1 \begin{bmatrix} x_1 \\ S(A_2x_1) \end{bmatrix}, x_1\rangle \\
&= \langle A_{\text{ext}} \begin{bmatrix} x_1 \\ S(A_2x_1) \end{bmatrix}, \begin{bmatrix} x_1 \\ 0 \end{bmatrix}\rangle \\
&= \langle A_{\text{ext}} \begin{bmatrix} x_1 \\ S(A_2x_1) \end{bmatrix}, \begin{bmatrix} x_1 \\ S(A_2x_1) \end{bmatrix}\rangle \\
&\quad -\langle A_{\text{ext}} \begin{bmatrix} x_1 \\ S(A_2x_1) \end{bmatrix}, \begin{bmatrix} 0 \\ S(A_2x_1) \end{bmatrix}\rangle \\
&= \langle A_{\text{ext}} \begin{bmatrix} x_1 \\ S(A_2x_1) \end{bmatrix}, \begin{bmatrix} x_1 \\ S(A_2x_1) \end{bmatrix}\rangle - \langle A_2x_1, S(A_2x_1)\rangle.
\end{aligned}$$

Using the fact that A_{ext} generates a contraction semigroup, we find

$$\begin{aligned}
\operatorname{Re}(\langle A_S x_1, x_1\rangle) &= \operatorname{Re}\left(\langle A_{\text{ext}} \begin{bmatrix} x_1 \\ S(A_2x_1) \end{bmatrix}, \begin{bmatrix} x_1 \\ S(A_2x_1) \end{bmatrix}\rangle\right) - \operatorname{Re}(\langle A_2x_1, S(A_2x_1)\rangle) \\
&\leq 0 - \operatorname{Re}(\langle A_2x_1, S(A_2x_1)\rangle) \leq 0,
\end{aligned}$$

where in the last step we used that S satisfies (7.37).

Next we prove the range condition (7.42) on A_S. That is, for a $\lambda > 0$ we have to show that for any given $f \in X$ we can find an $x \in D(A_S)$ such that

$$f = (\lambda I - A_S)x.$$

From (7.37) we find that

$$\operatorname{Re}\left(\langle S^{-1}z_2, z_2\rangle\right) \geq m_2\|S^{-1}z_2\|^2 \geq \frac{m_2}{\|S\|^2}\|z_2\|^2. \tag{7.43}$$

Choose λ such that $0 < \lambda < \frac{m_2}{\|S\|^2}$, and define the following mapping from $H_1 \times H_2$ to $H_1 \times H_2$

$$P = \begin{bmatrix} 0 & 0 \\ 0 & -S^{-1} + \lambda I \end{bmatrix}.$$

By our assumption we have that $\operatorname{Re}(\langle Px, x\rangle) \leq 0$ for all $x \in H_1 \times H_2$. Hence by Lemma 7.3.2, we conclude that $A_{\text{ext}} + P$ is generates a contraction semigroup. In particular, the range of $\lambda I - A_{\text{ext}} - P$ is the whole space. This implies that for all $\binom{f}{0} \in H_1 \times H_2$ there exists an $\binom{h_1}{h_2} \in D(A_{\text{ext}})$ such that

$$\begin{bmatrix} f \\ 0 \end{bmatrix} = (\lambda I - A_{\text{ext}} - P)\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}. \tag{7.44}$$

This is equivalent to

$$f = \lambda h_1 - A_1 \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

and
$$0 = \lambda h_2 - A_2 h_1 + (S^{-1} - \lambda) h_2.$$
This last relation, implies that $h_2 = S(A_2 h_1)$. Hence $h_1 \in D(A_S)$, and $f = (\lambda I - A_S)h_1$.

Concluding, we see that the range of $\lambda I - A_S$ is full, and so it generates a contraction semigroup on H_1. ∎

7.4. Exercises

7.5. Notes and references

This chapter is based on Chapter 6 of [28]. Theorem 7.2.9 can be extended to allow for non-linear dissipation terms.

Chapter A
Mathematical Background

A.1. Complex analysis

In this section, we present important facts from complex function theory that are used in this book. As main references, we have used Levison and Redheffer [17] and Rudin [24].

By a *domain* we shall mean a nonempty, open, and connected subset of $\mathbb{C}$; in some books the name *region* is used.

Definition A.1.1. Let Υ be a domain in $\mathbb{C}$, and let f be a function defined on Υ with values in $\mathbb{C}$. The function f is *holomorphic* on Υ if $\frac{df}{ds}(s_0)$ exists for every s_0 in Υ.

The function is said to be *entire* if it is holomorphic on $\mathbb{C}$.

The function g is *meromorphic* on Υ if g can be expressed as $g = \frac{f_1}{f_2}$, where f_1 and f_2 are holomorphic on Υ. ♣

We remark that some texts use the the term *analytic* instead of holomorphic. Examples of holomorphic functions are all polynomials and exponential powers; the latter are entire functions. Rational functions are meromorphic on $\mathbb{C}$ and holomorphic on every domain not containing the zeros of the denominator. It is easily verified that these classes of holomorphic functions have the property that the derivative is again a holomorphic function. In fact, this is a general property of holomorphic functions.

Proposition A.1.2. *A function f that is holomorphic on the domain Υ possesses the following properties:*

a. The derivative $\frac{df}{ds}$ is holomorphic on the same domain;

b. If $f \neq 0$ in Υ, then f^{-1} is holomorphic on Υ;

c. f is uniformly bounded on every compact subset of $\mathbb{C}$ contained in Υ. ♡

Proof *a.* See theorem 5.2 in Levison and Redheffer [17] or the corollary of theorem 10.16 in Rudin [24].

b. This follows easily from the fact that

$$\frac{df^{-1}}{ds} = -f^{-2}\frac{df}{ds}.$$

c. This follows directly from the fact that f is continuous on Υ. ■

The last property shows that holomorphic functions have similar properties to real-valued functions. However, in contrast to functions on $\mathbb{R}$, it cannot be bounded on the whole complex plane, unless it is a constant.

Theorem A.1.3. Liouville's Theorem. *If f is an entire function that is bounded on the whole complex plane, then it is a constant.*

Proof See theorem 5.5 of chapter 3 in Levison and Redheffer [17] or theorem 10.23 of Rudin [24]. ■

The zeros of a holomorphic function have very nice properties, as can be seen in the next theorem.

Theorem A.1.4. *The* zeros *of a function f that is holomorphic on the domain Υ have the following properties:*

a. *If s_0 is a zero of f, then $f(s)$ is either identically zero on Υ or the* order *of the zero is finite, that is, there exists an m such that $f(s) = (s - s_0)^m g(s)$, where g is holomorphic on Υ and $g(s_0) \neq 0$;*

b. *If f is not identically zero on Υ, then the zeros of f are isolated; that is, for every zero s_0 of f there exists a $\delta > 0$ such that $f(s) \neq 0$ for all s satisfying $0 < |s - s_0| < \delta$;*

c. *If the zeros of f have a limit point in Υ, then f is identically zero;*

d. *In every compact subset V of $\mathbb{C}$ with $V \subset \Upsilon$, there are only finitely many zeros, provided that f is not identically zero.*

Proof This result can be found in theorem 10.18 of Rudin [24] and in theorems 7.2–7.4 of Levison and Redheffer [17]. ■

A corollary of Theorem A.1.4.c is that two functions f_1, f_2, that are holomorphic on the domains Υ_1 and Υ_2, respectively, and are equal on a set containing a limit point in $\Upsilon_1 \cap \Upsilon_2$, are in fact equal on $\Upsilon_1 \cap \Upsilon_2$. Furthermore, there exists a unique function f that is holomorphic on $\Upsilon_1 \cup \Upsilon_2$ such that $f = f_1$ on Υ_1 and $f = f_2$ on Υ_2. This f is called the *holomorphic continuation.*

Definition A.1.5. A curve Γ in the complex plane is called a *rectifiable curve* if there exists an interval $[a, b] \subset \mathbb{R}$ and a continuously differentiable mapping γ from $[a, b]$ to $\mathbb{C}$ such that the image of γ equals Γ, that is, $\Gamma = \gamma([a, b])$. The rectifiable curve Γ is called *simple* if $\gamma(x) \neq \gamma(y)$ for all x and y in (a, b) such that $x \neq y$. It is called *closed* if $\gamma(a) = \gamma(b)$. By a *contour* Γ we shall mean a finite collection of rectifiable curves Γ_j, $j = 1, \ldots, n$, such that the final point of Γ_j is the initial point of Γ_{j+1} for $1 \leq j \leq n - 1$. The notions of simple and closed are the same for these curves. ♣

Theorem A.1.6. Rouché's Theorem. *Let f_1 and f_2 be functions that are holomorphic on the domain Υ, and suppose that Υ contains a simple, closed contour Γ. If $|f_1(s)| > |f_2(s)|$ for $s \in \Gamma$, then f_1 and $f_1 + f_2$ have the same number of zeros inside Γ. (A zero of order p counts for p zeros.)*

Proof See theorem 6.2 in Levison and Redheffer [17] or theorem 10.43 in Rudin [24]. ■

Definition A.1.7. For a function f that is continuous on the domain Υ we define its *integral* along the rectifiable curve $\Gamma \subset \Upsilon$ by

$$\int_{\Gamma} f(s)ds := \int_{a}^{b} f(\gamma(x))\frac{d\gamma}{dx}(x)dx. \tag{A.1}$$

Its integral over a contour Γ is defined by

$$\int_{\Gamma} f(s)ds = \sum_{j=1}^{n} \int_{\Gamma_j} f(s)ds, \tag{A.2}$$

where Γ_j, $1 \leq j \leq n$, are the curves that form the contour Γ. ♣

Before we can state one of the most important theorems of complex analysis, we need the concept of the *orientation* of a rectifiable, simple, closed contour. Let the contour be composed of the rectifiable curves $\Gamma_j = \gamma_j([a_j, b_j])$, and choose a point x_0 from (a_j, b_j) such that $\frac{d\gamma_j}{dx}(x_0) \neq 0$. If the vector obtained by rotating the tangent vector $\frac{d\gamma_j}{dx}(x_0)$ in a counterclockwise sense through an angle of $\frac{\pi}{2}$ points inside the interior bounded by the contour Γ, then the rectifiable, closed, simple contour is said to be *positively oriented*. For a circle it is easily seen that it is positively oriented if one transverses the circle in a counterclockwise sense going from a to b.

Theorem A.1.8. Cauchy's Theorem. *Consider the simply connected domain Υ that contains the positively oriented, closed, simple contour Γ. If f is holomorphic on Υ, then*

$$\int_{\Gamma} f(s)ds = 0,$$

and for any point s_0 inside Γ

$$\frac{1}{2\pi j}\int_{\Gamma} \frac{f(s)}{s - s_0}ds = f(s_0).$$

Proof See Levison and Redheffer [17, pp. 180 and 183] or theorem 10.35 in Rudin [24]. ■

Definition A.1.9. Let g be a function that is meromorphic on the domain Υ. A point s_0 in Υ is defined to be a *pole* of g if $\lim_{s \to s_0} |g(s)| = \infty$. The *order* of the pole is defined to be the smallest positive integer m such that $\lim_{s \to s_0} |(s - s_0)^m g(s)| < \infty$. ♣

It is easily seen that if g can be expressed as $g = \frac{f_1}{f_2}$, where f_1 and f_2 are holomorphic on Υ, then s_0 is a pole of g only if s_0 is a zero of f_2. Since the zeros have finite order (see Theorem A.1.4), so do the poles.

If g is a meromorphic function on the domain Υ with no poles on Γ, then it is continuous on Γ and hence (A.2) is well defined.

Theorem A.1.10. Cauchy's Residue Theorem. *Let g be a function that is meromorphic on the simply connected domain Υ with s_0 as its only pole inside the positively oriented, simple, closed contour Γ. Assume further that there are no poles on the contour Γ. Then*

$$\frac{1}{2\pi j}\int_{\Gamma} g(s)ds = \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{ds^{m-1}}(s - s_0)^m g(s)\right]_{s=s_0}, \tag{A.3}$$

where m is the order of the pole s_0.

Proof See theorem 2.1 in Levison and Redheffer [17] or theorem 10.42 in Rudin [24]. ∎

The value on the right-hand side of equation (A.3) is called the *residue* of g at s_0. If the meromorphic function f contains finitely many poles inside the contour Γ, then the integral in equation (A.3) equals the sum over all the residues.

In the next theorem, we see that it is possible to express a meromorphic function with a pole at s_0 as an infinite series of positive and negative powers of $s - s_0$.

Theorem A.1.11. *Let f be a holomorphic function on the punctured disc $\{s \in \mathbb{C} \mid 0 < |s-s_0| < R\}$ and let C be the circle $\{s \in \mathbb{C} \mid |s - s_0| = r\}$ for any r satisfying $0 < r < R$. If we define*

$$a_k := \frac{1}{2\pi\jmath} \int\limits_C \frac{f(s)}{(s - s_0)^{k+1}} ds$$

for $k \in \mathbb{Z}$, where C is transversed in a counterclockwise sense, then the Laurent series *given by*

$$f(s) = \sum_{k=-\infty}^{\infty} a_k (s - s_0)^k$$

converges uniformly to $f(s)$ in any closed annulus contained in the punctured disc $\{s \in \mathbb{C} \mid 0 < |s - s_0| < R\}$.

Proof See Levison and Redheffer [17, theorem 9.2]. ∎

We remark that if the function is holomorphic on the disc $\{s \in \mathbb{C} \mid |s - s_0| < R\}$, then $a_j = 0$ for negative values of j. Hence for every holomorphic function there exists a sequence of polynomials that approximate it on an open disc. In the next theorem, we shall see how good this approximation is on the closed disc.

Theorem A.1.12. *We define the disc $D(z_0, R) := \{z \in \mathbb{C} \mid |z - z_0| \le R\}$. If f is a holomorphic function on the interior of $D(z_0, R)$ and continuous on the boundary, then for every $\varepsilon > 0$ there exists a polynomial P_ε such that*

$$\sup_{z \in D(z_0,R)} |f(z) - P_\varepsilon(z)| < \varepsilon. \tag{A.4}$$

Proof See theorem 20.5 in Rudin [24]. ∎

We remark that if a sequence of polynomials converges to a function in the norm in equation (A.4), then this limit function is continuous on the boundary.

For the special case that the meromorphic function in Theorem A.1.10 is given by $f^{-1}\frac{df}{ds}$, we have the following result.

Theorem A.1.13. Principle of the Argument. *Let Υ be a simply connected domain and let Γ be a positively oriented, simple, closed contour contained in Υ. Let g be a function that is meromorphic on Υ with no zeros or poles on Γ, and let $N(\Gamma)$ and $P(\Gamma)$ denote the number of zeros and the number of poles, respectively, inside Γ. The following equalities hold*

$$\frac{1}{2\pi\jmath} \int\limits_\Gamma \frac{\frac{dg}{ds}(s)}{g(s)} ds = \frac{1}{2\pi} \arg(g(s))|_\Gamma = N(\Gamma) - P(\Gamma). \tag{A.5}$$

Furthermore, $N(\Gamma) - P(\Gamma)$ equals the number of times that $\{g(s) \mid s \in \Gamma\}$ winds around the origin as s transverses Γ once in a counterclockwise sense.

Proof See theorem 6.1 in Levison and Redheffer [17] or theorem 10.43 in Rudin [24]. ■

We would like to apply this theorem to the imaginary axis, but this is not a closed curve. To overcome this, we introduce an extra assumption on the functions.

Theorem A.1.14. Nyquist Theorem. *Let g be a function that is meromorphic on an open set containing $\overline{\mathbb{C}_0^+}$ and suppose that g has no poles or zeros on the imaginary axis. Furthermore, we assume that g has a* nonzero limit at ∞ in $\overline{\mathbb{C}_0^+}$*; that is, there exists a $g(\infty) \in \mathbb{C}$, $g(\infty) \neq 0$ such that*

$$\lim_{\rho\to\infty}\left[\sup_{\{s\in\overline{\mathbb{C}_0^+}\,|\,|s|>\rho\}} |g(s)-g(\infty)|\right] = 0. \tag{A.6}$$

Then g has at most finitely many poles and zeros in $\overline{\mathbb{C}_0^+}$ and

$$\begin{aligned}\frac{1}{2\pi\jmath}\int_{\infty}^{-\infty}\frac{\frac{dg}{ds}(\jmath\omega)}{g(\jmath\omega)}d\omega &= \frac{1}{2\pi}\lim_{\omega\to\infty}[\arg(g(-j\omega))-\arg(g(j\omega))] \\ &= N_0 - P_0,\end{aligned} \tag{A.7}$$

where N_0 and P_0 are the number of zeros and poles, respectively, in $\overline{\mathbb{C}_0^+}$. Furthermore, $N_0 - P_0$ equals the number of times that $\{g(\jmath\omega) \mid \omega \in \mathbb{R}\}$ winds around the origin as ω decreases from $+\infty$ to $-\infty$.

Proof This follows from Theorem A.1.13 by a limiting argument. ■

This theorem can be extended to allow for isolated poles or zeros on the imaginary axis in the following manner.

If g has a pole or a zero at $\jmath\omega_0$, then we integrate around this point via the half-circle in $\mathbb{C}_0^-$: $C_{\omega_0} = \{s \in \mathbb{C} \mid s = \jmath\omega_0 - \varepsilon e^{\jmath\theta};\ \frac{-\pi}{2} < \theta < \frac{\pi}{2},\ \varepsilon > 0\}$, and the principle of the argument also applies for this *indented imaginary axis*. Notice that the crucial requirement in Theorem A.1.14 has been the limit behavior of g as $|s| \to \infty$ in $\overline{\mathbb{C}_0^+}$.

This last version of the principle of the argument (A.7) motivates the following concept of the Nyquist index of a meromorphic, scalar, complex-valued function. As we have already noted, meromorphic functions have isolated poles and zeros (see Definitions A.1.1 and A.1.9 and Theorem A.1.4.b).

Definition A.1.15. Let g be a function that is meromorphic on $\mathbb{C}_{-\varepsilon}^+$ for some $\varepsilon > 0$ and suppose that g has a nonzero limit at ∞ in $\overline{\mathbb{C}_0^+}$ (see (A.6)). This implies that the graph of $g(s)$ traces out a closed curve in the complex plane, as s follows the indented imaginary axis. We define the number of times the plot of $g(s)$ encircles the origin in a counterclockwise sense as s decreases from $\jmath\infty$ to $-\jmath\infty$ over the indented imaginary axis to be its *Nyquist index*, which we denote by $\operatorname{ind}(g)$. Thus, by Theorem A.1.14 we have that

$$\operatorname{ind}(g) = \frac{1}{2\pi}\lim_{\omega\to\infty}[\arg(g(-j\omega))-\arg(g(j\omega))] = N_0 - P_0. \tag{A.8}$$

♣

If g has no poles or zeros on the imaginary axis, then the Nyquist index is just the number of times the plot of $g(\jmath\omega)$ encircles the origin in a counterclockwise sense as ω decreases from ∞ to $-\infty$.

In complex analysis books, the index for a curve is normally define as a winding number. Note that our Nyquist index is the winding number of the curve $g(s)$ with s on the indented imaginary axis.

From the properties of the argument, it follows that the Nyquist index has a similar property:

$$\mathrm{ind}(g_1 \times g_2) = \mathrm{ind}(g_1) + \mathrm{ind}(g_2). \tag{A.9}$$

The Nyquist index is a *homotopic invariant*, which basically means that deforming the closed curve $g(\jmath\mathbb{R})$ does not change the index, provided that the curve remains closed and does not pass through the origin. We recall the definition of homotopic maps.

Definition A.1.16. Let X be a topological space and let $\Gamma_1 = \gamma_1([0,1]), \Gamma_2 = \gamma_2([0,1])$ be two closed curves in X. Γ_1 and Γ_2 are *X-homotopic* if there exists a continuous map $\psi : [0,1]\times[0,1] \to X$ such that

$$\psi(y,0) = \gamma_1(y), \qquad \psi(y,1) = \gamma_2(y), \qquad \psi(0,t) = \psi(1,t) \tag{A.10}$$

for all $y, t \in [0,1]$. ♣

Theorem A.1.17. *If Γ_1 and Γ_2 are $\mathbb{C}\setminus\{0\}$-homotopic closed contours in the domain $\mathbb{C}\setminus\{0\}$, then the number of times that Γ_1 and Γ_2 wind around 0 is the same.*

Proof This follows from theorem 10.40 in Rudin [24]. ■

We apply this theorem to show that the indices of two functions that can be continuously transformed from the first into the second have the same Nyquist index.

Lemma A.1.18. *Let g_1 and g_2 be meromorphic functions on an open set containing $\overline{\mathbb{C}_0^+}$, with nonzero limits $g_1(\infty)$ and $g_2(\infty)$ at infinity in $\overline{\mathbb{C}_0^+}$. If there exists a continuous function $h(s,t) : (-\jmath\infty, \jmath\infty)\times[0,1] \to \mathbb{C}$ such that $h(\jmath\omega, 0) = g_1(\jmath\omega)$, $h(\jmath\omega,1) = g_2(\jmath\omega)$ and $h(\jmath\omega,t)$ and $h(\infty,t)$ are nonzero for all $t\in[0,1]$ and $\omega\in\mathbb{R}$, then the Nyquist indices of g_1 and g_2 are the same.*

Proof First we suppose that neither g_1 nor g_2 has poles or zeros on the imaginary axis. For $t\in[0,1]$ and $y\in(0,1)$ we define $\psi(y,t) := h(\jmath\tan(\pi y - \frac{\pi}{2}), t)$, $\gamma_1(y) := g_1(\jmath\tan(\pi y - \frac{\pi}{2}))$ and $\gamma_2(y) := g_2(\jmath\tan(\pi y - \frac{\pi}{2}))$. Furthermore, we define the end point of $\psi(\cdot,t)$ by $\psi(0,t) = \psi(1,t) = h(\infty,t)$ and the end points of γ_1,γ_2 by $\gamma_1(0) = \gamma_1(1) = g_1(\infty)$ and $\gamma_2(0) = \gamma_2(1) = g_2(\infty)$. By Definition A.1.16 we easily see that the closed curves $\gamma_1([0,1])$ and $\gamma_2([0,1])$ are $\mathbb{C}\setminus\{0\}$-homotopic, and so by Theorem A.1.17 the number of encirclements of 0 are the same. Since these curves are the same as $g_1(\jmath\omega)$ and $g_2(\jmath\omega)$, respectively, we have by Definition A.1.15 that their Nyquist indices are the same.

The proof for the case that g_1 or g_2 has poles and zeros on the imaginary axis is similar, replacing the imaginary axis with the indented version. ■

A.2. Normed linear spaces

The results in this section are well known in functional analysis and may be found in almost any book on this subject. The basic source is Kreyszig [16]; secondary sources are Kato [14], Naylor and Sell [19], Taylor [25], and Yosida [31].

A.2.1. General theory

The concept of normed linear spaces is fundamental to functional analysis and is most easily thought of as a generalization of the n-dimensional Euclidean vector space $\mathbb{R}^n$ with the euclidean length function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^+ = [0, \infty)$ given by

$$\|x\|^2 = \sum_{i=1}^{n} |x_i|^2.$$

In fact, it is just a linear vector space with a length function (norm) defined on it. First, we define a linear vector space; other terms are vector space or linear space.

Definition A.2.1. A *linear vector space* W over a scalar field $\mathcal{F}$ is a nonempty set W with a mapping: $(x_1, x_2) \to x_1 + x_2$ from $W \times W$ to W, which we call *addition*, and a mapping: $(\alpha, x) \to \alpha x$ from $\mathcal{F} \times W$ to W, which we call *scalar multiplication*. These mappings satisfy the following conditions for all x, y, z in W and all $\alpha, \beta \in \mathcal{F}$:

a. $x + y = y + x$ (the *commutative property*);

b. $(x + y) + z = x + (y + z)$ (the *associative property*);

c. There exists a unique element 0 in W such that $x + 0 = x$ (the existence of the *zero element*);

d. For each $x \in W$, there exists a unique element $-x \in W$ such that $x + -x = 0$ (the existence of an *inverse*);

e. $\alpha(\beta x) = (\alpha\beta)x$;

f. $(\alpha + \beta)x = \alpha x + \beta x$;

g. $\alpha(x + y) = \alpha x + \alpha y$;

h. $1x = x$, where 1 is the unit element of the scalar field $\mathcal{F}$. ■ ♣

In this book, $\mathcal{F}$ will be either the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$; W over $\mathbb{R}$ is called a *real vector space*, and W over $\mathbb{C}$ is called a *complex vector space*.

Definition A.2.2. If W is a linear vector space over the field $\mathcal{F}$, then a subset S of W is a *linear subspace* if $x, y \in S \Rightarrow \alpha x + \beta y \in S$ for all scalars $\alpha, \beta \in \mathcal{F}$ (i.e., S is closed under addition and scalar multiplication and so is itself a linear vector space over $\mathcal{F}$). ♣

Definition A.2.3. A *linear combination* of vectors $x_1, \ldots, x_n$ of a linear vector space W is an expression of the form $\alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_n x_n$, where the coefficients $\alpha_1, \ldots, \alpha_n$ are scalars. ♣

Definition A.2.4. For any nonempty subset M of the linear vector space W, the set of all linear combinations of vectors of M is called *span* of M and is denoted by

$$\operatorname{span}\{M\}.$$ ♣

Obviously, this is a linear subspace Y of W, and one can easily show that it is the smallest (with respect to inclusion) linear subspace that contains M. We say that Y is spanned by M.

Definition A.2.5. If $x_1, \ldots, x_n$ are elements of W, a linear vector space over $\mathcal{F}$, and there exist scalars $\alpha_1, \ldots, \alpha_n$, not all zero, such that the linear combination $\alpha_1 x_1 + \ldots + \alpha_2 x_n = 0$, then we say that $x_1, \ldots, x_n$ is a *linearly dependent* set. If no such set of scalars exist, then we say that $x_1, \ldots, x_n$ are *linearly independent*. ♣

Definition A.2.6. If the linear vector space W is the span of a finite set of linearly independent vectors $x_1, \ldots, x_n$, then we say that W has *dimension* n. If there exists no finite set M of vectors, such that $W = \text{span}\{M\}$, W is said to be *infinite-dimensional*. ♣

Definition A.2.7. A *norm* is a nonnegative set function on a linear vector space, $\|\cdot\|: W \to \mathbb{R}^+ = [0, \infty)$, such that:

a. $\|x\| = 0$ if and only if $x = 0$;

b. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in W$ (the *triangular inequality*);

c. $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in W$ and $\alpha \in \mathcal{F}$. ♣

Definition A.2.8. A *normed linear space* is a linear vector space X with a norm $\|\cdot\|_X$ on it and it is denoted by $(X, \|\cdot\|_X)$. If the meaning is clear from the context, we usually write simply X and use $\|\cdot\|$ for the norm. ♣

Example A.2.9 Let $p \geq 1$ be a fixed real number. By definition, each element in the space ℓ_p is a sequence $x = (\xi_j) = (\xi_1, \xi_2, \ldots)$ of numbers in $\mathbb{C}$ such that

$$\sum_{j=1}^{\infty} |\xi_j|^p < \infty.$$

This is a linear vector space over $\mathbb{C}$ with componentwise addition and scalar multiplication. It is also a normed linear space with the norm

$$\|x\| = \left(\sum_{j=1}^{\infty} |\xi_i|^p \right)^{1/p}.$$

The triangular inequality for the space ℓ_p is commonly know as the *Minkowski inequality* for sums. □

Example A.2.10 The space ℓ_∞ consists of all sequences $x = (\xi_1, \xi_2, \ldots)$, where $\xi_i \in \mathbb{C}$ and $\sup\limits_{i \geq 1} |\xi_i| < \infty$. This is a linear vector space over $\mathbb{C}$ with componentwise addition and scalar multiplication. Furthermore, it is a normed linear space with the norm

$$\|x\| = \sup_{i \geq 1} |\xi_i|.$$

□

Example A.2.11 Let $p \geq 1$ be a fixed real number and let $-\infty \leq a < b \leq \infty$. Consider the set of measurable functions $x(t)$ with $\int\limits_a^b |x(t)|^p dt$ finite, and with the norm

$$\|x\| = \left(\int_a^b |x(t)|^p dt \right)^{1/p}.$$

This is a linear vector space with addition and scalar multiplication defined by:

$$(x + y)(t) = x(t) + y(t);$$

$$(\alpha x)(t) = \alpha x(t). \qquad \square$$

However, it is not a normed linear space, since $\|x\| = 0$ only implies that $x(t) = 0$ almost everywhere. To make it into a normed linear space we have to consider (equivalence) classes of functions, $[x]$, where $[x]$ is the class of all functions that equal x almost everywhere. Clearly, these equivalence classes form a linear space and $\|[x]\| := \|x_1\|$ for any $x_1 \in [x]$ defines a norm; we call this normed linear space $L_p(a, b)$. Following usual practice, we write x_1 instead of $[x]$, where x_1 is any element of the equivalence class $[x]$.

The triangular inequality for $L_p(a, b)$ is called the *Minkowski inequality* for functions.

Example A.2.12 Let $-\infty \leq a < b \leq \infty$ and consider all measurable functions x from (a, b) to $\mathbb{C}$ with the property that $\text{ess}sup_{t\in(a,b)}|x(t)| < \infty$. As in Example A.2.11, we form equivalence classes $[x]$ that contain functions that equal x almost everywhere on (a, b). With the norm

$$\|[x]\|_\infty := \operatorname{ess\,sup}_{t\in(a,b)} |x_1(t)| \qquad \text{for any } x_1 \in [x],$$

this space is a normed linear space, which we denote by $L_\infty(a, b)$. As in Example A.2.11, we usually write x_1 instead of $[x]$, where x_1 is any element of $[x]$. $\square$

Definition A.2.13. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces. Then X and Y are *topologically isomorphic* if there exists a linear, bijective map $T : X \to Y$ and positive constants a, b such that

$$a\|x\|_X \leq \|Tx\|_Y \leq b\|x\|_X \text{ for all } x \in X.$$

The norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ are then called *equivalent norms.*

The normed linear spaces are *isometrically isomorphic* if there exists a linear, bijective map $T : X \to Y$ such that

$$\|Tx\|_Y = \|x\|_X. \qquad ♣$$

Definition A.2.14. A sequence $\{x_n\}$ in a normed linear space $(X, \|\cdot\|_X)$ *converges* to x if

$$\lim_{n\to\infty} \|x_n - x\|_X = 0. \qquad ♣$$

The series $\sum\limits_{i=0}^{\infty} x_i$ is said to converge to x, if the sequence $\sum\limits_{i=0}^{n} x_i$ converges to x as $n \to \infty$.

Definition A.2.15. A set V in a normed linear space X is *closed* if every convergent sequence in V has its limit point in V. A set V is *open* if its complement is closed. Alternatively, a set V is open if for any point $x \in V$, there exists an $\varepsilon > 0$ such that the sphere with centre x and radius ε, $B(x, \varepsilon) := \{y \in X \mid \|y - x\| < \varepsilon\}$ is contained entirely in V.

If we add to a set V all the limit points of sequences in V, we obtain the smallest closed set that contains V. This closed set is called the *closure* of V, which we write as $\overline{V}$.

A set V in a normed linear space $(X, \|\cdot\|_X)$ is *bounded* if $\sup\limits_{x\in\overline{V}} \|x\|_X < \infty$. A set V in a normed linear space is *compact* if every sequence in V contains a convergent subsequence with its limit point in V; V is *relatively compact* if its closure is compact.

Definition A.2.16. A subset V of a normed linear space is *dense* in X if its closure is equal to X. ♣

This important property means that every element x of X may be approximated as closely as we like by some element v of V, i.e., for any x in X and $\varepsilon > 0$ there exists a $v \in V$ such that $\|v - x\| < \varepsilon$.

