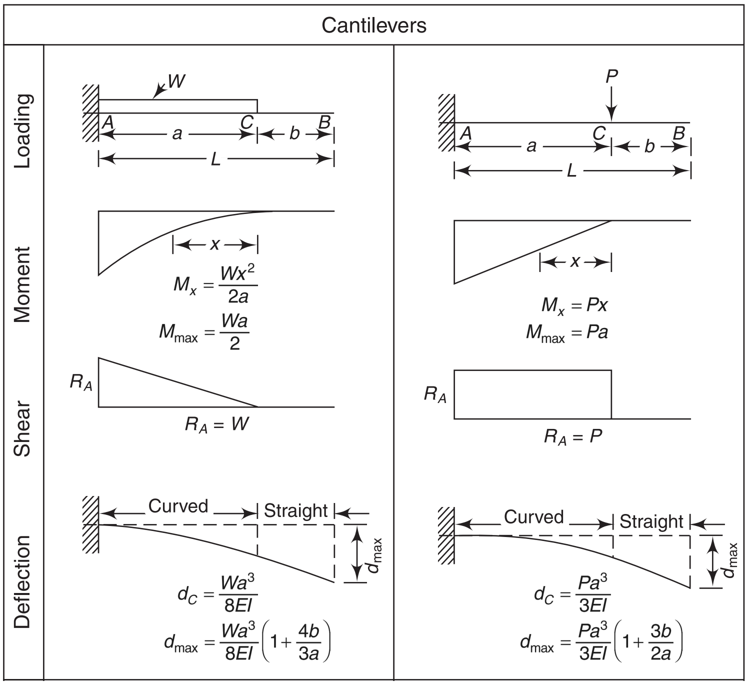
APPENDIX

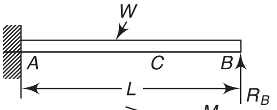
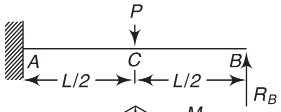
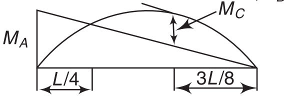
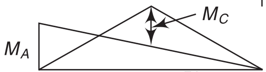
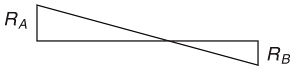
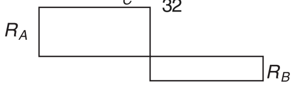
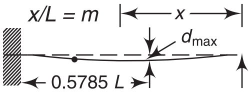
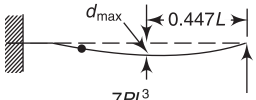
D

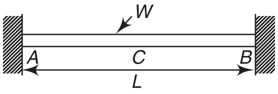
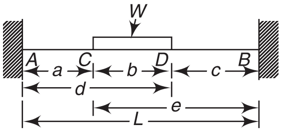
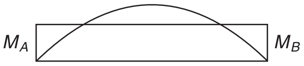
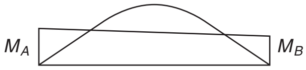
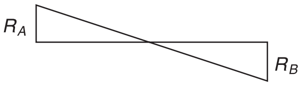
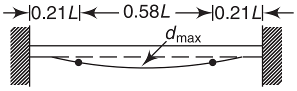
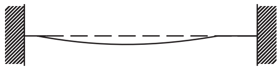
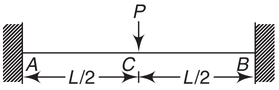
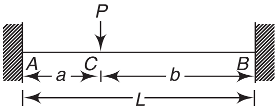
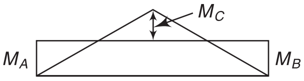
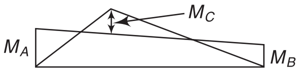
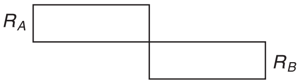
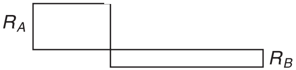
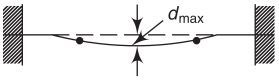
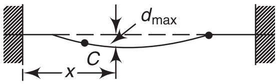
Bending Moment, Shear Force, and Deflection of Beams and Frames

The analysis methods described in Chapter 4 are to be used to find out the bending moments, shear force, and deflection of beams and frames. However, in this Appendix, formulae are provided for calculating the same for some beams and simple frames, which can be used for the design of these structures.