All normed linear spaces have dense subsets, but they need not be countable. Normed linear spaces that do have countable dense subsets have special properties that are important in applications.

Definition A.2.17. A normed linear space $(X, \|\cdot\|_X)$ is *separable* if it contains a dense subset that is countable. ♣

The concept of Cauchy sequence in $\mathbb{R}$ is very important, since even without evaluating the limit one can determine whether a sequence is convergent or not. We shall start by generalizing the concept of Cauchy sequences to general normed linear spaces.

Definition A.2.18. A sequence $\{x_n\}$ of elements in a normed linear space $(X, \|\cdot\|_X)$ is a *Cauchy sequence* if

$$\|x_n - x_m\|_X \to 0, \qquad \text{as } n, m \to \infty.$$

■ ♣

As stated above, every Cauchy sequence in $\mathbb{R}$ is convergent. Unfortunately, this does not hold for general normed linear spaces, as can be seen from the next example.

Example A.2.19 Let $X = C[0,1]$, the space of continuous functions on $[0,1]$ and as a norm we take $\|x\| = (\int_0^1 |x(t)|^2 dt)^{1/2}$. Now consider the sequence of functions $\{x_n\} \subset X$ given by

$$x_n(t) = \begin{cases} 0 & \text{for } 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ \frac{nt}{2} - \frac{n}{4} + \frac{1}{2} & \text{for } \frac{1}{2} - \frac{1}{n} \le t \le \frac{1}{2} + \frac{1}{n} \\ 1 & \text{for } \frac{1}{2} + \frac{1}{n} \le t \le 1. \end{cases}$$

$\{x_n\}$ is Cauchy, since for $n > m$ we have that

$$\begin{aligned} \|x_m - x_n\|^2 &= \int_0^1 |x_m(t) - x_n(t)|^2 dt \\ &= \int_{\frac{1}{2}-\frac{1}{m}}^{\frac{1}{2}-\frac{1}{n}} (\frac{mt}{2} - \frac{m}{4} + \frac{1}{2})^2 dt + \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}+\frac{1}{n}} (\frac{mt}{2} - \frac{m}{4} - \frac{nt}{2} + \frac{n}{4})^2 dt + \\ &\quad \int_{\frac{1}{2}+\frac{1}{n}}^{\frac{1}{2}+\frac{1}{m}} (\frac{mt}{2} - \frac{m}{4} - \frac{1}{2})^2 dt \\ &= \frac{1}{6}[\frac{m}{n^2} - 2\frac{1}{n} + \frac{1}{m}] \le \frac{1}{6m} - \frac{1}{6n}, \end{aligned}$$

since $n > m$.

Thus $\|x_m - x_n\|^2 \to 0$ as $m, n \to \infty$. Clearly, the pointwise limit of x_n is

$$x(t) = \begin{cases} 0 & \text{for } 0 \le t < \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} < t \le 1. \end{cases}$$

However, this function is not in $C[0,1]$, because of the discontinuity at $t = \frac{1}{2}$. □

This situation is clearly unsatisfactory and we prefer to work with spaces where Cauchy sequences always have limits in the same space. A normed linear space X is *complete* if every Cauchy sequence has a limit in X.

Definition A.2.20. A *Banach space* is a complete, normed linear space. ♣

The spaces ℓ_p, ℓ_∞, $L_p(a,b)$, and $L_\infty(a,b)$ introduced in Examples A.2.9 to A.2.12 are all Banach spaces.

Example A.2.19 was an example of a normed linear space that was not complete. Under a different norm it is complete.

Example A.2.21 Consider the space $C[0,1]$ of continuous functions on $[0,1]$ and define the sup norm

$$\|x\|_\infty := \sup_{t\in[0,1]} |x(t)|.$$

Clearly, with $\|\cdot\|_\infty$ $C[0,1]$ defines a normed linear space. It is complete, since it is known that uniformly convergent sequences of continuous functions converge to a continuous function. Hence $C[0,1]$ is a Banach space under this norm. □

Another way of producing a complete normed linear space is given in the next theorem.

Theorem A.2.22. *Let X be a normed linear space. Then there exists a Banach space $\hat{X}$ and a linear, injective map $T : X \to \hat{X}$ such that $T(X)$ is dense in $\hat{X}$ and $\|Tx\|_{\hat{X}} = \|x\|_X$ for all $x \in X$. The space $\hat{X}$ is called the* completion *of X.*

Proof See Kreyszig [16, theorem 2.3-2] or Yosida [31, section I.10] or Taylor [25, theorem 2.41-A]. ■

It is not hard to show that the completion of $C[0,1]$ with the norm as in Example A.2.19 is $L_2(0,1)$.

A.2.2. Hilbert spaces

A Banach space generalizes the notion of $\mathbb{R}^n$ as a linear space with a length function, but in order to generalize the useful geometric property of orthogonality we need some extra structure.

Definition A.2.23. An *inner product* on a linear vector space Z defined over the complex or real field $\mathcal{F}$ is a map

$$\langle\cdot,\cdot\rangle : Z \times Z \to \mathcal{F}$$ ♣

such that for all $x, y \in Z$ and $\alpha, \beta \in \mathcal{F}$ it holds that

a. $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$;

b. $\overline{\langle x, y\rangle} = \langle y, x\rangle$;

c. $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

Properties a and b imply that $\langle x, \alpha z + \beta y\rangle = \overline{\alpha}\langle x, z\rangle + \overline{\beta}\langle x, y\rangle$; we say that $\langle x, z\rangle$ is *semilinear* in z. A linear space Z with an inner product $\langle\cdot,\cdot\rangle$ is called an *inner product space.*

Using the inner product we can make an inner product space into a normed linear space $(Z, \|\cdot\|_Z)$ by defining the *induced norm* by

$$\|z\|_Z := \sqrt{\langle z, z\rangle}.$$

In general, Z will not be a Banach space, since it need not be complete. Complete inner product spaces have a special name.

Definition A.2.24. A *Hilbert space* is an inner product space that is complete as a normed linear space under the induced norm. ♣

Before we look at some examples of Hilbert spaces, we list some properties of inner products and their induced norms:

a. $\langle x, y\rangle = 0$ for all $x \in Z$ implies $y = 0$;

b. $|\langle x, y\rangle| \leq \|x\|\|y\|$, (*Cauchy-Schwarz inequality*);

c. $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$, (*parallelogram law*);

d. If the norm in a normed linear space satisfies the parallelogram law, then the following defines an inner product

$$\langle x, y\rangle = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 + \jmath\|x+\jmath y\|^2 - \jmath\|x-\jmath y\|^2\right],$$

and the norm is induced by this inner product.

Example A.2.25 The spaces ℓ_2 and $L_2(a,b)$ defined in Examples A.2.9 and A.2.11, respectively, are Hilbert spaces under the inner products

$$\langle x, y\rangle_{\ell_2} := \sum_{n=1}^{\infty} x_n \overline{y_n} \qquad \text{and} \qquad \langle x, y\rangle_{L_2} := \int_a^b x(t)\overline{y(t)}dt,$$

respectively. As in Example A.2.11, by x we really mean the equivalence class $[x]$. We remark that the much used Cauchy-Schwarz inequality on $L_2(a,b)$ becomes

$$|\int_a^b x(t)\overline{y(t)}dt|^2 \leq \int_a^b |x(t)|^2 dt \int_a^b |y(t)|^2 dt. \tag{A.1}$$

■ □

Using the Cauchy-Schwarz inequality one can show that functions in $L_2(0,\infty)$ with their derivative in $L_2(0,\infty)$ have zero limit at infinity.

Example A.2.26 Let f be an element of $L_2(0,\infty)$, and assume that f is differentiable with its derivative in $L_2(0,\infty)$. Then for all $t > s$ we have that

$$\begin{aligned} |f(t)|^2 - |f(s)|^2 &= \int_s^t \frac{d}{dt}|f(\tau)|^2 d\tau = \int_s^t f(\tau)\overline{\dot{f}(\tau)}d\tau + \int_s^t \dot{f}(\tau)\overline{f(\tau)}d\tau \\ &\leq 2\sqrt{\int_s^t |f(\tau)|^2 d\tau \int_s^t |\dot{f}(\tau)|^2 d\tau}. \end{aligned}$$

Since $f, \dot{f}$ are elements of $L_2(0,\infty)$ we see that $|f(t)|$ converges for $t \to \infty$. Using the fact that f is square integrable, we see that its limit can only be zero. □

We now illustrate how it is possible to define several inner products on the same linear vector space.

Example A.2.27 Consider $L_2(a,b)$ defined above with $-\infty < a < b < \infty$ and define the subspace

$$\begin{aligned} Z \quad := \quad & \{u \in L_2(a,b) \mid u \text{ is absolutely continuous on } (a,b) \\ & \text{with } \tfrac{du}{dt} \in L_2(a,b) \text{ and } u(a) = 0\}. \end{aligned}$$

We remark that an element in $L_2(a,b)$ is said to be absolutely continuous if there is an absolutely continuous function in the equivalence class (see Example A.2.11). One can easily show that there can at most be one absolutely continuous function in every equivalence class.

Z can be regarded as a subspace of $L_2(a,b)$ and it is in fact a dense subspace. On the other hand, we can introduce a different norm that is well defined for all $u, v \in Z$

$$\langle u, v\rangle_2 = \langle \frac{du}{dt}, \frac{dv}{dt}\rangle_{L_2(a,b)}.$$

With the above inner product we obtain the new Hilbert space Z_2. □

The above example brings us naturally to the following class of Hilbert spaces (see Yosida [31, sections I.9 and I.10] or Naylor and Sell [19, section 5.13]).

Definition A.2.28. For $-\infty < a < b < \infty$ we define the following subspace of $L_2(a,b)$

$$\begin{aligned} S_2^m(a,b) \quad := \quad & \{u \in L_2(a,b) \mid u, \ldots, \tfrac{d^{m-1}u}{dt^{m-1}} \text{ are absolutely} \\ & \text{continuous on } (a,b) \text{ with } \tfrac{d^m u}{dt^m} \in L_2(a,b)\}. \end{aligned}$$

This is a Hilbert space with respect to the inner product

$$\langle z_1, z_2\rangle_{S_2^m(a,b)} = \sum_{n=0}^{m} \langle \frac{d^n z_1}{dt^n}, \frac{d^n z_2}{dt^n}\rangle_{L_2}. \tag{A.2}$$

These Hilbert spaces are called *Sobolev spaces*[1]. ♣

One can show that $S_2^m(a,b)$ is the completion of $C^m[a,b]$ or $C^\infty[a,b]$ with respect to the norm induced by (A.2) (see Yosida [31, sections I.9 and I.10] or Naylor and Sell [19, section 5.13]). It is not difficult to show that $S_2^m(a,b)$ is topologically isomorphic to

$$\begin{aligned} & \{u \in L_2(a,b) \mid u, \ldots, \tfrac{d^{m-1}u}{dt^{m-1}} \text{ are absolutely continuous on } (a,b) \\ & \quad \text{with } \tfrac{d^m u}{dt^m} \in L_2(a,b)\} \end{aligned}$$

under the inner product

$$\langle z_1, z_2\rangle = \langle z_1, z_2\rangle_{L_2} + \langle \frac{d^m z_1}{dt^m}, \frac{d^m z_2}{dt^m}\rangle_{L_2}. \tag{A.3}$$

The inner product structure allows a simple generalization of the concept of orthogonality.

Definition A.2.29. We say that two vectors x and y in a Hilbert space Z are *orthogonal* if

$$\langle x, y\rangle = 0,$$

in which case we write $x \perp y$. ♣

[1]Another notation for S^m is H^m. However, in this book we use H^m for the Hardy spaces.

If $x \perp y$, then the parallelogram law reduces to a generalized statement of *Pythagoras' theorem*, namely,

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

Definition A.2.30. If V is a subspace of a Hilbert space Z, then the *orthogonal complement* $V^\perp$ is defined by

$$V^\perp = \{x \in Z \mid \langle x, y\rangle = 0 \text{ for all } y \in V\}. \qquad \clubsuit$$

It can be shown that $V^\perp$ is a closed linear subspace of Z and that Z can be uniquely decomposed as the direct sum

$$Z = \overline{V} \oplus V^\perp, \tag{A.4}$$

where $\overline{V}$ is the closure of V. This means that any $z \in Z$ has the unique representation

$$z = z_{\overline{V}} + z_{V^\perp},$$

where $z_{\overline{V}} \in \overline{V}$, $z_{V^\perp} \in V^\perp$, and $\|z\|^2 = \|z_{\overline{V}}\|^2 + \|z_{V^\perp}\|^2$.

Furthermore, we see that a subspace V is dense in the Hilbert space Z if and only if $V^\perp = \{0\}$.

Definition A.2.31. An *orthonormal set* in a Hilbert space Z is a nonempty subset $\{\phi_n, n \geq 1\}$ of Z such that

$$\langle \phi_n, \phi_m\rangle = \delta_{nm} := \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases} \qquad \clubsuit$$

In other words, $\{\phi_n, n \geq 1\}$ are mutually orthogonal unit vectors. Of course, any mutually orthogonal set $\{x_n, n \geq 1\}$ may be normalized by defining new vectors

$$\phi_n = \frac{x_n}{\|x_n\|}.$$

Definition A.2.32. Let $\{e_n, n \geq 1\}$ be a subset of Z. We say that it is *maximal* if

$$\overline{\operatorname{span}_{n\geq 1}\{e_n\}} = Z. \qquad \clubsuit$$

In $\mathbb{R}^n$ any element can be expressed as a linear combination of any set of n mutually orthonormal elements; such an orthonormal set is called a *basis*. For infinite-dimensional Hilbert spaces we have a similar property.

Definition A.2.33. We say that an orthonormal sequence of a separable Hilbert space Z is an *orthonormal basis* if it is maximal. Then for any $x \in Z$, we have the *Fourier expansion*

$$x = \sum_{n=1}^{\infty} \langle x, \phi_n\rangle \phi_n.$$

The terms $\langle x, \phi_n\rangle$ are called the *Fourier coefficients* of x with respect to ϕ_n. Furthermore, we have the important *Parseval equality*. Any two vectors x, y in Z satisfy

$$\langle x, y\rangle = \sum_{n=1}^{\infty} \langle x, \phi_n\rangle \overline{\langle y, \phi_n\rangle}.$$

In particular, for $x = y$ we have

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, \phi_n\rangle|^2. \qquad \clubsuit$$

Example A.2.34 Consider the Hilbert space $L_2(0,1)$ introduced in Examples A.2.11 and A.2.25. This space has several different orthonormal bases. The best known one is the following:

$$\{1, \sqrt{2}\sin(2\pi nt), \sqrt{2}\cos(2\pi nt),\ n \geq 1\}. \tag{A.5}$$

This is an orthonormal basis, so any $x \in L_2(0,1)$ may be represented in the form

$$x(t) = a_0 + \sqrt{2}\sum_{n=1}^{\infty} a_n \cos(2\pi nt) + \sqrt{2}\sum_{n=1}^{\infty} b_n \sin(2\pi nt),$$

where

$$\begin{aligned} a_0 &= \int_0^1 x(t)dt, \\ a_n &= \int_0^1 x(t)\sqrt{2}\cos(2\pi nt)dt \text{ for } n \geq 1, \\ b_n &= \int_0^1 x(t)\sqrt{2}\sin(2\pi nt)dt \text{ for } n \geq 1. \end{aligned}$$

This is the classical Fourier expansion, and a_n and b_n are the Fourier coefficients.

Other orthonormal bases are given by the sequences

$$\{1, \sqrt{2}\cos(\pi nt),\ n \geq 1\}, \tag{A.6}$$

see Example A.4.22, and

$$\{\sqrt{2}\sin(\pi nt),\ n \geq 1\}, \tag{A.7}$$

see Example A.4.21. A fourth orthonormal basis is given by the *Legendre polynomials*

$$P_n(t) = \frac{8^n}{2 \cdot n!}\frac{d^n}{dt^n}(t^2 - t)^n,\ n \geq 1.$$

□

We remark that the expansions in Example A.2.34 are not valid pointwise, but only in the sense of the $L_2(0,1)$ norm. For example, equality in the Fourier expansion means that

$$\|x - \sum_{i=1}^{N}\langle x, \phi_i\rangle\phi_i\| \to 0 \text{ as } N \to \infty.$$

Example A.2.35 Let $\{\phi_n, n \geq 1\}$ be an orthonormal basis of the Hilbert space Z, and let $\{\alpha_n, n \geq 1\}$ be a positive sequence with $\alpha_n \geq 1$. Now we define the following linear subspace of Z.

$$Z_\alpha := \{z \in Z \mid z = \sum_{n=1}^{\infty} z_n\phi_n, \text{ with } \sum_{n=1}^{\infty}\alpha_n|z_n|^2 < \infty\}. \tag{A.8}$$

It is clear that Z_α is a dense, linear subspace of Z. On this linear vector space we define the following inner product

$$\langle z_1, z_2\rangle_\alpha := \sum_{n=1}^{\infty}\alpha_n\langle z_1, \phi_n\rangle\overline{\langle z_2, \phi_n\rangle}. \tag{A.9}$$

□

Under this inner product, the space Z_α is a Hilbert space.

A recurring problem in infinite dimensions is the question of approximation. For example, we may ask how good an approximation $\sum_{n=1}^{N} \langle x, \phi_n \rangle \phi_n$ is to x and how one should improve upon this approximation by introducing extra terms. It turns out that there is a simple answer to this question if $\{\phi_n, n \geq 1\}$ is an orthonormal basis in the Hilbert space Z. It is based on the following generalization of "dropping a perpendicular" in the three-dimensional Euclidean space.

Theorem A.2.36. *Let Z be a Hilbert space and V a closed subspace of Z. Then, given $x \in Z$, there exists a unique $v_0 \in V$ such that*

$$\|x - v_0\| = \min_{v \in V} \|x - v\|.$$

Furthermore, a necessary and sufficient condition for $v_0 \in V$ to be the minimizing vector is that $(x - v_0) \perp V$.

Proof See Kreyszig [16, theorem 3.3-1 and lemma 3.3-2] or Naylor and Sell [19, theorem 5.14.4]. ∎

Notice that in the above theorem the vector $x - v_0$ is to be seen as that obtained by "dropping a perpendicular" onto V. We now apply this theorem to the approximation problem. Let $\phi_1, \phi_2, \ldots, \phi_N$ be an orthonormal sequence of vectors that span a finite-dimensional subspace V. For any given $x \in Z$ we seek the vector $\hat{x}$ in V such that $\|x - \hat{x}\|$ is minimized. By Theorem A.2.36, we see that

$$\langle x - \hat{x}, \phi_n \rangle = 0, \ n = 1, 2, \ldots, N.$$

Supposing that $\hat{x} = \sum_{n=1}^{N} \alpha_n \phi_n$, the above equality implies that

$$\langle x, \phi_n \rangle = \alpha_n.$$

So the best estimate of any vector $x \in Z$ using N orthonormal vectors $\phi_n, \ n = 1, \ldots, N$ is $\hat{x}$ given by

$$\hat{x} = \sum_{n=1}^{N} \langle x, \phi_n \rangle \phi_n.$$

To improve this estimate, all that is necessary is to add an extra term $\langle x, \phi_{N+1} \rangle \phi_{N+1}$. We remark that this would not be the case if the sequence $\{\phi_n\}$ were not orthonormal; then it would be necessary to recalculate all of the coefficients every time a better approximation were required.

A.3. Operators on normed linear spaces

The theory of operators on normed linear spaces is treated in any introductory book on functional analysis and most of the definitions, lemmas, theorems, and examples in this section are standard; useful references are Kato [14], Kreyszig [16], Rudin [23], and Naylor and Sell [19].

A.3.1. General theory

In this section we shall be concerned with transformations T from one normed linear space X to another Y. Usually, X and Y will be either Banach or Hilbert spaces and T will be linear. Later in this section we treat the special case where Y is the scalar field $\mathcal{F}$; there the transformations are called *functionals*. We start with the following fixed-point theorem.

Theorem A.3.1. Contraction Mapping Theorem. *Let X be a Banach space, T a mapping from X to X, $m \in \mathbb{N}$, and $\alpha < 1$. Suppose that T satisfies $\|T^m(x_1) - T^m(x_2)\| \leq \alpha \|x_1 - x_2\|$ for all $x_1, x_2 \in X$. Then there exists an unique $x^* \in X$ such that $T(x^*) = x^*$; x^* is the* fixed point *of T.*

Furthermore, for any $x_0 \in X$ the sequence $\{x_n, n \geq 1\}$ defined by $x_n := T^n(x_0)$ converges to x^* as $n \to \infty$.

Proof See Kreyszig [16, theorem 5.4-3] or Naylor and Sell [19, theorem 3.15.2 and corollary 3.15.3]. ■

In above theorem the mapping T does not need to be linear. However, in the rest of this section we only consider linear transformations.

Definition A.3.2. A *linear operator*, or simply an *operator*, T from a linear space X to a linear space Y over the same field $\mathcal{F}$ is a map $T : D(T) \subset X \to Y$, such that $D(T)$ is a subspace of X, and for all $x_1, x_2 \in D(T)$ and scalars α, it holds that

$$T(x_1 + x_2) = Tx_1 + Tx_2,$$

$$T(\alpha x_1) = \alpha T x_1. \qquad ♣$$

It follows immediately from this definition that if $\alpha_i \in \mathcal{F}$ and $x_i \in D(T)$ for $i = 1, \ldots, n$, then

$$T(\sum_{i=1}^{n} \alpha_i x_i) = \sum_{i=1}^{n} \alpha_i T x_i.$$

The set $D(T)$ is called the *domain* of T. In fact, changing the domain changes the operator; for example, the operator $T_1 : D(T_1) = \{x \in L_2(0,1) \mid x \text{ continuous}\} \to L_2(0,1)$, $T_1 x = 2x$ differs from the operator $T_2 : L_2(0,1) \to L_2(0,1), T_2 x = 2x$.

Example A.3.3 It is easy to see that the following mappings are all linear operators:
the shift operator defined by

$$\sigma : \ell_2 \to \ell_2,$$

where

$$(\sigma(x))_n = x_{n+1};$$

the integral operator

$$T_g : L_2(0,1) \to L_2(0,1)$$

defined by

$$T_g f = \int_0^1 f(t) g(t) dt \qquad \text{for a } g \in L_2(0,1);$$

the differentiation operator

$$T : D(T) = C^1(0,1) \subset L_2(0,1) \to L_2(0,1)$$

defined by

$$Tf = \frac{df}{dx}. \qquad \square$$

Definition A.3.4. The set of all possible images of the operator $T : D(T) \to Y$ is a subspace of Y, in general. It is called the *range* of T and we denote this by $\operatorname{ran} T$. If the range of an operator is finite-dimensional, then we say that the operator has *finite rank*. ♣

Operators for which the domain and the range are in one-to-one correspondences are called invertible.

Definition A.3.5. A operator $T : D(T) \subset X \to Y$ between two linear spaces X and Y is *invertible* if there exists a map $S : D(S) := \operatorname{ran} T \subset Y \to X$ such that

$$\begin{array}{rcll} STx & = & x, & x \in D(T), \\ TSy & = & y, & y \in \operatorname{ran} T. \end{array}$$ ♣

S is called the *algebraic inverse* of T and we write $T^{-1} = S$.

Lemma A.3.6. *Linear operators T from X to Y, where X and Y are linear vector spaces, have the following properties:*

a. T is invertible if and only if T is injective*, that is, $Tx = 0$ implies $x = 0$;*

b. If T is an operator and it is invertible, then its algebraic inverse is also linear.

Proof See Kreyszig [16, theorem 2.6-10] or Kato [14, section III.2]. ■

The set of all elements in the domain of T such that $Tx = 0$ is called the *kernel* of T and is denoted by $\ker T$. If T is a linear operator, then $\ker T$ is a linear subspace. From the above lemma we see that the linear operator T has an inverse if $\ker T = \{0\}$.

The continuity of a map from one normed linear space to another is another very important property, since it says that a small change in the original vectors gives rise to a corresponding small change in their images.

Definition A.3.7. A map $F : D(F) \subset X \to Y$ between two normed linear spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is said to be *continuous at* $x_0 \in X$ if, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|F(x) - F(x_0)\|_Y < \varepsilon$, whenever $\|x - x_0\|_X < \delta$. F is *continuous on* $D(F)$ if it is continuous at every point in $D(F)$. ♣

Definition A.3.8. Let T be a linear operator from $D(T) \subset X \to Y$, where X and Y are normed linear spaces. T is a *bounded linear operator* or T is *bounded* if there exists a real number c such that for all $x \in D(T)$

$$\|Tx\|_Y \leq c\|x\|_X.$$ ♣

The above formula shows that a bounded linear operator maps bounded sets in $D(T)$ into bounded sets in Y, and it leads naturally to the following definition of a norm.

Definition A.3.9. Let T be a bounded linear operator from $D(T) \subset X$ to Y. We define its *norm*, $\|T\|$, by

$$\|T\| = \sup_{\substack{x \in D(T), \\ x \neq 0}} \frac{\|Tx\|_Y}{\|x\|_X}.$$ ♣

If $D(T) = \{0\}$, we define $\|T\| = 0$, since in this uninteresting case from definition A.3.2 we have that $T0 = 0$.

An equivalent definition of $\|T\|$ is

$$\|T\| = \sup_{\substack{x \in D(T) \\ \|x\|_X = 1}} \|Tx\|_Y.$$

This norm satisfies the conditions for a norm (see definition A.2.7). An automatic consequence of definition A.3.9 is that

$$\|Tx\|_Y \leq \|T\|\|x\|_X; \tag{A.1}$$

this result will be used frequently.

Continuity and boundedefinitioness are equivalent concepts for linear operators.

Theorem A.3.10. *If $T : D(T) \subset X \to Y$ is a linear operator, where X and Y are normed linear spaces, then:*

a. T is continuous if and only if T is bounded;

b. If T is continuous at a single point, it is continuous on $D(T)$.

Proof See Kato [14, section III.2], Kreyszig [16, theorem 2.7-9], Naylor and Sell [19, theorem 5.6.4 and lemma 5.6.5], or Rudin [23, theorem 1.32]. ■

Bounded linear operators that map into a Banach space always have a unique extension to the closure of their domain.

Theorem A.3.11. *Let $T : D(T) \subset X \to Y$ be a bounded linear operator, where X is a normed linear space and Y is a Banach space. Then T has a unique bounded extension $\tilde{T} : \overline{D(T)} \to Y$. Furthermore, $\|\tilde{T}\| = \|T\|$.*

Proof See Kato [14, theorem 1.16] or Kreyszig [16, theorem 2.7-11]. ■

Of special interest are bounded linear operators whose domain is a normed linear space.

Definition A.3.12. If X and Y are normed linear spaces, we define the normed linear space $\mathcal{L}(X, Y)$ to be the space of bounded linear operators from X to Y with $D(T) = X$ and with norm given by definition A.3.9.

If it is necessary to distinguish between various norms, we shall write the norm as $\|\cdot\|_{\mathcal{L}(X,Y)}$. For the special case that $X = Y$ we denote $\mathcal{L}(X, X)$ by $\mathcal{L}(X)$. First we consider $\mathcal{L}(X, Y)$, where X and Y are finite-dimensional spaces.

Example A.3.13 Recall that matrices with k rows and m columns are linear mapping from $\mathbb{C}^m$ to $\mathbb{C}^k$. If we take the norm on $\mathbb{C}^k$ and $\mathbb{C}^m$ to be the Euclidian norm, then it is easy to see that this mapping is also bounded. We shall calculate the exact norm. Let T be a $k \times m$-matrix. Since the matrix T^*T is symmetric and nonnegative, we have that

$$T^*Tx = \sum_{i=1}^{m} \sigma_i^2 \langle x, \phi_i \rangle \phi_i, \tag{A.2}$$

where $\{\phi_i, 1 \leq i \leq m\}$ is an orthonormal basis of $\mathbb{C}^m$ and σ_i^2 are the eigenvalues of T^*T. σ_i are the *singular values* of T. Without loss of generality, we assume that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0$. Since $\{\phi_i, 1 \leq i \leq m\}$ is an orthonormal basis, we have that

$$\|x\|^2 = \sum_{i=1}^{m} |\langle x, \phi_i \rangle|^2. \tag{A.3}$$

Now from (A.2), we deduce

$$\begin{aligned} \|Tx\|^2 &= \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle = \sum_{i=1}^{m} \sigma_i^2 \langle x, \phi_i \rangle \langle \phi_i, x \rangle \\ &\leq \sigma_1^2 \|x\|^2 \qquad \text{using equation (A.3).} \end{aligned}$$ □

But $\|T\phi_1\|^2 = \langle T^*T\phi_1, \phi_1 \rangle = \sigma_1^2$, and so $\|T\| = \sigma_1$.

In the next lemma, we summarize some properties of the space of linear bounded operators, $\mathcal{L}(X, Y)$.

Lemma A.3.14. *Let $\mathcal{L}(X, Y)$ denote the space of bounded linear operators from X to Y. Then the following properties hold:*

a. If Y is a Banach space, then so is $\mathcal{L}(X, Y)$;

b. If X, Y, and Z are normed linear spaces, $T_1 \in \mathcal{L}(X, Y)$ and $T_2 \in \mathcal{L}(Y, Z)$, then T_3, defined by $T_3 x = T_2(T_1 x)$, is an element of $\mathcal{L}(X, Z)$ and $\|T_3\| \leq \|T_2\| \|T_1\|$;

c. For the special case that $X = Y$, $\mathcal{L}(X)$ is an algebra; that is, αT_1, $T_1 + T_2$ and $T_1 T_2$ are in $\mathcal{L}(X)$ for every T_1, T_2 in $\mathcal{L}(X)$; furthermore, $\|T_1 T_2\| \leq \|T_1\| \|T_2\|$.

Proof *a.* Proofs are given in the following texts: Kato [14, section III.3.1], where the notation $\mathcal{B}(X, Y)$ is used instead of $\mathcal{L}(X, Y)$; Kreyszig [16, theorem 2:10-2]; Naylor and Sell [19, theorem 5.8.6], where $Blt[X, Y]$ is used instead of $\mathcal{L}(X, Y)$; Rudin [23, theorem 4.1], where the notation $\mathcal{B}(X, Y)$ is used instead of $\mathcal{L}(X, Y)$; Taylor [25, theorem 4.1-A], where $[X, Y]$ is used for $\mathcal{L}(X, Y)$.

b. See Kreyszig [16, section 2.7, equation (7)], Yosida [31, proposition I.6.2], or Naylor and Sell [19, theorem 5.8.4], where the last reference uses $Blt[X, Y]$ instead of $\mathcal{L}(X, Y)$.

c. See Kreyszig [16, section 2.10] or Taylor [25, theorem 4.1-B], where $[X]$ is used instead of $\mathcal{L}(X)$. ■

Example A.3.15 Consider the Banach space Z with norm $\|\cdot\|_Z$ and let W be a linear subspace of Z. Suppose that another norm, $\|\cdot\|_W$, is also defined on W and that W is a Banach space under this norm.

Consider the linear operator from W to Z defined by

$$iw = w,$$

where on the left-hand side w is seen as an element of W and on the right-hand side as an element of Z. This mapping is called a *continuous embedding* if the operator i is an element of $\mathcal{L}(W, Z)$. In this case, we have that

$$\|w\|_Z \leq c\|w\|_W \tag{A.4}$$

for some positive constant c. If W is a dense subspace of Z (with respect to the norm $\|\cdot\|_Z$), we call i a *dense injection*. In the case that i is a continuous dense injection, we use the notation

$$W \hookrightarrow Z. \tag{A.5}$$

Let us now take W to be the Hilbert space Z_α with the norm induced by $\langle\cdot,\cdot\rangle_\alpha$ given by (A.9). It is easy to show that W is contained in Z with continuous, dense injection

$$Z_\alpha \hookrightarrow Z. \tag{A.6}$$

□

It is possible to introduce several different notions of convergence in the space of bounded linear operators. The natural one based on the norm in $\mathcal{L}(X,Y)$ is called *uniform convergence*, but this is a very strong property. Consequently, we find the following weaker concept very useful.

Definition A.3.16. Let $\{T_n, n \geq 1\}$ be a sequence of bounded linear operators in $\mathcal{L}(X,Y)$, where X and Y are normed linear spaces. If

$$\|T_n x - Tx\|_Y \to 0 \text{ as } n \to \infty \text{ for all } x \in X,$$

♣

then we say that T_n *converges strongly* to T.

Frequently, the bounded linear operator will depend on a parameter t, where t is usually from some interval in $\mathbb{R}$. We can define strong continuity and uniform continuity with respect to t in an analogous manner.

Definition A.3.17. If $T(t)$ is in $\mathcal{L}(X,Y)$ for every $t \in [a,b]$, where X and Y are normed linear spaces, then

a. $T(t)$ is *uniformly continuous at* t_0, if

$$\|T(t) - T(t_0)\|_{\mathcal{L}(X,Y)} \to 0 \text{ as } t \to t_0;$$

b. $T(t)$ is *strongly continuous at* t_0, if

$$\|T(t)x - T(t_0)x\|_Y \to 0 \text{ for all } x \in X \text{ as } t \to t_0.$$

♣

Using this notion of continuity, we can define the following linear space.

Definition A.3.18. Let X be a normed linear space, and suppose that $-\infty \leq a < b \leq \infty$. Let f be a function from $[a,b]$ to X that satisfies

$$\|f(s) - f(s_0)\|_X \to 0, \text{ as } s \to s_0$$

♣

for all $s_0 \in [a,b]$. This function is called *continuous* and we denote by $C([a,b];X)$ the space of continuous functions from $[a,b]$ to X. It is easy to show that $C([a,b];X)$ is a linear space.

Combining definitions A.3.17 and A.3.18 we see that $T(t) \in \mathcal{L}(X)$ is strongly continuous if and only if $T(t)x \in C([a,b];X)$ for every $x \in X$.

There are two very important theorems on linear operators that are used frequently in applications.

Theorem A.3.19. The Uniform Boundedefinitioness Theorem (Banach Steinhaus Theorem). *Let $\{T_n\}$ be a family of bounded linear operators in $\mathcal{L}(X, Y)$, where X is a Banach space and Y a normed linear space. If the family $\{T_n x\}$ is bounded for each x (that is,*

$$\|T_n x\|_Y \leq M_x,$$

where M_x depends on x, but is independent of n), then $\{\|T_n\|\}$ is uniformly bounded in n.

Proof See Kato [14, theorem III.1.26], Kreyszig [16, theorem 4.7-3], Rudin [23, theorem 2.5], Taylor [25, theorem 4.4-E], or Yosida [31, corollary II.1.1]. ■

Theorem A.3.20. The Open Mapping Theorem. *Let $T \in \mathcal{L}(X, Y)$, where both X and Y are Banach spaces and T maps X onto Y. Then T maps every open set of X onto an open set of Y.*

Proof See Kreyszig [16, theorem 4.12-3], Rudin [23, corollary 2.12(a)], or Yosida [31, theorem in section II.5]. ■

A special subclass of bounded linear operators with useful properties is the following.

Definition A.3.21. Let X and Y be normed linear spaces. An operator $T \in \mathcal{L}(X, Y)$ is said to be a *compact operator* if T maps bounded sets of X onto relatively compact sets of Y. An equivalent definition is that T is linear and for any bounded sequence $\{x_k\}$ in X, $\{Tx_k\}$ has a convergent subsequence in Y.

Compact operators have properties rather similar to those enjoyed by operators on finite-dimensional spaces.

Lemma A.3.22. *Let X and Y be normed linear spaces and let $T : X \to Y$ be a linear operator. Then the following assertions hold:*

a. *If T is bounded and $\dim(T(X)) < \infty$, then the operator T is compact;*

b. *If $\dim(X) < \infty$, then the operator T is compact;*

c. *The range of T is separable if T is compact;*

d. *If S, U are elements of $\mathcal{L}(X_1, X)$ and $\mathcal{L}(Y, Y_1)$, respectively, and $T \in \mathcal{L}(X, Y)$ is compact, then so is UTS;*

e. *If $\{T_n\}$ is a sequence of compact operators from X to the Banach space Y, that converge uniformly to T, then T is a compact operator;*

f. *The identity operator, I, on the Banach space X is compact if and only if $\dim(X) < \infty$;*

g. *If T is a compact operator in $\mathcal{L}(X, Y)$ whose range is a closed subspace of Y, then the range of T is finite-dimensional.*

Proof *a.* See Kreyszig [16, theorem 8.1-4(a)], Naylor and Sell [19, theorem 5.24.3], or Rudin [23, theorem 4.18(a)].

b. See Kreyszig [16, theorem 8.1-4(b)].

c. See Kato [14, theorem III.4.10], Kreyszig [16, theorem 8.2-3], or Taylor [25, theorem 5.5-A].

d. See Kato [14, theorem III.4.8], Naylor and Sell [19, theorem 5.24.7], Rudin [23, theorem 4.18(f)], or Yosida [31, part (*ii*) of the theorem in section X.2].

e. See Kato [14, theorem III.4.7], Kreyszig [16, theorem 8.1-5], Naylor and Sell [19, theorem 5.24.8], Rudin [23, theorem 4.18(c)], or Yosida [31, part (*iii*) of the theorem in section X.2].

f. See Kreyszig [16, lemma 8.1-2(b)].

g. See Rudin [23, theorem 4.18(b)]. ■

Parts a and e of this lemma are extremely useful in proving the compactness of operators, as seen in the next example.

Example A.3.23 Let $X = \ell_2$ and consider $T : \ell_2 \to \ell_2$ defined by

$$Tx = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots).$$

Clearly, T is linear. Now define T_n by

$$T_n x = (x_1, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, 0, 0 \ldots).$$

For every n, this operator is clearly linear and bounded and it has finite-dimensional range. So from Lemma A.3.22.a. we see that T_n is a compact operator. Now we prove that T_n converges uniformly to T

$$\|Tx - T_n x\|^2 = \sum_{i=n+1}^{\infty} \frac{1}{i^2}|x_i|^2 \leq \frac{1}{(n+1)^2} \sum_{i=n+1}^{\infty} |x_i|^2 \leq \frac{1}{(n+1)^2}\|x\|^2.$$

So we have that

$$\|T_n - T\| \leq \frac{1}{n+1},$$

and T_n converges uniformly to T. Lemma A.3.22.e shows that T is compact. □

An important class of compact operators on the space $L_2(a,b)$ are the *integral operators*.

Theorem A.3.24. *Let $k(t,s)$ be an element of $L_2([a,b] \times [a,b])$. Then the operator K from $L_2(a,b)$ to $L_2(a,b)$ defined by*

$$(Ku)(t) = \int_a^b k(t,s)u(s)ds$$

is a compact operator.

Proof See Naylor and Sell [19, example 6 of section 5.24], Taylor [25, example 2 of §5.5], or Yosida [31, example 2 of section X.2]. ■

We now consider linear operators from a normed linear space X to $\mathcal{F}$, the scalar field of X.

Definition A.3.25. A *linear functional* f is a linear operator from $D(f) \subset X$, a normed linear space, to $\mathcal{F}$, the scalar field of X. Thus

$$f : D(f) \subset X \to \mathcal{F},$$

where $\mathcal{F} = \mathbb{R}$ if X is real and $\mathcal{F}$ is $\mathbb{C}$ if X is complex. ♣

Definition A.3.26. A *bounded linear functional*, f, is a bounded linear operator from $D(f) \subset X$, a normed linear space, to $\mathcal{F}$, the scalar field of X.

Example A.3.27 Consider the following special case of the Hilbert space from Example A.2.25: $Z = \{z \in L_2(0,1) \mid z$ is absolutely continuous on $(0,1)$ with $\frac{dz}{dt} \in L_2(0,1)$ and $z(0) = 0\}$ with the inner product

$$\langle z_1, z_2 \rangle_Z = \langle \frac{dz_1}{dt}, \frac{dz_2}{dt} \rangle_{L_2(0,1)}.$$

Define the following functional on Z

$$f(z) = z(\frac{1}{2}).$$

Since z is absolutely continuous, this functional is well defined. We prove that it is also bounded.

$$\begin{aligned} |f(z)| &= |z(\tfrac{1}{2})| = |z(\tfrac{1}{2}) - z(0)| = |\int_0^{\frac{1}{2}} \dot{z}(s) ds| \leq \int_0^{\frac{1}{2}} |\dot{z}(s)| ds \\ &\leq \int_0^1 |\dot{z}(s)| ds \leq \left[\int_0^1 |\dot{z}(s)|^2 ds \right]^{1/2} = \|z\|_Z, \end{aligned}$$ $\square$

where in the last inequality we have used the Cauchy-Schwarz inequality, (A.1). So f is a bounded linear functional.

From Theorem A.3.11, we know that any bounded linear functional can be extended to the closure of its domain without increasing its norm. The following important theorem says that any bounded linear functional can be extended to the *whole* space without increasing its norm. A consequence of this theorem is the existence of nontrivial bounded linear functionals on any normed linear space.

Theorem A.3.28. The Hahn-Banach Theorem. *Every bounded linear functional $f : D(f) \to \mathcal{F}$ defined on a linear subspace $D(f)$ of a normed linear space X can be extended to a bounded linear functional F on all X with preservation of norm.*

Proof See Kato [14, theorem III.1.21], Kreyszig [16, theorem 4.3-2], Rudin [23, theorem 3.6] or Taylor [25, theorem 4.3-A], or Yosida [31, theorem IV.5.1]. ■

To see that this guarantees the existence of nontrivial continuous linear functionals, consider the subspace $D(f)$ =span$\{x_0\}$, where x_0 is an arbitrary nonzero element of X. A linear functional f defined on $D(f)$ is given by

$$f(y) = \alpha \|x_0\| \qquad \text{for } y = \alpha x_0.$$

We have

$$|f(y)| = \|y\|,$$

and so $\|f\| = 1$. Thus the Hahn-Banach Theorem A.3.28 says there exists an F defined on X with $F(x_0) = \|x_0\|$ and norm one.

Following our previous notation we can denote all bounded linear functionals by $\mathcal{L}(X, \mathcal{F})$, but it is customary to use the following notation.

Definition A.3.29. The (*topological*) *dual space* of a normed linear space X is the space of all bounded linear functionals on X with domain all of X. This space will be denoted by X'. ♣

Lemma A.3.30. *X' is a Banach space with norm*

$$\|f\|_{X'} = \sup_{\substack{x \in X \\ \|x\|_X = 1}} |f(x)|.$$

Furthermore, we have the following duality between $\|\cdot\|_X$ and $\|\cdot\|_{X'}$

$$\|x\|_X = \sup_{\substack{f \in X' \\ \|f\|_{X'} = 1}} |f(x)|.$$

Proof See Kato [14, section III.1.4], Kreyszig [16, theorem 2.10-4 and corollary 4.3-4], Rudin [23, theorem 4.3], Taylor [25, theorem 4.3-B], or theorem 1 in section IV.7 of Yosida [31]. ■

Example A.3.31 In this example, we shall show that the dual of ℓ_p is ℓ_q, where $\frac{1}{q} = 1 - \frac{1}{p}$. Let f be any element of $(\ell_p)'$; then since f is linear and bounded we have

$$f(x) = \sum_{k=1}^{\infty} x_k \gamma_k,$$

where $\gamma_k = f(e_k)$, $e_k = (\delta_{kj})$; i.e., all components are zero except that in position k, which equals one. Let q be $\frac{p}{p-1}$ and consider the following sequence in ℓ_p

$$(x^n)(k) = \begin{cases} \frac{|\gamma_k|^q}{\gamma_k} & \text{if } k \leq n \text{ and } \gamma_k \neq 0 \\ 0 & \text{if } k > n \text{ or } \gamma_k = 0. \end{cases}$$

So

$$f(x^n) = \sum_{k=1}^{n} |\gamma_k|^q.$$

Now we show that $\{\gamma_k\}$ is a sequence in ℓ_q

$$\begin{aligned} f(x^n) &= |f(x^n)| \leq \|f\| \|x^n\|_{\ell_p} \\ &= \|f\| \left[\sum_{k=1}^{n} \left(\frac{|\gamma_k|^q}{|\gamma_k|} \right)^p \right]^{1/p} = \|f\| \left[\sum_{k=1}^{n} (|\gamma_k|^{q-1})^p \right]^{1/p} \\ &= \|f\| \left[\sum_{k=1}^{n} |\gamma_k|^q \right]^{1/p}. \end{aligned}$$

Hence

$$\sum_{k=1}^{n} |\gamma_k|^q = f(x^n) \leq \|f\| \left[\sum_{k=1}^{n} |\gamma_k|^q \right]^{1/p}.$$

Dividing by the last factor and using $1 - \frac{1}{p} = \frac{1}{q}$, we obtain

$$\left[\sum_{k=1}^{n} |\gamma_k|^q \right]^{1/q} = \left[\sum_{k=1}^{n} |\gamma_k|^q \right]^{1 - \frac{1}{p}} \leq \|f\|.$$

Since n is arbitrary, we have

$$\|\gamma_k\|_{\ell_q} = \left[\sum_{k=1}^{\infty} |\gamma_k|^q\right]^{1/q} \leq \|f\|.$$

Thus $(\gamma_k) \in \ell_q$.

Conversely, for any $y = (y_n) \in \ell_q$ we get a bounded linear functional on ℓ_p, if we define

$$g(x) = \sum_{k=1}^{\infty} x_k y_k.$$

Then g is linear, and the boundedefinitioness follows from the *Hölder inequality*

$$|\sum_{k=1}^{\infty} x_k y_k| \leq \left[\sum_{k=1}^{\infty} |x_k|^p\right]^{1/p} \left[\sum_{k=1}^{\infty} |y_k|^q\right]^{1/q}. \tag{A.7}$$

So, finally, $(\ell_p)' = \ell_q$. □

The above results can be extended to the Lebesgue spaces to obtain

$$(L_p(a,b))' = L_q(a,b),$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$.

Example A.3.32 Consider the Hilbert space Z_α as defined in A.2.35. We want to calculate the dual space of Z_α.

Define $\{\beta_n, n \geq 1\}$ by $\beta_n = \frac{1}{\alpha_n}$, and consider the sequences $\{z_n, n \geq 1\}$ with $\sum\limits_{n=1}^{\infty} |z_n|^2 \beta_n < \infty$. With respect to these sequences we define the following linear space:

$$Z_\beta = \{z \mid z = \sum_{n=1}^{\infty} z_n \phi_n \text{ with } \sum_{n=1}^{\infty} |z_n|^2 \beta_n < \infty\}. \tag{A.8}$$

If we define on this (formal) space the inner product

$$\langle x, y\rangle_{Z_\beta} := \sum_{n=1}^{\infty} x_n \overline{y_n} \beta_n,$$

then Z_β is a Hilbert space. We shall show that Z_β can be identified with Z'_α. Let f be any element of Z'_α. Since f is linear and bounded, we have

$$f(z) = \sum_{n=1}^{\infty} z_n f_n,$$

where $f_n = f(\phi_n)$. Let β_n be $\frac{1}{\alpha_n}$ and consider the following sequence of elements in Z_α.

$$(z^N)(n) = \begin{cases} \overline{f_n}\beta_n & \text{if } n \leq N \\ 0 & \text{if } n > N. \end{cases}$$

So

$$f(z^N) = \sum_{n=1}^{N} |f_n|^2 \beta_n.$$

Now we shall show that $y = \sum_{n=1}^{\infty} f_n \phi_n$ is an element of Z_β.

$$\begin{aligned} f(z^N) &= |f(z^N)| \leq \|f\| \|z^N\|_{Z_\alpha} = \|f\| \left[\sum_{n=1}^{N} \alpha_n |f_n|^2 \beta_n^2 \right]^{1/2} \\ &= \|f\| \left[\sum_{n=1}^{N} |f_n|^2 \beta_n \right]^{1/2}. \end{aligned}$$

We have shown that

$$\sum_{n=1}^{N} |f_n|^2 \beta_n = f(z^N) = \|f\| \left[\sum_{n=1}^{N} |f_n|^2 \beta_n \right]^{1/2},$$

and dividing by the last factor, we obtain

$$\left[\sum_{n=1}^{N} |f_n|^2 \beta_n \right]^{1/2} \leq \|f\|.$$

Since N is arbitrary, we conclude that

$$\left[\sum_{n=1}^{\infty} |f_n|^2 \beta_n \right]^{1/2} \leq \|f\|.$$

So $y := \sum_{n=1}^{\infty} f_n \phi_n$ is an element of Z_β.

Conversely, for any $y = \sum_{n=1}^{\infty} y_n \phi_n \in Z_\beta$ we obtain a bounded linear functional on Z_α, by defining

$$g(z) = \sum_{n=1}^{\infty} z_n y_n. \tag{A.9}$$

This g is linear, and the boundedefinitioness follows from the Cauchy-Schwarz inequality

$$|\sum_{n=1}^{\infty} z_n y_n| = |\sum_{n=1}^{\infty} \sqrt{\alpha_n} z_n \sqrt{\beta_n} y_n| \leq \left[\sum_{n=1}^{\infty} \alpha_n |z_n|^2 \right]^{1/2} \left[\sum_{n=1}^{\infty} \beta_n |y_n|^2 \right]^{1/2}. \qquad \square$$

So Z_β can be identified with the space Z'_α.

In the previous example it is easily seen that $Z \hookrightarrow Z_\beta$; this holds more generally.

Lemma A.3.33. *Let X and Y be Hilbert spaces, such that $X \hookrightarrow Y$. Then $Y' \hookrightarrow X'$.*

Proof See proposition 3 in Aubin [1, chapter 3, section 5]. ■

Since the dual of a normed linear space X is a Banach space, we may consider the bounded linear functionals on X', which we shall denote by X''. Moreover, each element x in X gives rise to a bounded linear functional f_x in X', by

$$f_x'(f) = f(x), \ f \in X'.$$

It can be shown that the map $x \mapsto f_x'$ is an isometric isomorphism of X into X'', and it is called the *natural embedding* of X in X''. Sometimes it happens that X "equals" X''; these spaces have a special name.

Definition A.3.34. A space is *reflexive* if its second dual X'' is isometrically isomorphic to X under the natural embedding.

Examples of reflexive spaces are ℓ_p and $L_p(a,b)$ for $1 < p < \infty$ and Hilbert spaces; see Theorem A.3.52.

The introduction of the space X' leads to a new concept of convergence of a sequence.

Definition A.3.35. A sequence $\{x_n\}$ in the normed linear space X *converges weakly* to x if $f(x_n) \to f(x)$ as $n \to \infty$ for all $f \in X'$. ♣

Lemma A.3.36. *If $\{x_n\}$ is a weakly convergent sequence in a normed linear space with weak limit x, then $\{x_n\}$ is uniformly bounded in norm and*

$$\|x\| \leq \liminf \|x_n\| < \infty.$$

Proof See Kato [14, section III.1, equation (1.26)] or Yosida [31, section V.1, theorem 1]. ■

The next example will show that weak convergence is indeed weaker than strong convergence.

Example A.3.37 Consider $X = \ell_p$, $p > 1$ and the sequence $\{x^n\}$, where $x^n = (\delta_{nk})$. Then for $f \in X' = \ell_q$ and $f = (f_1, f_2, \ldots)$ we see that

$$f(x^n) = f_n,$$

and since $f \in \ell_q$, we have that $f_n \to 0$ as $n \to \infty$. Therefore x^n converges weakly to 0. However, $\|x^n - 0\|_{\ell_q} = 1$, so we see that x^n does not converge strongly. □

From Lemma A.3.36 we see that every weakly converging sequence is bounded. The following theorem shows that the converse is (almost) true for reflexive Banach spaces.

Theorem A.3.38. *A Banach space X is reflexive if and only if every bounded sequence in X contains a weakly convergent subsequence.*

Proof See Yosida [31, Eberlein-Shmulyan theorem]. ■

A consequence of this result is the following theorem.

Theorem A.3.39. *Let X_1 be a separable Banach space and let X_2 be a reflexive Banach space. Assume further that $\{T_n\} \subset \mathcal{L}(X_1, X_2)$ is a sequence of uniformly bounded operators. Then there exists a $T \in \mathcal{L}(X_1, X_2)$ and a subsequence $\alpha(n) \subset \mathbb{N}$ such that*

$$\lim_{n\to\infty} f\left(T_{\alpha(n)}x\right) \to f(Tx)$$

for every $x \in X_1$ and $f \in X_2'$.

Proof Let $\{e_n, n \in \mathbb{N}\}$ be a basis for X_1. Without loss of generality, we assume that for every n, $\|e_n\| = 1$. Since T_n are uniformly bounded, the sequence $\{T_n e_1\}$ is also uniformly bounded. Hence by Theorem A.3.38 there exists a subsequence $\alpha(1,n)$ such that $T_{\alpha(1,n)}e_1$ converges weakly to some $y_1 \in X_2$. Next we consider the sequence $T_{\alpha(1,n)}e_2$. This is again bounded; hence there exists a subsequence $\alpha(2,\cdot) \subset \alpha(1,\cdot)$ such that $T_{\alpha(2,n)}e_2$ converges weakly to some $y_2 \in X_2$. Repeating this argument, we obtain subsequences $\alpha(i,\cdot)$ and elements y_i such that $\alpha(i+1,\cdot) \subset \alpha(i,\cdot)$ and $T_{\alpha(i,n)}e_i$ converges weakly to y_i.

If we define $\alpha(n) := \alpha(n,n)$, that is, the nth element of the nth subsequence, then $\alpha(n) \in \alpha(i,\cdot)$ for $n > i$. Hence $T_{\alpha(n)}e_i$ converges weakly to y_i. Defining the linear operator T by $Te_i = y_i$ gives

$$T_{\alpha(n)} \sum_{i=1}^{N} \gamma_i e_i \to T \sum_{i=1}^{N} \gamma_i e_i, \tag{A.10}$$

where the convergence is in the weak sense. Combining (A.10) with Lemma A.3.36 gives

$$\|T \sum_{i=1}^{N} \gamma_i e_i\| \leq \liminf_{n\to\infty} \|T_{\alpha(n)} \sum_{i=1}^{N} \gamma_i e_i\| \leq M \|\sum_{i=1}^{N} \gamma_i e_i\|,$$

since $\{T_n\}$ is uniformly bounded. Hence T is an element of $\mathcal{L}(X_1, X_2)$ and $\|T\| \leq M$.

Choose an $f \in X_2'$ and an $x \in X_1$. For this x there exist an N and $\gamma_1, \ldots, \gamma_N$ such that

$$\|x - \sum_{i=1}^{N} \gamma_i e_i\| \leq \frac{\varepsilon}{3M\|f\|}.$$

Thus we obtain that

$$\begin{aligned}
&|f\left(T_{\alpha(n)}x\right) - f(Tx)| \\
&\leq \; |f\left(T_{\alpha(n)}x\right) - f\left(T_{\alpha(n)} \sum_{i=1}^{N} \gamma_i e_i\right)| + \\
&\quad\; \left| f\left(T_{\alpha(n)} \sum_{i=1}^{N} \gamma_i e_i\right) - f\left(T \sum_{i=1}^{N} \gamma_i e_i\right)\right| + \\
&\quad\; |f\left(T \sum_{i=1}^{N} \gamma_i e_i\right) - f(Tx)| \\
&\leq \; \varepsilon/3 + \left| f\left(T_{\alpha(n)} \sum_{i=1}^{N} \gamma_i e_i\right) - f\left(T \sum_{i=1}^{N} \gamma_i e_i\right)\right| + \varepsilon/3.
\end{aligned}$$

From (A.10) it follows that the last expression is smaller than ε for n sufficiently large. ε is arbitrary, and so we have proved the theorem. ■

On the dual spaces there exists a natural operator dual to a given operator.

Definition A.3.40. Let Q be an operator in $\mathcal{L}(X,Y)$, where X and Y are Banach spaces. The operator Q' from Y' to X', defined by

$$\left(Q'y'\right)(x) = y'(Qx), \tag{A.11}$$

is the *dual operator* of Q. ♣

Lemma A.3.41. *Let $Q \in \mathcal{L}(X, Y)$, where X and Y are Banach spaces. The dual operator Q' of Q has the following properties:*

a. $Q' \in \mathcal{L}(Y', X')$ *with* $\|Q'\| = \|Q\|$;

b. $(\alpha Q)' = \alpha Q'$.

Proof *a.* See Aubin [1, chapter 3, section 3, proposition 1], Kato [14, section III.3.3], Kreyszig [16, theorem 4.5-2], Rudin [23, theorem 4.10], Taylor [25, §4.5], or Yosida [31, theorem 2' in section VII.1].
b. See §4.5 in Taylor [25]. ■

Until now we have concentrated mainly on bounded linear operators. However, in applications one often comes across *unbounded* (not bounded) linear operators. Before we can introduce an important class of these operators, we need the concept of the graph of a linear operator.

Definition A.3.42. Let X and Y be normed linear spaces and $T : D(T) \subset X \to Y$ a linear operator. The *graph* $\mathcal{G}(T)$ is the set

$$\mathcal{G}(T) = \{(x, Tx) \mid x \in D(T)\}$$

in the product space $X \times Y$. ♣

Definition A.3.43. A linear operator T is said to be *closed* if its graph $\mathcal{G}(T)$ is a closed linear subspace of $X \times Y$. Alternatively, T is closed if whenever

$$x_n \in D(T), n \in \mathbb{N} \text{ and } \lim_{n\to\infty} x_n = x, \ \lim_{n\to\infty} Tx_n = y,$$

it follows that $x \in D(T)$ and $Tx = y$. ♣

From this definition, we see that the domain of definition is important for an operator to be closed. We shall illustrate this by the following example.

Example A.3.44 Let X be an infinite-dimensional normed linear space, and let V be a linear subspace of X that is not closed. If we consider the operator I on V, defined by

$$Ix = x \qquad \text{for } x \in V,$$

then I is trivially bounded, but it is not closed. If we take any x in $\overline{V}$ and not in V, there exists a sequence $\{x_n\}$ in V converging to x. So we have a sequence in V that converges and so does $\{Ix_n\}$. However, x is not in $D(I) = V$ so $I : V \subset X \to X$ is not closed. □

This example is rather special, since one can easily show that any bounded linear operator on a closed domain is closed. However, there are many unbounded linear operators that are closed, as in the following example.

Example A.3.45 Let Z be the Hilbert space $L_2(0, 1)$ and consider the following operator on $L_2(0, 1)$

$$T = \frac{d}{dx}$$

with

$$D(T) = \{z(x) \in Z \mid z \text{ is absolutely continuous with } z(0) = 0 \text{ and } \tfrac{dz}{dx} \in L_2(0, 1)\}.$$

We show that T with this domain is closed.

Let $\{z_n\} \subset D(T)$ be a sequence such that $z_n \to z$ and $\frac{dz_n}{dx} \to y$; we must show that $z \in D(T)$ and $\frac{dz}{dx} = y$. Define f by

$$f(\xi) = \int_0^\xi y(x)dx.$$

f is an element of $D(T)$ and $\frac{df}{dx} = y$. We show that $f = z$ by considering

$$\begin{aligned}
\|f - z\|_{L_2(0,1)} &= \|f - z_n + z_n - z\| \leq \|f - z_n\| + \|z_n - z\| \\
&\leq \|z_n - z\| + \left[\int_0^1 |\int_0^\xi y(x)dx - z_n(\xi)|^2 d\xi\right]^{1/2} \\
&\leq \|z_n - z\| + \left[\int_0^1 |\int_0^\xi y(x) - \frac{dz_n}{dx}(x)dx|^2 d\xi\right]^{1/2} \\
&\leq \|z_n - z\| + \left[\int_0^1 \|1_{[0,\xi]}\|_{L_2}^2 \|y - \frac{dz_n}{dx}\|_{L_2}^2 d\xi\right]^{1/2} \\
&\leq \|z_n - z\| + \frac{1}{3}\|y - \frac{dz_n}{dx}\|.
\end{aligned}$$

Since $z_n \to z$ and $\frac{dz_n}{dx} \to y$, this last expression can be made arbitrarily small, and so $z = f$. □

In many examples, it is rather difficult to prove that an operator is closed. The next theorem states that if the operator is the algebraic inverse of a bounded linear operator, then it is closed. With this theorem we can more easily prove the result in Example A.3.45 (see Example A.3.47).

Theorem A.3.46. *Assume that X and Y are Banach spaces and let T be a linear operator with domain $D(T) \subset X$ and range Y. If, in addition, T is invertible with $T^{-1} \in \mathcal{L}(Y, X)$, then T is a closed linear operator.*

Proof This follows from theorem 4.2-C of Taylor [25] with $f = T^{-1}$. ■

Example A.3.47 Let Z be the Hilbert space $L_2(0,1)$ and consider the operator of Example A.3.45 again, i.e.,

$$T = \frac{d}{dx}$$

with

$$D(T) = \{z(x) \in Z \mid z \text{ is absolutely continuous with } z(0) = 0 \text{ and } \tfrac{dz}{dx} \in L_2(0,1)\}.$$

We show that T with this domain is closed.

Define the following operator on Z:

$$(Sz)(x) = \int_0^x z(s)ds.$$

□

It is easy to see that $S \in \mathcal{L}(Z)$ and that $ST = I_{D(T)}$ and $TS = I_Z$. So $S = T^{-1}$ and from Theorem A.3.46 we conclude that T is a closed operator.

Example A.3.48 Let Z be the Hilbert space $L_2(0,1)$ and consider the following operator on Z:

$$T = \frac{d^2}{dx^2}$$

with domain

$$\begin{aligned} D(T) &= \{z \in L_2(0,1) \mid z, \tfrac{dz}{dx} \text{ are absolutely continuous} \\ &\quad \text{with } \tfrac{dz}{dx}(0) = \tfrac{dz}{dx}(1) = 0 \text{ and } \tfrac{d^2z}{dx^2} \in L_2(0,1)\}. \end{aligned}$$

Using Theorem A.3.46, we show that T with this domain is closed. Since $T1 = 0$, we have that T is not injective and thus is not invertible. Instead, we shall consider the operator $T + I$.

Define the following operator on Z:

$$(Sh)(x) = \int_0^x g(x,\xi)h(\xi)d\xi + \int_x^1 g(\xi,x)h(\xi)d\xi,$$

where

$$g(\xi, x) = \cot(1)\cos(x)\cos(\xi) + \sin(\xi)\cos(x).$$

This operator is clearly in $\mathcal{L}(Z)$, and by Theorem A.3.24 it is even compact. If we set $f(x) = (Sh)(x)$, then f is absolutely continuous and

$$\begin{aligned} \frac{df}{dx}(x) &= \int_0^x \left[-\cot(1)\cos(\xi)\sin(x) + \cos(\xi)\cos(x)\right] h(\xi)d\xi + \\ &\quad \int_x^1 \left[-\cot(1)\cos(\xi)\sin(x) - \sin(\xi)\sin(x)\right] h(\xi)d\xi. \end{aligned}$$

From this we see that $\frac{df}{dx}(0) = \frac{df}{dx}(1) = 0$ and $\frac{df}{dx}$ is absolutely continuous. Differentiating $\frac{df}{dx}$ once more, we obtain

$$\frac{d^2 f}{dx^2}(x) = h(x) - f(x).$$

□

Thus S is the bounded inverse of $T + I$. Thus, by Theorem A.3.46 $T + I$ is closed, and hence T is also closed.