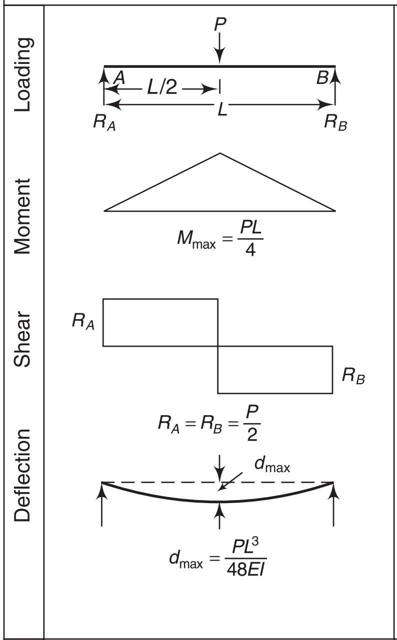


Propped Cantilevers

Loading		
Moment	 $M_A = -\frac{W_L}{8}$ $M_C = \frac{9WL}{128}$	 $M_A = -\frac{3PL}{16}$ $M_C = \frac{5PL}{32}$
Shear	 $R_A = \frac{5W}{8}$ $R_B = \frac{3W}{8}$	 $R_A = 11P/16$ $R_B = 5P/16$
Deflection	 $d = \frac{WL^3}{48EI}(m - 3m^3 + 2m^4)$ $d_{max} = \frac{WL^3}{I85EI}$	 $d_C = \frac{7PL^3}{768\ EI}$ $d_{max} = 0.00932\ \frac{PL^3}{EI}$

	Built-in-beams	
Loading		
Moment	 $M_A = M_B = -\frac{WL}{12}$ $M_C = \frac{WL}{24}$	 $M_A = \frac{-W}{12L^2b}\left[e^3(4L-3e)-c^3(4L-3c)\right]$ $M_B = \frac{-W}{12L^2b}\left[d^3(4L-3d)-a^3(4L-3a)\right]$
Shear	 $R_A = R_B = W/2$	 When r is simple support reaction $R_A = r_A + \frac{M_A - M_B}{L}$ $R_B = r_B + \frac{M_B - M_A}{L}$
Deflection	 $d_{\max} = \frac{WL^3}{384EI}$	 When $a = c$, $d_{\max}$ $= \frac{W}{384EI}(L^3 + 2L^2a + 4La^2 - 8a^3)$
Loading		
Moment	 $-M_A = -M_B = M_C = PL/8$	 $M_A = -\frac{Pab^2}{L^2}$ $M_B = -\frac{Pba^2}{L^2}$ $M_C = \frac{2Pa^2\,b^2}{L^3}$
Shear	 $R_A = R_B = P/2$	 $R_A = P\left(\frac{b}{L}\right)^2\left(1+2\frac{a}{L}\right)$ $R_B = P\left(\frac{a}{L}\right)^2\left(1+2\frac{b}{L}\right)$
Deflection	 $d_{\max} = \frac{PL^3}{192EI}$	 $d_C = \frac{Pa^3\,b^3}{3EIL^3}$ $d_{\max} = \frac{2Pa^2\,b^3}{3EI(3L-2a)^2}$ when $x = \frac{L^2}{3L-2a}$

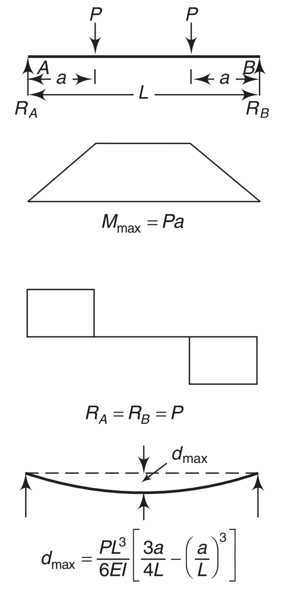
Simply supported beams



$$M_{\max} = \frac{PL}{4}$$

$$R_A = R_B = \frac{P}{2}$$

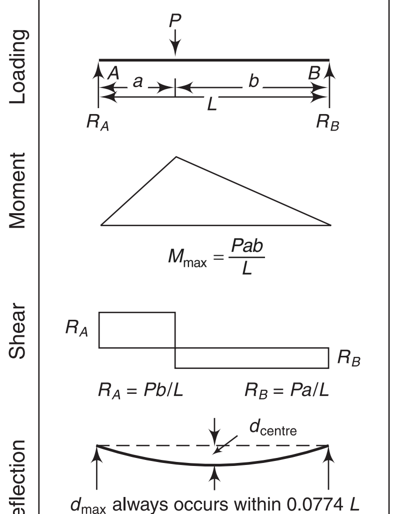
$$d_{\max} = \frac{PL^3}{48EI}$$



$$M_{\max} = Pa$$

$$R_A = R_B = P$$

$$d_{\max} = \frac{PL^3}{6EI}\left[\frac{3a}{4L} - \left(\frac{a}{L}\right)^3\right]$$



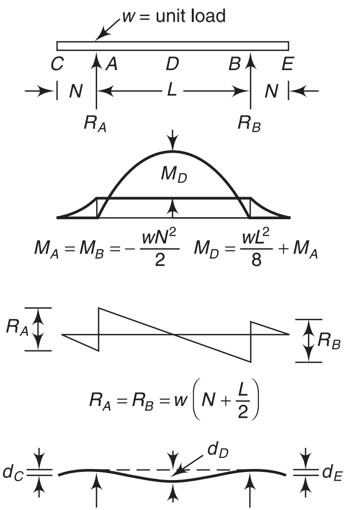
$$M_{\max} = \frac{Pab}{L}$$

$R_A = Pb/L$ $R_B = Pa/L$

$d_{\max}$ always occurs within 0.0774 L of the centre of the beam. When $b \geq a$,

$$d_{\text{centre}} = \frac{PL^3}{48EI}\left[\frac{3a}{L} - 4\left(\frac{a}{L}\right)^3\right]$$

This value is always within 2.5% of the maximum value.