Theorem A.3.46 gives an easy condition to check the closedefinitioness of an operator. The following theorem gives a similar result for the boundedefinitioness of a linear operator.

Theorem A.3.49. Closed Graph Theorem. *A closed linear operator defined on all of a Banach space X into a Banach space Y is bounded.*

Proof See Kato [14, theorem III.5.20], Kreyszig [16, theorem 4.13-2], Rudin [23, theorem 2.15], Taylor [25, theorem 4.2-I], or Yosida [31, theorem II.6.1]. ■

Corollary A.3.50. *If T is a closed linear operator from a Banach space X to a Banach space Y and T has an algebraic inverse T^{-1}, then T^{-1} is an element of $\mathcal{L}(Y,X)$ if and only if $D(T^{-1}) = \operatorname{ran} T = Y$.* ♠

Proof See theorem 4.7-A in Taylor [25]. ■

Many of the definitions that we gave for bounded linear operators have extensions to closed operators. One of these notions is that of the dual operator.

Definition A.3.51. Let A be a closed, densely defined operator from $D(A) \subset X$ to Y, where X and Y are Banach spaces. A' is constructed in the following way. $D(A')$ consists of all $g \in Y'$ such that there exists an $f \in X'$ with the property

$$g(Ax) = f(x) \qquad \text{for all } x \in D(A).$$

The *dual operator* $A'g$ is defined by

$$A'g = f \qquad \text{for } g \in D(A'). \qquad \clubsuit$$

A.3.2. Operators on Hilbert spaces

In the last subsection, we introduced linear operators on a normed linear space. A Hilbert space is a special normed linear space and so all the definitions made in that subsection are valid for Hilbert spaces. However, since we have additional structure on Hilbert spaces (the inner product), we can deduce extra properties of operators that exploit this structure.

One of the most important properties of a Hilbert space is that there is a particularly simple representation for its dual space.

Theorem A.3.52. Riesz Representation Theorem. *If Z is a Hilbert space, then every element in Z induces a bounded linear functional f defined by*

$$f(x) = \langle x, z \rangle_Z.$$

On the other hand, for every bounded linear functional f on Z, there exists a unique vector $z_0 \in Z$, such that

$$f(x) = \langle x, z_0 \rangle_Z \text{ for all } x \in Z,$$

and furthermore, $\|f\| = \|z_0\|$.

Proof See Kato [14, p. 252 and 253], Kreyszig [16, theorem 3.8-1], Naylor and Sell [19, theorem 5.21.1], Taylor [25, theorem 4.81-C], or Yosida [31, section III.6]. ∎

Using this theorem, one can easily give a representation of finite-rank bounded operators. In the next example, we do this for an operator of rank one.

Example A.3.53 Let Z be a Hilbert space and $T \in \mathcal{L}(Z)$ be an operator with one-dimensional range. This means that there exists a $v \in Z$ such that $Tz \in \operatorname{span}\{v\}$ for all $z \in Z$. Hence, $Tz = f(z)v$ for some mapping f. Since T is a linear and bounded operator, it follows directly that f is bounded linear functional. Thus by the Riesz Representation Theorem A.3.52 there exists a $z_0 \in Z$ such that $f(z) = \langle z, z_0 \rangle$, and so $Tz = \langle z, z_0 \rangle v$. □

The Riesz representation theorem gives an isometry between Z and Z'. Usually, we identify Z with its dual Z'.

Example A.3.54 In Example A.3.27, we showed that

$$f : Z \to \mathbb{C};\ f(z) = z(\frac{1}{2})$$

defines a bounded linear functional on the Hilbert space $Z := \{z \in L_2(0,1) \mid z \text{ is absolutely continuous on } (0,1) \text{ with } \frac{dz}{dx} \in L_2(0,1) \text{ and } z(0) = 0\}$. The Riesz representation theorem gives that there exists an element y of Z such that $\langle z, y\rangle_Z = f(z)$ for every z in Z. To determine this y, we consider

$$z(\frac{1}{2}) = \int_0^{\frac{1}{2}} \dot{z}(x)dx$$

and choose $\dot{y}(x) = 1_{[0,\frac{1}{2}]}(x)$, for then

$$\int_0^1 \dot{z}(x)\overline{\dot{y}(x)}dx = \int_0^{\frac{1}{2}} \dot{z}(x)dx = z(\frac{1}{2}).$$

So, if we define

$$y(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \leq x \leq 1, \end{cases}$$

then y is an element of Z and $\langle z, y\rangle_Z = z(\frac{1}{2})$. □

It is important to realize that a Hilbert space may be identified with several spaces, as can be seen in the next example.

Example A.3.55 Consider the Hilbert space Z_α as defined in Example A.2.35

$$Z_\alpha := \{z = (z_n) \mid \sum_{n=1}^{\infty} \alpha_n |z_n|^2 < \infty\},$$

with inner product

$$\langle z, w\rangle_\alpha := \sum_{n=1}^{\infty} \alpha_n z_n \overline{w_n}.$$

In Example A.3.32, we showed that the dual space can be identified with the Hilbert space

$$Z_\beta = \{z = (z_n) \mid \sum_{n=1}^{\infty} \beta_n |z_n|^2 < \infty\},$$

with inner product

$$\langle z, w\rangle_\beta = \sum_{n=1}^{\infty} \beta_n z_n \overline{w_n},$$

where $\beta_n = \frac{1}{\alpha_n}$. However, from the Riesz Representation Theorem A.3.52 we see that Z_α' can also be identified with itself. For every element of Z_β, we calculate the element of Z_α such that they define the same linear functional. For (y_n) in Z_β the corresponding functional is defined by

$$g(z) = \sum_{n=1}^{\infty} z_n y_n \qquad \text{see (A.9).}$$

An easy calculation shows that

$$g(z) = \sum_{n=1}^{\infty} \alpha_n z_n \frac{1}{\alpha_n} y_n = \langle z, w\rangle_{Z_\alpha},$$

where $w = (w_n) = (\frac{1}{\alpha_n}\overline{y_n})$. This is an element of Z_α, since

$$\sum_{n=1}^{\infty} \alpha_n |w_n|^2 = \sum_{n=1}^{\infty} \alpha_n |\frac{1}{\alpha_n}\overline{y_n}|^2 = \sum_{n=1}^{\infty} \frac{1}{\alpha_n} |y_n|^2 = \sum_{n=1}^{\infty} \beta_n |y_n|^2 < \infty.$$

□

Similarly, for every element of Z_α we can construct an element of Z_β such that their corresponding linear functionals are the same.

In the previous example, we saw that there is some freedom in identifying the dual of a Hilbert space. However, in the situation that there are two Hilbert spaces W and Z such that $W \hookrightarrow Z$, then we have from Lemma A.3.33 that $Z' \hookrightarrow W'$. If we could identify W with W' and Z with Z', then W would equal Z, but this is not true in general. For Hilbert spaces identified with their dual we use the term *pivot space*. So, if in the previous discussion Z is the pivot space, then

$$W \hookrightarrow Z = Z' \hookrightarrow W'. \tag{A.12}$$

This identification implies that if $w' \in W'$ is also an element of Z, then

$$w'(w) = \langle w, w' \rangle_Z. \tag{A.13}$$

It is usual to represent the action of the bounded linear functional $w' \in W'$ on $w \in W$ as a *duality pairing*

$$w'(w) := \langle w, w' \rangle_{W,W'}. \tag{A.14}$$

For more details about this we refer the reader to Aubin [1, chapter 3].

Example A.3.56 Consider the Hilbert space $Z = \ell_2$ and Z_α defined in Example A.2.35. Since $\alpha_n \geq 1$, we have

$$Z_\alpha \hookrightarrow Z$$

and if we choose Z as the pivot space, we obtain

$$Z_\alpha \hookrightarrow Z \hookrightarrow Z'_\alpha.$$

Consider the operator $T : Z \to Z_\alpha$ defined by

$$(Tz)_n = \frac{1}{n} z_n.$$

Clearly, T is linear and bounded, and its dual $T' : Z'_\alpha \to Z$. Since we have identified Z with its dual, by the Riesz Representation Theorem A.3.52, there exists a bounded bijective operator $J : Z'_\alpha \to Z_\alpha$ such that $z'(z) = \langle z, Jz' \rangle_\alpha$ for any $z' \in Z'_\alpha$ and any $z \in Z_\alpha$. Taking an arbitrary $z \in Z$, we have that

$$\begin{aligned} \langle z, T'z' \rangle_Z &= (T'z')(z) \\ &= z'(Tz) \qquad \text{by definition A.3.40} \\ &= \langle Tz, Jz' \rangle_\alpha \\ &= \sum_{n=1}^{\infty} \alpha_n \frac{1}{n} z_n \overline{(Jz')_n} = \langle z, w \rangle_Z, \end{aligned}$$

where $w_n = \frac{\alpha_n}{n}(Jz')_n$. So we have shown that

$$(T'z')_n = \frac{\alpha_n}{n}(Jz')_n.$$

■ □

Another consequence of the Riesz Representation Theorem A.3.52 is the existence of the adjoint operator.

Definition A.3.57. Let $T \in \mathcal{L}(Z_1, Z_2)$, where Z_1 and Z_2 are Hilbert spaces. Then there exists a unique operator $T^* \in \mathcal{L}(Z_2, Z_1)$ that satisfies

$$\langle Tz_1, z_2\rangle_{Z_2} = \langle z_1, T^* z_2\rangle_{Z_1} \qquad \text{for all } z_1 \in Z_1, z_2 \in Z_2.$$

This operator is called the *adjoint operator* of T. ♣

Example A.3.58 Let Z be a complex Hilbert space and define $Tz = \langle z, z_T\rangle$ for some $z_T \in Z$. It is easily seen that $T \in \mathcal{L}(Z, \mathbb{C})$. To calculate the adjoint of T, let $z \in Z$ and $\gamma \in \mathbb{C}$ be arbitrary, and consider $\langle Tz, \gamma\rangle_{\mathbb{C}} = \langle z, z_T\rangle_Z \overline{\gamma} = \langle z, \gamma z_T\rangle_Z$. Thus $T^*\gamma = z_T\gamma$.

Example A.3.59 Let $Z = L_2(a, b)$ and define $K : Z \to Z$ by $Kz(\cdot) = \int_a^b k(\cdot, s)z(s)ds$, where $k \in L_2([a, b] \times [a, b])$. Then from Theorem A.3.24, $K \in \mathcal{L}(Z)$, and for $z, w \in Z$ the following holds:

$$\begin{aligned} \langle Kz, w\rangle &= \int_a^b \int_a^b k(t, s)z(s)ds\overline{w(t)}dt \\ &= \int_a^b z(s) \int_a^b k(t, s)\overline{w(t)}dtds \\ &= \int_a^b z(s) \overline{\int_a^b \overline{k(t, s)}w(t)dt}ds. \end{aligned}$$

Hence $K^*w(\cdot) = \int_a^b \overline{k(t, \cdot)}w(t)dt$. □

Since for Hilbert spaces we may identify the dual space with the space itself, there is a relationship between the adjoint and the dual operator. We shall show that such a relationship exists for bounded linear operators (see also [16, section 4.5]). Let Z_1 and Z_2 be Hilbert spaces and suppose that $T \in \mathcal{L}(Z_1, Z_2)$. From definitions A.3.40 and A.3.57, we have

$$T' : Z_2' \to Z_1' \qquad \text{with } (T'z_2')(z_1) = z_2'(Tz_1),$$

and

$$T^* : Z_2 \to Z_1 \qquad \text{with } \langle z_1, T^*z_2\rangle_{Z_1} = \langle Tz_1, z_2\rangle_{Z_2}.$$

From the Riesz Representation Theorem A.3.52, we have that Z_1 is isometrically isomorphic to Z_1'. Thus there exists a bounded, bijective operator J_1 from Z_1' to Z_1 such that

$$z_1'(z) = \langle z, J_1 z_1'\rangle.$$

A similar relationship holds for the Hilbert space Z_2. All these operators between the spaces are given in Figure A.1. We remark that, for complex Hilbert spaces, J_1 and J_2 are not linear operators, since

$$\begin{aligned} \langle z, J_1(\alpha z_1 + \beta z_2)\rangle &= (\alpha z_1 + \beta z_2)(z) = \alpha z_1(z) + \beta z_2(z) \\ &= \langle z, \overline{\alpha} J_1 z_1\rangle + \langle z, \overline{\beta} J_1 z_2\rangle. \end{aligned}$$

Thus $J_1(\alpha z_1 + \beta z_2) = \overline{\alpha} J_1 z_1 + \overline{\beta} J_1 z_2$.

$$\begin{tikzcd}
\boxed{Z_1} \arrow[r, shift left, "T"] & \boxed{Z_2} \arrow[l, shift left, "T^*"] \\
\boxed{Z_1'} \arrow[u, "J_1"] & \boxed{Z_2'} \arrow[u, "J_2"'] \arrow[l, "T'"']
\end{tikzcd}$$

Figure A.1.: The relationship between T^* and T'

Using the definitions of T', T^*, J_1, and J_2, we see that for every $z_1 \in Z_1$ and $z_2 \in Z_2$ there holds

$$\begin{aligned}
\langle z_1, T^* z_2 \rangle_{Z_1} &= \langle T z_1, z_2 \rangle_{Z_2} = (J_2^{-1}(z_2))(T z_1) \\
&= (T'(J_2^{-1}(z_2)))(z_1) = \langle z_1, J_1 T' J_2^{-1} z_2 \rangle_{Z_1}.
\end{aligned}$$

So T^* has the following representation:

$$T^* = J_1 T' J_2^{-1}. \tag{A.15}$$

Usually, we identify Hilbert spaces with their dual, and in this case we use the adjoint and the notation T^*, as in Example A.3.59. Only in situations where we choose not to identify the Hilbert spaces do we use the dual notation T', as in Example A.3.56.

Many of the results presented in the next lemma can be proved using this relationship. Compare property a of Lemma A.3.41 with c of Lemma A.3.60.

Lemma A.3.60. *Let $T_1, T_2 \in \mathcal{L}(Z_1, Z_2)$ and $S \in \mathcal{L}(Z_2, Z_3)$, where Z_1, Z_2, and Z_3 are Hilbert spaces. The adjoint has the following properties:*

a. $I^* = I$;

b. $(\alpha T_1)^* = \overline{\alpha} T_1^*$;

c. $\|T_1^*\| = \|T_1\|$;

d. $(T_1 + T_2)^* = T_1^* + T_2^*$;

e. $(S T_1)^* = T_1^* S^*$;

f. $\|T_1^* T_1\| = \|T_1\|^2$.

Proof See Kreyszig [16, theorem 3.9-4], Naylor and Sell [19, theorem 5.22.2 and corollary 5.22.3], Rudin [23, section 12.9], Taylor [25, section 4.9], or Yosida [31, section VII.2]. ■

Theorem A.3.61. *If $T \in \mathcal{L}(Z_1, Z_2)$, where Z_1 and Z_2 are Hilbert spaces, then we have the following equalities:*

a. $\operatorname{ran}(T)^{\perp} = \ker(T^*)$;

b. $\overline{\operatorname{ran}(T)} = \ker(T^*)^{\perp}$;

c. $\operatorname{ran}(T^*)^\perp = \ker T$;

d. $\overline{\operatorname{ran}(T^*)} = \ker(T)^\perp$.

Proof See proposition 1 and corollary 2 in Aubin [1, chapter 3, section 4], Naylor and Sell [19, theorem 5.22.6], or Rudin [23, theorem 12.10]. ■

Definition A.3.62. A bounded linear operator T on a Hilbert space Z is:

a. *normal* if $TT^* = T^*T$;

b. *unitary* if $TT^* = T^*T = I$.

In the theory of this book we also need the adjoint of an unbounded linear operator.

Definition A.3.63. Let A be a linear operator on a Hilbert space Z. Assume that the domain of A, $D(A)$, is dense in Z. Then the *adjoint operator* $A^* : D(A^*) \subset Z \to Z$ of A is defined as follows. The domain $D(A^*)$ of A^* consists of all $y \in Z$ such that there exists a $y^* \in Z$ satisfying

$$\langle Ax, y\rangle = \langle x, y^*\rangle \qquad \text{for all } x \in D(A).$$

For each such $y \in D(A^*)$ the *adjoint operator* A^* is then defined in terms of y^* by

$$A^* y = y^*.$$

■ ♣

In addition, it can be shown that if A is a closed, densely defined operator, then $D(A^*)$ is dense in Z and A^* is closed. Furthermore, one can show the same relationship exists between A^* and A' as in (A.15). In the following example we shall calculate the adjoint of an unbounded operator heuristically.

Example A.3.64 Let $Z = L_2(0,1)$ and consider the operator A given by

$$(Az)(x) = \frac{dz}{dx}(x),$$

where $D(A) = \{z \in L_2(0,1) \mid z \text{ is absolutely continuous with } \frac{dz}{dx} \in L_2(0,1), z(0) = 0\}$.

In Example A.3.45 we have shown that A is a closed linear operator. Now we calculate its adjoint heuristically as follows:

$$\begin{aligned}\langle Az, y\rangle &= \int_0^1 \frac{dz}{dx}(x)\overline{y(x)}dx = \Big[z(x)\overline{y(x)}\Big]_0^1 - \int_0^1 z(x)\overline{\frac{dy}{dx}}(x)dx \\ &= z(1)\overline{y(1)} - \int_0^1 z(x)\overline{\frac{dy}{dx}}(x)dx.\end{aligned}$$

This can be written in the form $\langle z, y^*\rangle$ if and only if $y(1) = 0$ and $\frac{dy}{dx} \in Z$. So the logical choice is $D(A^*) = \{y \in Z \mid y \text{ is absolutely continuous with } \frac{dy}{dx} \in Z \text{ and } y(1) = 0\}$ and $A^* y = -\frac{dy}{dx}$. □

In order to justify the above we need the following results.

Lemma A.3.65. *Let A be an arbitrary, densely defined operator and let T be a bounded linear operator defined on the whole of the Hilbert space Z. The following holds:*

a. $(\alpha A)^* = \overline{\alpha} A^*$; $D((\alpha A)^*) = D(A^*)$ *if* $\alpha \neq 0$ *and* Z *if* $\alpha = 0$;

b. $(A+T)^* = A^* + T^*$, *with domain* $D((A+T)^*) = D(A^*)$;

c. If A *has a bounded inverse, i.e., there exists an* $A^{-1} \in \mathcal{L}(Z)$ *such that* $AA^{-1} = I_Z$; $A^{-1}A = I_{D(A)}$, *then* A^* *also has a bounded inverse and* $(A^*)^{-1} = (A^{-1})^*$.

Proof *a.* This is shown in Rudin [23, theorem 13.2].

b. Suppose that there exist y, y^* such that $\langle (A+T)x, y\rangle = \langle x, y^*\rangle$ for all $x \in D(A+T)$. This implies that

$$\langle Ax, y\rangle = \langle x, y^* - T^*y\rangle \qquad \text{for all } x \in D(A+T) = D(A).$$

Hence $y \in D(A^*)$ and $A^*y = y^* - T^*y$, and so we conclude that $D((A+T)^*) \subset D(A^*)$ and $(A+T)^* = A^* + T^*$ on $D((A+T)^*)$. The inclusion $D(A^*) \subset D((A+T)^*)$ follows similarly.

c. See Kato [14, theorem III.5.30] or Kreyszig [16, theorem 10.2-2]. ■

Example A.3.66 Let Z be the Hilbert space $L_2(0,1)$ and consider the operator of Example A.3.64 again, i.e.,

$$A = \frac{d}{dx}$$

with

$$D(A) = \{z \in Z \mid z \text{ is absolutely continuous with } z(0) = 0 \text{ and } \tfrac{dz}{dx} \in L_2(0,1)\}.$$

From Example A.3.47, we have that the algebraic inverse is bounded and given by

$$(A^{-1}z)(x) = \int_0^x z(s)ds.$$

We calculate A^* via $(A^{-1})^*$; so we consider

$$\begin{aligned}
\langle A^{-1}z_1, z_2\rangle &= \int_0^1 \int_0^x z_1(s)ds\overline{z_2(x)}dx \\
&= \int_0^1 \int_s^1 z_1(s)\overline{z_2(x)}dxds \quad \text{by Fubini's Theorem A.5.22} \\
&= \int_0^1 z_1(s)\overline{\int_s^1 z_2(x)dx}ds = \langle z_1, (A^{-1})^* z_2\rangle,
\end{aligned}$$

where $[(A^{-1})^* z_2](s) = \int_s^1 z_2(x)dx$. From this it is easy to see that

$$A^*z = -\frac{dz}{dx}$$

with domain

$$D(A) = \{z \in Z \mid z \text{ is absolutely continuous with } z(1) = 0 \text{ and } \tfrac{dz}{dx} \in L_2(0,1)\}.$$

Thus we see that this is the same as in Example A.3.64. The difference here is that we have proven it rigorously. □

Example A.3.67 Let Z be the Hilbert space $L_2(0,1)$ and consider the operator of Example A.3.48 on $L_2(0,1)$

$$A = \frac{d^2}{dx^2}$$

with domain

$$\begin{aligned} D(A) \quad = \quad & \{z \in L_2(0,1) \mid z, \tfrac{dz}{dx} \text{ are absolutely continuous} \\ & \text{with } \tfrac{dz}{dx}(0) = \tfrac{dz}{dx}(1) = 0 \text{ and } \tfrac{d^2z}{dx^2} \in L_2(0,1)\}. \end{aligned}$$

From Example A.3.48, we have that $A+I$ has a bounded, algebraic inverse given by

$$((I+A)^{-1}h)(x) = \int_0^x g(x,\xi)h(\xi)d\xi + \int_x^1 g(\xi,x)h(\xi)d\xi,$$

where

$$g(\xi,x) = \cot(1)\cos(x)\cos(\xi) + \sin(\xi)\cos(x).$$

If we calculate the adjoint of $(I+A)^{-1}$, then we have that

$$\begin{aligned} &\langle (I+A)^{-1}h, z\rangle \\ &\quad = \int_0^1 \int_0^x g(x,\xi)h(\xi)d\xi\overline{z(x)}dx + \int_0^1 \int_x^1 g(\xi,x)h(\xi)d\xi\overline{z(x)}dx \\ &\quad = \int_0^1 \int_\xi^1 g(x,\xi)h(\xi)\overline{z(x)}dxd\xi + \int_0^1 \int_0^\xi g(\xi,x)h(\xi)\overline{z(x)}dxd\xi \\ &\quad = \int_0^1 h(\xi)\overline{\int_\xi^1 g(x,\xi)z(x)dxd\xi} + \int_0^1 h(\xi)\overline{\int_0^\xi g(\xi,x)z(x)dxd\xi} \\ &\quad = \langle h, (I+A)^{-1}z\rangle. \end{aligned}$$

□

So we see that $((I+A)^{-1})^* = (I+A)^{-1}$. Thus from Lemma A.3.65.c it follows that $(I+A)^* = I+A$, and from Lemma A.3.65.b we conclude that $A^* = A$.

Example A.3.67 belongs to a special class of operators.

Definition A.3.68. We say that a densely defined, linear operator A is *symmetric* if for all $x, y \in D(A)$

$$\langle Ax, y\rangle = \langle x, Ay\rangle.$$

A symmetric operator is *self-adjoint* if $D(A^*) = D(A)$. ♣

All bounded, symmetric operators are self-adjoint. It can be shown that the adjoint of an operator is always closed, so, in particular, every self-adjoint operator is closed. Furthermore, we have from Lemma A.3.65 that an invertible operator is self-adjoint if and only if its inverse is (see Example A.3.67). For a self-adjoint operator, we always have that $\langle Az, z\rangle = \langle z, Az\rangle$. Thus from property b of definition A.2.23 we conclude that $\langle Az, z\rangle$ must be real for all $z \in D(A)$. The converse is also true.

Lemma A.3.69. *Let T be an element of $\mathcal{L}(Z)$, with Z a complex Hilbert space. T is self-adjoint if and only if $\langle Tz, z\rangle$ is real for all $z \in Z$.*

Proof See Kreyszig [16, theorem 3.10-3] or Naylor and Sell [19, theorem 5.23.6]. ■

Lemma A.3.70. *Let T be a self-adjoint operator in $\mathcal{L}(Z)$, where Z is a Hilbert space. We have the following relation between the norm and the inner product:*

$$\|T\| = \sup_{\|z\|=1} |\langle Tz, z\rangle|.$$

Proof See Kreyszig [16, theorem 9.2-2], Naylor and Sell [19, theorem 5.23.8], or theorem 3 in Yosida [31, section VII.3]. ■

So for every self-adjoint operator A the range of $\langle Az, z\rangle$ is real. Operators for which this range is nonnegative have a special name.

Definition A.3.71. A self-adjoint operator A on the Hilbert space Z is *nonnegative* if

$$\langle Az, z\rangle \geq 0 \qquad \text{for all } z \in D(A);$$

A is *positive* if

$$\langle Az, z\rangle > 0 \qquad \text{for all nonzero } z \in D(A);$$

and A is *coercive* if there exists an $\varepsilon > 0$ such that

$$\langle Az, z\rangle \geq \varepsilon \|z\|^2 \qquad \text{for all } z \in D(A).$$ ♣

We shall use the notation $A \geq 0$ for nonnegativity of the self-adjoint operator A, and $A > 0$ for positivity. Furthermore, If T, S are self-adjoint operators in $\mathcal{L}(Z)$, then we shall write $T \geq S$ for $T - S \geq 0$.

With this new notation, it is easy to see that A is coercive if and only if $A \geq \varepsilon I$, for some $\varepsilon > 0$. Some of the special properties of self-adjoint, nonnegative operators are collected in the following theorem and lemmas.

Theorem A.3.72. *Let Z be a complex Hilbert space, and let T_n be a sequence of bounded, nonnegative, self-adjoint operators on Z such that $T_{n+1} \geq T_n$ and $\alpha I \geq T_n$, for some positive $\alpha \in \mathbb{R}$. Then the sequence $\{T_n\}$ is strongly convergent; that is, there exists a $T \in \mathcal{L}(Z)$ such that $T_n z \to Tz$ for every $z \in Z$. Furthermore, T is nonnegative, self-adjoint, and $\alpha I \geq T \geq T_n$ for all n.*

Proof See Kreyszig [16, theorem 9.3-3]. ■

Lemma A.3.73. *If A is self-adjoint and nonnegative, then A has a unique nonnegative* square root $A^{\frac{1}{2}}$, *so that $D(A^{\frac{1}{2}}) \supset D(A)$, $A^{\frac{1}{2}} z \in D(A^{\frac{1}{2}})$ for all $z \in D(A)$, and $A^{\frac{1}{2}} A^{\frac{1}{2}} z = Az$ for $z \in D(A)$. Furthermore, if A is positive, then $A^{\frac{1}{2}}$ is positive too.*

Proof See Kato [14, theorem V-3.35] for a general proof. Kreyszig [16, theorem 9.4-2] and Rudin [23, theorem 12.33] only prove the bounded case. ■

Lemma A.3.74. *Let T be a nonnegative, self-adjoint operator in $\mathcal{L}(Z)$, where Z is a Hilbert space. It has the following properties:*

a. $\|T^{\frac{1}{2}}\| = \|T\|^{\frac{1}{2}}$;

b. $|\langle Tz_1, z_2\rangle|^2 \leq \langle Tz_1, z_1\rangle \langle Tz_2, z_2\rangle$ *for all* $z_1, z_2 \in Z$;

c. $\|Tz\|^2 \le \|T\|\langle z, Tz\rangle$ *for all* $z \in Z$.

Note that b is a generalization of the Cauchy-Schwarz inequality.

Proof *a.* This follows from Lemma A.3.60.f with $T_1 = T^{\frac{1}{2}}$.
b. For $z_1, z_2 \in Z$ we have

$$\begin{aligned} |\langle Tz_1, z_2\rangle|^2 &= |\langle T^{\frac{1}{2}} z_1, T^{\frac{1}{2}} z_2\rangle|^2 \\ &\le \|T^{\frac{1}{2}} z_1\|^2 \|T^{\frac{1}{2}} z_2\|^2 \qquad \text{by the Cauchy-Schwarz inequality} \\ &= \langle Tz_1, z_1\rangle\langle Tz_2, z_2\rangle. \end{aligned}$$

c. It is easy to see that $\|Tz\| = \sup_{\|y\|=1}\langle Tz, y\rangle$, and thus using part b we obtain

$$\|Tz\|^2 = \sup_{\|y\|=1} |\langle Tz, y\rangle|^2 \le \sup_{\|y\|=1} \langle Ty, y\rangle\langle Tz, z\rangle = \|T\|\langle Tz, z\rangle,$$

where we used Lemma A.3.70 and the fact that T is nonnegative. ■

One of the most important classes of nonnegative operators is the orthogonal projections.

Definition A.3.75. An operator $P \in \mathcal{L}(Z)$ is a *projection* if $P^2 := PP = P$, and a projection operator is called *orthogonal* if $P^* = P$.

In Appendix A.2, we have seen that given a closed linear subspace V we can decompose the Hilbert space Z into $Z = V \oplus V^\perp$. Let z be any element of Z, then there exist $z_V \in V$ and $z_{V^\perp} \in V^\perp$ such that $z = z_V + z_{V^\perp}$.

Define the operator $P : Z \to Z$ by $Pz = z_V$. Then, since $\|z\|^2 = \|z_V\|^2 + \|z_{V^\perp}\|^2$, we have that $P \in \mathcal{L}(Z)$, $\ker P = V^\perp$, and P is an orthogonal projection. We call P the *orthogonal projection on* V. On the other hand, if P is an orthogonal projection, then $Z = \operatorname{ran} P \oplus \ker P$ with $\operatorname{ran} P \perp \ker P$. So an orthogonal projection is naturally associated with an orthogonal decomposition of the Hilbert space.

We close this section with an important lemma concerning minimalization problems that complements Theorem A.2.36.

Lemma A.3.76. Orthogonal Projection Lemma. *Let Z be a Hilbert space and V a closed subspace of Z. Then, given $z_0 \in Z$, there exists a unique v_0 in V such that*

$$\|z_0 - v_0\| = \min_{v \in V} \|z_0 - v\|.$$

Furthermore, the element v_0 is given by $v_0 = P_V z$, where P_V is the orthogonal projection on V. We see that $z_0 - v_0 = P_{V^\perp} z_0$.

Proof See Kato [14, page 252], Kreyszig [16, theorem 3.3-1 and lemma 3.3-2], or Naylor and Sell [19, theorem 5.14.4]. ■

Corollary A.3.77. *Let Z be a Hilbert space, V a closed subspace of Z, $z_0 \in Z$, and define the affine set*

$$V_{z_0} := \{z \in Z \mid z = z_0 + v \text{ for some } v \in V\}.$$

There exists a unique element z_V in V_{z_0} such that

$$\|z_V\| = \min_{z \in V_{z_0}} \|z\|.$$

This element is given by $z_V := P_{V^\perp} z$, where z is an arbitrary element of V_{z_0}. ♠

Proof This follows from Lemma A.3.76, since

$$\min_{z \in V_{z_0}} \|z\| = \min_{v \in V} \|z_0 + v\| = \min_{v \in V} \|z_0 - v\|.$$

■

A.4. Spectral theory

A.4.1. General spectral theory

In this section, we consider abstract equations of the form

$$(\lambda I - A)x = y, \tag{A.1}$$

where A is a closed linear operator on a complex Banach space X with $D(A) \subset X$, $x, y \in X$, and $\lambda \in \mathbb{C}$. As an example of this formulation, which will be considered in more detail later in this appendix, we consider the boundary value problem

$$\begin{aligned}&\frac{d^2 z}{dx^2}(x) + \lambda z(x) = v(x) \quad \text{on } L_2(0,1),\\ &z(0) = 0 = z(1),\end{aligned}$$

where v is a given function in $L_2(0,1)$.

The solutions of these problems are reduced to asking under what conditions $(\lambda I - A)$ has a bounded inverse on the particular Banach space X. When X is finite-dimensional, it is well known that this depends on whether λ is an eigenvalue of A. For the infinite-dimensional case, we need to generalize the concept of eigenvalues. We shall generalize this to the class of closed linear operators, and a study of these will give useful information about the existence and uniqueness of solutions to (A.1). This abstract approach to studying linear equations on a Banach space is what is known as spectral theory. This theory can be found in almost any book on operator theory. In the finite-dimensional case, not every matrix on a real space has eigenvalues and eigenvectors. To overcome this situation, one has to consider the matrix on a complex space. In this section, we shall therefore only consider complex normed spaces.

As our motivation is the study of linear equations of the form

$$y = (\lambda I - A)x$$

on a complex normed linear space X, where $A : D(A) \subset X \to X$ is a closed linear operator, we are interested in those $\lambda \in \mathbb{C}$ for which (A.1) has a unique solution for all $y \in X$, which, following definition A.3.5, we may write as

$$x = (\lambda I - A)^{-1} y, \tag{A.2}$$

where $(\lambda I - A)^{-1}$ is the algebraic inverse of $\lambda I - A$. Here we also require that this inverse is bounded.

Definition A.4.1. Let A be a closed linear operator on a (complex) normed linear space X. We say that λ is in the *resolvent set* $\rho(A)$ of A, if $(\lambda I - A)^{-1}$ exists and is a bounded linear operator on a dense domain of X. ♣

Now, from the fact that $(\lambda I - A)$ is closed and invertible, $(\lambda I - A)^{-1}$ must be a closed operator. Since it is also a bounded linear operator, its domain must be a closed subspace. So its domain is both closed and dense, which by the Closed Graph Theorem A.3.49 means that $(\lambda I - A)^{-1} \in \mathcal{L}(Z)$. So $\lambda \in \rho(A)$ if and only if $(\lambda I - A)^{-1} \in \mathcal{L}(Z)$. We shall call $(\lambda I - A)^{-1}$ the *resolvent operator* of A. Other names that are used are: *bounded algebraic inverse* or *bounded inverse.*

Example A.4.2 Let Z be a Hilbert space. Consider the positive, self-adjoint operator A on Z that is coercive, i.e., $\langle Az, z\rangle \geq \alpha\|z\|^2$ for all $z \in D(A)$ and a given $\alpha > 0$. From this it is clear that A is injective, and so from Lemma A.3.6.a we obtain that the algebraic inverse exists. For $z \in \operatorname{ran} A$ we have

$$\|A^{-1}z\|^2 \leq \frac{1}{\alpha}\langle AA^{-1}z, A^{-1}z\rangle \leq \frac{1}{\alpha}\|z\|\|A^{-1}z\|,$$

where we have used the Cauchy-Schwarz inequality. This implies that

$$\|A^{-1}z\| \leq \frac{1}{\alpha}\|z\|,$$

and A^{-1} is bounded on its range. If $\operatorname{ran} A$ is dense in Z, then 0 is in the resolvent set of A and $A^{-1} \in \mathcal{L}(Z)$.

Let x be in the orthogonal complement to the range of A, i.e., for all $z \in D(A)$ the following holds:

$$\langle Az, x\rangle = 0. \qquad \square$$

By definition A.3.63, this implies that $x \in D(A^*)$ and $A^*x = 0$. Since A is self-adjoint, we conclude that $Ax = A^*x = 0$. The positivity of A shows that this can only happen if $x = 0$, and so $\operatorname{ran} A$ is dense in Z.

Example A.4.3 Let Z be a Hilbert space and consider the positive, self-adjoint operator A on Z that is coercive. In the previous example, we saw that A is boundedly invertible. This inverse is positive, since for every nonzero $z \in Z$

$$\langle A^{-1}z, z\rangle = \langle y, Ay\rangle > 0, \qquad \text{where } y = A^{-1}z.$$

Since A and A^{-1} are positive operators, they have a positive square root (see Lemma A.3.73). We show that $\left(A^{\frac{1}{2}}\right)^{-1} = \left(A^{-1}\right)^{\frac{1}{2}}$.

Define the operator $Q = A^{\frac{1}{2}}A^{-1}$. Since $\operatorname{ran} A^{-1} = D(A)$ and $D(A^{\frac{1}{2}}) \supset D(A)$ (see Lemma A.3.73), we have that Q is a well defined linear operator. If Q is closed, then by the Closed Graph Theorem A.3.49 $Q \in \mathcal{L}(Z)$. Let $z_n \to z$ and $Qz_n \to y$. Then $x_n := A^{-1}z_n \to A^{-1}z$ and $A^{\frac{1}{2}}x_n \to y$. From the fact that $A^{\frac{1}{2}}$ is closed (see the remark after definition A.3.68), we conclude that $A^{\frac{1}{2}}A^{-1}z = y$. So Q is closed and hence bounded. It is easy to see that $\operatorname{ran} Q \subset D(A^{\frac{1}{2}})$ and

$$A^{\frac{1}{2}}Q = I_Z. \tag{A.3}$$

Define $x = QA^{\frac{1}{2}}z$, for $z \in D(A^{\frac{1}{2}})$. Then

$$A^{\frac{1}{2}}x = A^{\frac{1}{2}}QA^{\frac{1}{2}}z = A^{\frac{1}{2}}z \qquad \text{by (A.3).}$$

The operator $A^{\frac{1}{2}}$ is positive, and so $z = x$. In other words, $QA^{\frac{1}{2}} = I_{D(A^{\frac{1}{2}})}$. Thus $A^{\frac{1}{2}}$ is invertible, and

$$\left(A^{\frac{1}{2}}\right)^{-1} = Q = A^{\frac{1}{2}}A^{-1}. \tag{A.4}$$

To see that $\left(A^{\frac{1}{2}}\right)^{-1}$ is positive, consider the following for $z \in Z$

$$\langle \left(A^{\frac{1}{2}}\right)^{-1} z, z\rangle = \langle y, A^{\frac{1}{2}} y\rangle > 0, \qquad \text{where } y = \left(A^{\frac{1}{2}}\right)^{-1} z.$$

Multiplying both sides of (A.4) by $\left(A^{\frac{1}{2}}\right)^{-1}$ gives

$$\left(A^{\frac{1}{2}}\right)^{-1} \left(A^{\frac{1}{2}}\right)^{-1} = A^{-1}. \qquad \square$$

Thus $\left(A^{\frac{1}{2}}\right)^{-1}$ is a positive square root of A^{-1}. Since the positive square root is unique, $\left(A^{\frac{1}{2}}\right)^{-1} = \left(A^{-1}\right)^{\frac{1}{2}}$. We shall denote this operator by $A^{-\frac{1}{2}}$.

Definition A.4.4. Let A be a closed linear operator on a (complex) normed linear space X. The *spectrum* of A is defined to be

$$\sigma(A) = \mathbb{C} \backslash \rho(A).$$

The *point spectrum* is

$$\sigma_p(A) = \{\lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is not injective}\}.$$

The *continuous spectrum* is

$$\begin{aligned} \sigma_c(A) &= \{\lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is injective}, \overline{\operatorname{ran}(\lambda I - A)} = X, \text{ but} \\ &\qquad (\lambda I - A)^{-1} \text{ is unbounded}\} \\ &= \{\lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is injective}, \overline{\operatorname{ran}(\lambda I - A)} = X, \text{ but} \\ &\qquad \operatorname{ran}(\lambda I - A) \neq X\}. \end{aligned}$$

The *residual spectrum* is

$$\sigma_r(A) = \{\lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is injective, but } \operatorname{ran}(\lambda I - A) \text{ is not dense in } X\}.$$

So $\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$.

A point $\lambda \in \sigma_p(A)$ is an *eigenvalue*, and $x \neq 0$ such that $(\lambda I - A)x = 0$, an *eigenvector*. ♣

For eigenvalues, we have natural generalizations of the finite-dimensional concepts.

Definition A.4.5. Let λ_0 be an eigenvalue of the closed linear operator A on the Banach space X. Suppose further that this eigenvalue is *isolated*; that is, there exists an open neighbourhood O of λ_0 such that $\sigma(A) \cap O = \{\lambda_0\}$. We say that λ_0 has *order* ν_0 if for every $x \in X$

$$\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{\nu_0} (\lambda I - A)^{-1} x$$

exists, but there exists an x_0 such that the following limit does not

$$\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{\nu_0 - 1} (\lambda I - A)^{-1} x_0.$$

If for every $\nu \in \mathbb{N}$ there exists an $x_\nu \in X$ such that the limit

$$\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{\nu} (\lambda I - A)^{-1} x_\nu \qquad ♣$$

does not exist, then the order of λ_0 is infinity.

For the isolated eigenvalue λ_0 of finite order ν_0, its (*algebraic*) *multiplicity* is defined as $\dim(\ker(\lambda_0 I - A)^{\nu_0})$. The elements of $\ker(\lambda_0 - A)^{\nu_0}$ are called the *generalized eigenvectors* corresponding to λ_0.

We remark that if the kernel of $(\lambda_0 I - A)$ is finite-dimensional, then so is the kernel of $(\lambda_0 I - A)^\nu$ for any $\nu \geq 1$.

In finite-dimensional spaces, we always have that $\sigma_c(A)$ and $\sigma_r(A)$ are empty, but this is not the case if X is infinite-dimensional, as can be seen from the following example.

Example A.4.6 Let $X = \ell_1$ and let $T : X \to X$ be given by

$$Tx = (x_1, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, \ldots).$$

Consider

$$(\lambda I - T)x = y.$$

Now

$$(\lambda I - T)x = ((\lambda - 1)x_1, \ldots, (\lambda - \frac{1}{n})x_n, \ldots).$$

So $\lambda = \frac{1}{n}$, $n = 1, 2, \ldots$, are the eigenvalues of the operator T with associated eigenvectors $e_n := (0, \ldots, 0, 1, 0 \ldots)$. Let $\lambda \neq \frac{1}{n}$, $n = 1, 2, \ldots$, and $\lambda \neq 0$; then

$$x = (\lambda I - T)^{-1} y = ((\lambda - 1)^{-1} y_1, (\lambda - \frac{1}{2})^{-1} y_2, \ldots, (\lambda - \frac{1}{n})^{-1} y_n, \ldots).$$

This defines a bounded linear operator. Thus $\rho(T) \supset \mathbb{C} \setminus \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$.

So we only have to investigate in which part of the spectrum the point 0 lies. If $\lambda = 0$, then $x = (-y_1, -2y_2, \ldots, -ny_n, \ldots)$, and for $y = (\frac{1}{n^2})$ we have that $\|x\|_1 = \sum_{i=1}^{\infty} \frac{1}{n}$ is not finite, and so $0 \notin \rho(T)$. We know that T is one-one, so we must determine the range of T to decide whether $0 \in \sigma_c(T)$ or $\sigma_r(T)$. Now $T(ne_n) = e_n$, and so $e_n \in \operatorname{ran} T$. Since $\overline{\operatorname{span}}\{e_n\}$ is X, we have $\overline{\operatorname{ran} T} = X$ and so $0 \in \sigma_c(T)$.

Summarizing, we have shown that

$$\sigma_c(T) = \{0\};\ \sigma_r(T) = \emptyset,$$

$$\sigma_p(T) = \{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}.$$

□

One of the most important relations for the resolvent operator is the *resolvent equation*

$$(\mu I - A)^{-1} = (\lambda I - A)^{-1} + (\lambda - \mu)(\lambda I - A)^{-1}(\mu I - A)^{-1}, \tag{A.5}$$

for $\mu, \lambda \in \rho(A)$.

Repeatedly applying the resolvent equation gives

$$\begin{aligned}(\mu I - A)^{-1} &= \sum_{k=0}^{n} (\lambda - \mu)^k (\lambda I - A)^{-k-1} + \\ &\quad (\lambda - \mu)^{n+1} (\lambda I - A)^{-n-1} (\mu I - A)^{-1}.\end{aligned}$$

If $\|(\lambda - \mu)(\lambda I - A)^{-1}\| < 1$, then the term $(\lambda - \mu)^{n+1}(\lambda I - A)^{-n-1}$ converges uniformly to zero, and we obtain the following result.

Lemma A.4.7. *Suppose that $\mu, \lambda \in \rho(A)$, and $\|(\lambda - \mu)(\lambda I - A)^{-1}\| < 1$. Then*

$$(\mu I - A)^{-1} = \sum_{k=0}^{\infty} (\lambda - \mu)^k (\lambda I - A)^{-k-1}. \tag{A.6}$$

Proof See Kreyszig [16, theorem 7.3-3] or Taylor [25, theorem 5.1-C]. ■

A direct consequence of this lemma is the following result.

Lemma A.4.8. *For a closed linear operator, A, on a Banach space X the following hold:*

a. If $\lambda \in \rho(A)$ and $\mu \in \mathbb{C}$ are such that $\|(\lambda - \mu)(\lambda I - A)^{-1}\| < 1$, then $\mu \in \rho(A)$, and $(\mu I - A)^{-1}$ is given by (A.6);

b. The resolvent set of A is open;

c. The resolvent operator is a holomorphic function on $\rho(A)$, and the derivative is given by $\frac{d}{d\lambda}(\lambda I - A)^{-1} = -(\lambda I - A)^{-2}$; see definition A.5.32.

Proof *a.* See Kreyszig [16, theorem 7.3-3], Naylor and Sell [19, theorem 6.7.3], or Taylor [25, theorem 5.1-A].

b. See Kato [14, theorem III.6.7], Kreyszig [16, theorem 7.3-2], Naylor and Sell [19, theorem 6.7.3], Taylor [25, theorem 5.1-B], or theorem 1 in Yosida [31, section VIII.2].

c. See Kato [14, theorem III.6.7], Kreyszig [16, theorem 7.5-2], or Example A.5.33, or Taylor [25, theorem 5.1-C]. ■

The order of the isolated eigenvalue as defined in definition A.4.5 is the same as the order of the pole that $(\lambda I - A)^{-1}$ has as a holomorphic function.

So for closed linear operators we have information about the resolvent set and the resolvent operator. For bounded linear operators we can prove even stronger results. A useful tool is the following theorem.

Theorem A.4.9. *Let X be a Banach space and $T \in \mathcal{L}(X)$ such that $\|T\| < 1$. Then $(I-T)^{-1}$ exists and is in $\mathcal{L}(X)$ with*

$$(I-T)^{-1} = I + T + T^2 + \ldots + T^n + \ldots,$$

where the convergence is in the uniform topology on $\mathcal{L}(X)$ and

$$\|(I-T)^{-1}\| \leq (1 - \|T\|)^{-1}.$$

Proof See Kreyszig [16, theorem 7.3-1], Naylor and Sell [19, theorem 6.7.2], Rudin [23, theorem 10.7], or Taylor [25, theorem 5.2-A]. ■

If we rewrite $(\lambda I - T) = \lambda(I - \frac{1}{\lambda}T)$, then we have the following corollary of Theorem A.4.9.

Corollary A.4.10. *Let $T \in \mathcal{L}(X)$, where X is a Banach space. If $|\lambda| > \|T\|$, then $\lambda \in \rho(T)$. Moreover, we have*

$$(\lambda I - T)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} T^n$$

and

$$\|(\lambda I - T)^{-1}\| \leq (|\lambda| - \|T\|)^{-1}.$$ ♠

A consequence of this corollary and Lemma A.4.8.c is the following lemma.

Lemma A.4.11. *If $T \in \mathcal{L}(X)$, where X is a Banach space, then $\sigma(T)$ is nonempty.*

Proof See Kato [14, section III.6.2], Kreyszig [16, theorem 7.5-4], Rudin [23, theorem 10.13], or Taylor [25, theorem 5.2-B]. ■

We summarize results concerning the spectrum of a bounded linear operator in the following theorem.

Theorem A.4.12. *If X is a complex Banach space and $T \in \mathcal{L}(X)$, then the spectrum of T is a nonempty compact subset of the complex plane lying in the closed ball $\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\}$.*

Proof See Kato [14, section III.6.2], Kreyszig [16, theorem 7.3-4], Naylor and Sell [19, theorem 6.7.4], or Rudin [23, theorem 10.13]. ■

Example A.4.13 Let Z denote the Hilbert space $\ell_2(\mathbb{Z})$ and let A be the right shift operator given by

$$(Az)_k = z_{k-1} \qquad \text{for } k \in \mathbb{Z},$$

where $z = (\ldots, z_{-1}, z_0, z_1, \ldots)$.

It is easy to see that $\|A\| = 1$, and so by Theorem A.4.12 we may conclude that $\sigma(A) \subset \{s \in \mathbb{C} \mid |s| \leq 1\}$. Furthermore, we have that A^{-1} exists and, it is given by the left shift operator

$$(A^{-1}z)_k = z_{k+1} \qquad \text{for } k \in \mathbb{Z}.$$

From this it follows that $\|A^{-1}\| = 1$, and so by Theorem A.4.12 we deduce that $\sigma(A^{-1}) \subset \{s \in \mathbb{C} \mid |s| \leq 1\}$.

For $0 \neq s \in \mathbb{C}$ the following equality holds

$$(sI - A) = sA(A^{-1} - \frac{1}{s}I).$$

For $0 < |s| < 1$ the inverse of the right-hand side exists in $\mathcal{L}(Z)$, and so the inverse of the left-hand side is in $\mathcal{L}(Z)$ too. Hence we see that $\sigma(A) \subset \{s \in \mathbb{C} \mid |s| = 1\}$. We show that, in fact, equality holds.

Suppose that λ is an eigenvalue with $|\lambda| = 1$. Then there exists a $z \in Z$ satisfying $Az = \lambda z$. From the definition of A, we obtain

$$z_{k-1} = \lambda z_k \qquad \text{for } k \in \mathbb{Z}.$$

The unique solution of this equation is given by $z_k = \lambda^{-k} z_0$. However, since $|\lambda| = 1$, we have that this is in $Z = \ell_2(\mathbb{Z})$ if and only if $z_0 = 0$, and this implies that $z = 0$. Thus a λ with modulus one cannot be an eigenvalue. Next we shall show that every λ on the unit circle is in the spectrum of A. Let e^n denote the element of Z, which is defined as follows

$$e^n = (e^n_k) = \delta_{n,k} = \begin{cases} 1 & \text{for } k = n \\ 0 & \text{for } k \neq n. \end{cases}$$

Consider the equation $(A - \lambda I)z = e^0$, or equivalently,

$$z_{k-1} - \lambda z_k = e^0_k \qquad \text{for } k \in \mathbb{Z}.$$

This has the unique solution $z_k = 0$ for $k < 0$ and $z_k = \lambda^{-1-k}$ for $k \geq 0$. Again, since $|\lambda| = 1$, this is not an element of Z, and thus we have shown that $\operatorname{ran}(\lambda I - A) \neq Z$. The remark after definition A.4.1 shows that $\lambda \notin \rho(A)$. Combining these results and using Theorem A.4.12, we conclude that $\sigma(A) = \{s \in \mathbb{C} \mid |s| = 1\}$. For completeness we shall show that $\sigma(A) = \sigma_c(A)$. Let

$z \in Z$ be any element in the orthogonal complement of ran $(A - \lambda I)$. This implies that for every $n \in \mathbb{Z}$ we have that

$$\begin{aligned} \langle z, (A - \lambda I)e^n \rangle &= 0 \qquad \text{for all } n \in \mathbb{Z} \qquad \Leftrightarrow \\ \langle z, e^{n+1} - \lambda e^n \rangle &= 0 \qquad \text{for all } n \in \mathbb{Z} \qquad \Leftrightarrow \\ z^{n+1} - \lambda z^n &= 0 \qquad \text{for all } n \in \mathbb{Z}, \end{aligned} \qquad \square$$

where $z = (z_n)$. This equation has the unique solution $z_n = \lambda^n z_0$. However, since $|\lambda| = 1$, this is in Z if and only if $z_0 = 0$. Hence $z = 0$ is the only element in the orthogonal complement of ran $(A - \lambda I)$, and thus $\overline{\text{ran}\,(A - \lambda I)} = Z$. From definition A.4.4 it follows that $\sigma_c(A) = \{s \in \mathbb{C} \mid |s| = 1\} = \sigma(A)$.

In the following lemma, we characterize the exact radius of the smallest ball containing the spectrum of T. The *spectral radius* $r_\sigma(T)$ of the operator $T \in \mathcal{L}(X)$ on a complex Banach space X is defined by

$$r_\sigma(T) := \sup_{\lambda \in \sigma(T)} |\lambda|.$$

We have the following result.

Lemma A.4.14. *For $T \in \mathcal{L}(X)$ on the complex Banach space X, we have*

$$r_\sigma(T) = \lim_{n \to \infty} \sqrt[n]{\|T^n\|}.$$

Proof See Kato [14, section III.6.2, equation (6.13)], Kreyszig [16, theorem 7.5-5], Rudin [23, theorem 10.13], Taylor [25, theorem 5.2-E], or Yosida [31, theorems 3 and 4 in section VIII.2]. ■

With this lemma we can easily prove the following result.

Lemma A.4.15. *Let T, S be bounded operators on the Banach space X. The following relation holds:*

$$r_\sigma(TS) = r_\sigma(ST).$$

Proof If S or T is the zero operator, then the result is trivial. Suppose that they are nonzero. We have that

$$\begin{aligned} r_\sigma(TS) &= \lim_{n \to \infty} \sqrt[n]{\|[TS]^n\|} \\ &\leq \lim_{n \to \infty} \sqrt[n]{\|T\| \|[ST]^{n-1}\| \|S\|} \qquad \text{by Lemma A.3.14} \\ &= \lim_{n \to \infty} \sqrt[n]{\|T\|} \lim_{n \to \infty} \sqrt[n]{\|[ST]^{n-1}\|} \lim_{n \to \infty} \sqrt[n]{\|S\|} \\ &= 1 \cdot r_\sigma(ST) \cdot 1. \end{aligned}$$

Hence we have shown that $r_\sigma(TS) \leq r_\sigma(ST)$. Similarly, one can show that the reverse inequality holds, and so we have proved the assertion. ■

Lemma A.4.14 gives information about the size of the spectrum of an operator. For self-adjoint operators, we have more information.

Lemma A.4.16. *If A is a self-adjoint operator on the Hilbert space Z, then $\sigma(A) \subset \mathbb{R}$. Furthermore, if $A \in \mathcal{L}(Z)$, then we have the following additional properties:*

a. $\sigma(A) \subset [m, M]$, *where* $m = \inf_{\|z\|=1} \langle Az, z\rangle$ *and* $M = \sup_{\|z\|=1} \langle Az, z\rangle$*;*

b. $m, M \in \sigma(A)$*;*

c. $\|A\| = \max\{|m|, |M|\}$*;*

d. $r_\sigma(A) = \|A\|$.

Proof *a.* See Kreyszig [16, theorem 9.2-1] or Taylor [25, theorem 6.2-B].

b. See Kreyszig [16, theorem 9.2-3] or Taylor [25, theorem 6.2-B].

c. See Kreyszig [16, theorem 9.2-2], Taylor [25, theorem 6.11-C], or Yosida [31, theorem 3 in section VII.3].

d. This follows from parts a, b, and c. ■

Lemma A.4.17. *Consider the densely defined, closed, linear operator A on the Banach space X. The following relation holds between the spectrum of A and of its adjoint*

$$\sigma(A^*) = \overline{\sigma(A)}, \tag{A.7}$$

where the bar denotes the complex conjugate.

Proof See Kato [14, theorem III.6.22]. ■

A.4.2. Spectral theory for compact normal operators

From the previous subsection it is clear that the spectral properties of infinite-dimensional operators are much more complicated than those for finite-dimensional operators. However, compact operators have a simple spectrum, and the following theorem shows that we can expect the theory for solutions of linear equations with compact operators to be similar to that for (finite-dimensional) operators on $\mathbb{C}^n$.

Theorem A.4.18. *If T is a compact operator on a Banach space X, then $\lambda \neq 0$ is in either the point spectrum or the resolvent set of T. The point spectrum of T is, at most, countably infinite with $\lambda = 0$ the only possible point of accumulation. Furthermore, the order of every nonzero eigenvalue is finite, and so is its multiplicity.*

Proof See Kato [14, theorem III.6.26], Kreyszig [16, theorems 8.3-1 and 8.3-3], Naylor and Sell [19, corollary 6.10.5 and theorem 6.10.1], Rudin [23, theorem 4.25], Taylor [25, theorems 5.5-C and 5.5-G], or Yosida [31, theorem 2, section X.5]. ■

In Section A.3 we have seen that if the Banach space is infinite-dimensional and the compact operator T is injective, then 0 is an element of the spectrum of T; see Lemma A.3.22.

The following general result is very useful in the applications.

Lemma A.4.19. *Let A be a closed linear operator with $0 \in \rho(A)$ and A^{-1} compact. The spectrum of A consists of only isolated eigenvalues with finite multiplicity.*

Proof See Kato [14, theorem III.6.29]. ■

Compact, normal operators on a Hilbert space do not have generalized eigenvectors but they have a spectral decomposition analogous to normal matrices.

Theorem A.4.20. *If $T \in \mathcal{L}(Z)$ is a compact, normal operator on a Hilbert space Z, then there exists an orthonormal basis of eigenvectors $\{\phi_i, i \geq 1\}$ corresponding to the eigenvalues $\{\lambda_i, i \geq 1\}$ such that*

$$Tz = \sum_{i=1}^{\infty} \lambda_i \langle z, \phi_i \rangle \phi_i \qquad \text{for all } z \in Z.$$

Proof See Kato [14, theorem V.2.10] or Naylor and Sell [19, theorem 6.11.2]. ■

A consequence of this theorem is that every compact, normal operator induces an orthonormal basis for the Hilbert space. We shall illustrate this by the next classical examples.

Example A.4.21 Let $Z = L_2(0,1)$ and let T be given by

$$(Tv)(x) = \int_0^1 g(x,\tau) v(\tau) d\tau, \tag{A.8}$$

where

$$g(x,\tau) = \begin{cases} (1-\tau)x & \text{for } 0 \leq x \leq \tau \leq 1, \\ (1-x)\tau & \text{for } 0 \leq \tau \leq x \leq 1. \end{cases}$$

Since $g(x,\tau) = g(\tau, x)$, we see from Example A.3.59 that T is self-adjoint. Furthermore, T is a compact operator, by Theorem A.3.24. So we may calculate the eigenvalues and eigenvectors. We can rewrite (A.8) as

$$(Tv)(x) = \int_0^x (1-x)\tau v(\tau) d\tau + \int_x^1 (1-\tau) x v(\tau) d\tau.$$

So (Tv) is absolutely continuous and $(Tv)(0) = 0 = (Tv)(1)$. Let $\lambda \in \mathbb{C}$ be such that $Tv = \lambda v$. Then

$$\int_0^x \tau v(\tau) d\tau + \int_x^1 x v(\tau) d\tau - \int_0^1 x\tau v(\tau) d\tau = \lambda v(x) \quad \text{for } x \in [0,1]. \tag{A.9}$$

Since the left-hand side is absolutely continuous, we may differentiate (A.9) to obtain

$$x v(x) - x v(x) + \int_x^1 v(\tau) d\tau - \int_0^1 \tau v(\tau) d\tau = \lambda \dot{v}(x) \quad \text{for } x \in [0,1]. \tag{A.10}$$

The left-hand side is again absolutely continuous and so we may differentiate (A.10) to obtain

$$-v(x) = \lambda \ddot{v}(x). \tag{A.11}$$

So $\lambda = 0$ is not an eigenvalue. The general solution of (A.11) is given by

$$v(x) = a \sin(\lambda^{-\frac{1}{2}} x) + b \cos(\lambda^{-\frac{1}{2}} x). \tag{A.12}$$

Using the fact that $v(0) = \frac{1}{\lambda}(Tv)(0) = 0$ and $v(1) = \frac{1}{\lambda}(Tv)(1) = 0$ gives $\lambda = \frac{1}{n^2\pi^2}$ and $v(x) = a\sin(n\pi x)$. So the eigenvalues are $\{\frac{1}{n^2\pi^2}, n \geq 1\}$ and the eigenvectors are $\{\sin(n\pi x), n \geq 1\}$. By Theorem A.4.20, we now have that $\{\sqrt{2}\sin(n\pi x), n \geq 1\}$ is an orthonormal basis for $L_2(0,1)$. □

Example A.4.22 Let $Z = L_2(0,1)$and let S be the operator defined in Example A.3.48

$$(Sh)(x) = \int_0^x g(x,\xi)h(\xi)d\xi + \int_x^1 g(\xi,x)h(\xi)d\xi,$$

where

$$g(\xi,x) = \cot(1)\cos(x)\cos(\xi) + \sin(\xi)\cos(x).$$

This operator is clearly in $\mathcal{L}(Z)$; by Theorem A.3.24 it is even compact. From Example A.3.67 we have that S is self-adjoint, and so it is certainly normal. From Example A.3.48, we have that S is the bounded inverse of $I + A$, where

$$A = \frac{d^2}{dx^2},$$

with domain

$$\begin{aligned} D(A) \quad = \quad & \{z \in L_2(0,1) \mid z, \tfrac{dz}{dx} \text{ are absolutely continuous} \\ & \text{with } \tfrac{dz}{dx}(0) = \tfrac{dz}{dx}(1) = 0 \text{ and } \tfrac{d^2z}{dx^2} \in L_2(0,1)\}. \end{aligned}$$

We shall calculate the eigenvalues and eigenvectors of S. If $Sz = \lambda z$, then by applying the inverse we obtain that $z = \lambda(I + A)z$. So we have to solve

$$\frac{d^2z}{dx^2} = \frac{1-\lambda}{\lambda} z. \tag{A.13}$$

Using the boundary conditions ($z \in D(A)$), this has a nonzero solution if $\frac{1-\lambda}{\lambda} = -n^2\pi^2$ for some $n \geq 0$, and then z is given by $\cos(n\pi\cdot)$. So the eigenvalues are given by $\{\frac{1}{1-n^2\pi^2}, n \geq 0\}$ and the eigenvectors are $\{\cos(n\pi\cdot), n \geq 0\}$. From Theorem A.4.20, we obtain that $\{1, \sqrt{2}\cos(n\pi\cdot), n \geq 1\}$ is an orthonormal basis of Z. □

From Theorem A.4.20, we see that every compact, normal operator has a nice representation. In the next theorem we shall show that every compact operator has a similar representation.

Theorem A.4.23. *If $T \in \mathcal{L}(Z_1, Z_2)$ is a compact operator, where Z_1 and Z_2 are Hilbert spaces, then it has the following representation:*

$$Tz_1 = \sum_{i=1}^{\infty} \sigma_i \langle z_1, \psi_i \rangle \phi_i, \tag{A.14}$$

where $\{\psi_i\}$, $\{\phi_i\}$ are the eigenvectors of T^*T and TT^*, respectively, and $\sigma_i \geq 0$ are the square roots of the eigenvalues. $\{\psi_i\}$ form an orthonormal basis for Z_1 and $\{\phi_i\}$ form an orthonormal basis for Z_2. (ψ_i, ϕ_i) are the Schmidt pairs *of T, σ_i the* singular values *and (A.14) is its* Schmidt decomposition.

Furthermore, the norm of T equals its largest singular value.