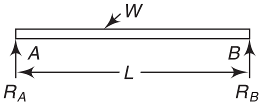
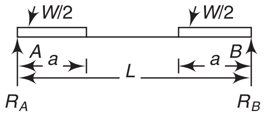
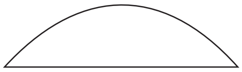
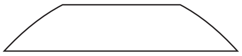
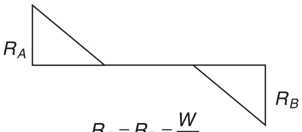
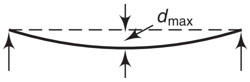
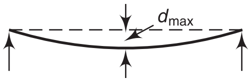
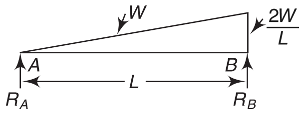
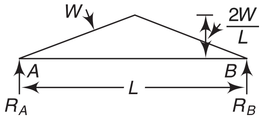
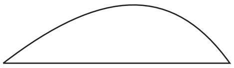
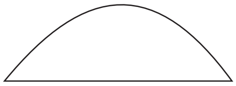
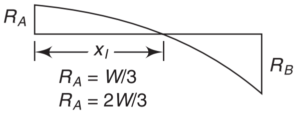
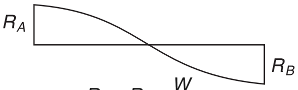
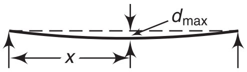
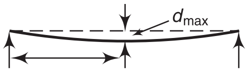
$$M_A = M_B = -\frac{wN^2}{2} \quad M_D = \frac{wL^2}{8} + M_A$$

$$R_A = R_B = w\left(N + \frac{L}{2}\right)$$

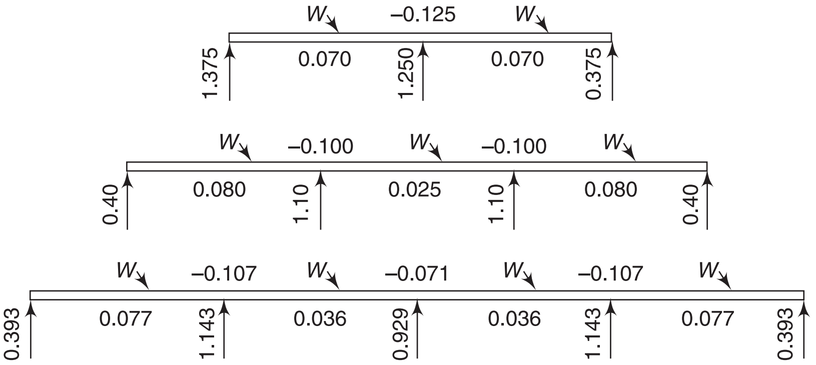
$$d_C = d_E = \frac{wL^3N}{24EI}(1 - 6n^2 - 3n^3)$$

$$d_D = \frac{wL^4}{384EI}(5 - 24n^2)$$

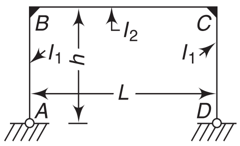
$$\text{where } n = \frac{N}{L}$$

	Simply supported beams	
Loading		
Moment	 $M_x = \frac{Wx}{2}\left(1 - \frac{x}{L}\right)$ $M_{\max} = \frac{WL}{8}$	 $M_{\max} = \frac{Wa}{4}$
Shear	 $R_A = R_B = \frac{W}{2}$	 $R_A = R_B = \frac{W}{2}$
Deflection	 $d_{\max} = \frac{5}{384} \bullet \frac{WL^3}{EI}$	 $d_{\max} = \frac{Wa\,(3L^2 - 2a^2)}{96EI}$
Loading		
Moment	 $M_x = \frac{Wx}{3}\left(1 - \frac{x^2}{L^2}\right)$ $M_{\max} = 0.128WL$ $when\ x_1 = 0.5774L$	 $M_x = Wx\left(\frac{1}{2} - \frac{2x^2}{3L^2}\right)$ $M_{\max} = WL/6$
Shear	 $R_A = W/3$ $R_A = 2W/3$	 $R_A = R_B = \frac{W}{2}$
Deflection	 $d_{\max} = \frac{0.01304WL^3}{EI}$ $when\ x = 0.5193L$	 $d_{\max} = \frac{WL^3}{60EI}$

Equal Span Continuous Beams
Uniformly Distributed Loads (UDL)
Moment = coefficient × $W \times L$
Reaction = coefficient × W
where W is the UDL on one span only and L is one span