Proof *a.* T^*T is clearly a self-adjoint, nonnegative operator and by Lemma A.3.22 it is compact. By Theorem A.4.20 T^*T has the following representation in terms of its eigenvalues σ_i^2, ($\sigma_i \geq 0$) and its eigenvectors ψ_i, which form an orthonormal basis for Z_1

$$T^*Tz = \sum_{i=1}^{\infty} \sigma_i^2 \langle z, \psi_i \rangle \psi_i = \sum_{i \in \mathbb{J}} \sigma_i^2 \langle z, \psi_i \rangle \psi_i,$$

where $\mathbb{J}$ is the index set that contains all indices for which $\sigma_i > 0$. For $i \in \mathbb{J}$ we define

$$\phi_i = \frac{1}{\sigma_i} T\psi_i,$$

and we easily obtain

$$T^*\phi_i = \frac{1}{\sigma_i} T^*T\psi_i = \sigma_i\psi_i \qquad \text{for } i \in \mathbb{J}.$$

Notice that

$$TT^*\phi_i = \sigma_i T\psi_i = \sigma_i^2\phi_i \qquad \text{for } i \in \mathbb{J}, \tag{A.15}$$

which shows that ϕ_i is the eigenvector of TT^* corresponding to σ_i^2. They form an orthonormal set, since

$$\langle \phi_i, \phi_j \rangle = \frac{1}{\sigma_i\sigma_j}\langle T\psi_i, T\psi_j\rangle = \frac{1}{\sigma_i\sigma_j}\langle T^*T\psi_i, \psi_j\rangle = \frac{\sigma_i}{\sigma_j}\langle \psi_i, \psi_j\rangle = \delta_{i,j}.$$

We shall show that we can extend the set $\{\phi_i; i \in \mathbb{J}\}$ to an orthonormal basis for Z_2 by adding an orthonormal basis for the kernel of T^*. For this we need the following observation. If ψ_i is an eigenvector of T^*T corresponding to the eigenvalue zero, then

$$\|T\psi_i\|^2 = \langle T\psi_i, T\psi_i\rangle = \langle T^*T\psi_i, \psi_i\rangle = 0. \tag{A.16}$$

Let z_2 be orthogonal to every ϕ_i with $i \in \mathbb{J}$. For $i \in \mathbb{J}$ we have that

$$\langle \psi_i, T^*z_2\rangle = \langle T\psi_i, z_2\rangle = \sigma_i\langle \phi_i, z_2\rangle = 0.$$

For i not an element of $\mathbb{J}$ it follows directly from (A.16) that $\langle \psi_i, T^*z_2\rangle = 0$. Since $\{\psi_i\}$ forms an orthonormal basis, we have that $T^*z_2 = 0$. So we have shown that the orthogonal complement of $\{\phi_i; i \in \mathbb{J}\}$ equals the kernel of T^*. This implies that we can decompose the Hilbert space Z_2 into the direct sum of the closure of the span of $\{\phi_i; i \in \mathbb{J}\}$ and the kernel of T^* (A.4). Choosing an orthonormal basis for the kernel of T^* produces an orthonormal basis for Z_2. We shall denote this basis by $\{\phi_i; i \in \mathbb{N}\}$. For $i \in \mathbb{J}$ we have seen that ϕ_i is an eigenvector for TT^*. However, for i not an element of $\mathbb{J}$ we have that $TT^*\phi_i = 0$, and so ϕ_i is also an eigenvector for all $i \in \mathbb{N}$.

b. We now show that T has the Schmidt decomposition

$$Tz = \sum_{i \in \mathbb{J}} \sigma_i \langle z, \psi_i\rangle \phi_i.$$

From Theorem A.4.20 $\{\psi_i\}$ is an orthonormal basis in Z_1, and so

$$z = \sum_{i=1}^{\infty} \langle z, \psi_i\rangle \psi_i \qquad \text{for all } z \in Z_1.$$

Since T is bounded, the following holds:

$$\begin{aligned} Tz &= \sum_{i=1}^{\infty} \langle z, \psi_i\rangle T\psi_i = \sum_{i \in \mathbb{J}} \langle z, \psi_i\rangle T\psi_i && \text{by (A.16)} \\ &= \sum_{i \in \mathbb{J}} \sigma_i \langle z, \psi_i\rangle \phi_i. && \text{by (A.15)} \end{aligned}$$

c. Let us number the singular values so that $\sigma_1 \geq \sigma_2 \geq \cdots$. Since $\{\phi_i\}$ is a orthonormal set in Z_2, from (A.14), it follows that

$$\|Tz\|^2 = \sum_{i=1}^{\infty} \sigma_i^2 |\langle z, \psi_i \rangle|^2 \leq \sigma_1^2 \|z\|^2.$$

But $\|T\psi_1\| = \|\sigma_1 \phi_1\| = \sigma_1$ and so $\|T\| = \sigma_1$. ■

So the class of compact operators has very special properties. Another class of operators with useful properties is the following.

Definition A.4.24. Let A be a linear operator on a Hilbert space Z. We say that A has *compact, normal resolvent* if there exists a $\lambda_0 \in \rho(A)$ for which $(\lambda_0 I - A)^{-1}$ is compact and normal. ♣

With the resolvent equation one can easily prove that definition A.4.24 is independent of the particular λ_0, i.e., if $(\lambda_0 I - A)^{-1}$ is compact and normal, then $(\lambda I - A)^{-1}$ is compact and normal for all $\lambda \in \rho(A)$.

Theorem A.4.25. *Let A be a linear operator on the Hilbert space Z with domain $D(A)$ and let $0 \in \rho(A)$ with A^{-1} compact and normal. From Theorem A.4.20, it follows that for $z \in Z$ we have the representation*

$$A^{-1} z = \sum_{i=1}^{\infty} \lambda_i^{-1} \langle z, \phi_i \rangle \phi_i,$$

where λ_i^{-1} and ϕ_i are the eigenvalues and the eigenvectors of A^{-1}, respectively, and $\{\phi_i, i \geq 1\}$ is an orthonormal basis. Moreover, for $z \in D(A)$, A has the decomposition

$$Az = \sum_{i=1}^{\infty} \lambda_i \langle z, \phi_i \rangle \phi_i,$$

with $D(A) = \{z \in Z \mid \sum_{i=1}^{\infty} |\lambda_i|^2 |\langle z, \phi_i \rangle|^2 < \infty\}$, and A is a closed linear operator.

Proof Define

$$A_1 z = \sum_{i=1}^{\infty} \lambda_i \langle z, \phi_i \rangle \phi_i,$$

with domain $D(A_1) = \{z \in Z \mid \sum_{i=1}^{\infty} |\lambda_i|^2 |\langle z, \phi_i \rangle|^2 < \infty\}$. We shall show that A_1 is a closed operator, and that A_1 equals A.

Since $\{\phi_i, i \geq 1\}$ is an orthonormal basis, we have that $z \in Z$ if and only if $\sum_{i=1}^{\infty} |\langle z, \phi_i \rangle|^2 < \infty$. Next we prove that A_1 is a closed linear operator.

Assume that $z_n \to z$ and $A_1 z_n \to y$ as $n \to \infty$. Since $z_n \to z$, we have that $\langle z_n, \phi_i \rangle \to \langle z, \phi_i \rangle$ for all $i \in \mathbb{N}$. Furthermore, since $A_1 z_n \to y$ we have that $\lambda_i \langle z_n, \phi_i \rangle \to \langle y, \phi_i \rangle$. So $\lambda_i \langle z, \phi_i \rangle = \langle y, \phi_i \rangle$, and since $y \in Z$, this implies that $z \in D(A_1)$ and $A_1 z = y$. Thus A_1 is closed. For $z \in D(A_1)$ it is easy to show that $A^{-1} A_1 z = z$, and since $\operatorname{ran}(A^{-1}) = D(A)$, this implies that $D(A_1) \subset D(A)$. Now we prove that $A_1 A^{-1} z = z$ for all z in Z. If z is in Z, then $x_n := \sum_{i=1}^{n} \langle z, \phi_i \rangle \phi_i$ converges to z as $n \to \infty$, and $A^{-1} z_n = \sum_{i=1}^{n} \lambda_i^{-1} \langle z, \phi_i \rangle \phi_i$ converges to $A^{-1} z$ as $n \to \infty$ (A^{-1} is continuous). So $A^{-1}z_n$ converges and so does $z_n = A_1A^{-1}z_n$. Thus by the closedefinitioness of A_1 we have that $A^{-1}z \in D(A_1)$ and $A_1A^{-1}z = z$ for all $z \in Z$. Since $\operatorname{ran}(A^{-1}) = D(A)$, we have $D(A) \subset D(A_1)$. So $D(A) = D(A_1)$ and $A_1z = AA^{-1}A_1z = Az$ for $z \in D(A_1) = D(A)$. ■

We conclude with an illustration of this result.

Example A.4.26 Let $Z = L_2(0,1)$ and let A be given by

$$Az = -\frac{d^2z}{dx^2} \qquad \text{for } z \in D(A),$$

where $D(A) = \{z \in Z \mid z, \frac{dz}{dx}$ absolutely continuous and $\frac{d^2z}{dx^2} \in L_2(0,1)$ with $z(0) = 0 = z(1)\}$. It is easy to verify that the inverse of A may be expressed by

$$(A^{-1}z)(x) = \int_0^1 g(x,\tau)z(\tau)d\tau,$$

where

$$g(x,\tau) = \begin{cases} (1-\tau)x & \text{for } 0 \le x \le \tau \le 1 \\ (1-x)\tau & \text{for } 0 \le \tau \le x \le 1. \end{cases}$$

So A^{-1} equals the operator from Example A.4.21. In that example, we showed that A^{-1} was self-adjoint and compact with eigenvalues $\{\frac{1}{n^2\pi^2}, n \ge 1\}$ and eigenvectors $\{\sin(n\pi x), n \ge 1\}$. Now, applying Theorem A.4.25 we see that A is closed and has the representation

$$Az = \sum_{n=1}^{\infty} n^2\pi^2 \langle z, \sqrt{2}\sin(n\pi\cdot)\rangle \sqrt{2}\sin(n\pi\cdot), \qquad \text{with}$$

$$D(A) = \{z \in L_2(0,1) \mid \sum_{n=1}^{\infty} n^4\pi^4 |\langle z, \sqrt{2}\sin(n\pi\cdot)\rangle|^2 < \infty\}.$$

■ □

A.5. Integration and differentiation theory

A.5.1. Integration theory

In this section, we wish to extend the ideas of Lebesgue integration of complex-valued functions to vector-valued and operator-valued functions, which take their values in a separable Hilbert space Z or in the Banach space $\mathcal{L}(Z_1, Z_2)$, where Z_1, Z_2 are separable Hilbert spaces. As main references, we have used Diestel and Uhl [6], Dunford and Schwartz [8], and Hille and Phillips [12].

Throughout this section, we use the notation Ω for a closed subset of $\mathbb{R}$, and $(\Omega, \mathcal{B}, dt)$ for the measure space with measurable subsets $\mathcal{B}$ and the Lebesgue measure dt. It is possible to develop a Lebesgue integration theory based on various measurability concepts.

Definition A.5.1. Let X be a Banach space. A function $f : \Omega \to X$ is called *simple* if there exist $x_1, x_2, \ldots, x_n \in X$ and $E_1, E_2, \ldots, E_n \in \mathcal{B}$ such that $f = \sum_{i=1}^n x_i 1_{E_i}$, where $1_{E_i}(t) = 1$ if $t \in E_i$ and 0 otherwise.

Let Z_1, Z_2 be two separable Hilbert spaces, and let $F : \Omega \to \mathcal{L}(Z_1, Z_2)$ and $f : \Omega \to Z_1$.

a. F is *uniformly (Lebesgue) measurable* if there exists a sequence of simple functions $F_n : \Omega \to \mathcal{L}(Z_1, Z_2)$ such that

$$\lim_{n\to\infty} \|F - F_n\|_{\mathcal{L}(Z_1,Z_2)} = 0 \text{ almost everywhere.}$$

b. f is *strongly (Lebesgue) measurable* if there exists a sequence of simple functions $f_n : \Omega \to Z_1$ such that

$$\lim_{n\to\infty} \|f - f_n\|_{Z_1} = 0 \text{ almost everywhere.}$$

F is *strongly measurable* if Fz_1 is strongly measurable for every $z_1 \in Z_1$.

c. f is *weakly (Lebesgue) measurable* if $\langle f, z_1\rangle$ is measurable for every $z_1 \in Z_1$.

F is *weakly measurable* if Fz_1 is weakly measurable for every $z_1 \in Z_1$. ♣

It is easy to see that uniform measurability implies strong measurability, which implies weak measurability. For the case that Z is a separable Hilbert space, the concepts weak and strong measurability coalesce.

Lemma A.5.2. *For the case that Z is a separable Hilbert space the concepts of weak and strong measurability in definition A.5.1 coincide.*

Proof See Hille and Phillips [12, theorem 3.5.3] or Yosida [31, theorem in Section V.4]. ∎

We often consider the inner product of two weakly measurable functions.

Lemma A.5.3. *Let Z be a separable Hilbert space, and let $f_1, f_2 : \Omega \to Z$ be two weakly measurable functions. The complex-valued function $\langle f_1(t), f_2(t)\rangle$ defined by the inner product of these functions is a measurable function.*

Proof This follows directly from Lemma A.5.2 and definition A.5.1. ∎

The notion of the Lebesgue integral follows naturally from the measurability concepts given in definition A.5.1.

Definition A.5.4. Suppose that $(\Omega, \mathcal{B}, dt)$ is the Lebesgue measure space and that $E \in \mathcal{B}$.

a. Let X be a Banach space and let $f : \Omega \to X$ be a simple function given by $f = \sum_{i=1}^n x_i 1_{E_i}$, where the E_i are disjoint. We define f to be *Lebesgue integrable* over E if $\|f\|$ is Lebesgue integrable over E, that is, $\sum_{i=1}^n \|x_i\|\lambda(E_i \cap E) < \infty$, where $\lambda(\cdot)$ denotes the Lebesgue measure of the set and we follow the usual convention that $0 \cdot \infty = 0$. The *Lebesgue integral* of f over E is given by $\sum_{i=1}^n x_i\lambda(E_i \cap E)$ and will be denoted by $\int_E f(t)dt$.

b. Let Z_1 and Z_2 be two separable Hilbert spaces. The uniformly measurable function $F : \Omega \to \mathcal{L}(Z_1, Z_2)$ is *Lebesgue integrable* over E if there exists a sequence of simple integrable functions F_n converging almost everywhere to F and such that

$$\lim_{n\to\infty} \int_E \|F(t) - F_n(t)\|_{\mathcal{L}(Z_1,Z_2)} dt = 0.$$

We define the *Lebesgue integral* by

$$\int_E F(t)dt = \lim_{n\to\infty} \int_E F_n(t)dt.$$

c. Let Z be a separable Hilbert space. The strongly measurable function $f : \Omega \to Z$ is *Lebesgue integrable* over E if there exists a sequence of simple integrable functions f_n converging almost everywhere to f and such that

$$\lim_{n\to\infty} \int_E \|f(t) - f_n(t)\|_Z dt = 0.$$

We define the *Lebesgue integral* by

$$\int_E f(t)dt = \lim_{n\to\infty} \int_E f_n(t)dt.$$

♣

These integrals in the above definition are also called *Bochner integrals* in the literature. For functions from $\mathbb{R}$ to a separable Hilbert space Z, there is a simple criterion to test whether a function is Lebesgue integrable.

Lemma A.5.5. *Let $f(t) : \Omega \to Z$, where Z is a separable Hilbert space Z. $\int_E f(t)dt$ is well defined as a Lebesgue integral for $E \in \mathcal{B}$ if and only if the function $\langle z, f(t)\rangle$ is measurable for every $z \in Z$ and $\int_E \|f(t)\|dt < \infty$.*

Proof See Hille and Phillips [12, theorem 3.7.4], noting that weak and strong measurability are the same for separable Hilbert spaces (Lemma A.5.2). ■

In the case of operator-valued functions $F(t) : \Omega \to \mathcal{L}(Z_1, Z_2)$, where Z_1 and Z_2 are separable Hilbert spaces, we need to distinguish between the Lebesgue integral $\int_E F(t)dt$ for the case that $F(t)$ is uniformly (Lebesgue) measurable and the Lebesgue integral $\int_E F(t)zdt$ for the case that $F(t)$ is only strongly (Lebesgue) measurable.

Example A.5.6 Let $T(t)$ be a C_0-semigroup on a separable Hilbert space Z. Since $T(t)$ is strongly continuous, it is strongly measurable. In fact, Hille and Phillips [12, theorem 10.2.1] show that the C_0-semigroup is uniformly measurable if and only if it is uniformly continuous. Now the only uniformly continuous semigroups are those whose infinitesimal generator is a bounded operator, Hille and Phillips [12, theorem 9.4.2], and so $T(t)$ will only be strongly measurable in general. Thus $\int_0^1 T(t)zdt$ is a well defined Lebesgue integral for any $z \in Z$, but $\int_0^1 T(t)dt$ is not. □

Example A.5.7 Next consider $\int_0^\tau T(\tau - s)F(s)ds$, where $T(t)$ is a C_0-semigroup on a separable Hilbert space Z, $F(\cdot) \in \mathcal{L}(U, Z)$, U is a Hilbert space, F is weakly measurable, and $\|F\| \in L_1(0, \tau)$. Since $T^*(t)$ is also a C_0-semigroup, $T^*(t)z$ is continuous and so strongly measurable. Furthermore, by definition, we have that $F(s)u$ is weakly measurable. Hence Lemma A.5.3 shows that $\langle z, T(\tau - s)F(s)u\rangle = \langle T^*(\tau - s)z, F(s)u\rangle$ is measurable in s for all $z \in Z$, $u \in U$. So from Lemma A.5.5 we have that for each $u \in U$ $\int_0^\tau T(\tau - s)F(s)uds$ is a well defined Lebesgue integral. However, $\int_0^\tau T(\tau - s)F(s)ds$ need not be a well defined Lebesgue integral, since the integrand will not to be uniformly measurable in general.

This example motivates the need for a weaker concept of integration based on weak measurability. We now introduce the Pettis integral, which satisfies this requirement.

Lemma A.5.8. *Let Z_1 and Z_2 be separable Hilbert spaces, and let $F(t) : \Omega \to \mathcal{L}(Z_1, Z_2)$. Assume furthermore, that for every $z_1 \in Z_1$ and $z_2 \in Z_2$ the function $\langle z_2, F(t)z_1\rangle$ is an element of $L_1(\Omega)$. Then for each $E \in \mathcal{B}$, there exists a unique $z_{F,E}(z_1) \in Z_2$ satisfying*

$$\langle z_2, z_{F,E}(z_1)\rangle = \int_E \langle z_2, F(t)z_1\rangle dt.$$

Proof Set
$$G(z_2) = \int_E \langle z_2, F(t)z_1\rangle dt.$$
It is clear that G is well defined for every $z_2 \in Z_2$, and that it is linear on Z_2. It remains to show that G is bounded. To do this, we define the following operator from Z_2 to $L_1(E)$: $Q(z_2) = \langle z_2, F(t)z_1\rangle$ and we show that it is closed. This follows since if $z_2^n \to z_2$ and $\langle z_2^n, F(t)z_1\rangle \to h(t)$ in $L_1(E)$, we have $\langle z_2^n, F(t)z_1\rangle \to \langle z_2, F(t)z_1\rangle$ everywhere on E, and so $\langle z_2, F(t)z_1\rangle = h(t)$. Thus Q is a closed linear operator with domain Z_2; so with the Closed Graph Theorem A.3.49, we conclude that Q is a bounded linear operator. Thus
$$\mid G(z_2) \mid \leq \int_E \mid \langle z_2, F(t)z_1\rangle \mid dt \leq \|Q\|\|z_2\|,$$
and G is bounded. Applying the Riesz Representation Theorem A.3.52, we obtain the existence of a $z_{F,E}(z_1)$ such that
$$\langle z_2, z_{F,E}(z_1)\rangle = G(z_2) = \int_E \langle z_2, F(t)z_1\rangle dt. \tag{A.1}$$
■

In the next lemma, we shall show that $z_{F,E}(z_1)$ defines a bounded linear operator from Z_1 to Z_2.

Lemma A.5.9. *The mapping $z_{F,E}(z_1)$ in (A.1) is a linear function of z_1, and $z_{F,E}$ defines a bounded linear operator from Z_1 to Z_2.*

Proof The linearity of $z_{F,E}(z_1)$ in z_1 follows easily from the uniqueness of $z_{F,E}(z_1)$. The boundedefinitioness will follow from the closedefinitioness of the operator
$$z_1 \mapsto z_{F,E}(z_1).$$
If $z_1^n \to z_1$ in Z_1 and $z_{F,E}(z_1^n) \to z^E$, then for all $z_2 \in Z_2$ we have
$$\langle z_2, z_{F,E}(z_1^n)\rangle \to \langle z_2, z_2^E\rangle,$$
and
$$\begin{aligned}
\langle z_2, z_{F,E}(z_1^n)\rangle &= \int_E \langle z_2, F(t)z_1^n\rangle dt = \overline{\int_E \langle z_1^n, F^*(t)z_2\rangle dt} \\
&= \overline{\langle z_1^n, z_{F^*,E}(z_2)\rangle} \\
&\to \overline{\langle z_1, z_{F^*,E}(z_2)\rangle} \qquad \text{as } n \to \infty \\
&= \int_E \langle z_2, F(t)z_1\rangle dt = \langle z_2, z_{F,E}(z_1)\rangle,
\end{aligned}$$
where we have used Lemma A.5.8. Thus $z^E = z_{F,E}(z_1)$. ■

In fact, we have established the following result.

Theorem A.5.10. *Let Z_1 and Z_2 be separable Hilbert spaces, and let $F(t) : \Omega \to \mathcal{L}(Z_1, Z_2)$. Assume further that for all $z_1 \in Z_1$ and $z_2 \in Z_2$, the function $\langle z_2, F(t)z_1\rangle$ is an element of $L_1(\Omega)$. Then for each $E \in \mathcal{B}$, there exists a bounded linear operator, denoted by $z_{F,E}$, satisfying*
$$\langle z_2, z_{F,E}z_1\rangle = \int_E \langle z_2, F(t)z_1\rangle dt.$$

This last result leads naturally to the definition of the Pettis integral.

Definition A.5.11. Let Z_1 and Z_2 be separable Hilbert spaces and let $F : \Omega \to \mathcal{L}(Z_1, Z_2)$. If for all $z_1 \in Z_1$ and $z_2 \in Z_2$ the function $\langle z_2, F(t)z_1\rangle \in L_1(\Omega)$, then we say that $F(\cdot)$ is *Pettis integrable*. Furthermore, for all $E \in \mathcal{B}$, we call $\int_E F(t)dt$ defined by

$$\langle z_2, \int_E F(t)dtz_1\rangle := \langle z_2, z_{F,E}z_1\rangle = \int_E \langle z_2, F(t)z_1\rangle dt. \tag{A.2}$$

the *Pettis integral* of $F(t)$ over E and $\int_E F(t)z_1dt$ the *Pettis integral* of $F(t)z_1$ over E. ♣

One can easily prove the usual properties such as linearity of the integral

$$\int_E (\alpha F_1(t) + \beta F_2(t))\, dt = \alpha \int_E F_1(t)dt + \beta \int_E F_2(t)dt. \tag{A.3}$$

From the definition of the Pettis integral, we always have that

$$\int_E |\langle z_2, F(t)z_1\rangle|dt < \infty. \tag{A.4}$$

In particular, if $\int_E \|F(t)\|dt < \infty$, then the condition (A.4) is satisfied. Furthermore, it is easy to see that if F is an integrable simple function, then the Pettis integral equals the Lebesgue integral. From the definition of the Lebesgue integral, it follows easily that if the Lebesgue integral of a function exists, then the Pettis integral also exists, and they are equal.

In the next example we shall show that a function may be Pettis integrable, but not Lebesgue integrable.

Example A.5.12 Let Z be ℓ_2 from Example A.2.9 and define e_n to be the nth-basis vector. Define the function

$$f(t) = \frac{1}{n}e_n \qquad \text{for } n-1 \leq t < n.$$

It is easy to show that

$$\int_0^\infty \|f(t)\|dt = \sum_{n=1}^\infty \frac{1}{n} = \infty.$$

So by Lemma A.5.5 we see that f is not Lebesgue integrable. On the other hand, we have that for any $z \in \ell_2$,

$$\int_0^\infty |\langle z, f(t)\rangle|dt = \sum_{n=1}^\infty \frac{1}{n}|\langle z, e_n\rangle| \leq \sqrt{\sum_{n=1}^\infty \frac{1}{n^2} \sum_{n=1}^\infty |\langle z, e_n\rangle|^2} = c\|z\|.$$

So $\langle z, f(t)\rangle \in L_1(0, \infty)$, and we conclude that the Pettis integral exists. □

In the following example, we re-examine Examples A.5.6 and A.5.7, which we considered as Lebesgue integrals.

Example A.5.13 We recall from Example A.5.6 that the C_0-semigroup $T(t)$ on the separable Hilbert space Z is in general only strongly measurable and so while $\int_0^1 T(t)zdt$ exists as a Lebesgue integral $\int_0^1 T(t)dt$ does not. We show that it does exist as a Pettis integral. Since $T(t)$ is strongly continuous, we have that $\langle z_1, T(t)z_2\rangle$ is measurable for every $z_1, z_2 \in Z$. From Theorem 2.5.1

we have that $\int_0^1 \|T(t)\|dt < \infty$. Thus by definition A.5.11 the Pettis integral $\int_0^1 T(t)dt$ is well defined. If the infinitesimal generator A of $T(t)$ is invertible, then using Theorem 2.5.2 we can even calculate this Pettis integral to obtain

$$\int_0^1 T(t)dt = A^{-1}T(1) - A^{-1}.$$

□

Example A.5.14 From Example A.5.7 we recall that $\int_0^\tau T(\tau - s)F(s)ds$ was not a well defined Lebesgue integral. There we already showed that $\langle z, T(\tau - s)F(s)u\rangle$ is Lebesgue measurable for all $z \in Z$, $u \in U$. Furthermore, we see that

$$\int_0^\tau \|T(\tau - s)F(s)\|ds \leq M_\omega e^{\omega\tau} \int_0^\tau \|F(s)\|ds < \infty.$$

So by definition A.5.11 the integrals $\int_0^\tau T(\tau - s)F(s)ds$ and $\int_0^\tau T(\tau - s)F(s)uds$ are well defined as Pettis integrals. □

Most of the integrals we use in this text satisfy the conditions in Lemma A.5.5, and so we may speak about the integral, as in the following example.

Example A.5.15 Consider $\int_0^\tau T(\tau - s)Bu(s)ds$, where $T(t)$ is a C_0-semigroup on a separable Hilbert space Z, $B \in \mathcal{L}(U, Z)$, U is a separable Hilbert space and $u \in L_1([0, \tau]; U)$ (see definition A.5.16). Then, as in Example A.5.14, $\langle z, T(\tau - s)Bu(s)\rangle$ is measurable in s for all $z \in Z$ and $\int_0^\tau \|T(\tau - s)Bu(s)\|ds \leq M_\omega e^{\omega\tau}\|B\| \int_0^\tau \|u(s)\|ds < \infty$. So by Lemma A.5.5, the integral is well defined as a Pettis or as a Lebesgue integral. □

To avoid confusion between the Pettis and Lebesgue integrals we introduce the following notation.

Definition A.5.16. Let Z_1, Z_2, and Z be separable Hilbert spaces, and let Ω be a closed subset of $\mathbb{R}$. We define the following spaces:

$$\begin{aligned}
P(\Omega; \mathcal{L}(Z_1, Z_2)) \quad &:= \quad \{F : \Omega \to \mathcal{L}(Z_1, Z_2) \mid \langle z_2, F(\cdot)z_1\rangle \text{ is mea-} \\
&\qquad \text{surable for every } z_1 \in Z_1 \text{ and } z_2 \in Z_2\}. \\
P_p(\Omega; \mathcal{L}(Z_1, Z_2)) \quad &:= \quad \{F \in P(\Omega; \mathcal{L}(Z_1, Z_2)) \mid \|F\|_p := \\
&\qquad \left(\int_\Omega \|F(t)\|^p_{\mathcal{L}(Z_1,Z_2)}\right)^{1/p} < \infty\};\ 1 \leq p < \infty. \\
P_\infty(\Omega; \mathcal{L}(Z_1, Z_2)) \quad &:= \quad \{F \in P(\Omega; \mathcal{L}(Z_1, Z_2)) \mid \|F\|_\infty := \\
&\qquad \operatorname{ess\,sup}_\Omega \|F(t)\|_{\mathcal{L}(Z_1,Z_2)} < \infty\}.
\end{aligned}$$

$$\begin{aligned}
L(\Omega; Z) \quad &:= \quad \{f : \Omega \to Z \mid \langle z, f(\cdot)\rangle \text{ is measurable for all } z \in Z\}. \\
L_p(\Omega; Z) \quad &:= \quad \{f \in L(\Omega; Z) \mid \|f\|_p := \left(\int_\Omega \|f(t)\|^p_Z dt\right)^{1/p} < \infty\}; \\
&\qquad 1 \leq p < \infty. \\
L_\infty(\Omega; Z) \quad &:= \quad \{f \in L(\Omega; Z) \mid \|f\|_\infty := \operatorname{ess\,sup}_\Omega \|f(t)\|_Z < \infty\}.
\end{aligned}$$

♣

The reason for using the "L" notation is that these integrals are also defined in the Lebesgue sense. For example, if $T(t)$ is a strongly continuous semigroup, then $T(t)z \in L_p([0,\tau];Z)$ for all $z \in Z$, but we only have that $T(t) \in P_p([0,\tau];\mathcal{L}(Z))$ instead of the Lebesgue space $L_p([0,\tau];\mathcal{L}(Z))$ (see Example A.5.13).

We remark that if Z_1 and Z_2 are finite-dimensional, then $\mathcal{L}(Z_1, Z_2)$ is also finite-dimensional, and so $L_\infty(\Omega;\mathcal{L}(Z_1,Z_2))$ is well defined as a Lebesgue space (see Lemma A.5.5) and equals $P_\infty(\Omega;\mathcal{L}(Z_1,Z_2))$.

Lemma A.5.17. *If we do not distinguish between two functions that differ on a set of measure zero, then the spaces $P_p(\Omega;\mathcal{L}(Z_1,Z_2))$, $P_\infty(\Omega;\mathcal{L}(Z_1,Z_2))$, $L_p(\Omega;Z)$, and $L_\infty(\Omega;Z)$ are Banach spaces.*

Furthermore, $L_2(\Omega;Z)$ is a Hilbert space with inner product

$$\langle h, f\rangle = \int_\Omega \langle h(t), f(t)\rangle_Z \, dt. \tag{A.5}$$

Proof See Thomas [26] or [27].

The completeness property of L_p is also shown in theorem III.6.6 of Dunford and Schwartz [8].

In Section 3.5 of Balakrishnan [2] it is shown that $L_2(\Omega, Z)$ is a Hilbert space. ∎

It is interesting to remark that P_p is not a Banach space under the norm

$$\|F\|_p := \left(\sup_{\|z_1\|=1, \|z_2\|=1} \int_\Omega |\langle F(t)z_1, z_2\rangle| dt \right)^{1/p};$$

see Thomas [27].

From Lemmas A.5.8 and A.5.9 it should be clear that the integrals share the usual properties of their finite-dimensional Lebesgue counterparts.

Theorem A.5.18. *If $f \in P_1(\Omega;\mathcal{L}(Z_1,Z_2))$, where Z_1 and Z_2 are separable Hilbert spaces, then the following hold:*

a. $\| \int_\Omega f(t)dt\| \leq \int_\Omega \|f(t)\|dt$;

b. $\lim_{\lambda(E)\to 0} \int_E f(t)dt = 0$, *where $\lambda(E)$ denotes the Lebesgue measure of $E \in \mathcal{B}$.*

Proof *a.* This follows easily from definitions A.5.16 and A.3.9 and Lemma A.3.30, since from these last results, it follows that

$$\begin{aligned}
\| \int_\Omega f(t)dt\| &= \sup_{z_1\in Z_1, z_2 \in Z_2} \frac{|\langle z_2, \int_\Omega f(t)dt z_1\rangle|}{\|z_1\|\|z_2\|} \\
&= \sup_{z_1\in Z_1, z_2 \in Z_2} \left| \int_\Omega \frac{\langle z_2, f(t)z_1\rangle}{\|z_1\|\|z_2\|} dt \right| \\
&\leq \sup_{z_1\in Z_1, z_2 \in Z_2} \int_\Omega \frac{|\langle z_2, f(t)z_1\rangle|}{\|z_1\|\|z_2\|} dt \\
&\leq \int_\Omega \sup_{z_1\in Z_1, z_2 \in Z_2} \frac{|\langle z_2, f(t)z_1\rangle|}{\|z_1\|\|z_2\|} dt = \int_\Omega \|f(t)\|dt.
\end{aligned}$$

b. This follows directly from part a and the standard Lebesgue theory. ∎

Lemma A.5.19. *Let Z be a separable Hilbert space and let a and b be real numbers such that $-\infty < a < b < \infty$. $L_p([a,b];Z)$ has the following dense subspaces:*

a. The space of all continuous functions on $[a,b]$, $C([a,b];Z)$;

b. The space of all piecewise constant functions *that are functions of the form $f(x) = \sum_{i=0}^{n} z_i 1_{[a_i,b_i]}(x)$ with $a = a_0 < b_0 \le a_1 \dots b_n = b$ and $z_i \in Z$.*

Proof See page 86 of Hille and Phillips [12]. ∎

Lemma A.5.20. *Let Z be a separable Hilbert space and let $1 \le p < \infty$. $L_p((-\infty,\infty);Z)$ has the following dense subspaces:*

a. The functions in $L_p((-\infty,\infty);Z)$ that are zero outside some finite interval;

b. $L_p((-\infty,\infty);Z) \cap L_q((-\infty,\infty);Z)$ for every $q \ge 1$.

Proof *a.* Let f be an arbitrary function in $L_p((-\infty,\infty);Z)$. For sufficiently large N we have that $f_N(t) := f(t)1_{[-N,N]}(t)$ is arbitrarily close to f in the L_p-norm, since

$$\|f - f_N\|_p = \left[\int_{\infty}^{-N} \|f(t)\|^p dt + \int_{N}^{\infty} \|f(t)\|^p dt\right]^{1/p}$$

converges to zero as N approaches ∞.
b. Let $\varepsilon > 0$ and let N be chosen such that $\|f - f_N\|_p < \varepsilon$, where f and f_N be the same as in part a. Now we have that $f_N \in L_p([-N,N];Z)$, so by Lemma A.5.19 there exists a function $g_N \in C([-N,N];Z)$ such that $\|f_N - g_N\|_{L_p([-N,N],Z)} < \varepsilon$. Since g_N is continuous, it is easy to see that it is an element of $L_q((-N,N);Z)$. Now we define

$$g(t) = \begin{cases} g_N(t) & \text{for } t \in [-N,N] \\ 0 & \text{for } t \notin [-N,N] \end{cases}$$

and so $g \in L_p((-\infty,\infty);Z) \cap L_q((-\infty,\infty);Z)$. Furthermore,

$$\|g - f\|_p \le \|g - f_N\|_p + \|f_N - f\|_p < 2\varepsilon.$$

ε was arbitrary, so we have proved the result. ∎

We remark that part a of this lemma is false for L_∞.

Theorem A.5.21. Lebesgue-Dominated Convergence Theorem. *Let Z be a separable Hilbert space and let f_n be a sequence in $L_1(\Omega;Z)$. Suppose that f_n converges almost everywhere to f, i.e., $\lim_{n\to\infty} \|f_n(t) - f(t)\| = 0$ except for t in a set of measure zero. Assume further that there exists a fixed function $g \in L_1(\Omega)$ such that $\|f_n(t)\| \le g(t)$ for all n and almost all $t \in \Omega$. Then $f \in L_1(\Omega;Z)$ and*

$$\lim_{n\to\infty} \int_E f_n(t)dt = \int_E f(t)dt$$

for all $E \in \mathcal{B}$.

Proof See theorem 3 on page 45 of Diestel and Uhl [6] or theorem III.3.7 in Dunford and Schwartz [8]. ∎

Theorem A.5.22. Fubini's Theorem. *Let $(\Omega_1, \mathcal{B}_1, dt)$, $(\Omega_2, \mathcal{B}_2, dt)$ be two Lebesgue measure spaces with $\lambda(\Omega_1) < \infty$ and $\lambda(\Omega_2) < \infty$. We denote by $\mathcal{B}_1 \times \mathcal{B}_2$ the σ-algebra of subsets of $\Omega_1 \times \Omega_2$ generated by the class of all rectangular sets of the form $E \times F$, where $E \in \mathcal{B}_1$, $F \in \mathcal{B}_2$ and the product measure is denoted by $dt \times ds$.*

For $f(\cdot,\cdot) \in L_1(\Omega_1 \times \Omega_2, Z)$ the functions

$$\int_{\Omega_1} f(t,\cdot)dt \text{ and } \int_{\Omega_2} f(\cdot,s)ds$$

are in $L_1(\Omega_2; Z)$ and $L_1(\Omega_1; Z)$, respectively, and

$$\int_{\Omega_1 \times \Omega_2} f(t,s)dt \times ds = \int_{\Omega_1} (\int_{\Omega_2} f(t,s)dt)ds = \int_{\Omega_2} (\int_{\Omega_1} f(t,s)dt)ds.$$

Proof See Hille and Phillips [12, theorem 3.7.13]. ■

Theorem A.5.23. *Let Z_1 and Z_2 be separable Hilbert spaces, and let A be a closed linear operator from $D(A) \subset Z_1$ to Z_2. If $f \in L_1(\Omega; Z_1)$ with $f \in D(A)$ almost everywhere and $Af \in L_1(\Omega; Z_2)$, then*

$$A \int_E f(t)dt = \int_E Af(t)dt$$

for all $E \in \mathcal{B}$.

Proof See Hille and Phillips [12, theorem 3.7.12]. ■

Example A.5.24 Let $-\infty \leq a < b \leq \infty$ and let Z be a separable Hilbert space. Assume further that $\{f_n, n \geq 1\}$ and $\{e_m, m \geq 1\}$ are orthonormal bases for $L_2(a,b)$ and Z, respectively. We show that $\{\phi_{n,m}, n, m \geq 1\}$ with $\phi_{n,m} := f_n e_m$ is an orthonormal basis for $L_2([a,b]; Z)$. From the definition of the inner product on $L_2([a,b]; Z)$, (A.5), we have that

$$\begin{aligned}
\langle \phi_{n,m}, \phi_{i,j} \rangle &= \int_a^b \langle f_n(t)e_m, f_i(t)e_j \rangle_Z dt = \int_a^b f_n(t)\overline{f_i(t)} \langle e_m, e_j \rangle_Z dt \\
&= \int_a^b f_n(t)\overline{f_i(t)} \delta_{mj} dt = \delta_{mj} \langle f_n, f_i \rangle_{L_2(a,b)} = \delta_{mj}\delta_{ni}.
\end{aligned}$$

Thus $\{\phi_{n,m},\ n, m \geq 1\}$ is an orthonormal set. Next we show that it is maximal. If z is orthogonal to every $\phi_{n,m}$, then

$$\int_a^b \langle f_n(t)e_m, z(t) \rangle_Z dt = 0 \qquad \text{for all } n, m \geq 1.$$

If we fix m, then we see that for all $n \geq 1$,

$$\int_a^b f_n(t) \langle e_m, z(t) \rangle_Z dt = 0.$$

But f_n is maximal in $L_2(a,b)$, and so $\langle e_m, z(t) \rangle_Z = 0$ almost everywhere. This holds for all $m \geq 1$. Now using the fact that e_m is maximal in Z, we obtain that $z(t) = 0$ almost everywhere. Thus $z = 0$ in $L_2([a,b]; Z)$, which concludes the proof. □

A.5.2. Differentiation theory

In the previous subsection, we concentrated on the integration of Hilbert-space-valued functions. However, as is known from standard calculus, integration is naturally related to differentiation, and in this subsection we summarize standard results on differential calculus for Hilbert-space-valued functions. We start with the concept of the *Fréchet derivative*.

Definition A.5.25. Consider the mapping U from the Banach space X to the Banach space Y. Given $x \in X$, if a linear bounded operator $dU(x)$ exists such that

$$\lim_{\|h\|_X \to 0} \frac{\|U(x+h) - U(x) - dU(x)h\|_Y}{\|h\|_X} = 0,$$ ♣

then U is *Fréchet differentiable at* x, and $dU(x)$ is said to be the *Fréchet differential at* x.

It is easy to see that if U is identical to a bounded linear operator, then $dU(x) = 0$ for every $x \in X$.

One of the most important applications of the derivative is the determination of the maxima and minima of functionals.

Theorem A.5.26. *Let O be an open subset of the Banach space X. If the mapping $f : O \to \mathbb{R}$ has a minimum or a maximum at $x \in O$, and $df(x)$ exists, then $df(x) = 0$.*

Proof We shall only give the proof for the case that f has a minimum. The proof for the other case follows easily by replacing f by $-f$.

For sufficiently small h we have that $x + h$ and $x - h$ are in O. Furthermore, we have that

$$f(x+h) - f(x) \approx df(x)h$$

and

$$f(x-h) - f(x) \approx -df(x)h.$$

Since x is a minimum, the left-hand side of both equations is nonnegative. Looking at the right-hand side of these equations we conclude that $df(x)$ must be zero. ■

Most of the applications of differential calculus in this book are to functions from $\mathbb{R}$ or $\mathbb{C}$ to the Banach space X. Since this is frequently used we shall give a special definition for functions of this class

Definition A.5.27. A function $f : \mathbb{R} \to X$ is *differentiable* if f is Fréchet differentiable

$$\lim_{h \to 0} \frac{\|f(t+h) - f(t) - df(t)h\|}{|h|} = 0. \tag{A.6}$$

We shall denote the *derivative* of f at t_0 by $\frac{df}{dt}(t_0)$ or $\dot{f}(t_0)$. ♣

In applications, we apply definition A.5.27 to a function $f(x,t)$ of two variables by considering it to be a function of t taking its values in an appropriate function space, corresponding to the Banach space X. However, this Fréchet derivative may exist, whereas the usual partial derivative does not, as the following example shows.

Example A.5.28 Consider the function $f(x,t) : [0,1] \times [-1,1] \to \mathbb{R}$ defined by

$$f(x,0) = 0$$

$$f(x,t) = \begin{cases} t & \text{for } |x + [\frac{1}{t}] - \frac{1}{t}| < \frac{1}{2}|t| \\ 0 & \text{elsewhere,} \end{cases}$$

where $[\frac{1}{t}]$ denotes the integer part of $\frac{1}{t}$, that is, the largest integer smaller than or equal to $\frac{1}{t}$. We have that $f(\cdot,t) \in L_2(0,1)$ for every $t \in [-1,1]$ and

$$\int_0^1 |f(x,h) - f(x,0)|^2 dx = \int_0^1 |f(x,h)|^2 dx = \int_{\max\{0,-\frac{1}{2}|h|-[\frac{1}{h}]+\frac{1}{h}\}}^{\min\{1,\frac{1}{2}|h|-[\frac{1}{h}]+\frac{1}{h}\}} |h|^2 dx \leq |h|^3,$$

since the length of the integration interval is smaller than $|h|$. From this it is easy to see that the Fréchet derivative at $t = 0$ exists and equals 0. Now we shall show that the partial derivative of f with respect to t at $t = 0$ does not exist for any $x \in [0,1]$. Let x be an element of $[0,1]$ and consider the sequences $\{t_n, n \geq 1\}$ with $t_n = \frac{1}{n+x}$ and $\{\tau_n, n \geq 1\}$ with $\tau_n = \frac{1}{n}$. Both sequences converge to zero, and for sufficiently large n $f(x,t_n) = t_n$ and $f(x,\tau_n) = 0$. So we have that

$$\lim_{n\to\infty} \frac{f(x,t_n) - f(x,0)}{t_n} = 1,$$

and

$$\lim_{n\to\infty} \frac{f(x,\tau_n) - f(x,0)}{\tau_n} = 0.$$

Hence the partial derivative with respect to t does not exist at $t = 0$. □

The next theorem concerns differentiation of integrals.

Theorem A.5.29. *Let $u : [0,\infty) \to Z$ be such that $u \in L([0,\infty), Z)$, where Z is a separable Hilbert space. If $v(t) = \int_0^t u(s)ds$, then v is differentiable for almost all t, and*

$$\frac{dv}{dt}(t) = u(t) \qquad \textit{almost everywhere.}$$

Proof See corollary 2 on page 88 in Hille and Phillips [12]. ■

For operator-valued functions we can define three types of differentiability.

Definition A.5.30. Let $U(\cdot)$ be functions from $\mathbb{C}$ or $\mathbb{R}$ to $\mathcal{L}(Z_1, Z_2)$, where Z_1 and Z_2 are Hilbert spaces. Then

a. $U(\cdot)$ is *uniformly differentiable at* t_0 if there exists a $\frac{dU}{dt}(t_0) \in \mathcal{L}(Z_1, Z_2)$ such that

$$\lim_{h\to 0} \frac{\|U(t_0+h) - U(t_0) - h\frac{dU}{dt}(t_0)\|_{\mathcal{L}(Z_1,Z_2)}}{|h|} = 0.$$

b. $U(\cdot)$ is *strongly differentiable at* t_0 if there exists a $\frac{dU}{dt}(t_0) \in \mathcal{L}(Z_1, Z_2)$ such that

$$\lim_{h\to 0} \frac{\|U(t_0+h)z_1 - U(t_0)z_1 - h\frac{dU}{dt}(t_0)z_1\|_{Z_2}}{|h|} = 0,$$

for every $z_1 \in Z_1$

c. $U(\cdot)$ is *weakly differentiable at* t_0 if there exists a $\frac{dU}{dt}(t_0) \in \mathcal{L}(Z_1, Z_2)$ such that

$$\lim_{h\to 0} \frac{|\langle z_2, U(t_0+h)z_1\rangle - \langle z_2, U(t_0)z_1\rangle - h\langle z_2, \dfrac{dU}{dt}(t_0)z_1\rangle|_{\mathbb{C}}}{|h|} = 0, \qquad \clubsuit$$

for every $z_1 \in Z_1$ and $z_2 \in Z_2$

One can easily show that uniform implies strong, which implies weak differentiability, with the same derivative. Furthermore, by the Riesz representation Theorem A.3.52 one can easily show that $U(\cdot)$ is weakly differentiable at t_0 if and only if the complex-valued functions $f(t) = \langle z_2, U(t)z_1\rangle$ are differentiable at t_0, for every $z_1 \in Z_1$ and $z_2 \in Z_2$.

The next example shows that strong differentiability does not imply uniform differentiability.

Example A.5.31 Let Z be a Hilbert space with orthonormal basis $\{e_n, n \geq 1\}$, and let V_n denote the orthogonal complement of span$\{e_1, \ldots, e_n\}$. Define the operator-valued function $U(\cdot)$ by

$$U(t) = \begin{cases} 0 & \text{if} \quad t \leq 0 \text{ or } t \geq 1 \\ \\ tP_{V_n} & \text{if} \quad \dfrac{1}{n+1} \leq t \leq \dfrac{1}{n}, \end{cases}$$

where P_{V_n} denotes the orthogonal projection on V_n. Then for $\frac{1}{1+n} \leq h < \frac{1}{n}$ we have

$$U(h)z = h \sum_{i=n+1}^{\infty} \langle z, e_i\rangle e_i.$$

Thus

$$\frac{\|U(h)z - U(0)z\|}{|h|} = \|\sum_{i=n+1}^{\infty} \langle z, e_i\rangle e_i\| = \left[\sum_{i=n+1}^{\infty} |\langle z, e_i\rangle|^2\right]^{1/2},$$

and so $U(\cdot)$ is strongly differentiable at 0, with derivative 0. However,

$$\frac{\|U(h) - U(0) - \dfrac{dU}{dt}(0)\|}{|h|} = \frac{\|U(h)\|}{|h|} = \|P_{V_n}\| = 1.$$

So $U(\cdot)$ is not uniformly differentiable at zero. ■ □

The situation is different for operator-valued functions of a complex variable. As in finite dimensions, we define holomorphicity of a complex-valued function as differentiability.

Definition A.5.32. Let Z_1 and Z_2 be Hilbert spaces, and let $U : \Upsilon \to \mathcal{L}(Z_1, Z_2)$, where Υ is a domain in $\mathbb{C}$. Then U is *holomorphic* on Υ if U is weakly differentiable on Υ. ♣

Example A.5.33 Let A be a closed linear operator on the Hilbert space Z. Define $U(\lambda) : \rho(A) \to \mathcal{L}(Z)$ by $U(\lambda) = (\lambda I - A)^{-1}$. We shall prove that this is holomorphic on $\rho(A)$. We have from the resolvent equation (A.5) that

$$\begin{aligned} &\langle z_1, ((\lambda+h)I - A)^{-1}z_2\rangle - \langle z_1, (\lambda I - A)^{-1}z_2\rangle \\ &= \quad \langle z_1, -h(\lambda I - A)^{-1}((\lambda+h)I - A)^{-1}z_2\rangle. \end{aligned} \qquad \square$$

This implies that $U(\lambda)$ is weakly differentiable with $\frac{dU}{d\lambda}(\lambda) = -(\lambda I - A)^{-2}$. Thus the resolvent operator is holomorphic, and this proves Lemma A.4.8.c.

The following important theorem shows the equivalence of uniform and weak holomorphicity.

Theorem A.5.34. *Let Z_1 and Z_2 be separable Hilbert spaces and let $U(\cdot) : \Upsilon \to \mathcal{L}(Z_1, Z_2)$, where Υ is a domain of $\mathbb{C}$. If U is holomorphic, then $U(\cdot)$ is uniformly differentiable in Υ and furthermore,*

$$\frac{d}{dt}\langle z_2, U(t)z_1\rangle = \langle z_2, \frac{dU}{dt}z_1\rangle,$$

for every $z_1 \in Z_1$, $z_2 \in Z_2$.

Proof See Hille and Phillips [12, theorem 3.10.1]. ■

We remark that the above result is also valid in a general Banach space; for more details see Hille and Phillips [12]. With this result is easy to extend results that hold for holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ to Hilbert-space-valued holomorphic functions $f : \mathbb{C} \to Z$.

Example A.5.35 Let Υ be a domain in $\mathbb{C}$, and let Γ be a positively oriented, closed, simple contour in Υ. Consider a holomorphic function f on Υ with values in a separable Hilbert space Z. Then we have the following relation

$$\frac{1}{2\pi\jmath}\int_{\Gamma}\frac{f(s)}{s-\lambda}ds = f(\lambda),$$

where λ is any point inside Γ.

First, we have to say what we mean by the integral on the left-hand side. Since Γ is a rectifiable, closed, simple curve there exists a differentiable mapping k from $[0, 1]$ onto Γ. The integral is then defined to be

$$\frac{1}{2\pi\jmath}\int_{0}^{1}\frac{f(k(t))}{k(t)-\lambda}\dot{k}(t)dt.$$

This is well defined as a Pettis or Lebesgue integral by Lemma A.5.5.

From Theorem A.5.10, for every $z \in Z$ the following holds:

$$\begin{aligned}\langle\frac{1}{2\pi\jmath}\int_{\Gamma}\frac{f(s)}{s-\lambda}ds, z_1\rangle &= \frac{1}{2\pi\jmath}\int_{\Gamma}\langle\frac{f(s)}{s-\lambda}, z_1\rangle ds\\ &= \frac{1}{2\pi\jmath}\int_{\Gamma}\frac{1}{s-\lambda}\langle f(s), z_1\rangle ds = \langle f(\lambda), z_1\rangle, \qquad \square\end{aligned}$$

since $\langle f(s), z_1\rangle$ is a holomorphic function. This proves the assertion. This result is known as *Cauchy's theorem.*

A.6. Frequency-domain spaces

A.6.1. Laplace and Fourier transforms

In this book, we consider both state- and frequency-domain representations. The relation between these two representations is provided by the Laplace or Fourier transform. In this section, we take Z to be a separable Hilbert space.

Definition A.6.1. Let $h : [0, \infty) \to Z$ have the property that $e^{-\beta t}h(t) \in L_1([0,\infty); Z)$ for some real β. We call these *Laplace-transformable functions* and we define their *Laplace transform* $\hat{h}$ by

$$\hat{h}(s) = \int_0^\infty e^{-st}h(t)dt \tag{A.1}$$

for $s \in \overline{\mathbb{C}_\beta^+} := \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq \beta\}$. ■ ♣

A good reference for Laplace transforms of scalar functions is Doetsch [7], and for vector-valued functions a good reference is Hille and Phillips [12], where it is shown that $\hat{h}$ has the following properties.

Proposition A.6.2. *Laplace transformable functions $h : [0, \infty) \to Z$ have the following properties:*

a. *If $e^{-\beta t}h(t) \in L_1([0,\infty); Z)$ for some real β, then $\hat{h}$ is holomorphic and bounded on $\mathbb{C}_\beta^+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) > \beta\}$ and so $\hat{h}(\cdot + \beta) \in H_\infty(Z)$ (see definition A.6.14); furthermore, the following inequality holds:*

$$\sup_{\mathrm{Re}(s)\geq 0} \|\hat{h}(s+\beta)\| \leq \|e^{-\beta \cdot}h(\cdot)\|_{L_1([0,\infty);Z)}; \tag{A.2}$$

b. Uniqueness of the Laplace transform*: if h_1 and h_2 are Laplace transformable functions such that $\hat{h}_1(s) = \hat{h}_2(s)$ in $\mathbb{C}_\beta^+$, for some $\beta \in \mathbb{R}$, then $h_1 = h_2$;*

c. *If $e^{-\beta t}h(t) \in L_1([0,\infty); Z)$ for some real β, then $\hat{h}(\beta + \jmath\omega)$ is continuous in ω for $\omega \in \mathbb{R}$ and $\|\hat{h}(\beta + \jmath\omega)\| \to 0$ as $|\omega| \to \infty$;*

d. *If h is differentiable for $t > 0$ and $\frac{dh}{dt}$ is Laplace-transformable, then*

$$\left[\widehat{\frac{dh}{dt}}\right](s) = s\hat{h}(s) - h(0^+); \tag{A.3}$$

e. *For $\alpha \in \mathbb{R}$ it holds that*

$$[\widehat{e^{-\alpha t}h(t)}](s) = \hat{h}(s+\alpha); \tag{A.4}$$

f. *If $e^{-\beta t}h(t) \in L_1([0,\infty); Z)$ for some real β, then the derivative of $\hat{h}$ in $\mathbb{C}_\beta^+$ equals the Laplace transform of $-th(t)$;*

g. *If $e^{-\beta t}h(t) \in L_1([0,\infty); Z)$ for some real β, then $\hat{h}(s) \to 0$ as $|s| \to \infty$ in $\overline{\mathbb{C}_\beta^+}$, i.e.,*

$$\lim_{\rho\to\infty}\left[\sup_{\{s\in\overline{\mathbb{C}_\beta^+} \,|\, |s|>\rho\}} \|\hat{h}(s)\|\right] = 0.$$

♡

Proof *a.* See Doetsch [7] for the scalar case. For the general case, consider the scalar functions $h_z(t) := \langle h(t), z\rangle$. It is easy to see that $\hat{h}_z(s) = \langle \hat{h}(s), z\rangle$. This function is holomorphic for every $z \in Z$ and so by definition A.5.32, h is holomorphic. Furthermore, for $\mathrm{Re}(s) > 0$,

$$\|\hat{h}(s+\beta)\| \leq \int_0^\infty \|e^{-(s+\beta)t}h(t)\|dt \leq \int_0^\infty \|e^{-\beta t}h(t)\|dt,$$

where we have used Theorem A.5.18.

b. See Hille and Phillips [12, theorem 6.2.3].

c. For the scalar case see Bochner and Chandrasekharan [3, theorem 1]. This proof is based on the denseness of the simple functions that are zero outside some interval. However, this fact also holds for $L_1([0,\infty);Z)$ (see Lemma A.5.20) and so a similar proof is valid for the vector-valued case.

This property is known as the *Riemann-Lebesgue lemma*.

d and e. The proof of these properties is similar in the scalar and nonscalar cases; see Doetsch [7] theorems 9.1 and 7.7, respectively.

f. For the scalar case see theorem 6.1 of Doetsch [7]. The general case is proven again by introducing the functions $h_z(t) := \langle h(t), z\rangle$ with Laplace transform $\hat{h}_z(s) = \langle \hat{h}, z\rangle$. From the scalar case we know that the derivative of $\hat{h}_z$ equals the Laplace transform of $-th_z(t)$. Since $\hat{h}$ is holomorphic we know by Theorem A.5.34 that the derivative of $\hat{h}_z$ equals $\langle \frac{d}{ds}\hat{h}(s), z\rangle$. Hence $\langle \frac{d}{ds}\hat{h}(s), z\rangle$ is the Laplace transform of $\langle -th(t), z\rangle$. Since this holds for any $z \in Z$, the assertion is proved.

g. This follows essentially from part c, see Doetsch [7, theorem 23.7]. ∎

The Laplace transform can be seen as a special case of the Fourier transform.

Definition A.6.3. For $h \in L_1((-\infty,\infty);Z)$ we define the *Fourier transform* of h by

$$\check{h}(\jmath\omega) := \int_{-\infty}^{\infty} e^{-\jmath\omega t} h(t)dt. \tag{A.5}$$

♣

In fact, the Fourier transform can be extended to functions in $L_2(\mathbb{R};Z)$; see Theorem A.6.13. If h has support on $\mathbb{R}^+$ and its Fourier transform exists, then it is equal to its Laplace transform

$$\check{h}(\jmath\omega) = \hat{h}(\jmath\omega). \tag{A.6}$$

Sometimes it is convenient to introduce the *two-sided Laplace transform* for functions h defined on all of $\mathbb{R}$, that is, $\hat{h}(s) := \int_{-\infty}^{\infty} e^{-st}h(t)dt$. This then coincides with the Fourier transform using (A.6). This connection with the Fourier transform makes it easy to deduce some further properties of the Laplace transform.

The reason for using Laplace transforms in linear differential equations lies in Property A.6.2.d and in the simple property for the convolution product. In this book we shall only need the scalar version.

Definition A.6.4. For two functions h, g in $L_1(-\infty,\infty)$, we define the *convolution product*

$$(h * g)(t) := \int_{-\infty}^{\infty} h(t-s)g(s)ds. \tag{A.7}$$

♣

Note that if h and g have their support on $[0,\infty)$, then $h * g$ also has its support on $[0,\infty)$, and for $t \geq 0$ the convolution product is given by

$$(h * g)(t) := \int_{0}^{t} h(t-s)g(s)ds. \tag{A.8}$$

The following are very useful properties of the convolution product.

Lemma A.6.5. *For two functions h, g from $\mathbb{R}$ to $\mathbb{C}$ the following hold:*

a. If $h \in L_1(-\infty,\infty)$, $g \in L_p(-\infty,\infty)$, then $h * g \in L_p(-\infty,\infty)$ and

$$\|h * g\|_p \le \|h\|_1 \|g\|_p \tag{A.9}$$

for $1 \le p \le \infty$;

b. If $h \in L_1([0,\infty))$ and $g \in L_p([0,\infty))$, then $h * g \in L_p([0,\infty))$ and

$$\|h * g\|_p \le \|h\|_1 \|g\|_p \tag{A.10}$$

for $1 \le p \le \infty$;

c. If h and g are zero on $(-\infty, 0)$ and are Laplace transformable, then $h * g$ is Laplace transformable and

$$\widehat{h * g} = \hat{h}\hat{g}; \tag{A.11}$$

d. If h and g are in $L_1(-\infty,\infty) \cap L_2(-\infty,\infty)$, then

$$\overset{\vee}{h * g} = \check{h}\check{g}. \tag{A.12}$$

Proof *a.* See theorem 53 of Bochner and Chandrasekharan [3].
b. This follows from part a by defining extended functions on $(-\infty,\infty)$

$$h_e(t) = \begin{cases} h(t) & t \ge 0, \\ 0 & t < 0 \end{cases}$$

and g_e similarly. See also the remark made after definition A.6.4.
c and d. See theorems 10.2 and 31.3 of Doetsch [7]. ■

The definition of the convolution product for real- or complex-valued functions as given in definition A.6.4 can easily be extended to vector-valued functions, and similar results to those given in Lemma A.6.5 hold.

Lemma A.6.6. *Let Z_1 and Z_2 be separable Hilbert spaces, $H \in P_1((-\infty,\infty); \mathcal{L}(Z_1, Z_2))$, and $g \in L_p((-\infty,\infty); Z_1)$ for a p, $1 \le p \le \infty$. The convolution product of H and g is defined by*

$$(H * g)(t) := \int_{-\infty}^{\infty} H(t-s)g(s)ds. \tag{A.13}$$

Furthermore, $H * g \in L_p((-\infty,\infty); Z_2)$ and

$$\|H * g\|_p \le \|H\|_1 \|g\|_p. \tag{A.14}$$

Proof Let $z \in Z_2$. Consider the function in s defined by $\langle z, H(t-s)g(s)\rangle$. This function is the same as $\langle H^*(t-s)z, g(s)\rangle$, which is measurable by Lemma A.5.3. Furthermore, since $\|H(t-s)g(s)\| \le \|H(t-s)\|\|g(s)\|$, we have that integral (A.13) is well defined; see Lemma A.5.5. Now inequality (A.14) follows directly from Lemma A.6.5. ■

Inequalities (A.9) and (A.10), together with the Cauchy-Schwarz inequality, are used frequently in this book. Another useful inequality is given in the next lemma.

Lemma A.6.7. Gronwall's Lemma. *Let $a \in L_1(0,\tau)$, $a(t) \geq 0$. If for some $\beta \geq 0$ the function $z \in L_\infty(0,\tau)$ satisfies*

$$z(t) \leq \beta + \int_0^t a(s)z(s)ds,$$

then

$$z(t) \leq \beta \exp(\int_0^t a(s)ds).$$

Proof See the lemma on page 169 and problem 8 on page 178 of Hirsch and Smale [13]. ∎

A.6.2. Frequency-domain spaces

In the text we frequently make use of the following Lebesgue frequency-domain spaces (see Appendix A.5).

Definition A.6.8. Let Z, U, and Y be separable Hilbert spaces. We define the following frequency-domain spaces:

$$\begin{aligned} &P_\infty((-\jmath\infty, \jmath\infty); \mathcal{L}(U,Y)) \\ &\quad := \{G : (-\jmath\infty,\jmath\infty) \to \mathcal{L}(U,Y) \mid \langle y, Gu\rangle \text{ is measurable for every } u \in U \\ &\qquad \text{and } y \in Y \text{ and } G \text{ is bounded almost everywhere on } (-\jmath\infty,\jmath\infty)\}; \end{aligned} \tag{A.15}$$

$$L_2((-\jmath\infty,\jmath\infty); Z) := \{z : (-\jmath\infty,\jmath\infty) \to Z \mid \langle z, x\rangle \text{ is measurable for all } x \in Z \text{ and } \int_{-\infty}^{\infty} \|z(\jmath\omega)\|_Z^2 d\omega < \infty\}. \tag{A.16}$$

♣

We remark that in the case that U and Y are finite-dimensional spaces, we may write $L_\infty((-\jmath\infty,\jmath\infty); \mathcal{L}(U,Y))$ instead of $P_\infty((-\jmath\infty,\jmath\infty); \mathcal{L}(U,Y))$; see also the remark after definition A.5.16.