Rigid frames

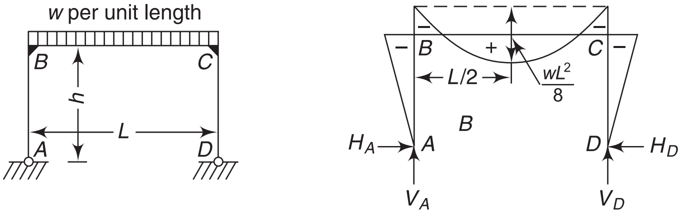


Frame data

Coefficients

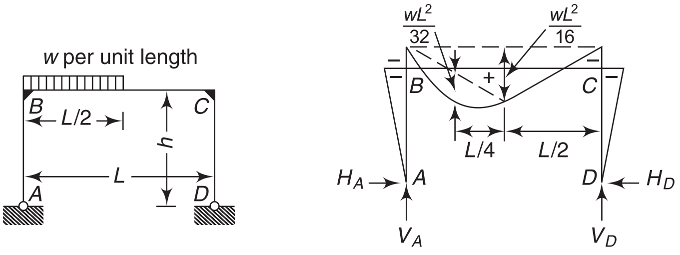
$$k = \frac{I_2}{I_1} \bullet \frac{h}{L}$$

$$N = 2k + 3$$



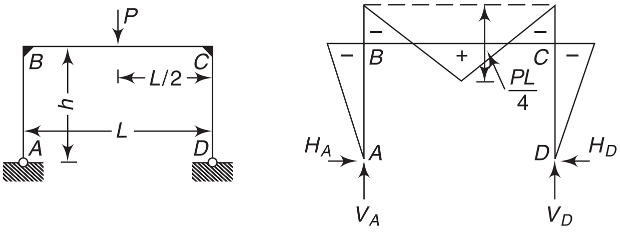
$$M_B = M_C = -\frac{wL^2}{4N} \qquad M_{\max} = \frac{wL^2}{8} + M_B$$

$$V_A = V_D = \frac{wL}{2} \qquad H_A = H_D = -\frac{M_B}{h}$$

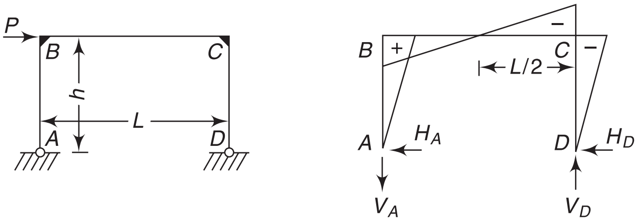


$$M_B = M_C = -\frac{wL^2}{8N}$$

$$V_A = \frac{3wL}{8} \qquad V_D = \frac{wL}{8} \qquad H_A = H_D = -\frac{M_B}{h}$$

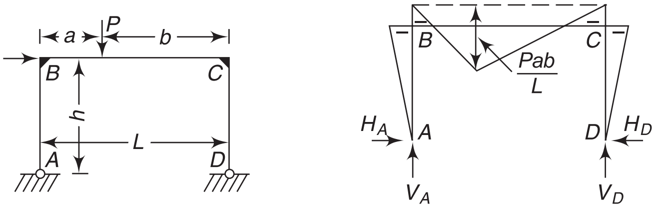


$$M_B = M_C = -\frac{3PL}{8N} \quad V_A = V_D = \frac{P}{2} \quad H_A = H_D = -\frac{M_B}{h}$$



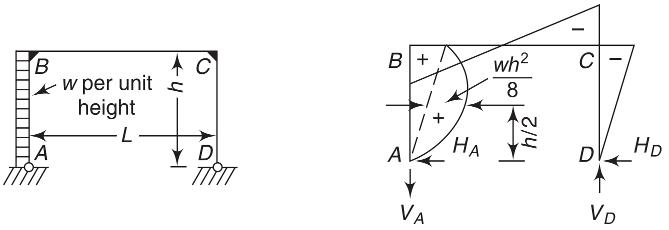
$$M_B = -M_C = +\frac{Ph}{2}$$

$$V_A = -V_D = -\frac{Ph}{L} \quad H_A = -H_D = -\frac{P}{2}$$



$$M_B = M_C = -\frac{Pab}{L} \cdot \frac{3}{2N}$$

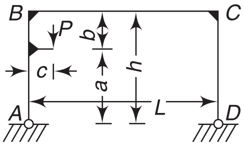
$$V_A = \frac{Pb}{L} \quad V_D = \frac{Pa}{L} \quad H_A = H_D = -\frac{M_B}{h}$$

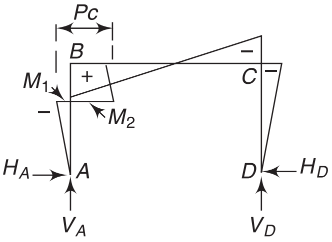


$$M_B = \frac{wh^2}{4}\left[-\frac{k}{2N}+1\right] \qquad H_D = -\frac{M_C}{h}$$

$$M_C = \frac{wh^2}{4}\left[-\frac{k}{2N}-1\right] \qquad H_A = -(wh - H_D)$$

$$V_A = -V_D = -\frac{wh^2}{2L}$$





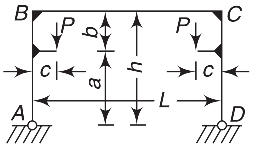
Constant: $a_1 = \dfrac{a}{h}$

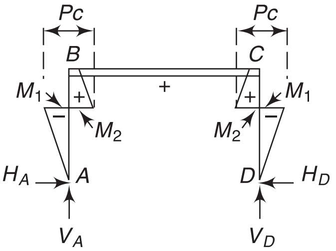
$$M_B = \frac{Pc}{2}\left[\frac{(3a_1^2 - 1)k}{N} + 1\right] \qquad H_A = H_D = -\frac{Mc}{h}$$

$$M_C = \frac{Pc}{2}\left[\frac{(3a_1^2 - 1)k}{N} - 1\right]$$

$$V_D = \frac{Pc}{L} \qquad V_A = P - V_D$$

$$M_1 = -H_A a \quad M_2 = Pc - H_A a$$



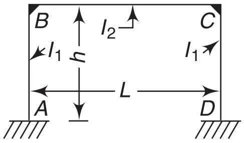


Constant: $a_1 = \dfrac{a}{h}$

$$M_B = M_C = \frac{Pc(3a_1^2 - 1)k}{N}$$

$$H_A = H_D = \frac{Pc - M_B}{h} \qquad V_A = V_D = P$$

$$M_1 = -H_A a \qquad M_2 = Pc - H_A a$$

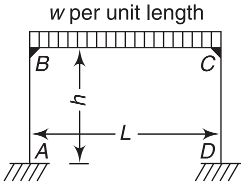


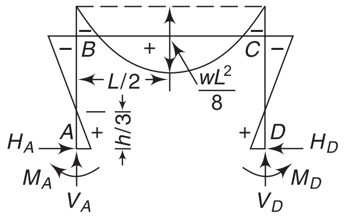
Frame data

Coefficients:

$$k = \frac{I_2}{I_1} \cdot \frac{h}{L}$$

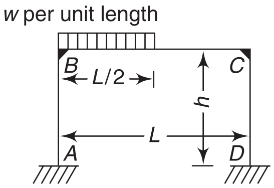
$$N_1 = k + 2 \quad N_2 = 6k + 1$$

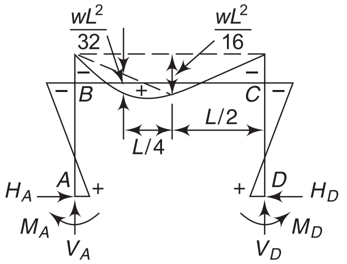




$$M_A = M_D = \frac{wL^2}{12N_1} \quad M_B = M_C = -\frac{wL^2}{6N_1} = -2M_A$$

$$M_{\max} = \frac{wL^2}{8} + M_B \quad V_A = V_D = \frac{wL}{2} \quad H_A = H_D = \frac{3M_A}{h}$$

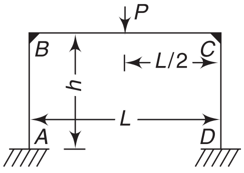


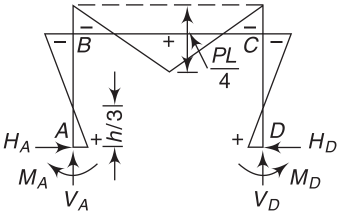


$$M_A = \frac{wL^2}{8}\left[\frac{1}{3N_1} - \frac{1}{8N^2}\right] \quad M_B = -\frac{wL^2}{8}\left[\frac{2}{3N_1} + \frac{1}{8N_2}\right]$$

$$M_D = \frac{wL^2}{8}\left[\frac{1}{3N_1} + \frac{1}{8N_2}\right] \quad M_C = -\frac{wL^2}{8}\left[\frac{2}{3N_1} - \frac{1}{8N_2}\right]$$

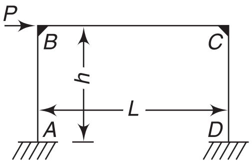
$$V_D = \frac{wL}{8}\left[1 - \frac{1}{4N_2}\right] \quad V_A = \frac{wL}{2} - V_D \quad H_A = H_D = \frac{wL^2}{8hN_1}$$