From Lemma A.5.17 we deduce the following results.

Lemma A.6.9. *If Z, U, and Y are separable Hilbert spaces, $P_\infty((-\jmath\infty,\jmath\infty); \mathcal{L}(U,Y))$ is a Banach space under the P_∞-norm defined by*

$$\|G\|_\infty = \operatorname{ess\,sup}_{-\infty<\omega<\infty} \|G(\jmath\omega)\|_{\mathcal{L}(U,Y)} \tag{A.17}$$

and $L_2((-\jmath\infty,\jmath\infty); Z)$ is a Hilbert space under the inner product

$$\langle z_1, z_2\rangle_2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle z_1(\jmath\omega), z_2(\jmath\omega)\rangle_Z d\omega. \tag{A.18}$$

We remark that the inner product on $L_2((-\jmath\infty,\jmath\infty); Z)$ differs by a constant from the inner product defined in Lemma A.5.17. This is important in the theory of Fourier transforms, as can be seen from equation (A.24).

If U and Y are finite-dimensional, then we have a more explicit expression for the P_∞-norm.

Lemma A.6.10. *Let U and Y be finite-dimensional Hilbert spaces. The P_∞-norm for the space $P_\infty((-\jmath\infty,\jmath\infty);\mathcal{L}(U,Y))$ is given by*

$$\|G\|_\infty = \operatorname{ess\,sup}_{-\infty<\omega<\infty} \sigma_{\max}(G(\jmath\omega)) = \operatorname{ess\,sup}_{-\infty<\omega<\infty}\{\lambda_{\max}^{1/2}(G(\jmath\omega)^*G(\jmath\omega))\}, \tag{A.19}$$

where $\sigma_{\max}$ and $\lambda_{\max}$ denote the maximum singular value and eigenvalue, respectively (see Example A.3.13).

Continuous functions in $L_\infty((-\jmath\infty,\jmath\infty);\mathcal{L}(U,Y))$ can be approximated by strictly proper rational ones. By a *strictly proper* rational matrix we mean a quotient of polynomials such that it has the limit zero at infinity.

Lemma A.6.11. *Let U and Y be finite-dimensional Hilbert spaces. If $F \in L_\infty((-\jmath\infty,\jmath\infty);\mathcal{L}(U,Y))$ is continuous and*

$$\lim_{\omega\to\infty} F(\jmath\omega) = \lim_{\omega\to-\infty} F(\jmath\omega) = 0,$$

then F can be approximated in the L_∞-norm by strictly proper rational matrices with no poles on the imaginary axis.

Proof We reduce this to an equivalent problem on the unit circle, by introducing the bilinear transformation θ: $\mathbb{C}\to\mathbb{C}$ defined by

$$\theta(z) := \frac{1+z}{1-z} \qquad \text{for } z \in \mathbb{C}\backslash\{1\}.$$

It is easy to see that it maps the unit circle excluding the point 1 onto the imaginary axis.

Furthermore, it is easy to see that $F_d(z) := F(\theta(z))$ is bounded and continuous on the unit circle except for the point 1. For the point 1, we have

$$\lim_{|z|=1, z\to 1} F_d(z) = \lim_{\omega\in\mathbb{R}, |\omega|\to\infty} \hat{F}(\jmath\omega) = 0.$$

Hence F_d is continuous on the unit circle.

From the Weierstrass Theorem [22, theorem 7.26], we have that for every $\varepsilon > 0$ there exists a polynomial Q_ε such that

$$\sup_{|z|=1} |F_d(z) - Q_\varepsilon(z)| < \varepsilon.$$

Since $F_d(1) = 0$, $|Q_\varepsilon(1)| < \varepsilon$. Defining $P_\varepsilon := Q_\varepsilon - Q_\varepsilon(1)$ gives $P_\varepsilon(1) = 0$ and

$$\sup_{|z|=1} |F_d(z) - P_\varepsilon(z)| < 2\varepsilon.$$

Now using the bilinear transformation again, we see that

$$\sup_{\omega\in\mathbb{R}} |F(\jmath\omega) - P_\varepsilon(\theta^{-1}(\jmath\omega))| = \sup_{|z|=1} |F_d(z) - P_\varepsilon(z)| < 2\varepsilon.$$

The function $P_\varepsilon(\theta^{-1}(\cdot))$ is a rational function with no poles on the imaginary axis. Furthermore, we have that

$$\lim_{\omega\in\mathbb{R}, |\omega|\to\infty} P_\varepsilon(\theta^{-1}(\jmath\omega)) = \lim_{|z|=1, z\to 1} P_\varepsilon(z) = 0,$$

and so $P_\varepsilon(\theta^{-1}(\cdot))$ is strictly proper. ∎

The following lemma enables us to reduce many properties of the vector-valued function space $L_2(\Omega; Z)$ to analogous ones of the scalar function space $L_2(\Omega)$.

Lemma A.6.12. *Let Z be a separable Hilbert space with an orthonormal basis $\{e_n, n \geq 1\}$. For every $f \in L_2(\Omega, Z)$ there exists a sequence of functions $f_n \in L_2(\Omega)$ such that*

$$f(t) = \sum_{n=1}^{\infty} f_n(t) e_n \tag{A.20}$$

and

$$\|f\|^2_{L_2(\Omega,Z)} = \sum_{n=1}^{\infty} \|f_n\|^2_{L_2(\Omega)}. \tag{A.21}$$

On the other hand, if $\{f_n, n \geq 1\}$ is a sequence of functions in $L_2(\Omega)$ such that $\sum\limits_{n=1}^{\infty} \|f_n\|^2_{L_2(\Omega)} < \infty$, then f defined by (A.20) is in $L_2(\Omega, Z)$ and (A.21) holds.

Proof For $f \in L_2(\Omega, Z)$, it is easy to see that the function f_n defined by $f_n(t) := \langle f_n(t), e_n \rangle$ is in $L_2(\Omega)$. Furthermore, (A.20) holds, since $\{e_n, n \geq 1\}$ is an orthonormal basis for Z (see definition A.2.33). The norm equality follows from this because

$$\begin{aligned}
\|f\|^2_{L_2(\Omega,Z)} &= \int_\Omega \|f(t)\|^2_Z dt = \int_\Omega \sum_{n=1}^{\infty} \|f_n(t)\|^2 dt \\
&\quad \text{since the summant is positive} \\
&= \sum_{n=1}^{\infty} \int_\Omega \|f_n(t)\|^2 dt = \sum_{n=1}^{\infty} \|f_n\|^2_{L_2(\Omega)}.
\end{aligned}$$

For the other assertion, define the sequence of functions $g_n := \sum_{k=1}^{n} f_k e_k$. It is easy to see that $g_n \in L_2(\Omega, Z)$, and furthermore, for $N > n$,

$$\|g_N - g_n\|^2_{L_2(\Omega,Z)} = \sum_{k=n+1}^{N} \|f_k\|^2_{L_2(\Omega)}.$$

Since $\sum_{k=1}^{\infty} \|f_k\|^2_{L_2(\Omega)} < \infty$, this implies that g_n is a Cauchy sequence in $L_2(\Omega, Z)$. Thus g_n converges, and now it is easy to show that (A.21) holds. ∎

We are interested in the relationships between the frequency-domain spaces $L_2((-\jmath\infty, \jmath\infty); Z)$ and $P_\infty((-\jmath\infty, \jmath\infty); \mathcal{L}(U, Y))$ defined in definition A.6.8 and their time-domain counterparts $L_2((-\infty, \infty); Z)$ and $P_\infty((-\infty, \infty); \mathcal{L}(U, Y))$, respectively (see Appendix A.5).

Theorem A.6.13. *The frequency-domain space $L_2((-\jmath\infty, \jmath\infty); Z)$ is isomorphic to the time-domain space $L_2((-\infty, \infty); Z)$ via the Fourier transform. So the Fourier transform gives an isometry from $L_2((-\infty, \infty); Z)$ to $L_2((-\jmath\infty, \jmath\infty); Z)$. If $\check{h}$ is the Fourier transform of h, then h can be recovered via the inversion formula*

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\jmath\omega t} \check{h}(\jmath\omega) d\omega. \tag{A.22}$$

From the isometry we know that the norms are equivalent, and this equivalence is usually known as Parseval's equality

$$\int_{-\infty}^{\infty} \|h(t)\|^2_Z dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\check{h}(\jmath\omega)\|^2_Z d\omega. \tag{A.23}$$

In addition, for $h, g \in L_2((-\infty,\infty); Z)$,

$$\int_{-\infty}^{\infty} \langle h(t), g(t)\rangle_Z dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle \check{h}(\jmath\omega), \check{g}(\jmath\omega)\rangle_Z d\omega. \tag{A.24}$$

So $\langle h, g\rangle = \langle \check{h}, \check{g}\rangle_2$, where the first inner product is defined in Lemma A.5.17 and the second in Lemma A.6.9.

Proof *a.* First we have to show that we can extend the Fourier transform to functions in $L_2(-\infty,\infty); Z)$. Let $h \in L_1((-\infty,\infty); Z) \cap L_2((-\infty,\infty); Z)$ and let $\{e_n, n \geq 1\}$ be an orthonormal basis of Z. From (A.5) it is easy to see that

$$\check{h}_n(\jmath\omega) := \langle \check{h}(\jmath\omega), e_n\rangle = \overset{\vee}{\langle h, e_n\rangle}(\jmath\omega).$$

Furthermore, we know from Rudin [24, theorem 9.13] that

$$\int_{-\infty}^{\infty} |\langle h, e_n\rangle(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\check{h}_n(\jmath\omega)|^2 d\omega.$$

So with Lemma A.6.12 we see that $\check{h} \in L_2((-\jmath\infty,\jmath\infty); Z)$ and (A.23) holds. This implies that we can extend the definition of the Fourier transform to functions in $L_2((-\infty,\infty); Z)$ and that equality (A.23) still holds.

So we have established that the Fourier transform maps the time-domain space $L_2((-\infty,\infty); Z)$ isometrically into the frequency-domain space $L_2((-\jmath\infty,\jmath\infty); Z)$. It remains to show that this mapping is onto, and that inversion formula (A.22) holds.

b. As in part a, one can show that for $\check{h} \in L_1((-\jmath\infty,\jmath\infty); Z) \cap L_2((-\jmath\infty,\jmath\infty); Z)$ the mapping

$$\Phi\check{h} \mapsto \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\jmath\omega t}\check{h}(\jmath\omega) d\omega$$

is well defined and that

$$\int_{-\infty}^{\infty} \|\left(\Phi\check{h}\right)(t)\|_Z^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|\check{h}(\jmath\omega)\|_Z^2 d\omega.$$

So this mapping can be extended to hold for all $\check{h} \in L_2((-\jmath\infty,\jmath\infty); Z)$. Furthermore, we know from Rudin [24, theorem 913] that Φ is the inverse Fourier transform for functions of the form $\check{h}_n e_n$. Since, Φ is linear this implies that Φ is the inverse of the Fourier transform for all $\check{h}$. Thus the Fourier transform is an isometry between the time- and frequency-domain L_2 spaces.

Equality (A.24) follows from (A.23) and property d. after definition A.2.24. ∎

A.6.3. The Hardy spaces

In this subsection, we consider some special classes of frequency-domain functions that are holomorphic on the open half-plane.

Good general references for this section are Kawata [15] and Helson [11] for the scalar case and Thomas [26] and Rosenblum and Rovnyak [21] for the vector-valued case.

Definition A.6.14. For a Banach space X and a separable Hilbert space Z we define the following *Hardy spaces*:

$$H_\infty(X) := \left\{ G : \mathbb{C}_0^+ \to X \mid G \text{ is holomorphic, and } \sup_{\mathrm{Re}(s)>0} \|G(s)\| < \infty \right\};$$

$$H_2(Z) := \left\{ f : \mathbb{C}_0^+ \to Z \mid f \text{ is holomorphic; and} \right.$$

$$\left. \|f\|_2^2 = \sup_{\zeta>0}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|f(\zeta + \jmath\omega)\|^2 d\omega\right) < \infty \right\}. \tag{A.25}$$

♣

When the Banach space X or the Hilbert space Z equals $\mathbb{C}$, we shall use the notation H_∞ and H_2 for $H_\infty(\mathbb{C})$ and $H_2(\mathbb{C})$, respectively. In most of the literature, Hardy spaces on the disc are usually treated; see, for example, Rosenblum and Rovnyak [21]. The following lemma shows this is equivalent to considering Hardy spaces on the open right half-plane.

Lemma A.6.15. *Denote by θ the bilinear transformation $\theta(z) = \frac{1+z}{1-z}$. A function G is an element of $H_\infty(X)$ if and only if $f\circ\theta$ is holomorphic and bounded on the unit disc $\mathbb{D}$. Furthermore, $\sup_{s\in\mathbb{C}_0^+} \|G(s)\| = \sup_{z\in\mathbb{D}} \|G(\theta(z))\|$.*

Lemma A.6.16. *If X is a Banach space, then $H_\infty(X)$ from definition A.6.14 is a Banach space under the H_∞-norm*

$$\|G\|_\infty := \sup_{Re(s)>0} \|G(s)\|_X. \tag{A.26}$$

Proof Combine Lemma A.6.16 with theorem D of Rosenblum and Rovnyak [21, section 4.7]. ■

We now collect several important results in the following lemma.

Lemma A.6.17. *The following are important properties of $H_\infty(\mathcal{L}(U,Y))$, where U, Y are separable Hilbert spaces:*

a. For every $F \in H_\infty(\mathcal{L}(U,Y))$ there exists a unique function $\tilde{F} \in P_\infty((-\jmath\infty, \jmath\infty); \mathcal{L}(U,Y))$ such that

$$\lim_{x\downarrow 0} F(x + \jmath\omega)u = \tilde{F}(\jmath\omega)u \qquad \text{for all } u \in U \text{ and almost all } \omega \in \mathbb{R}$$

(i.e., $F \in H_\infty(\mathcal{L}(U,Y))$ has a well defined extension to the boundary);

b. The mapping $F \mapsto \tilde{F}$ is linear, injective and norm preserving, i.e.,

$$\sup_{\mathrm{Re}(s)>0} \|F(s)\|_{\mathcal{L}(U,Y)} = \operatorname{ess\,sup}_{\omega\in\mathbb{R}} \|\tilde{F}(\jmath\omega)\|_{\mathcal{L}(U,Y)}$$

(we identify $F \in H_\infty(\mathcal{L}(U,Y)$ with its boundary function $\tilde{F} \in P_\infty((-\jmath\infty, \jmath\infty); \mathcal{L}(U,Y))$ and we can regard $H_\infty(\mathcal{L}(U,Y))$ as a closed subspace of $P_\infty((-\jmath\infty, \jmath\infty); \mathcal{L}(U,Y))$);

c. Identifying F with $\tilde{F}$, the following holds:

$$\sup_{\mathrm{Re}(s)\geq 0} \|F(s)\|_{\mathcal{L}(U,Y)} = \operatorname{ess\,sup}_{\omega\in\mathbb{R}} \|F(\jmath\omega)\|_{\mathcal{L}(U,Y)} < \infty.$$

Proof Combine Lemma A.6.15 with theorems A of sections 4.6 and 4.7 of Rosenblum and Rovnyak [21]. ■

We remark that Rosenblum and Rovnyak [21] use L^∞ for P_∞. In general, the boundary function $\tilde{F}$ will not have the property that $\tilde{F}$ is uniformly measurable in the $\mathcal{L}(U,Y)$ topology; see Rosenblum and Rovnyak [21, exercise 1 of chapter 4] or Thomas [26].

Lemma A.6.18. *$H_2(Z)$ is a Banach space under the H_2-norm defined by (A.25), and the following important properties hold:*

a. For each $f \in H_2(Z)$ there exists a unique function $\tilde{f} \in L_2((-\jmath\infty, \jmath\infty); Z)$ such that

$$\lim_{x \downarrow 0} f(x + \jmath\omega) = \tilde{f}(\jmath\omega) \qquad \text{for almost all } \omega \in \mathbb{R}$$

and

$$\lim_{x \downarrow 0} \|f(x + \cdot) - \tilde{f}(\cdot)\|_{L_2((-\jmath\infty, \jmath\infty); Z)} = 0;$$

b. The mapping $f \to \tilde{f}$ is linear, injective, and norm preserving.
(we identify the function $f \in H_2(Z)$ with its boundary function $\tilde{f} \in L_2((-\jmath\infty, \jmath\infty); Z)$ and regard $H_2(Z)$ as a closed subspace of $L_2((-\jmath\infty, \jmath\infty); Z)$);

c. For any $f \in H_2(Z)$ and any $\alpha > 0$ we have that

$$\lim_{\rho \to \infty} \left(\sup_{s \in \overline{\mathbb{C}_\alpha^+};\ |s| > \rho} |f(s)| \right) = 0 \tag{A.27}$$

(sometimes the terminology $f(s) \to 0$ as $|s| \to \infty$ in $\overline{\mathbb{C}_\alpha^+}$ is used).

Proof *a and b.* The proof for the scalar case as given by Kawata [15, theorem 6.5.1] is based on Theorem A.6.13. Since this theorem holds for vector-valued function as well, the proof of parts a and b is similar to that for the scalar case.

c. See Hille and Phillips [12, theorem 6.4.2]. ■

We remark that in general part c is not true for $\alpha = 0$. From this lemma we deduce the following result.

Corollary A.6.19. *If Z is a separable Hilbert space, then $H_2(Z)$ is a Hilbert space under the inner product*

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle f(\jmath\omega), g(\jmath\omega) \rangle d\omega.$$

♠

$H_2(Z)$ is a very special Hilbert space, as is apparent from the following lemma and the Paley-Wiener theorem.

Lemma A.6.20. *Let Z be a separable Hilbert space and let $f \in H_2(Z)$ be different from the zero function. Then f is nonzero almost everywhere on the imaginary axis.*

Proof Suppose that there is a subset V of the imaginary axis with positive measure such that f is zero on this set. Then for every $z \in Z$, we have that $\langle f, z \rangle \in H_2$ and it is zero on V. This implies that

$$\int_{\infty}^{\infty} \frac{|\log(\langle f(\jmath\omega), z \rangle)|}{1 + \omega^2} d\omega = \infty.$$

By Theorem 6.6.1 of Kawata [15] this can only happen if $\langle f, z\rangle$ is the zero function. Since $z \in Z$ was arbitrary, this would imply that $f = 0$. This is in contradiction to our assumption, and so the set V cannot have positive measure. ■

Theorem A.6.21. Paley-Wiener Theorem. *If Z is a separable Hilbert space, then under the Laplace transform $L_2([0,\infty); Z)$ is isomorphic to $H_2(Z)$ and it preserves the inner products.*

Proof See Thomas [26]. ■

An important consequence is the orthogonal decomposition of $L_2((-\jmath\infty, \jmath\infty); Z)$.

Theorem A.6.22. *The following holds:*

$$L_2((-\jmath\infty, \jmath\infty); Z) = H_2(Z) \oplus H_2(Z)^{\perp},$$

where $H_2(Z)^{\perp}$ is the orthogonal complement in $L_2((-\jmath\infty, \jmath\infty); Z)$ of $H_2(Z)$ as in definition A.2.30. $H_2(Z)^{\perp}$ is given by

$$H_2(Z)^{\perp} = \Bigg\{ f : \mathbb{C}_0^- \to Z \mid f \text{ is holomorphic, and}$$

$$\|f\|_2^2 = \sup_{\zeta<0}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|f(\zeta + \jmath\omega)\|^2 d\omega\right) < \infty \Bigg\}. \tag{A.28}$$

Proof It is easy to see that

$$L_2((-\infty,\infty); Z) = L_2((0,\infty); Z) \oplus L_2((-\infty, 0); Z) \tag{A.29}$$

and that $L_2((0,\infty); Z)^{\perp} = L_2((-\infty, 0); Z)$. Applying the Fourier transform to (A.29) and appealing to Theorems A.6.13 and A.6.21, we obtain that

$$L_2((-\jmath\infty, \jmath\infty); Z) = H_2(Z) \oplus \mathcal{F}\left(L_2((0,\infty); Z)^{\perp}\right).$$

By Parseval's equality (A.24) we have that $\mathcal{F}\left(L_2((0,\infty); Z)^{\perp}\right) = H_2(Z)^{\perp}$. So it remains to show equality (A.25).

For $h \in L_2((0,\infty); Z)^{\perp} = L_2((-\infty, 0); Z)$ it is easy to see that the two-sided Laplace transform of h satisfies

$$\hat{h}(s) = \widehat{h_-}(-s) \qquad \text{for } s \in \mathbb{C}_0^-,$$

where $h_-(t) := h(-t)$ and $\widehat{h_-}$ denotes the one-sided Laplace transform of h_-. Now h_- is an element of $L_2((0,\infty); Z)$ and thus by the Paley-Wiener Theorem A.6.21 $\widehat{h_-}(-s)$ is an element of the space defined in (A.28). On the other hand, if f is an element of the reflected $H_2(Z)$ space defined in (A.28), then $f(-s)$ is an element of $H_2(Z)$. Thus by Theorem A.6.21 $f(-s) = \widehat{h_-}(s)$ for some $\widehat{h_-} \in L_2((0,\infty); Z)$. Defining $h(t) := h_-(-t)$, it is easy to see that $h \in L((-\infty, 0); Z)$ and that $\hat{h}(s) = f(s)$ for $s \in \mathbb{C}_0^-$. ■

Another consequence of the Paley-Wiener Theorem is the following corollary.

Corollary A.6.23. *If $f(\cdot+\beta)$ is in $H_2(Z)$ for some real β, then there exists a function $h : \mathbb{R}^+ \to Z$ such that $e^{-\beta\cdot}h(\cdot) \in L_2([0,\infty); Z)$ and $\hat{h}(s) = f(s)$ almost everywhere.* ♠

Proof By Theorem A.6.21 $f_\beta(\cdot) := f(\cdot + \beta) \in H_2(Z)$ is the image under the Laplace transform of a $z \in L_2([0,\infty); Z)$, $\hat{z}(s) = f_\beta(s)$. Thus from (A.4) we have that $h(t) = e^{\beta t} z(t)$ satisfies $\hat{h}(s) = f(s)$. ■

It is often important to know when a function in $H_\infty(Z)$ corresponds to a Laplace-transformable function. In general, it will only correspond to a distribution; for example, $e^{-\alpha s}$ corresponds to the delta distribution centred at α, $\delta(t-\alpha)$. In this direction we have the following result from Mossaheb [18]; it is a direct consequence of Theorem A.6.21.

Lemma A.6.24. *Let g be a holomorphic function on $\mathbb{C}_\alpha^+$ such that $sg(s)$ is bounded on $\mathbb{C}_\alpha^+$. Then for any $\varepsilon > 0$ there exists an h such that $g = \hat{h}$ on $\mathbb{C}_{\alpha+\varepsilon}^+$, and $\int_0^\infty e^{-(\alpha+\varepsilon)t}|h(t)|dt < \infty$.*

Other sufficient conditions for a holomorphic function to be the Laplace transform of a function in $L_1(0,\infty)$ can be found in Gripenberg, Londen, and Staffans [10].

The Paley-Wiener Theorem A.6.21 gives a complete characterization of all functions with Laplace transforms in $H_2(Z)$. The following theorem gives a complete characterization in the frequency domain of all $L_2([0,\infty); Z)$ functions with compact support. This result is also called the Paley-Wiener Theorem.

Theorem A.6.25. Paley-Wiener Theorem. *If $h \in L_2([0,\infty); Z)$ has the Laplace transform $\hat{h}$, then necessary and sufficient conditions for h to be zero almost everywhere on (t_0,∞) are*

a. $\hat{h}$ is an entire function,

b. $\hat{h} \in H_2(Z)$, and

c. $\|\hat{h}(-x+\jmath y)\| \leq Ce^{t_0 x}$ for $x \geq 0$.

Proof *Necessity*: By the Cauchy-Schwarz inequality (A.1) we see that

$$\begin{aligned}\left\|\int_0^\infty e^{-st}h(t)dt\right\| &\leq \int_0^{t_0} \|e^{-st}h(t)\|dt \leq \sqrt{\int_0^{t_0} e^{-2\mathrm{Re}(s)t}dt \int_0^{t_0} \|h(t)\|^2 dt} \\ &= \sqrt{\frac{e^{-2\mathrm{Re}(s)t_0}-1}{-2\mathrm{Re(s)}}}\|h\|_{L_2((0,\infty);Z)}.\end{aligned}$$

So $e^{-\beta t}h(t) \in L_1([0,\infty); Z)$ for all $\beta \in \mathbb{R}$, and by Property A.6.2.a, $\hat{h}$ is holomorphic on every $\mathbb{C}_\beta^+$. In other words, $\hat{h}$ is entire. Furthermore, we see that for $\mathrm{Re}(s) < 0$, the inequality as stated in c holds. Part b follows directly from Theorem A.6.21.

Sufficiency: From part b and Theorem A.6.21 we know that $\hat{h}$ is the Laplace transform of an $L_2([0,\infty); Z)$ function. Thus it remains to prove that it is zero on (t_0,∞). For $z \in Z$ consider the scalar function $\hat{h}_z := \langle \hat{h}, z\rangle$. This clearly satisfies a, b, and c with $H_2(Z)$ replaced by H_2. Now by Theorem 19.3 of Rudin [24] it follows that h_z is zero almost everywhere on (t_0,∞). Since $h_z = \langle h, z\rangle$ and Z is separable, we obtain that h is zero almost everywhere on (t_0,∞). ■

The following theorem gives a characterization of bounded operators between frequency-domain spaces.

Theorem A.6.26. *Suppose that U and Y are separable Hilbert spaces.*

a. *If* $F \in P_\infty((-\jmath\infty, \jmath\infty); \mathcal{L}(U,Y))$ *and* $u \in L_2((-\jmath\infty, \jmath\infty); U)$, *then* $Fu \in L_2((-\jmath\infty, \jmath\infty); Y)$. *Moreover, the multiplication map* $\Lambda_F : u \mapsto Fu$ *defines a bounded linear operator from* $L_2((-\jmath\infty, \jmath\infty); U)$ *to* $L_2((-\jmath\infty, \jmath\infty); Y)$, *and*

$$\|\Lambda_F u\|_{L_2((-\jmath\infty,\jmath\infty);Y)} \leq \|F\|_\infty \|u\|_{L_2((-\jmath\infty,\jmath\infty);U)},$$

where $\|\cdot\|_\infty$ *denotes the norm on* $P_\infty((-\jmath\infty, \jmath\infty); \mathcal{L}(U,Y))$. *In fact,*

$$\|\Lambda_F\| = \sup_{u \neq 0} \frac{\|\Lambda_F u\|_{L_2((-\jmath\infty,\jmath\infty);Y)}}{\|u\|_{L_2((-\jmath\infty,\jmath\infty);U)}} = \|F\|_\infty.$$

b. *If* $F \in H_\infty(\mathcal{L}(U,Y))$ *and* $u \in H_2(U)$, *then* $Fu \in H_2(Y)$. *Moreover, the multiplication map* $\Lambda_F : u \mapsto Fu$ *defines a bounded linear operator from* $H_2(U)$ *to* $H_2(Y)$, *and*

$$\|\Lambda_F u\|_{H_2(Y)} \leq \|F\|_\infty \|u\|_{H_2(U)},$$

where $\|\cdot\|_\infty$ *denotes the norm on* $H_\infty(\mathcal{L}(U,Y))$. *In fact,*

$$\|\Lambda_F\| = \sup_{u \neq 0} \frac{\|\Lambda_F u\|_{H_2(Y)}}{\|u\|_{H_2(U)}} = \|F\|_\infty.$$

c. $F \in P_\infty((-\jmath\infty, \jmath\infty); \mathcal{L}(U,Y))$ *is in* $H_\infty(\mathcal{L}(U,Y))$ *if and only if* $\Lambda_F H_2(U) \subset H_2(Y)$.

Proof *a.* See Thomas [26].

b. It is easy to show that for $F \in H_\infty(\mathcal{L}(U,Y))$ the first inequality holds. So $\|\Lambda_F\| \leq \|F\|_\infty$. To prove the other inequality, let $\lambda \in \mathbb{C}_0^+$, $y_0 \in Y$ and $f \in H_2$. Consider

$$\begin{aligned}
\langle f, \Lambda_F^* \frac{y_0}{s+\lambda} \rangle_{H_2} &= \langle \Lambda_F f, \frac{y_0}{s+\lambda} \rangle_{H_2} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle F(\jmath\omega) f(\jmath\omega), \frac{y_0}{\jmath\omega + \lambda} \rangle_Y d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle F(\jmath\omega) f(\jmath\omega), y_0 \rangle_Y \frac{-1}{\jmath\omega - \overline{\lambda}} d\omega \\
&= -\jmath \langle F(\overline{\lambda}) f(\overline{\lambda}), y_0 \rangle_Y \qquad \text{by Cauchy's Theorem A.1.8} \\
&= -\jmath \langle f(\overline{\lambda}), F(\overline{\lambda})^* y_0 \rangle_Y \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle f(\jmath\omega), F(\overline{\lambda})^* y_0 \rangle_Y \frac{-1}{\jmath\omega - \lambda} d\omega \\
&\quad \text{using Cauchy's Theorem A.1.8 again} \\
&= \langle f, F(\overline{\lambda})^* \frac{y_0}{s+\lambda} \rangle_{H_2}.
\end{aligned}$$

Since the above equality holds for every $f \in H_2$, we have that

$$\Lambda_F^* \frac{y_0}{s+\lambda} = F(\overline{\lambda})^* \frac{y_0}{s+\lambda}.$$

This implies that $\|\Lambda_F^*\| \geq \|F^*\|$, and Lemma A.3.60 concludes the proof.

c. See Thomas [26]. ■

The proof of part b was communicated by George Weiss.

These frequency-domain operators have time-domain counterparts that are isometrically isomorphic under the Fourier or Laplace transform; see, for example, Thomas [26, section 5].

Theorem A.6.27. *Suppose that U and Y are separable Hilbert spaces.*

a. *To every $F \in P_\infty((-\jmath\infty, \jmath\infty); \mathcal{L}(U,Y))$ there corresponds a unique shift-invariant, bounded linear operator $\mathcal{F}$ from $L_2((-\infty,\infty);U)$ to $L_2((-\infty,\infty);U)$. Moreover, F and $\mathcal{F}$ are isometrically isomorphic via the Fourier transform*

$$\check{y}(\jmath\omega) = (\check{\mathcal{F}}u)(\jmath\omega) = F(\jmath\omega)\check{u}(\jmath\omega),$$

which holds for every $u \in L_2((-\infty,\infty);U)$.

b. *Moreover, we have in part a that $F \in H_\infty(\mathcal{L}(U,Y))$ if and only if*

$$\mathcal{F} \in \mathcal{L}(L_2([0,\infty);U), L_2([0,\infty);Y)).$$

Theorem A.6.28. *Suppose that Q is a bounded,* shift-invariant, *linear operator from $L_2(-\jmath\infty, \jmath\infty)$ to itself; i.e.,*

$$Q\left(e^{\jmath\theta\cdot} f(\cdot)\right) = e^{\jmath\theta\cdot} Q f(\cdot).$$

Then there exists a function $q \in L_\infty(-\jmath\infty, \jmath\infty)$ such that $(Qf)(\jmath\omega) = q(\jmath\omega) f(\jmath\omega)$.

Proof See theorem 72 of Bochner and Chandrasekharan [3]. ∎

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