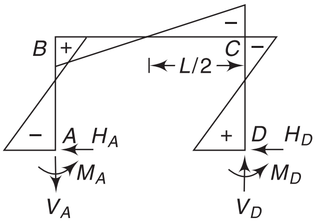




$$M_A = M_D = +\frac{PL}{8N_1} \quad M_B = M_C = -2M_A$$

$$V_A = V_D = \frac{P}{2} \quad H_A = H_D = \frac{3M_A}{h}$$

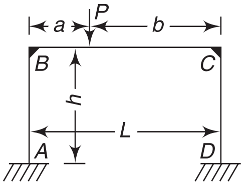


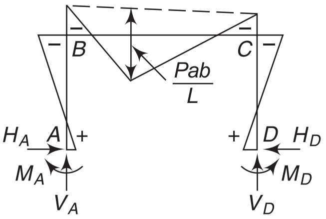


$$M_A = -\frac{Ph}{2} \bullet \frac{3k+1}{N_2} \qquad M_B = +\frac{Ph}{2} \bullet \frac{3k}{N_2}$$

$$M_D = +\frac{Ph}{2} \bullet \frac{3k+1}{N_2} \qquad M_C = -\frac{Ph}{2} \bullet \frac{3k}{N_2}$$

$$H_A = -H_D = -\frac{P}{2} \qquad V_A = -V_D = -\frac{2M_B}{L}$$



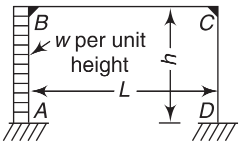


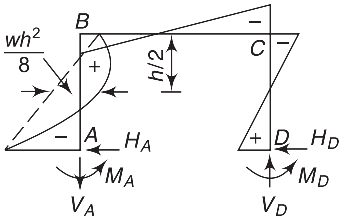
Constants: $a_1 = a/L$ $\qquad b_1 = b/L$

$$M_A = +\frac{Pab}{L}\left[\frac{1}{2N_1} - \frac{b_1 - a_1}{2N_2}\right] \qquad M = -\frac{Pab}{L}\left[\frac{1}{N_1} + \frac{b_1 - a_1}{2N_2}\right]$$

$$M_D = +\frac{Pab}{L}\left[\frac{1}{2N_1} + \frac{b_1 - a_1}{2N_2}\right] \qquad M_C = -\frac{Pab}{L}\left[\frac{1}{N_1} - \frac{b_1 - a_1}{2N_2}\right]$$

$$V_A = Pb_1\left[1 + \frac{a_1(b_1 - a_1)}{N_2}\right] \qquad V_D = P - V_A \qquad H_A = H_D = \frac{3Pab}{2LhN_1}$$

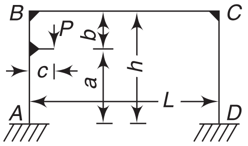


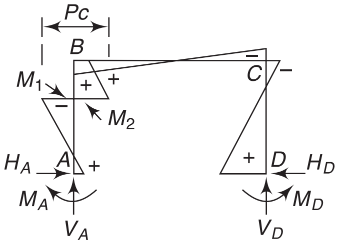


$$M_A = \frac{wh^2}{4}\left[-\frac{k+3}{6N_1} - \frac{4k+1}{N_2}\right] \qquad M_B = \frac{wh^2}{4}\left[-\frac{k}{6N_1} + \frac{2k}{N_2}\right]$$

$$M_D = \frac{wh^2}{4}\left[-\frac{k+3}{6N_1} + \frac{4k+1}{N_2}\right] \qquad M_C = \frac{wh^2}{4}\left[-\frac{k}{6N_1} - \frac{2k}{N_2}\right]$$

$$H_D = \frac{wh(2k+3)}{8N_1} \qquad H_A = -(wh - H_D) \qquad V_A = -V_D = -\frac{wh^2 k}{LN_2}$$





$$\text{Constants: } a_1 = \frac{a}{h} \quad b_1 = \frac{b}{h}$$

$$X_1 = \frac{Pc}{2N_1}[1 + 2b_1k - 3b_1^2(k+1)] \qquad X_2 = \frac{Pcka_1(3a_1 - 2)}{2N_1}$$

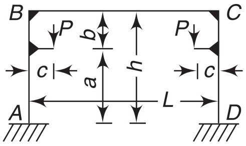
$$X_3 = \frac{3Pcka_1}{N_2}$$

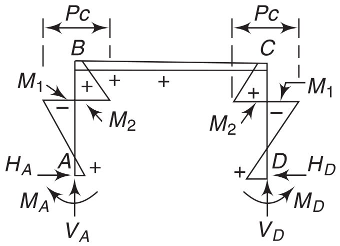
$$M_A = +X_1 - \left(\frac{Pc}{2} - X_3\right) \qquad M_B = +X_2 + X_3$$

$$M_D = +X_1 + \left(\frac{Pc}{2} - X_3\right) \qquad M_C = +X_2 - X_3$$

$$H_A = H_D = \frac{Pc}{2h} + \frac{X_1 - X_2}{h} \qquad V_D = \frac{2X_3}{L} \qquad V_A = P - V_D$$

$$M_1 = M_A - H_Aa \qquad M_2 = M_B + H_Db$$





$$\text{Constants: } a_1 = \frac{a}{h} \qquad b_1 = \frac{b}{h}$$

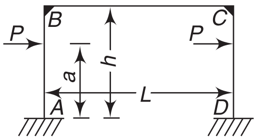
$$X_1 = \frac{Pc}{2N_1}[1 + 2b_1k - 3b_1^2(k+1)] \qquad X_2 = \frac{Pcka_1(3a_1 - 2)}{2N_1}$$

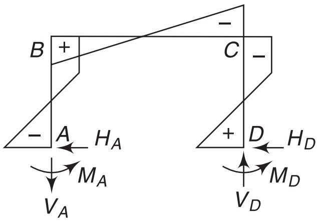
$$M_A = M_D = \frac{Pc}{N_1}[1 + 2b_1k - 3b_1^2(k+1)] = 2X_1$$

$$M_B = M_C = \frac{Pcka_1(3a_1 - 2)}{N_1} = 2X_2$$

$$V_A = V_D = P \qquad H_A = H_D = \frac{Pc + M_A - M_B}{h}$$

$$M_1 = M_A - H_Aa \qquad M_2 = M_B + H_Db$$



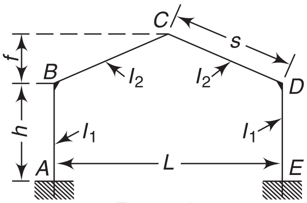


Constants: $a_1 = \dfrac{a}{h}$ $\qquad X_1 = \dfrac{3Paa_1k}{N_2}$

$M_A = -Pa + X_1 \qquad M_B = X_1$

$M_D = +\,Pa - X_1 \qquad M_C = -X_1$

$V_A = -V_D = -\dfrac{2X_1}{L} \qquad H_A = -H_D = -P$



Frame data

Coefficients:

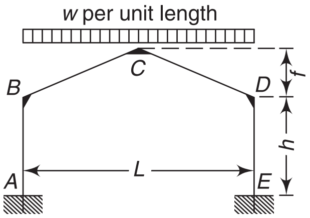
$k = \dfrac{I_2}{I_1} \cdot \dfrac{h}{s} \qquad \phi = \dfrac{f}{h}$

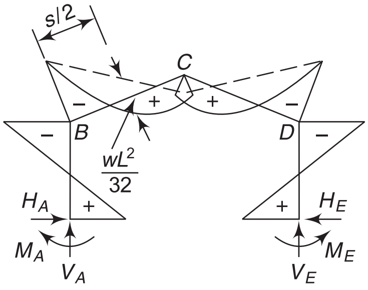
$m = 1 + \phi$

$B = 3k + 2 \qquad C = 1 + 2m$

$K_1 = 2(k + 1 + m + m^2) \qquad K_2 = 2(k + \phi^2)$

$R = \phi C - k \qquad N_1 = K_1K_2 - R^2 \qquad N_2 = 3k + B$



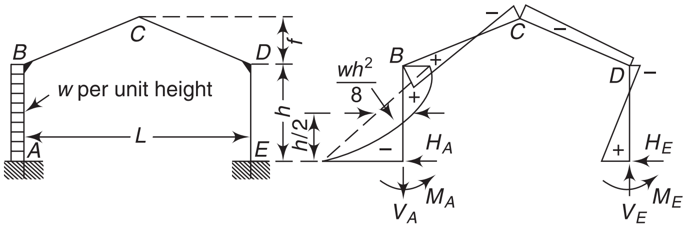


$$M_A = M_E = \frac{wL^2}{16} \cdot \frac{k(8 + 15\phi) + \phi(6 - \phi)}{N_1}$$

$$M_B = M_D = -\frac{wL^2}{16} \cdot \frac{k(16 + 15\phi) + \phi^2}{N_1}$$

$$M_C = \frac{wL^2}{8} - \phi M_A + mM_B$$

$$V_A = V_E = \frac{wL}{2} \qquad H_A = H_E = \frac{M_A - M_B}{h}$$



$$\text{Constants: } X_1 = \frac{wh^2}{8} \bullet \frac{k(k+6) + k\phi(15+16\phi) + 6\phi^2}{N_1}$$

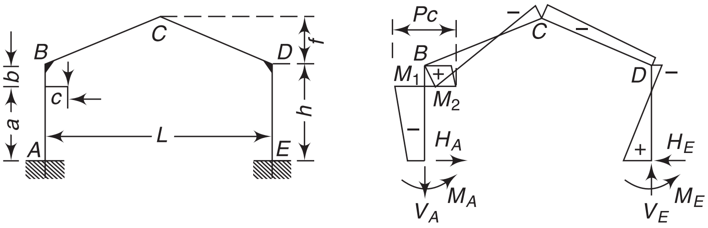
$$X_2 = \frac{wh^2 k(9\phi + 8\phi^2 - k)}{8N_1} \qquad X_3 = \frac{wh^2(2k+1)}{2N_2}$$

$$M_A = -X_1 - X_3 \qquad M_B = +X_2 + \left(\frac{wh^2}{4} - X_3\right)$$

$$M_E = -X_1 + X_3 \qquad M_D = +X_2 - \left(\frac{wh^2}{4} - X_3\right)$$

$$M_C = -\frac{whf}{4} + \phi X_1 + mX_2$$

$$V_A = -V_E = -\frac{wh^2}{2L} + \frac{2X_3}{L} \qquad H_E = \frac{wh}{4} - \frac{X_1 + X_2}{h} \qquad H_A = -(wh - H_E)$$



$$\text{Constants: } a_1 = \frac{a}{h} \qquad b_1 = \frac{b}{h}$$

$$Y_1 = Pc[2\phi^2 - (1 - 3b_1^2)k] \qquad Y_2 = Pc[\phi c - (3a_1^2 - 1)k]$$

$$X_1 = \frac{Y_1 K_1 - Y_2 R}{2N_1} \qquad X_2 = \frac{Y_2 K_2 - Y_1 R}{2N_1} \qquad X_3 = \frac{Pc}{2} \bullet \frac{b - 3(a_1 - b_1)k}{N_2}$$

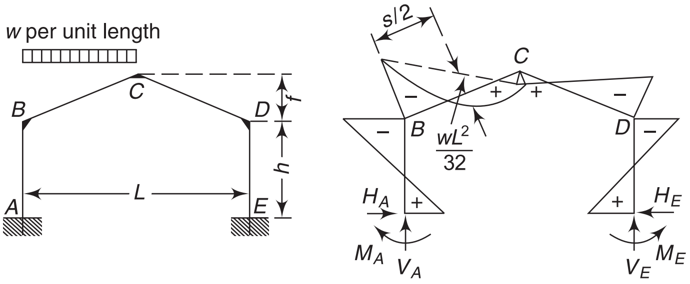
$$M_A = -X_1 - X_3 \qquad M_B = +X_2 + \left(\frac{Pc}{2} - X_3\right)$$

$$M_E = -X_1 + X_3 \qquad M_D = +X_2 - \left(\frac{Pc}{2} - X_3\right)$$

$$M_C = -\frac{\phi Pc}{4} + \phi X_1 + mX_2$$

$$M_1 = M_A - H_A a \qquad M_2 = M_B + H_E b$$

$$V_E = \frac{Pc - 2X_3}{L} \qquad V_A = P - V_E \qquad H_A = H_E = \frac{Pc}{2h} - \frac{X_1 + X_2}{h}$$



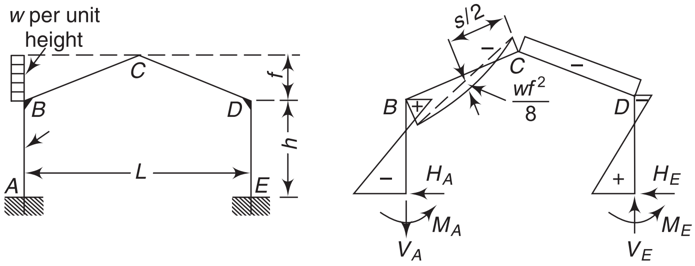
$$\text{Constants: } X_1 = \frac{wL^2}{32} \bullet \frac{k\,(8 + 15\,\phi) + \phi\,(6 - \phi)}{N_1}$$

$$X_2 = \frac{wL^2}{32} \bullet \frac{k\,(16 + 15\,\phi) + \phi^2}{N_1} \qquad X_3 = \frac{wL^2}{32\,N_2}$$

$$M_A = +X_1 - X_3 \qquad M_B = -X_2 - X_3 \qquad M_E = +X_1 + X_3 \qquad M_D = -X_2 + X_3$$

$$M_C = \frac{wL^2}{16} - \phi X_1 - mX_2$$

$$V_E = \frac{wL}{8} - \frac{2\,X_3}{L} \qquad V_A = \frac{wL}{2} - V_E \qquad H_A = H_E = \frac{X_1 + X_2}{h}$$



$$\text{Constants: } X_1 = \frac{wf^2}{8} \cdot \frac{k\,(9\,\phi + 4) + \phi\,(6 + \phi)}{N_1}$$

$$X_2 = \frac{wf^2}{8} \bullet \frac{k\,(8 + 9\,\phi) - \phi^2}{N_1} \qquad X_3 = \frac{wfh}{8} \bullet \frac{4\,b + \phi}{N_2}$$

$$M_A = -X_1 - X_3 \qquad M_B = +X_2 + \left(\frac{wfh}{2} - X_3\right)$$

$$M_E = -X_1 + X_3 \qquad M_D = +X_2 - \left(\frac{wfh}{2} - X_3\right)$$

$$M_C = -\frac{wf^2}{4} + \phi X_1 + mX_2$$

$$V_A = -V_E = -\frac{wfh\,(2 + \phi)}{2\,L} + \frac{2\,X_3}{L} \qquad H_E = \frac{wf}{2} - \frac{X_1 + X_2}{h} \qquad H_A = -(wf - H_E)$$

Source: Kleinlogel, A., *Rahmenformeln*, 11th edn, Verlag von Wilhelm Ernst & Sohn., Berlin, 1949.